Quadratic Finite Element Methods for Unilateral Contact Problems

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The present paper is concerned with the frictionless unilateral contact problem between two elastic bodies in a bidimensional context. We consider a mixed formulation in which the unknowns are the displacement field and the contact pressure. We introduce a finite element method using quadratic elements and continuous piecewise quadratic multipliers on the contact zone. The discrete unilateral non-interpenetration condition is either an exact non-interpenetration condition or only a nodal condition. In both cases, we study the convergence of the finite element solutions and a priori error estimates are given. Finally, we perform the numerical comparison of the quadratic approach with linear finite elements.

Keywords : Quadratic finite elements, unilateral contact problem, a priori error estimates.

1. Introduction

The numerical approximation of contact and impact problems occurring in solid mechanics is generally accomplished using finite elements (see [12, 17, 22]). In the engineering context, such simulations involve an increasing number of difficulties due to a more precise modelling of complex phenomena so that the choice of improved and accurate finite element methods is often investigated. The present paper is concerned with quadratic finite element methods for unilateral contact problems (i.e. the Signorini problem in elasticity). Our aim is to study if such methods should be chosen for a more precise approximation of contact problems formulated through a variational inequality or an equivalent mixed formulation.

The first convergence analysis of a finite element method for the unilateral contact problem written as a variational inequality was achieved by Haslinger and Hlaváček in [11] (see also [12]) in the case of linear finite elements. More recently, Ben Belgacem completes the previous studies by considering in [2] a wider class of regularity assumptions and Coorevits, Hild, Lhalouani and Sassi obtain in [7] a first error estimate in the L^2-norm for the displacements. In reference [13], Haslinger and Lovišek accomplish the initial error analysis for a mixed method using linear finite elements for the displacement field and discontinuous piecewise constant multipliers approximating the pressure on the contact zone (see also [12]). In [7], a mixed method using continuous and piecewise linear multipliers (on the same mesh as the displacements) is analyzed. In fact, the latter choice of multipliers allows to prove a uniform inf-sup condition.

Using second order finite elements for variational inequality problems has already been achieved by Brezzi, Hager and Raviart for the obstacle problem in [4, 5]. In these references, the authors obtain optimal convergence rates for both the variational inequality approximation and a mixed formulation. For unilateral contact problems, Kikuchi and Song used a penalized finite element approach in [18] (see also [17]) to perform the analysis of a second order method as well as some numerical experiments.

Let us mention that there are significant differences in the finite element error analyses of the variational inequalities. Although the basic tool is always an adaptation of Falk's lemma [9], the handling of the approximation terms involves specific techniques leading generally to different error bounds (see [10]). A particularity of the unilateral contact model (in which the test functions lie in a convex set and not in a space) comes from the location of the inequality condition holding only on (a part of) the boundary and not on the entire domain. That leads to investigate positivity preserving approximation properties in trace spaces. Since the usual positivity preserving approximation operators do not satisfy optimal approximation properties in such spaces (see [14]), the error analysis using linear finite elements leads to a convergence rate of only $h^{\frac{3}{4}}$ in the H^1-norm under H^2 regularity assumptions (h stands for the mesh size and H^m denotes the standard Sobolev space, see [11, 12]). An optimal error estimate of order h is recovered under supplementary hypotheses on the exact solution, in particular the finiteness of the set of points where the change from contact to separation occurs (see [4, 11, 12]).

When considering quadratic finite elements, the error analysis of the variational inequality or the mixed formulation issued from unilateral contact shows error terms and difficulties which disappear in the case of linear finite elements. Let us briefly describe them. If one chooses a (classic) discrete non-interpenetration condition holding at the nodes only, then some interpenetration can occur and it must be estimated in the error analysis. Another possibility consists in choosing a (non-classic) discrete non-interpenetration condition which holds everywhere on the contact part. In such a case the classical interpolation operator of degree two becomes inefficient in the analysis because it does not preserve positivity. Our aim in the error analysis is to overcome these difficulties specific to the quadratic case.

An outline of the paper is as follows. Section 2 deals with the continuous setting of the unilateral contact problem. In section 3, we define two mixed finite element methods using quadratic finite elements with multipliers which are continuous and piecewise of degree two on the contact part. The difference between both approaches is that the non-interpenetration conditions are either of linear type (i.e. hold at the discretization nodes of the method) or of quadratic type (i.e. hold everywhere on the contact part). The link of the mixed methods with the corresponding variational inequality formulation is given. Section 4 is concerned with the convergence study of the methods for which we prove identical convergence rates under various regularity hypotheses. Finally, in section 5, we carry out numerical experiments where quadratic finite elements and linear finite elements are compared.

Preliminaries and notations. We begin with introducing some useful notation and several functional spaces. In what follows, bold letters like $\boldsymbol{u}, \boldsymbol{v}$, indicate vector valued quantities, while the capital ones (e.g. $\mathbf{V}, \mathbf{K}, \ldots$) represent functional sets involving vector fields.

Let Ω be an open bounded subset of $\mathbb{R}^2$ whose generic point is denoted $\boldsymbol{x} = (x_1, x_2)$ and denote by $L^2(\Omega)$ the Hilbert space of square integrable functions endowed with the inner product

$$(\varphi, \psi) = \int_\Omega \varphi(\boldsymbol{x})\psi(\boldsymbol{x})\, d\Omega.$$

Given $m \in \mathbb{N}$, introduce the Sobolev space

$$H^m(\Omega) = \Big\{\psi \in L^2(\Omega), D^\alpha\psi \in L^2(\Omega), |\alpha| \leq m\Big\},$$

where $\alpha = (\alpha_1, \alpha_2)$ represents a multi–index in $\mathbb{N}^2$ and $|\alpha| = \alpha_1 + \alpha_2$. The notation D^α denotes a partial derivative and the convention $H^0(\Omega) = L^2(\Omega)$ is adopted. The spaces $H^m(\Omega)$ are equipped with the norm

$$\|\psi\|_{H^m(\Omega)} = \Big(\sum_{|\alpha|\leq m} \|D^\alpha\psi\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}.$$

The Sobolev space of fractional order $H^\tau(\Omega)$, $\tau \in \mathbb{R}_+ \setminus \mathbb{N}$ is then defined by the norm (see [1]):

$$\|\psi\|_{H^\tau(\Omega)} = \left(\|\psi\|^2_{H^m(\Omega)} + \sum_{|\alpha|=m} \int_\Omega\int_\Omega \frac{(D^\alpha\psi(\boldsymbol{x}) - D^\alpha\psi(\boldsymbol{y}))^2}{|\boldsymbol{x}-\boldsymbol{y}|^{2+2\theta}}\, d\Omega\, d\Omega\right)^{\frac{1}{2}},$$

where m is the integer part of τ and θ its decimal part. Let γ be a connected portion of the boundary of Ω. For any $\tau \in \mathbb{R}_+ \setminus \mathbb{N}$, the Hilbert space $H^\tau(\gamma)$ is assigned with the norm

$$\|\psi\|_{H^\tau(\gamma)} = \left(\|\psi\|^2_{H^m(\gamma)} + \int_\gamma\int_\gamma \frac{(D^m\psi(\boldsymbol{x}) - D^m\psi(\boldsymbol{y}))^2}{|\boldsymbol{x}-\boldsymbol{y}|^{1+2\theta}}\, d\gamma d\gamma\right)^{\frac{1}{2}}.$$

In the previous integral, $D^m\psi$ stands for the m–order derivative of ψ along γ and $d\gamma$ denotes the linear measure on γ. The norm on the topological dual space of $H^{\frac{1}{2}}(\gamma)$ is

$$\|\psi\|_{H^{-\frac{1}{2}}(\gamma)} = \sup_{\varphi \in H^{\frac{1}{2}}(\gamma)} \frac{\Big\langle \psi, \varphi \Big\rangle_{-\frac{1}{2},\frac{1}{2},\gamma}}{\|\varphi\|_{H^{\frac{1}{2}}(\gamma)}},$$

where the notation $\langle \ , \ \rangle_{-\frac{1}{2},\frac{1}{2},\gamma}$ represents the duality pairing between $H^{-\frac{1}{2}}(\gamma)$ and $H^{\frac{1}{2}}(\gamma)$. We will also make use of norm $\|\psi\|_{L^\infty(\gamma)} = \operatorname{ess\,sup}_{\boldsymbol{x}\in\gamma} |\psi(\boldsymbol{x})|$ as well as the Hölder spaces $\mathscr{C}^{m,\nu}(\gamma),\ m = 0, 1,\ 0 < \nu \leq 1$ defined by the norm

$$\|\psi\|_{\mathscr{C}^{m,\nu}(\gamma)} = \left(\max_{\alpha\leq m} \sup_{\boldsymbol{x}\in\gamma} |D^\alpha \psi(\boldsymbol{x})| + \max_{\alpha\leq m} \sup_{\boldsymbol{x},\boldsymbol{y}\in\gamma} \frac{|D^\alpha\psi(\boldsymbol{x}) - D^\alpha\psi(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\nu} \right).$$

