Analyse Numérique/*Numerical Analysis*

Approximation of the unilateral contact problem by the mortar finite element method.

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Abstract — The purpose of this note is to extend the mortar finite element method to handle the unilateral contact model between two deformable bodies. The corresponding variational inequality is approximated using finite elements with meshes which do not fit on the contact zone. The mortar technique allows to match (independent) discretizations within each solid and to express in a satisfying way the contact conditions. Then, we realize a numerical analysis of the algorithm and, using a bootstrapping argument, we give an upper bound of the convergence rate similar to that already obtained for compatible grids.

Approximation du problème de contact unilatéral par la méthode des éléments finis avec joints.

Résumé — Dans cette note on étend la méthode des éléments finis avec joints à l'inéquation variationnelle du contact unilatéral entre deux solides élastiques déformables. On effectue l'analyse numérique de la méthode et on donne un résultat de convergence.

Version française abrégée (Les numéros d'équations renvoient à la version anglaise.)

La méthode de décomposition de domaines des éléments avec joints, appliquée aux équations variationnelles du second ordre (cf. [3]), s'est avérée performante car elle permet de raccorder des maillages avec les avantages pratiques qu'on peut imaginer, en particulier la possibilité de générer des maillages globalement non structurés/localement structurés. Dans la présente note, on tente d'élargir le champ d'application de cette technique à une inéquation variationnelle provenant de la mécanique de contact entre deux solides déformables.

Les deux corps occupent, dans la configuration initiale, deux domaines bornés Ω^1 et Ω^2 dans $\mathbb{R}^2$ et sont initialement en contact sur une partie commune de leur frontière $\Gamma_c = \Gamma_c^1 = \Gamma_c^2$. On notera $\boldsymbol{n}^\ell$ le vecteur normal unitaire sortant de Ω^ℓ. Le problème modèle est explicité dans (1)–(4). La formulation faible débouche sur une inéquation variationnelle du type (6).

Pour calculer une solution approchée par éléments finis, on suppose que chacun des deux solides Ω^ℓ (supposé polygonal pour simplifier) est maillé en une famille de triangulations régulières $\mathscr{T}_h^\ell$ d'éléments dont le diamètre n'excède pas h_ℓ. Le paramètre de la discrétisation est le couple $h = (h_1, h_2)$, destiné à tendre vers zéro. Sur chaque Ω^ℓ on construit l'espace $\boldsymbol{V}_h^\ell$ de type éléments finis de degré 1 définis en (7). Sachant que les deux maillages $\mathscr{T}_h^\ell$ ne coïncident pas sur Γ_c, le point le plus délicat consiste à exprimer l'analogue discret de la condition de contact $\boldsymbol{v}^1.\boldsymbol{n}^1 + \boldsymbol{v}^2.\boldsymbol{n}^2 \le 0$ sur Γ_c, condition incorporée dans la définition du convexe $\boldsymbol{K}$ des déplacements admissibles défini en (5). Et ce, de façon à obtenir un taux de convergence le meilleur possible. Comme pour les équations variationnelles (en particulier pour le contact bilatéral), le désanvantage d'une traduction point par point de cette condition nécessite le recours à un opérateur de projection π_h^1 (cf. (8)) introduit dans [3]. Dans ce cas, le convexe approché $\boldsymbol{K}_h$ des déplacements admissibles est formé des champs $\boldsymbol{v}_h = (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) \in \boldsymbol{V}_h^1 \times \boldsymbol{V}_h^2$, satisfaisant la condition de contact $\boldsymbol{v}_h^1.\boldsymbol{n}^1 + \pi_h^1(\boldsymbol{v}_h^2.\boldsymbol{n}^2) \le 0$ sur Γ_c, ce qui conduit au problème discret (10). L'analyse numérique de l'algorithme des éléments avec joints est réalisé au moyen d'une adaptation du lemme de Falk qui joue un rôle comparable au deuxième lemme de Strang pour le problème de contact bilatéral (cf. [2]). Une majoration de l'erreur $(\boldsymbol{u} - \boldsymbol{u}_h)$ commise sur la solution

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exacte est la somme des contributions des erreurs de la meilleure approximation ainsi que de l'erreur de consistance (due à l'incompatibilité des maillages). L'étude des ces différentes contributions montre, avec le recours à un argument de "bootstrap", que le taux de convergence est au pire de l'ordre de $(h_1^{\frac{3}{4}} + h_2)$. Le fait de considérer des maillages incompatibles n'altère pas la vitesse de convergence qui est identique à celle obtenue dans le cas de maillages compatibles (cf. [5]). Dans le cas "symétrique", où le raccord est décrit par l'opérateur π_h^2, l'erreur diminue au moins comme $(h_1 + h_2^{\frac{3}{4}})$.