2. The Signorini problem in elasticity

Let $\Omega^\ell,\ \ell = 1, 2$ denote two elastic bodies in $\mathbb{R}^2$. The boundary $\partial\Omega^\ell$ is supposed to be “regular” and it consists of three nonoverlapping parts Γ_D^ℓ, Γ_N^ℓ and Γ_C^ℓ with $\operatorname{meas}(\Gamma_D^\ell) > 0$. The normal unit outward vector on $\partial\Omega^\ell$ is denoted $\boldsymbol{n}^\ell = (n_1^\ell, n_2^\ell)$. In the initial stage, the bodies are in contact on their common boundary part $\Gamma_C^1 = \Gamma_C^2$ which we shall denote by Γ_C and we suppose that the unknown final contact zone after deformation will be included in Γ_C. The bodies, clamped on Γ_D^ℓ, are subjected to volume forces $\boldsymbol{f}^\ell = (f_1^\ell, f_2^\ell) \in (L^2(\Omega^\ell))^2$ and to surface forces $\boldsymbol{g}^\ell = (g_1^\ell, g_2^\ell) \in (L^2(\Gamma_N^\ell))^2$.

The Signorini problem in elasticity (or unilateral contact problem) consists of finding the displacement fields $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2)$ (where the notation $\boldsymbol{u}^\ell$ stands for $\boldsymbol{u}|_{\Omega^\ell}$) with $\boldsymbol{u}^\ell = (u_1^\ell, u_2^\ell),\ 1 \leq \ell \leq 2$ verifying the following equations (2.1)-(2.7):

$$\mathbf{div}\ \boldsymbol{\sigma}^\ell + \boldsymbol{f}^\ell = 0 \qquad \text{in } \Omega^\ell, \tag{2.1}$$

where $\mathbf{div}$ denotes the divergence operator of tensor valued functions and $\boldsymbol{\sigma}^\ell = (\sigma_{ij}^\ell),\ 1 \leq i, j \leq 2,\ 1 \leq \ell \leq 2$ stands for the stress tensor field. The latter is obtained from the displacement field by the constitutive law of linear elasticity

$$\boldsymbol{\sigma}^\ell = \mathbf{A}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) \qquad \text{in } \Omega^\ell, \tag{2.2}$$

where $\mathbf{A}^\ell$ is a fourth order symmetric and elliptic tensor and $\boldsymbol{\varepsilon}(\boldsymbol{v}) = \frac{1}{2}(\nabla\boldsymbol{v} + {}^t\nabla\boldsymbol{v})$ represents the linearized strain tensor field. On Γ_D^ℓ and Γ_N^ℓ, the conditions are as follows:

$$\boldsymbol{u}^\ell = 0 \qquad \text{on } \Gamma_D^\ell, \tag{2.3}$$

$$\boldsymbol{\sigma}^\ell \boldsymbol{n}^\ell = \boldsymbol{g}^\ell \qquad \text{on } \Gamma_N^\ell. \tag{2.4}$$

It remains to specify the conditions modelling unilateral contact on Γ_C:

$$(\boldsymbol{\sigma}^1\boldsymbol{n}^1).\boldsymbol{n}^1 = (\boldsymbol{\sigma}^2\boldsymbol{n}^2).\boldsymbol{n}^2 = \sigma_{\boldsymbol{n}}, \tag{2.5}$$

$$[\boldsymbol{u}.\boldsymbol{n}] \leq 0, \qquad \sigma_{\boldsymbol{n}} \leq 0, \qquad \sigma_{\boldsymbol{n}}[\boldsymbol{u}.\boldsymbol{n}] = 0, \tag{2.6}$$

$$\boldsymbol{\sigma}_{\boldsymbol{t}}^1 = \boldsymbol{\sigma}_{\boldsymbol{t}}^2 = 0, \tag{2.7}$$

where $\sigma_{\boldsymbol{n}}$ denotes the normal constraint (or contact pressure), $[\boldsymbol{u}.\boldsymbol{n}] = \boldsymbol{u}^1.\boldsymbol{n}^1 + \boldsymbol{u}^2.\boldsymbol{n}^2$ stands for the jump of the relative normal displacement across Γ_C and $\boldsymbol{\sigma}_{\boldsymbol{t}}^\ell = \boldsymbol{\sigma}^\ell\boldsymbol{n}^\ell - \sigma_{\boldsymbol{n}}\boldsymbol{n}^\ell$ represents the tangential constraints equal to zero because friction is omitted.

In order to derive the variational formulation of (2.1)-(2.7), we consider the Hilbert space $\mathbf{V} = \mathbf{V}(\Omega^1) \times \mathbf{V}(\Omega^2)$ where

$$\mathbf{V}(\Omega^\ell) = \Big\{ \boldsymbol{v}^\ell \in \Big(H^1(\Omega^\ell) \Big)^2, \quad \boldsymbol{v}^\ell = 0 \text{ on } \Gamma_D^\ell \Big\},$$

endowed with the broken norm

$$\|\boldsymbol{v}\| = \Big(\sum_{\ell=1}^{2} \|\boldsymbol{v}^\ell\|^2_{(H^1(\Omega^\ell))^2} \Big)^{\frac{1}{2}}, \qquad \forall \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \mathbf{V}.$$

The forthcoming mixed variational formulation uses the following convex cone of multipliers on Γ_C

$$M = \Big\{ \mu \in H^{-\frac{1}{2}}(\Gamma_C), \ \Big\langle \mu, \psi \Big\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_C} \geq 0 \ \text{ for all } \psi \in H^{\frac{1}{2}}(\Gamma_C), \psi \geq 0 \ \text{ a.e. on } \Gamma_C \Big\}.$$

Define

$$a(\boldsymbol{u}, \boldsymbol{v}) = \sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathbf{A}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell)\, \boldsymbol{\varepsilon}(\boldsymbol{v}^\ell)\, d\Omega^\ell, \qquad b(\boldsymbol{v}, \mu) = \Big\langle \mu, [\boldsymbol{v}.\boldsymbol{n}] \Big\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_C},$$

$$L(\boldsymbol{v}) = \sum_{\ell=1}^{2} \Big(\int_{\Omega^\ell} \boldsymbol{f}^\ell.\boldsymbol{v}^\ell \, d\Omega^\ell + \int_{\Gamma_N^\ell} \boldsymbol{g}^\ell.\boldsymbol{v}^\ell \, d\Gamma^\ell \Big),$$

for any $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2)$ and $\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2)$ in $\mathbf{V}$ and μ in $H^{-\frac{1}{2}}(\Gamma_C)$. We recall that the notation $[\boldsymbol{v}.\boldsymbol{n}]$ stands for $\boldsymbol{v}^1.\boldsymbol{n}^1 + \boldsymbol{v}^2.\boldsymbol{n}^2$.

The mixed formulation of the unilateral contact problem without friction consists then of finding $\boldsymbol{u} \in \mathbf{V}$ and $\lambda \in M$ such that (see [12]):

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, \lambda) = L(\boldsymbol{v}), & \forall\, \boldsymbol{v} \in \mathbf{V}, \\ b(\boldsymbol{u}, \mu - \lambda) \leq 0, & \forall\, \mu \in M, \end{cases} \tag{2.8}$$

and it can be easily verified that (2.1)-(2.7) implies (2.8). An equivalent formulation of (2.8) consists of finding $(\boldsymbol{u}, \lambda) \in \mathbf{V} \times M$ satisfying

$$\mathscr{L}(\boldsymbol{u}, \mu) \leq \mathscr{L}(\boldsymbol{u}, \lambda) \leq \mathscr{L}(\boldsymbol{v}, \lambda), \qquad \forall \boldsymbol{v} \in \mathbf{V},\ \forall \mu \in M,$$

where $\mathscr{L}(\boldsymbol{v}, \mu) = \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) - L(\boldsymbol{v}) + b(\boldsymbol{v}, \mu)$. Another classical formulation of problem (2.1)-(2.7) is a variational inequality (see [8, 19]): find $\boldsymbol{u}$ such that

$$\boldsymbol{u} \in \mathbf{K}, \qquad a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \geq L(\boldsymbol{v} - \boldsymbol{u}), \quad \forall \boldsymbol{v} \in \mathbf{K}, \tag{2.9}$$

where $\mathbf{K}$ denotes the closed convex cone of admissible displacement fields satisfying the non-interpenetration conditions

$$\mathbf{K} = \big\{ \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \mathbf{V}, \qquad [\boldsymbol{v}.\boldsymbol{n}] \leq 0 \text{ on } \Gamma_C \big\}.$$

The existence and uniqueness of $(\boldsymbol{u}, \lambda)$ solution to (2.8) has been stated in [12]. Moreover, the first argument $\boldsymbol{u}$ solution to (2.8) is also the unique solution of problem (2.9) and $\lambda = -\sigma_{\boldsymbol{n}}$.

3. Approximation with quadratic finite elements

We suppose that the subdomains Ω^1 and Ω^2 are polygons and that Γ_C is a straight line segment for the sake of simplicity. Moreover, we assume afterwards that $\overline{\Gamma_C} \cap \overline{\Gamma_D^\ell} = \emptyset$, $(\ell = 1, 2)$.

With each subdomain Ω^ℓ, we then associate a regular family of triangulations (the extension to quadrangular finite elements is straightforward) $\mathscr{T}_h^\ell$ made of quadratic elements denoted κ such that $\overline{\Omega^\ell} = \bigcup_{\kappa \in \mathscr{T}_h^\ell} \overline{\kappa}$. The index $h = \max(h_1, h_2)$ is defined from the discretization parameter h_ℓ on Ω^ℓ which is given by $h_\ell = \max_{\kappa \in \mathscr{T}_h^\ell} h_\kappa$ where h_κ denotes the diameter of the triangle κ. We suppose that the end points $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ of the contact zone Γ_C are common nodes of the triangulations $\mathscr{T}_h^1$ and $\mathscr{T}_h^2$ and that the monodimensional traces of triangulations of $\mathscr{T}_h^1$ and $\mathscr{T}_h^2$ on Γ_C are uniformly regular. The set of nodes on Γ_C belonging to triangulation $\mathscr{T}_h^\ell$ is denoted ξ_h^ℓ and generally one has $\xi_h^1 \neq \xi_h^2$ so that our study takes also into account nonmatching meshes on Γ_C (see [3, 15, 7]). For any integer $q \geq 0$, the notation $\mathbb{P}_q(\kappa)$ denotes the space of the polynomials defined on κ whose degree is lower or equal to q.

The finite element space on Ω^ℓ is then defined as (see [6]):

$$\mathbf{V}_h(\Omega^\ell) = \left\{ \boldsymbol{v}_h^\ell \in (\mathscr{C}(\overline{\Omega^\ell}))^2, \qquad \boldsymbol{v}_h^\ell|_\kappa \in (\mathbb{P}_2(\kappa))^2, \qquad \forall \kappa \in \mathscr{T}_h^\ell, \quad \boldsymbol{v}_h^\ell|_{\Gamma_D^\ell} = 0 \right\},$$

and the following approximation space of $\mathbf{V}$ is written

$$\mathbf{V}_h = \mathbf{V}_h(\Omega^1) \times \mathbf{V}_h(\Omega^2).$$

Next, we define the space $W_h^\ell(\Gamma_C)$ of continuous functions which are piecewise of degree two on the mesh of Ω^ℓ on Γ_C.

$$W_h^\ell(\Gamma_C) = \left\{ \psi_h \in \mathscr{C}(\overline{\Gamma_C}),\ \exists \boldsymbol{v}_h^\ell \in \mathbf{V}_h(\Omega^\ell) \text{ such that } \boldsymbol{v}_h^\ell . \boldsymbol{n}^\ell = \psi_h \text{ on } \Gamma_C \right\}.$$

Let us now approximate the closed convex cone M by a subset of $W_h^\ell(\Gamma_C)$. It is straightforward that the key point lies in the translation of the nonnegativity condition. We first introduce the set Q_h^ℓ where nonnegativity holds everywhere on Γ_C:

$$Q_h^\ell = \left\{ \mu_h \in W_h^\ell(\Gamma_C),\ \mu_h \geq 0 \text{ on } \Gamma_C \right\},$$

which corresponds to convex constraints of quadratic type. We then define another set denoted L_h^ℓ where nonnegativity holds only at the nodes of the finite element method (i.e. the nodes of the mesh and the midpoints) that lead to convex constraints of linear type:

$$L_h^\ell = \left\{ \mu_h \in W_h^\ell(\Gamma_C),\ \mu_h(\boldsymbol{a}) \geq 0,\ \forall \boldsymbol{a} \in \xi_h^\ell \right\}.$$

Note that the difference between Q_h^ℓ and L_h^ℓ would disappear if piecewise affine instead of piecewise quadratic functions had been used in the definition of $W_h^\ell(\Gamma_C)$ since nonnegativity on Γ_C is then equivalent to nonnegativity at the nodes of the triangulation.

Next, we define the positive polar cones (see [16], p. 119) $Q_h^{\ell,*}$ and $L_h^{\ell,*}$ of Q_h^ℓ and L_h^ℓ respectively:

$$Q_h^{\ell,*} = \Big\{ \mu_h \in W_h^\ell(\Gamma_C),\ \int_{\Gamma_C} \mu_h \psi_h \, d\Gamma \geq 0,\ \forall \psi_h \in Q_h^\ell \Big\},$$

$$L_h^{\ell,*} = \Big\{ \mu_h \in W_h^\ell(\Gamma_C),\ \int_{\Gamma_C} \mu_h \psi_h \, d\Gamma \geq 0,\ \forall \psi_h \in L_h^\ell \Big\}.$$

From the inclusion $Q_h^\ell \subset L_h^\ell$, one immediately gets by polarity $L_h^{\ell,*} \subset Q_h^{\ell,*}$.

We then choose a discretized mixed formulation which uses either $Q_h^{\ell,*}$ or $L_h^{\ell,*}$ as approximation of M. The discrete problem is: find $\boldsymbol{u}_h \in \mathbf{V}_h$ and $\lambda_h \in M_h$ satisfying:

$$\left\{ \begin{array}{ll} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \lambda_h) = L(\boldsymbol{v}_h), & \forall\, \boldsymbol{v}_h \in \mathbf{V}_h, \\ b(\boldsymbol{u}_h, \mu_h - \lambda_h) \leq 0, & \forall\, \mu_h \in M_h, \end{array} \right. \tag{3.1}$$

where $M_h = Q_h^{\ell,*}$ or $M_h = L_h^{\ell,*}$ with $\ell = 1$ or 2.

From the obvious relation

$$\Big\{ \mu_h \in W_h^\ell(\Gamma_C), \qquad b(\boldsymbol{v}_h, \mu_h) = 0, \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h \Big\} = \{0\} \tag{3.2}$$

and the $\mathbf{V}_h$-ellipticity of $a(.,.)$, we immediately get the following proposition.

Proposition 3.1 *Let $M_h = Q_h^{\ell,*}$ or $M_h = L_h^{\ell,*}$ with $\ell = 1$ or 2. Then problem* (3.1) *admits a unique solution $(\boldsymbol{u}_h, \lambda_h) \in \mathbf{V}_h \times M_h$.*

Remark 3.2 *It can be easily checked that the compatibility relation* (3.2) *implies the existence of a constant β_h such that*

$$\inf_{\mu_h \in W_h^\ell(\Gamma_C)} \sup_{\boldsymbol{v}_h \in \mathbf{V}_h} \frac{b(\boldsymbol{v}_h, \mu_h)}{\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \|\boldsymbol{v}_h\|} \geq \beta_h > 0.$$

In fact, the constant β_h does not depend on h (see Proposition 3.5 hereafter).