1. Introduction

In structural mechanics the contact models between rigid and/or deformable bodies are of high interest. Developing numerical algorithms that take into account in an efficient way the contact constraints is of current concern. The mortar element domain decomposition method seems naturally fitted to such problems. Indeed, this technique offers a great facility for coupling different variational approximations and for using meshes that do not match at the interfaces of the subdomains (see e.g. [3]). This allows to make independant discretizations within each body that are well adapted to their own particularities (geometry, constitutive equations, etc ...). So far the method was applied to partial differential equations expressed under weak formulation. When translated under a weak formulation, the unilateral contact model without friction leads to a variational inequality. In the present note, we intend to apply the mortar finite element concept to approximate such a problem. We need an adaptation of Falk's lemma for the numerical analysis of the error committed on the exact solution. Finally, we prove the convergence of the mortar procedure and, using a bootstrap argument, we provide a convergence rate similar to that already obtained for compatible grids (see [5]).

2. The unilateral contact problem and the mortar finite element approximation

Let us consider two elastic bodies occupying, in the initial unconstrained configuration, two bounded domains Ω^ℓ, $\ell = 1, 2$, of the space $\mathbb{R}^2$. For $\ell = 1, 2$, the boundary $\Gamma^\ell = \partial\Omega^\ell$ is assumed to be "smooth" and is the union of three nonoverlapping given portions $\Gamma^\ell_{\boldsymbol{u}}$, $\Gamma^\ell_{\boldsymbol{g}}$ and Γ^ℓ_c. The two bodies have as contact region $\Gamma_c = \Gamma^1_c = \Gamma^2_c$. The measure of $\Gamma^\ell_{\boldsymbol{u}}$ does not vanish and the outward unit normal vector on $\partial\Omega^\ell$ is $\boldsymbol{n}^\ell$. Each of the bodies is subjected to body forces $\boldsymbol{f}^\ell \in (L^2(\Omega^\ell))^2$ and to surface forces $\boldsymbol{g}^\ell \in (L^2(\Gamma^\ell_{\boldsymbol{g}}))^2$ on $\Gamma^\ell_{\boldsymbol{g}}$. The unilateral contact problem consists of finding the displacements fields $\boldsymbol{u} = (\boldsymbol{u}^\ell)_\ell$, and stress tensors fields $(\sigma^\ell(\boldsymbol{u}^\ell))_\ell$ satisfying the following equations for $\ell = 1, 2$:

$$\mathbf{div}\, \sigma^\ell(\boldsymbol{u}^\ell) + \boldsymbol{f}^\ell = 0 \quad \text{in } \Omega^\ell, \qquad \sigma^\ell(\boldsymbol{u}^\ell)\boldsymbol{n}^\ell = \boldsymbol{g}^\ell \quad \text{on } \Gamma^\ell_{\boldsymbol{g}}, \qquad \boldsymbol{u}^\ell = 0 \quad \text{on } \Gamma^\ell_{\boldsymbol{u}}, \tag{1}$$

where the symbol $\mathbf{div}$ denotes the divergence operator and is defined by $\mathbf{div}\ \sigma = \left(\frac{\partial \sigma_{ij}}{\partial x_j}\right)_i$. The sum convention of the repeated indices is adopted. The stress tensor is linked to the displacement by the constitutive law $\sigma^\ell(\boldsymbol{u}^\ell) = A^\ell\, \varepsilon(\boldsymbol{u}^\ell)$. The symbol $\varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ stands for the linearized strain and $A^\ell = (a^\ell_{ij,kh})_{1\le i,j,k,h\le 2} \in (L^\infty(\Omega^\ell))^{16}$ is a symmetrical and elliptic fourth order tensor. The contact conditions on Γ_c are:

$$(\sigma^1(\boldsymbol{u}^1)\boldsymbol{n}^1).\boldsymbol{n}^1 = (\sigma^2(\boldsymbol{u}^2)\boldsymbol{n}^2).\boldsymbol{n}^2 = \sigma_{\boldsymbol{n}}(\boldsymbol{u}), \tag{2}$$

$$[\boldsymbol{u}.\boldsymbol{n}] \le 0, \qquad \sigma_{\boldsymbol{n}}(\boldsymbol{u}) \le 0, \qquad \sigma_{\boldsymbol{n}}(\boldsymbol{u})[\boldsymbol{u}.\boldsymbol{n}] = 0, \tag{3}$$

$$\sigma^1_{\boldsymbol{t}}(\boldsymbol{u}^1) = \sigma^2_{\boldsymbol{t}}(\boldsymbol{u}^2) = 0, \tag{4}$$

The notation $[\boldsymbol{u}.\boldsymbol{n}]$ represents the jump $(\boldsymbol{u}^1.\boldsymbol{n}^1 + \boldsymbol{u}^2.\boldsymbol{n}^2)$ of the normal displacement on Γ_c and $\sigma^\ell_{\boldsymbol{t}}(\boldsymbol{u}^\ell) = \sigma^\ell(\boldsymbol{u}^\ell)\boldsymbol{n}^\ell - \sigma_{\boldsymbol{n}}(\boldsymbol{u})\boldsymbol{n}^\ell$. Conditions (3) mean that there is unilateral contact between the solids, they allow the

two bodies to leave each other on a portion of the contact zone Γ_c. While (2) expresses the action and the reaction principle and finally (4) represents a contact without friction.
We define the spaces $\boldsymbol{V}^\ell = \{\boldsymbol{v}^\ell \in \Big(H^1(\Omega^\ell)\Big)^2,\ \boldsymbol{v}^\ell = 0 \text{ on } \Gamma^\ell_{\boldsymbol{u}}\},\ \ell = 1,2$. The generating vector field of the product space $\boldsymbol{V}^1 \times \boldsymbol{V}^2$ is denoted $\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2)$. This space is endowed with the Hilbertian broken norm $\|\boldsymbol{v}\|_* = \Big(\|\boldsymbol{v}^1\|^2_{(H^1(\Omega^1))^2} + \|\boldsymbol{v}^2\|^2_{(H^1(\Omega^2))^2}\Big)^{\frac{1}{2}}$. Then, the appropriate closed convex set $\boldsymbol{K}$ where we look for the solutions is contained in $\boldsymbol{V}^1 \times \boldsymbol{V}^2$ and incorporates the contact condition

$$\boldsymbol{K} = \{\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \boldsymbol{V}^1 \times \boldsymbol{V}^2, \qquad [\boldsymbol{v}.\boldsymbol{n}] \le 0 \text{ on } \Gamma_c\}. \tag{5}$$

The variational formulation of problem (1)–(4) is: *find $\boldsymbol{u} \in \boldsymbol{K}$ such that:*

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \ge L(\boldsymbol{v} - \boldsymbol{u}), \qquad \forall \boldsymbol{v} \in \boldsymbol{K}. \tag{6}$$

In (6) we set, for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}^1 \times \boldsymbol{V}^2$:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \sum_{\ell=1}^{2} \int_{\Omega^\ell} A^\ell\, \varepsilon(\boldsymbol{u}^\ell).\, \varepsilon(\boldsymbol{v}^\ell)\, d\Omega^\ell, \quad \text{and} \quad L(\boldsymbol{v}) = \sum_{\ell=1}^{2} \Big(\int_{\Omega^\ell} \boldsymbol{f}^\ell.\boldsymbol{v}^\ell\, d\Omega^\ell + \int_{\Gamma^\ell_g} \boldsymbol{g}^\ell.\boldsymbol{v}^\ell\, d\Gamma^\ell\Big).$$

According to Stampacchia's theorem, problem (6) has a unique solution in $\boldsymbol{K}$.

We intend to build up a convex $\boldsymbol{K}_h$ which will be a "good" finite element approximation of $\boldsymbol{K}$. We assume, for simplicity, that Ω^1 and Ω^2 are polygonally shaped (otherwise we use curved finite elements). To each subdomain Ω^ℓ is associated a regular family of triangulations $\mathscr{T}^\ell_h$, made of triangular elements κ, the diameter of which does not exceed h_ℓ, and, let $h = (h_1, h_2)$ be the given parameters that are expected to decay to 0. For every κ, $\mathbb{P}_q(\kappa)$ is the space of the polynomials on κ whose global degree is $\le q$. The finite element space set in each subdomain Ω^ℓ is

$$\boldsymbol{V}^\ell_h = \{\boldsymbol{v}^\ell_h \in \mathscr{C}(\overline{\Omega}^\ell)^2, \quad \boldsymbol{v}^\ell_h|_\kappa \in (\mathbb{P}_1(\kappa))^2 \quad \forall \kappa \in \mathscr{T}^\ell_h, \quad \boldsymbol{v}^\ell_h|_{\Gamma^\ell_{\boldsymbol{u}}} = 0\ \}. \tag{7}$$