The next result gives the link between the mixed problem (3.1) and a discretized variational inequality issued from (2.9) when $M_h = Q_h^{\ell,*}$ or $M_h = L_h^{\ell,*}$. To get it, we need to introduce the projection operator on $W_h^\ell(\Gamma_C)$, denoted π_h^ℓ, and defined for any function $\varphi \in L^2(\Gamma_C)$ as follows:

$$\pi_h^\ell \varphi \in W_h^\ell(\Gamma_C), \qquad \int_{\Gamma_C} (\pi_h^\ell \varphi - \varphi)\, \mu_h \, d\Gamma = 0, \qquad \forall\, \mu_h \in W_h^\ell(\Gamma_C). \tag{3.3}$$

Proposition 3.3 *Let $M_h = Q_h^{\ell,*}$ or $M_h = L_h^{\ell,*}$ with $\ell = 1$ or 2 and let $(\boldsymbol{u}_h, \lambda_h) \in \mathbf{V}_h \times M_h$ be the solution of* (3.1)*. Then $\boldsymbol{u}_h$ is also solution of the variational inequality:*

$$\boldsymbol{u}_h \in \mathbf{K}_h, \qquad a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) \geq L(\boldsymbol{v}_h - \boldsymbol{u}_h), \qquad \forall \boldsymbol{v}_h \in \mathbf{K}_h, \tag{3.4}$$

where $\mathbf{K}_h = \mathbf{K}_h^Q$ if $M_h = Q_h^{\ell,*}$, $\mathbf{K}_h = \mathbf{K}_h^L$ if $M_h = L_h^{\ell,*}$ with

$$\mathbf{K}_h^Q = \big\{\boldsymbol{v}_h = (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) \in \mathbf{V}_h, \qquad \pi_h^\ell[\boldsymbol{v}_h.\boldsymbol{n}] \leq 0 \text{ on } \Gamma_C\big\}, \tag{3.5}$$

$$\mathbf{K}_h^L = \big\{\boldsymbol{v}_h = (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) \in \mathbf{V}_h, \qquad (\pi_h^\ell[\boldsymbol{v}_h.\boldsymbol{n}])(\boldsymbol{a}) \leq 0 \ \ \forall \boldsymbol{a} \in \xi_h^\ell\ \big\}. \tag{3.6}$$

Proof. Let us first notice that $\mathbf{K}_h^Q$ and $\mathbf{K}_h^L$ depend on ℓ which has been omitted to lighten the notations. Taking $\mu_h = 0$ and $\mu_h = 2\lambda_h$ in (3.1) leads to $b(\boldsymbol{u}_h, \lambda_h) = 0$ and to

$$b(\boldsymbol{u}_h, \mu_h) = \int_{\Gamma_C} \mu_h [\boldsymbol{u}_h.\boldsymbol{n}] \, d\Gamma = \int_{\Gamma_C} \mu_h \, \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] \, d\Gamma \leq 0, \quad \forall \mu_h \in M_h,$$

where the definition (3.3) of π_h^ℓ has been used. The latter inequality implies by polarity that $\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] \in -M_h^*$ (the notation X^* stands for the positive polar cone of X).

- If $M_h = Q_h^{\ell,*}$ then $M_h^* = (Q_h^{\ell,*})^* = Q_h^\ell$ since Q_h^ℓ is a closed convex cone. Hence $\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] \in -Q_h^\ell$ and $\boldsymbol{u}_h \in \mathbf{K}_h^Q$. Consequently (3.1) and $b(\boldsymbol{u}_h, \lambda_h) = 0$ lead to

$$a(\boldsymbol{u}_h, \boldsymbol{u}_h) = L(\boldsymbol{u}_h) \tag{3.7}$$

and for any $\boldsymbol{v}_h \in \mathbf{K}_h^Q$, we get

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}_h) - L(\boldsymbol{v}_h) = -b(\boldsymbol{v}_h, \lambda_h) &= -\int_{\Gamma_C} \lambda_h [\boldsymbol{v}_h.\boldsymbol{n}] \, d\Gamma \\ &= -\int_{\Gamma_C} \lambda_h \, \pi_h^\ell[\boldsymbol{v}_h.\boldsymbol{n}] \, d\Gamma \geq 0, \end{aligned} \tag{3.8}$$

owing to $\lambda_h \in Q_h^{\ell,*}$.

Putting together (3.7) and (3.8) implies that $\boldsymbol{u}_h$ is solution to the variational inequality (3.4) (with $\mathbf{K}_h = \mathbf{K}_h^Q$) which admits a unique solution according to Stampacchia's theorem.

- The case $M_h = L_h^{\ell,*}$ is treated similarly to the previous one. □

Remark 3.4 *1. The projection operator π_h^ℓ comes from the presence of nonmatching meshes and the projection operator reduces to identity when matching meshes are used. As a matter of fact, suppose that the meshes fit together on the contact zone which means that $\xi_h^1 = \xi_h^2$ or equivalently $W_h^1(\Gamma_C) = W_h^2(\Gamma_C)$. The choice of the multipliers set $Q_h^{\ell,*}$ leads then to the quadratic sign condition on the displacements $[\boldsymbol{v}_h.\boldsymbol{n}] \leq 0$ on Γ_C in* (3.5)*. The other choice of $L_h^{\ell,*}$ corresponds to the linear node-on-node sign condition on the displacements $[\boldsymbol{v}_h.\boldsymbol{n}](\boldsymbol{a}) \leq 0 \ \ \forall \boldsymbol{a} \in \xi_h^\ell$ in* (3.6)*. Note that the latter condition is commonly used in engineering computations.*

2. In the definitions of $\mathbf{K}_h^L$ and $\mathbf{K}_h^Q$ one can also write $\boldsymbol{v}_h^\ell.\boldsymbol{n}^\ell + \pi_h^\ell(\boldsymbol{v}_h^{3-\ell}.\boldsymbol{n}^{3-\ell})$ instead of $\pi_h^\ell[\boldsymbol{v}_h.\boldsymbol{n}]$.

We are now interested in obtaining a uniform inf-sup condition for $b(.,.)$ over $\mathbf{V}_h \times W_h^\ell(\Gamma_C)$. The result is given in the following proposition. The proof of the proposition is the same as in the case of linear finite elements with continuous linear multipliers (see [7], Proposition 3.3) and consists essentially of proving the stability of the projection operator π_h^ℓ in the $H^{\frac{1}{2}}(\Gamma_C)$-norm.

Proposition 3.5 *Suppose that $\overline{\Gamma_C} \cap \overline{\Gamma_D^\ell} = \emptyset$ for $\ell = 1, 2$. The following inf-sup condition holds*

$$\inf_{\mu_h \in W_h^\ell(\Gamma_C)} \sup_{\boldsymbol{v}_h \in \mathbf{V}_h} \frac{b(\boldsymbol{v}_h, \mu_h)}{\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \|\boldsymbol{v}_h\|} \geq \beta > 0, \tag{3.9}$$

where β does not depend on h.

Remark 3.6 *The choice of the mixed continuous and discrete formulations* (2.8) *and* (3.1) *will allow us to obtain more information compared to the variational inequality approach* (2.9) *and* (3.4)*. In particular, the forthcoming error estimates of the two mixed methods hold also for the variational inequality problem according to Proposition 3.3.*

4. Error estimates

Now we intend to analyze the convergence of both quadratic finite element approaches: discrete non-interpenetration condition of quadratic type (3.5) or of linear type (3.6). Before considering separately both methods, we begin with a common result.

4.1. *An abstract lemma*

Proposition 4.1 *Let $(\boldsymbol{u}, \lambda)$ be the solution of* (2.8) *and let $(\boldsymbol{u}_h, \lambda_h) \in \mathbf{V}_h \times M_h$ (where $M_h = Q_h^{\ell,*}$ or $M_h = L_h^{\ell,*}$ with $\ell = 1$ or 2) be the solution of* (3.1)*. Then, there exists a positive constant C independent of h satisfying:*

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}_h\| + \|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq C \Bigg\{ &\inf_{\boldsymbol{v}_h \in \mathbf{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\| + \inf_{\mu_h \in W_h^\ell(\Gamma_C)} \|\lambda - \mu_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \\ &+ \Big(\max(b(\boldsymbol{u}, \lambda_h), 0)\Big)^{\frac{1}{2}} + \Big(\max(b(\boldsymbol{u}_h, \lambda), 0)\Big)^{\frac{1}{2}} \Bigg\}.\end{aligned} \tag{4.1}$$

Proof. The proof is divided into three parts. First, an upper bound of $\|\boldsymbol{u} - \boldsymbol{u}_h\|$ will be obtained in (4.2). Then the inf-sup condition (3.9) will lead to an upper bound of $\|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}$ in (4.3). Both estimates will allow us to get estimate (4.1).