The contact zone Γ_c inherits two regular families of independent (1D)-meshes denoted $\mathcal{T}^\ell_h$. We shall assume that these families are uniformly regular so that the inverse inequalities in Sobolev spaces are available (see [4]). Moreover, for technical needs, we consider the scale of (h_1, h_2) so that $\frac{h_1}{h_2}$ is bounded. We denote by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ the end points of Γ_c and we suppose them to be nodes of both meshes $\mathscr{T}^\ell_h$. To express the contact constraints, we need to introduce some functional spaces over Γ_c. Let $W^\ell_h(\Gamma_c)$ be the range of $\boldsymbol{V}^\ell_h$ by the normal traces operator on Γ_c, it is the mortar space

$$W^\ell_h(\Gamma_c) = \{\varphi_h = \boldsymbol{v}^\ell_h|_{\Gamma_c}.\boldsymbol{n}^\ell, \qquad \boldsymbol{v}^\ell_h \in \boldsymbol{V}^\ell_h\ \},$$

which coincides with the space of the continuous and piecewise linear functions on $\mathcal{T}^\ell_h$. We introduce the space of the Lagrange multipliers

$$M^\ell_h(\Gamma_c) = \{q_h \in W^\ell_h(\Gamma_c), \quad q^\ell_h|_T \in \mathbb{P}_0(T), \quad \text{if } \boldsymbol{a}_1 \text{ or } \boldsymbol{a}_2 \in T\}.$$

Next, π^ℓ_h stands for the projection operator on $W^\ell_h(\Gamma_c)$ defined for any function $\varphi \in \mathscr{C}(\overline{\Gamma}_c)$ by

$$\begin{aligned} (\pi^\ell_h\varphi)(\boldsymbol{a}_i) &= \varphi(\boldsymbol{a}_i) && \text{for } i = 1 \text{ and } 2,\\ \int_{\Gamma_c} (\varphi - \pi^\ell_h\varphi) q_h\, d\Gamma &= 0 && \forall q_h \in M^\ell_h(\Gamma_c). \end{aligned} \tag{8}$$

The properties of π_h^ℓ are enumerated in [1] and we just recall the following

LEMMA 1. – *Let* μ *such that* $\frac{1}{2} < \mu \leq 2$; *then, for all* $\varphi \in H^\mu(\Gamma_c)$

$$\|\varphi - \pi_h^\ell \varphi\|_{H_{00}^{\frac{1}{2}}(\Gamma_c)} \leq C h_\ell^{\mu-1/2} \|\varphi\|_{H^\mu(\Gamma_c)}. \tag{9}$$

We are in a position to define the discrete convex $\boldsymbol{K}_h$ approximating $\boldsymbol{K}$,

$$\boldsymbol{K}_h = \big\{ \boldsymbol{v}_h = (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) \in \boldsymbol{V}_h^1 \times \boldsymbol{V}_h^2, \qquad \boldsymbol{v}_h^1.\boldsymbol{n}^1 + \pi_h^1(\boldsymbol{v}_h^2.\boldsymbol{n}^2) \leq 0 \text{ on } \Gamma_c \big\}.$$

In this case, $W_h^2(\Gamma_c)$ stands for the mortar space. Of course, it is possible to define the symmetrical convex by taking as mortar space $W_h^1(\Gamma_c)$ and using the projection π_h^2. Besides, it is straightforward that $\boldsymbol{K}_h \not\subset \boldsymbol{K}$ unless the meshes are compatible: the approximation is then not "Hodge" conforming. The finite element problem issued from (6) is: *find* $\boldsymbol{u}_h \in \boldsymbol{K}_h$ *such that:*

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) \geq L(\boldsymbol{v}_h - \boldsymbol{u}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{K}_h. \tag{10}$$

Using again Stamppachia's theorem, we conclude to the existence and uniqueness of the solution $\boldsymbol{u}_h \in \boldsymbol{K}_h$ satisfying the stability condition $\|\boldsymbol{u}_h\|_* \leq C \sum_{\ell=1}^2 \Big(\|\boldsymbol{f}^\ell\|_{(L^2(\Omega^\ell))^2} + \|\boldsymbol{g}^\ell\|_{(L^2(\Gamma_g^\ell))^2} \Big)$.