Let $\boldsymbol{v}_h \in \mathbf{V}_h$. According to (2.8) and (3.1), we have

$$\begin{aligned} a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h) &= a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{v}_h) + a(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}_h) - a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) \\ &= a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{v}_h) - b(\boldsymbol{v}_h - \boldsymbol{u}_h, \lambda) + b(\boldsymbol{v}_h - \boldsymbol{u}_h, \lambda_h) \\ &= a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{v}_h) - b(\boldsymbol{v}_h - \boldsymbol{u}, \lambda - \lambda_h) - b(\boldsymbol{u} - \boldsymbol{u}_h, \lambda - \lambda_h). \end{aligned}$$

Besides, the inequality of (2.8) implies $b(\boldsymbol{u},\lambda)=0$. Similarly, (3.1) leads to $b(\boldsymbol{u}_h,\lambda_h)=0$. Therefore

$$a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{u}_h)=a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{v}_h)+b(\boldsymbol{u}-\boldsymbol{v}_h,\lambda-\lambda_h)+b(\boldsymbol{u}_h,\lambda)+b(\boldsymbol{u},\lambda_h).$$

Denoting by α the ellipticity constant of $a(.,.)$ on $\mathbf{V}$, by M the continuity constant of $a(.,.)$ on $\mathbf{V}$ and using the trace theorem, we obtain

$$\begin{aligned}\alpha\|\boldsymbol{u}-\boldsymbol{u}_h\|^2\leq{}& M\|\boldsymbol{u}-\boldsymbol{u}_h\|\|\boldsymbol{u}-\boldsymbol{v}_h\|+C\|\lambda-\lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\|\boldsymbol{u}-\boldsymbol{v}_h\|\\&+b(\boldsymbol{u}_h,\lambda)+b(\boldsymbol{u},\lambda_h).\end{aligned}\tag{4.2}$$

Now, let us consider problem (2.8). The inclusion $\mathbf{V}_h\subset\mathbf{V}$ implies

$$a(\boldsymbol{u},\boldsymbol{v}_h)+b(\boldsymbol{v}_h,\lambda)=L(\boldsymbol{v}_h),\qquad\forall\,\boldsymbol{v}_h\in\mathbf{V}_h.$$

The latter equality together with (3.1) yields

$$a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h)+b(\boldsymbol{v}_h,\lambda-\lambda_h)=0,\qquad\forall\,\boldsymbol{v}_h\in\mathbf{V}_h.$$

Inserting $\mu_h\in W_h^{\ell}(\Gamma_C)$, using the continuity of $a(.,.)$ as well as the trace theorem gives

$$\begin{aligned}b(\boldsymbol{v}_h,\lambda_h-\mu_h)&=a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h)+b(\boldsymbol{v}_h,\lambda-\mu_h)\\&\leq M\|\boldsymbol{u}-\boldsymbol{u}_h\|\|\boldsymbol{v}_h\|+C\|\lambda-\mu_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\|\boldsymbol{v}_h\|,\\&\qquad\qquad\forall\,\mu_h\in W_h^{\ell}(\Gamma_C),\forall\,\boldsymbol{v}_h\in\mathbf{V}_h.\end{aligned}$$

This estimate and condition (3.9) allow us to write

$$\beta\|\lambda_h-\mu_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\leq\sup_{\boldsymbol{v}_h\in\mathbf{V}_h}\frac{b(\boldsymbol{v}_h,\lambda_h-\mu_h)}{\|\boldsymbol{v}_h\|}\leq M\|\boldsymbol{u}-\boldsymbol{u}_h\|+C\|\lambda-\mu_h\|_{H^{-\frac{1}{2}}(\Gamma_C)},$$

for any $\mu_h\in W_h^{\ell}(\Gamma_C)$. Since

$$\|\lambda-\lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\leq\|\lambda-\mu_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}+\|\mu_h-\lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)},\quad\forall\,\mu_h\in W_h^{\ell}(\Gamma_C),$$

we finally come to the conclusion that there exists $C>0$ such that

$$\|\lambda-\lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\leq C\Big(\|\boldsymbol{u}-\boldsymbol{u}_h\|+\inf_{\mu_h\in W_h^{\ell}(\Gamma_C)}\|\lambda-\mu_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\Big).\tag{4.3}$$

Putting together (4.3) and (4.2), we obtain for any $\boldsymbol{v}_h\in\mathbf{V}_h$

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|^2\leq C\bigg\{&\|\boldsymbol{u}-\boldsymbol{u}_h\|\|\boldsymbol{u}-\boldsymbol{v}_h\|+\inf_{\mu_h\in W_h^{\ell}(\Gamma_C)}\|\lambda-\mu_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\|\boldsymbol{u}-\boldsymbol{v}_h\|\\&+b(\boldsymbol{u}_h,\lambda)+b(\boldsymbol{u},\lambda_h)\bigg\}.\end{aligned}$$

Using estimate $ab \leq \gamma a^2 + \dfrac{1}{4\gamma} b^2$ (with $\gamma > 0$) leads to the bound

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|^2 \leq C \Bigg\{ \inf_{\boldsymbol{v}_h \in \mathbf{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|^2 + \inf_{\mu_h \in W_h^\ell(\Gamma_C)} \|\lambda - \mu_h\|^2_{H^{-\frac{1}{2}}(\Gamma_C)} + b(\boldsymbol{u}_h, \lambda) + b(\boldsymbol{u}, \lambda_h) \Bigg\}.$$

Taking the square root of this inequality which is then combined with (4.3) terminates the proof of (4.1). □

As a consequence the convergence error $\|\boldsymbol{u} - \boldsymbol{u}_h\| + \|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}$ can be divided into four quantities. The first two parts are the classical approximation terms measuring the "quality" of the space $\mathbf{V}_h$ approximating $\mathbf{V}$ and of $W_h^\ell(\Gamma_C)$ approximating $H^{-\frac{1}{2}}(\Gamma_C)$. The term $\big(\max(b(\boldsymbol{u}_h, \lambda), 0)\big)^{\frac{1}{2}}$ takes into account the interpenetration of the bodies (i.e. when $[\boldsymbol{u}_h.\boldsymbol{n}] > 0$) which is possible if $M_h = L_h^{\ell,*}$ or if $M_h = Q_h^{\ell,*}$ and nonmatching meshes are used. Finally, the term $\big(\max(b(\boldsymbol{u}, \lambda_h), 0)\big)^{\frac{1}{2}}$ measures the possible negativity of λ_h coming from the properties $L_h^{\ell,*} \not\subset M$ and $Q_h^{\ell,*} \not\subset M$.

Next, we consider separately both finite element approaches. Our purpose is to prove error estimates under H^s regularity hypotheses on the displacements (with $\frac{3}{2} < s < \frac{5}{2}$). First of all, we recall some useful basic error estimates issued from [6]. Let I_h^ℓ be the Lagrange interpolation operator with values in $\mathbf{V}_h(\Omega^\ell)$ and let $1 < r \leq 3$. Then

$$\forall \boldsymbol{v}^\ell \in (H^r(\Omega^\ell))^2, \quad \|\boldsymbol{v}^\ell - I_h^\ell \boldsymbol{v}^\ell\|_{(H^1(\Omega^\ell))^2} \leq C\, h_\ell^{r-1} \|\boldsymbol{v}^\ell\|_{(H^r(\Omega^\ell))^2}. \tag{4.4}$$

The projection operator π_h^ℓ defined in (3.3) satisfies the following estimates for any $0 \leq r \leq 3$:

$$\forall \varphi \in H^r(\Gamma_C), \quad h_\ell^{-\frac{1}{2}} \|\varphi - \pi_h^\ell \varphi\|_{H^{-\frac{1}{2}}(\Gamma_C)} + \|\varphi - \pi_h^\ell \varphi\|_{L^2(\Gamma_C)} \leq C\, h_\ell^r\, \|\varphi\|_{H^r(\Gamma_C)}. \tag{4.5}$$

We suppose that the elasticity coefficients incorporated in the operator $\mathbf{A}^\ell$ in (2.2) are regular enough so that the trace theorem implies for any $r > \frac{3}{2}$

$$\|\lambda\|_{H^{r-\frac{3}{2}}(\Gamma_C)} \leq C \|\boldsymbol{u}^\ell\|_{(H^r(\Omega^\ell))^2}, \quad \ell = 1, 2.$$

4.2. *The quadratic discrete non-interpenetration conditions*

Theorem 4.2 *Set $M_h = Q_h^{\ell,*}$ with $\ell = 1$ or 2 and let $(\boldsymbol{u}_h, \lambda_h)$ be the solution of* (3.1).