3. Error estimation

We give the basic tool, an adaptation of Falk's lemma, that allows to issue an upper bound of the error. The proof of such a result may be found in [2].

LEMMA 2. – *Let* $\boldsymbol{u} \in \boldsymbol{K}$ *be the solution of the exact problem* (6) *and* $\boldsymbol{u}_h \in \boldsymbol{K}_h$ *be the solution of the discrete one* (10). *Then there exists a constant* C *independent of* h *such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_*^2 \leq C \Big\{ \inf_{\boldsymbol{v}_h \in \boldsymbol{K}_h} \Big(\|\boldsymbol{u} - \boldsymbol{v}_h\|_*^2 + \int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\boldsymbol{u})[(\boldsymbol{v}_h - \boldsymbol{u}).\boldsymbol{n}] \, d\Gamma \Big) + \inf_{\boldsymbol{v} \in \boldsymbol{K}} \int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\boldsymbol{u})[(\boldsymbol{v} - \boldsymbol{u}_h).\boldsymbol{n}] \, d\Gamma \Big\}.$$

We recognize in the first term, where the infimum is taken on $\boldsymbol{K}_h$, the best approximation error. The boundary integral is due to the nature of the problem and does not disappear even when $\boldsymbol{K}_h \subset \boldsymbol{K}$. The second infimum (on $\boldsymbol{K}$) is the consistency error and is generated by the non conformity. We begin by providing an upper bound of the approximation error. It is obtained by using the estimate (9) on π_h^1.

LEMMA 3. – *Assume that* $(\boldsymbol{u}|_{\Omega^1}, \boldsymbol{u}|_{\Omega^2}) \in (H^2(\Omega^1))^2 \times (H^2(\Omega^2))^2$, *then there exists* $\boldsymbol{v}_h \in \boldsymbol{K}_h$ *such that*

$$\|\boldsymbol{u} - \boldsymbol{v}_h\|_*^2 \leq C(\boldsymbol{u})(h_1^2 + h_2^2),$$
$$\int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\boldsymbol{u})[(\boldsymbol{v}_h - \boldsymbol{u}).\boldsymbol{n}] \, d\Gamma \leq C(\boldsymbol{u})(h_1^{\frac{3}{2}} + h_2^2).$$

This is coherent with the results of [5] proven in the conforming case where π_h^1 restricted to $W_h^2(\Gamma_c)(= W_h^1(\Gamma_c))$ coincides with the identity. The next step consists of deriving a first (rough) evaluation of the consistency error.

LEMMA 4. – *Assume that* $\boldsymbol{u}$ *satisfies the hypotheses of Lemma 3. Then, the following estimate holds*

$$\inf_{\boldsymbol{v} \in \boldsymbol{K}} \int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\boldsymbol{u})[(\boldsymbol{v} - \boldsymbol{u}_h).\boldsymbol{n}] \, d\Gamma \leq C(\boldsymbol{u}) h_1.$$

In the proof, the only information used on $\boldsymbol{u}_h$ is the boundedness of $\|\boldsymbol{u}_h\|_*$. Putting together the results of the three last Lemmas yields the following bound of the global error: $\|\boldsymbol{u} - \boldsymbol{u}_h\|_* \leq C(\boldsymbol{u})(h_1^{\frac{1}{2}} + h_2)$. This (first) estimate allows, by using a bootstrap argument, to improve the bound of the consistency error and to obtain the final result.

THEOREM 5. – *Assume that the exact solution* $\boldsymbol{u} \in \boldsymbol{K}$ *of the weak problem* (6) *belongs to* $(H^2(\Omega^1))^2 \times (H^2(\Omega^2))^2$. *Then, the discrete solution* $\boldsymbol{u}_h \in \boldsymbol{K}_h$ *is such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_* \leq C(\boldsymbol{u})(h_1^{\frac{3}{4}} + h_2),$$

where $C(\boldsymbol{u}) > 0$ *is independent of* h *and depends linearly on* $\|\boldsymbol{u}^1\|_{(H^2(\Omega^1))^2}$ *and* $\|\boldsymbol{u}^2\|_{(H^2(\Omega^2))^2}$.

In other respects, the study of a model taking into account friction forces along the contact zone is under investigation and will be addressed in a forthcoming paper (see [6]).

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