(i) Let $0 < \nu < 1$. Suppose that the solution $(\boldsymbol{u}, \lambda)$ of (2.8) *satisfies the regularity assumption $\boldsymbol{u}^1 \in (H^{\frac{3}{2}+\nu}(\Omega^1))^2$, $\boldsymbol{u}^2 \in (H^{\frac{3}{2}+\nu}(\Omega^2))^2$. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| + \|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq C(\boldsymbol{u}) h^{\frac{1}{2}+\frac{\nu}{2}}, \tag{4.6}$$

where the constant $C(\boldsymbol{u})$ depends linearly on $\|\boldsymbol{u}^1\|_{(H^{\frac{3}{2}+\nu}(\Omega^1))^2}$ and $\|\boldsymbol{u}^2\|_{(H^{\frac{3}{2}+\nu}(\Omega^2))^2}$.

(ii) Let $\frac{1}{2} < \nu < 1$. *Suppose that the solution* $(\boldsymbol{u}, \lambda)$ *of* (2.8) *satisfies the regularity assumption* $\boldsymbol{u}^1 \in (H^{\frac{3}{2}+\nu}(\Omega^1))^2$, $\boldsymbol{u}^2 \in (H^{\frac{3}{2}+\nu}(\Omega^2))^2$. *Assume that the set of points of* Γ_C *where the change from* $[\boldsymbol{u}.\boldsymbol{n}] < 0$ *to* $[\boldsymbol{u}.\boldsymbol{n}] = 0$ *occurs is finite. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| + \|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq C(\boldsymbol{u}) h^{\frac{1}{2}+\nu}, \tag{4.7}$$

where the constant $C(\boldsymbol{u})$ *depends linearly on* $\|\boldsymbol{u}^1\|_{(H^{\frac{3}{2}+\nu}(\Omega^1))^2}$ *and* $\|\boldsymbol{u}^2\|_{(H^{\frac{3}{2}+\nu}(\Omega^2))^2}$.

Remark 4.3 *Estimate* (4.7) *is optimal under the considered assumptions. The regularity* $H^{\frac{5}{2}}$ *can not generally be passed beyond for problems governed by variational inequalities (see [20], [10] and the references quoted therein). The assumption on the finite set of points where the change from contact to separation occurs is needed to recover optimality (as in the case of linear finite elements, see [12]). From an engineering point of view, the latter hypothesis is not restrictive when considering realistic frictionless contact problems.*

Proof of the theorem. Hereafter, the notation $C(\boldsymbol{u})$ represents a generic constant which depends linearly on $\|\boldsymbol{u}^\ell\|_{(H^{\frac{3}{2}+\nu}(\Omega^\ell))^2}$, $\ell = 1, 2$. Let us choose $\boldsymbol{v}_h = I_h \boldsymbol{u} = (I_h^1 \boldsymbol{u}^1, I_h^2 \boldsymbol{u}^2)$ and $\mu_h = \pi_h^\ell \lambda$ in (4.1). Using (4.4) and (4.5) yields

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}_h\| + \|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq C\Big\{ & C(\boldsymbol{u}) h^{\frac{1}{2}+\nu} + \big(\max(b(\boldsymbol{u}, \lambda_h), 0)\big)^{\frac{1}{2}} \\ & + \big(\max(b(\boldsymbol{u}_h, \lambda), 0)\big)^{\frac{1}{2}} \Big\}. \end{aligned} \tag{4.8}$$

It remains then to estimate both terms $b(\boldsymbol{u}, \lambda_h)$ and $b(\boldsymbol{u}_h, \lambda)$.

Step 1. Estimation of $b(\boldsymbol{u}, \lambda_h)$.

Let us denote by j_h^ℓ the Lagrange interpolation operator of order one on the mesh of Ω^ℓ on Γ_C. Such an operator satisfies for any $\frac{1}{2} < r \leq 2$:

$$\forall \varphi \in H^r(\Gamma_C), \quad \|\varphi - j_h^\ell \varphi\|_{L^2(\Gamma_C)} + h_\ell^{\frac{1}{2}} \|\varphi - j_h^\ell \varphi\|_{H^{\frac{1}{2}}(\Gamma_C)} \leq C\, h_\ell^r \, \|\varphi\|_{H^r(\Gamma_C)}. \tag{4.9}$$

We write

$$\begin{aligned} b(\boldsymbol{u}, \lambda_h) &= \int_{\Gamma_C} \lambda_h [\boldsymbol{u}.\boldsymbol{n}] d\Gamma \\ &= \int_{\Gamma_C} \lambda_h ([\boldsymbol{u}.\boldsymbol{n}] - j_h^\ell [\boldsymbol{u}.\boldsymbol{n}]) d\Gamma + \int_{\Gamma_C} \lambda_h \, j_h^\ell [\boldsymbol{u}.\boldsymbol{n}] d\Gamma. \end{aligned}$$

Obviously $j_h^\ell [\boldsymbol{u}.\boldsymbol{n}] \leq 0$ on Γ_C. From $\lambda_h \in Q_h^{\ell,*}$, $-j_h^\ell [\boldsymbol{u}.\boldsymbol{n}] \in Q_h^\ell$ and (4.9), we deduce

$$\begin{aligned} b(\boldsymbol{u}, \lambda_h) &\leq \int_{\Gamma_C} \lambda_h ([\boldsymbol{u}.\boldsymbol{n}] - j_h^\ell [\boldsymbol{u}.\boldsymbol{n}]) d\Gamma \\ &\leq \int_{\Gamma_C} \lambda ([\boldsymbol{u}.\boldsymbol{n}] - j_h^\ell [\boldsymbol{u}.\boldsymbol{n}]) d\Gamma + \|\lambda_h - \lambda\|_{H^{-\frac{1}{2}}(\Gamma_C)} \|[\boldsymbol{u}.\boldsymbol{n}] - j_h^\ell [\boldsymbol{u}.\boldsymbol{n}]\|_{H^{\frac{1}{2}}(\Gamma_C)} \\ &\leq \int_{\Gamma_C} \lambda ([\boldsymbol{u}.\boldsymbol{n}] - j_h^\ell [\boldsymbol{u}.\boldsymbol{n}]) d\Gamma + C h^{\frac{1}{2}+\nu} \|[\boldsymbol{u}.\boldsymbol{n}]\|_{H^{1+\nu}(\Gamma_C)} \|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}. \end{aligned} \tag{4.10}$$

The remaining integral term is estimated using (4.9):

$$\begin{aligned}\int_{\Gamma_C} \lambda([\boldsymbol{u}.\boldsymbol{n}] - j_h^\ell[\boldsymbol{u}.\boldsymbol{n}])d\Gamma &\le \|\lambda\|_{L^2(\Gamma_C)}\|[\boldsymbol{u}.\boldsymbol{n}] - j_h^\ell[\boldsymbol{u}.\boldsymbol{n}]\|_{L^2(\Gamma_C)} \\ &\le Ch^{1+\nu}\|[\boldsymbol{u}.\boldsymbol{n}]\|_{H^{1+\nu}(\Gamma_C)}\|\lambda\|_{L^2(\Gamma_C)}. \qquad (4.11)\end{aligned}$$

Putting (4.10) and (4.11) and using the trace theorem gives

$$b(\boldsymbol{u}, \lambda_h) \le C(\boldsymbol{u})(h^{\frac{1}{2}+\nu}\|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} + C(\boldsymbol{u})h^{1+\nu}). \qquad (4.12)$$

Consider again estimate (4.10) and suppose now that $\frac{1}{2} < \nu < 1$. Let $N(h)$ represent the number of (1D)-segments denoted T_i ($1 \le i \le N(h)$), of the triangulation of Ω^ℓ on Γ_C where the change from $[\boldsymbol{u}.\boldsymbol{n}] < 0$ to $[\boldsymbol{u}.\boldsymbol{n}] = 0$ occurs. We obtain

$$\begin{aligned}\int_{\Gamma_C} \lambda([\boldsymbol{u}.\boldsymbol{n}] - j_h^\ell[\boldsymbol{u}.\boldsymbol{n}])d\Gamma &= -\int_{\Gamma_C} \lambda\, j_h^\ell[\boldsymbol{u}.\boldsymbol{n}]d\Gamma \\ &\le \sum_{i=1}^{N(h)} \int_{T_i} |\lambda|\ |j_h^\ell[\boldsymbol{u}.\boldsymbol{n}]|d\Gamma \\ &\le h_\ell \sum_{i=1}^{N(h)} \|\lambda\|_{L^\infty(T_i)}\ \|j_h^\ell[\boldsymbol{u}.\boldsymbol{n}]\|_{L^\infty(T_i)} \\ &\le h_\ell \sum_{i=1}^{N(h)} \|\lambda\|_{L^\infty(T_i)}\ \|[\boldsymbol{u}.\boldsymbol{n}]\|_{L^\infty(T_i)}. \qquad (4.13)\end{aligned}$$

From the definition of the segment T_i, we deduce that $\|\lambda\|_{L^\infty(T_i)} \le h_\ell^{\nu-\frac{1}{2}}\|\lambda\|_{\mathscr{C}^{0,\nu-\frac{1}{2}}(T_i)}$ and $\|D^1[\boldsymbol{u}.\boldsymbol{n}]\|_{L^\infty(T_i)} \le h_\ell^{\nu-\frac{1}{2}}\|D^1[\boldsymbol{u}.\boldsymbol{n}]\|_{\mathscr{C}^{0,\nu-\frac{1}{2}}(T_i)} \le h_\ell^{\nu-\frac{1}{2}}\|[\boldsymbol{u}.\boldsymbol{n}]\|_{\mathscr{C}^{1,\nu-\frac{1}{2}}(T_i)}$. So

$$\begin{aligned}\int_{\Gamma_C} \lambda([\boldsymbol{u}.\boldsymbol{n}] - j_h^\ell[\boldsymbol{u}.\boldsymbol{n}])d\Gamma &\le h_\ell \sum_{i=1}^{N(h)} h_\ell^{\nu-\frac{1}{2}}\|\lambda\|_{\mathscr{C}^{0,\nu-\frac{1}{2}}(T_i)}\ h_\ell\|D^1[\boldsymbol{u}.\boldsymbol{n}]\|_{L^\infty(T_i)} \\ &\le h_\ell^{1+2\nu} \sum_{i=1}^{N(h)} \|\lambda\|_{\mathscr{C}^{0,\nu-\frac{1}{2}}(T_i)}\ \|[\boldsymbol{u}.\boldsymbol{n}]\|_{\mathscr{C}^{1,\nu-\frac{1}{2}}(T_i)} \\ &\le N(h)h_\ell^{1+2\nu}\|\lambda\|_{\mathscr{C}^{0,\nu-\frac{1}{2}}(\Gamma_C)}\ \|[\boldsymbol{u}.\boldsymbol{n}]\|_{\mathscr{C}^{1,\nu-\frac{1}{2}}(\Gamma_C)} \\ &\le N(h)h_\ell^{1+2\nu}\|\lambda\|_{H^\nu(\Gamma_C)}\ \|[\boldsymbol{u}.\boldsymbol{n}]\|_{H^{1+\nu}(\Gamma_C)}, \qquad (4.14)\end{aligned}$$

where the imbedding properties of Sobolev and Hölder spaces (see [1] or [21] p.24) have been used. If $N(h)$ is uniformly bounded in h, we obtain thanks to the trace theorem, (4.10) and (4.14):

$$b(\boldsymbol{u}, \lambda_h) \le C(\boldsymbol{u})(h^{\frac{1}{2}+\nu}\|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} + C(\boldsymbol{u})h^{1+2\nu}). \qquad (4.15)$$

<u>Step 2</u>. Estimation of $b(\boldsymbol{u}_h, \lambda)$.

Noting that $\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] \le 0$ on Γ_C (see Proposition 3.3), we get

$$\begin{aligned} b(\boldsymbol{u}_h,\lambda) &= \int_{\Gamma_C} \lambda[\boldsymbol{u}_h.\boldsymbol{n}]d\Gamma \\ &= \int_{\Gamma_C} \lambda([\boldsymbol{u}_h.\boldsymbol{n}] - \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}])d\Gamma + \int_{\Gamma_C} \lambda\, \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}]d\Gamma \\ &\le \int_{\Gamma_C} \lambda([\boldsymbol{u}_h.\boldsymbol{n}] - \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}])d\Gamma. \end{aligned}$$

Let ℓ' such that $\ell + \ell' = 3$. We can write

$$\begin{aligned} b(\boldsymbol{u}_h,\lambda) &\le \int_{\Gamma_C} \lambda(\boldsymbol{u}_h^{\ell'}.\boldsymbol{n}^{\ell'} - \pi_h^\ell(\boldsymbol{u}_h^{\ell'}.\boldsymbol{n}^{\ell'}))d\Gamma \\ &= \int_{\Gamma_C} (\lambda - \pi_h^\ell\lambda)(\boldsymbol{u}_h^{\ell'}.\boldsymbol{n}^{\ell'} - \pi_h^\ell(\boldsymbol{u}_h^{\ell'}.\boldsymbol{n}^{\ell'}))d\Gamma \\ &= \int_{\Gamma_C} (\lambda - \pi_h^\ell\lambda)((\boldsymbol{u}_h^{\ell'} - \boldsymbol{u}^{\ell'}).\boldsymbol{n}^{\ell'} - \pi_h^\ell((\boldsymbol{u}_h^{\ell'} - \boldsymbol{u}^{\ell'}).\boldsymbol{n}^{\ell'}))d\Gamma \\ &\quad + \int_{\Gamma_C} (\lambda - \pi_h^\ell\lambda)(\boldsymbol{u}^{\ell'}.\boldsymbol{n}^{\ell'} - \pi_h^\ell(\boldsymbol{u}^{\ell'}.\boldsymbol{n}^{\ell'}))d\Gamma. \end{aligned}$$

Then, the approximation properties of π_h^ℓ in the $L^2(\Gamma_C)$ norm and the trace theorem yield:

$$b(\boldsymbol{u}_h,\lambda) \le C(\boldsymbol{u})(h^{\frac{1}{2}+\nu}\|\boldsymbol{u} - \boldsymbol{u}_h\| + C(\boldsymbol{u})h^{1+2\nu}). \tag{4.16}$$

Step 3. End of the proof.

Let us insert results (4.12) and (4.16) into (4.8) and use estimate $ab \le \gamma a^2 + \dfrac{1}{4\gamma}b^2$. The first convergence result (4.6) of the theorem is then obtained.

Similarly, the second bound (4.7) of the theorem is proved by combining (4.15), (4.16) and (4.8). □

4.3. *The linear discrete non-interpenetration conditions*

The next theorem states convergence results in the case of (more classical) linear discretized non-interpenetration conditions. Notice that the convergence rate proved is exactly the same as in the quadratic case but the techniques used in the proof are not identical.

Theorem 4.4 *Set $M_h = L_h^{\ell,*}$ with $\ell = 1$ or 2 and let $(\boldsymbol{u}_h, \lambda_h)$ be the solution of* (3.1).

(i) Let $0 < \nu < 1$. Suppose that the solution $(\boldsymbol{u}, \lambda)$ of (2.8) *satisfies the regularity assumption $\boldsymbol{u}^1 \in (H^{\frac{3}{2}+\nu}(\Omega^1))^2$, $\boldsymbol{u}^2 \in (H^{\frac{3}{2}+\nu}(\Omega^2))^2$. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| + \|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \le C(\boldsymbol{u})h^{\frac{1}{2}+\frac{\nu}{2}}, \tag{4.17}$$

where the constant $C(\boldsymbol{u})$ depends linearly on $\|\boldsymbol{u}^1\|_{(H^{\frac{3}{2}+\nu}(\Omega^1))^2}$ and $\|\boldsymbol{u}^2\|_{(H^{\frac{3}{2}+\nu}(\Omega^2))^2}$.

(ii) Let $\frac{1}{2} < \nu < 1$. Suppose that the solution $(\boldsymbol{u}, \lambda)$ of (2.8) satisfies the regularity assumption $\boldsymbol{u}^1 \in (H^{\frac{3}{2}+\nu}(\Omega^1))^2$, $\boldsymbol{u}^2 \in (H^{\frac{3}{2}+\nu}(\Omega^2))^2$. Assume that the set of points of Γ_C where the change from $[\boldsymbol{u}.\boldsymbol{n}] < 0$ to $[\boldsymbol{u}.\boldsymbol{n}] = 0$ occurs is finite. Then

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| + \|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq C(\boldsymbol{u}) h^{\frac{1}{2}+\nu}, \tag{4.18}$$

where the constant $C(\boldsymbol{u})$ depends linearly on $\|\boldsymbol{u}^1\|_{(H^{\frac{3}{2}+\nu}(\Omega^1))^2}$ and $\|\boldsymbol{u}^2\|_{(H^{\frac{3}{2}+\nu}(\Omega^2))^2}$.

Proof. As in the quadratic case, it is obvious that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| + \|\lambda - \lambda_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq C\Big\{C(\boldsymbol{u}) h^{\frac{1}{2}+\nu} + \big(\max(b(\boldsymbol{u}, \lambda_h), 0)\big)^{\frac{1}{2}} + \big(\max(b(\boldsymbol{u}_h, \lambda), 0)\big)^{\frac{1}{2}}\Big\}, \tag{4.19}$$

and that the proof consists of estimating $b(\boldsymbol{u}, \lambda_h)$ and $b(\boldsymbol{u}_h, \lambda)$. We recall that the notation $C(\boldsymbol{u})$ stands for a generic constant depending on $\|\boldsymbol{u}^\ell\|_{(H^{\frac{3}{2}+\nu}(\Omega^\ell))^2}$, $(\ell = 1, 2)$ in a linear way.

Step 1. Estimation of $b(\boldsymbol{u}, \lambda_h)$.

The proof of Step 1 in the previous theorem is still valid so that we obtain again estimates (4.12) and (4.15) depending on the assumptions of the theorem.

Step 2. Estimation of $b(\boldsymbol{u}_h, \lambda)$.

Let $X_h^\ell(\Gamma_C)$ be the space of the piecewise continuous functions on Γ_C which are constant on the meshes of Ω^ℓ on Γ_C. Define Π_h^ℓ as the projection operator for the $L^2(\Gamma_C)$ inner product on $X_h^\ell(\Gamma_C)$. Such an operator satisfies the following estimate for any $0 \leq r \leq 1$:

$$\forall \varphi \in H^r(\Gamma_C), \quad \|\varphi - \Pi_h^\ell \varphi\|_{L^2(\Gamma_C)} \leq C\, h_\ell^r\, \|\varphi\|_{H^r(\Gamma_C)}. \tag{4.20}$$

According to Proposition 3.3, we have $(\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}])(\boldsymbol{a}) \leq 0$ for any $\boldsymbol{a} \in \xi_h^\ell$. This implies that

$$\Pi_h^\ell(\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}]) \leq 0 \quad \text{on } \Gamma_C.$$

As a consequence

$$\begin{aligned}
b(\boldsymbol{u}_h, \lambda) &= \int_{\Gamma_C} \lambda[\boldsymbol{u}_h.\boldsymbol{n}] d\Gamma \\
&= \int_{\Gamma_C} \lambda([\boldsymbol{u}_h.\boldsymbol{n}] - \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}]) d\Gamma + \int_{\Gamma_C} \lambda(\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] - \Pi_h^\ell \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}]) d\Gamma \\
&\quad + \int_{\Gamma_C} \lambda\, \Pi_h^\ell \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] d\Gamma \\
&\leq \int_{\Gamma_C} \lambda([\boldsymbol{u}_h.\boldsymbol{n}] - \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}]) d\Gamma + \int_{\Gamma_C} \lambda(\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] - \Pi_h^\ell \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}]) d\Gamma.
\end{aligned}$$

The term $\int_{\Gamma_C} \lambda([\boldsymbol{u}_h.\boldsymbol{n}] - \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}])d\Gamma$ has already been estimated in step 2 of Theorem 4.2 and bounded in (4.16). The remaining term is developed as follows:

$$\begin{aligned}
&\int_{\Gamma_C} \lambda(\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] - \Pi_h^\ell \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}])d\Gamma \\
&= \int_{\Gamma_C} \lambda((\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] - [\boldsymbol{u}_h.\boldsymbol{n}]) - \Pi_h^\ell(\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] - [\boldsymbol{u}_h.\boldsymbol{n}]))d\Gamma \\
&\quad + \int_{\Gamma_C} \lambda([\boldsymbol{u}_h.\boldsymbol{n}] - \Pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}])d\Gamma \\
&= \int_{\Gamma_C} (\lambda - \Pi_h^\ell\lambda)((\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] - [\boldsymbol{u}_h.\boldsymbol{n}]) - \Pi_h^\ell(\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] - [\boldsymbol{u}_h.\boldsymbol{n}]))d\Gamma \\
&\quad + \int_{\Gamma_C} (\lambda - \Pi_h^\ell\lambda)([\boldsymbol{u}_h.\boldsymbol{n}] - \Pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}])d\Gamma.
\end{aligned}$$

Using the (obvious) stability of Π_h^ℓ in the $L^2(\Gamma_C)$-norm and developing more the last integral term gives

$$\begin{aligned}
&\int_{\Gamma_C} \lambda(\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] - \Pi_h^\ell \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}])d\Gamma \\
&\le 2\|\lambda - \Pi_h^\ell\lambda\|_{L^2(\Gamma_C)}\|\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] - [\boldsymbol{u}_h.\boldsymbol{n}]\|_{L^2(\Gamma_C)} \\
&\quad + \int_{\Gamma_C} (\lambda - \Pi_h^\ell\lambda)(([\boldsymbol{u}_h.\boldsymbol{n}] - [\boldsymbol{u}.\boldsymbol{n}]) - \Pi_h^\ell([\boldsymbol{u}_h.\boldsymbol{n}] - [\boldsymbol{u}.\boldsymbol{n}]))d\Gamma \\
&\quad + \int_{\Gamma_C} (\lambda - \Pi_h^\ell\lambda)([\boldsymbol{u}.\boldsymbol{n}] - \Pi_h^\ell[\boldsymbol{u}.\boldsymbol{n}])d\Gamma.
\end{aligned}$$

Now, we use the approximation properties (4.20) and (4.5) of Π_h^ℓ and π_h^ℓ in the $L^2(\Gamma_C)$-norm. That gives

$$\begin{aligned}
&\int_{\Gamma_C} \lambda(\pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}] - \Pi_h^\ell \pi_h^\ell[\boldsymbol{u}_h.\boldsymbol{n}])d\Gamma \\
&\le C(\boldsymbol{u})h^\nu\Big(\|\pi_h^\ell([\boldsymbol{u}_h.\boldsymbol{n}] - [\boldsymbol{u}.\boldsymbol{n}]) - ([\boldsymbol{u}_h.\boldsymbol{n}] - [\boldsymbol{u}.\boldsymbol{n}])\|_{L^2(\Gamma_C)} + \|\pi_h^\ell[\boldsymbol{u}.\boldsymbol{n}] - [\boldsymbol{u}.\boldsymbol{n}]\|_{L^2(\Gamma_C)}\Big) \\
&\quad + C(\boldsymbol{u})h^{\frac{1}{2}+\nu}\|\boldsymbol{u} - \boldsymbol{u}_h\| + \int_{\Gamma_C} (\lambda - \Pi_h^\ell\lambda)([\boldsymbol{u}.\boldsymbol{n}] - \Pi_h^\ell[\boldsymbol{u}.\boldsymbol{n}])d\Gamma \\
&\le C(\boldsymbol{u})(h^{\frac{1}{2}+\nu}\|\boldsymbol{u} - \boldsymbol{u}_h\| + C(\boldsymbol{u})h^{1+2\nu}) + \int_{\Gamma_C} (\lambda - \Pi_h^\ell\lambda)([\boldsymbol{u}.\boldsymbol{n}] - \Pi_h^\ell[\boldsymbol{u}.\boldsymbol{n}])d\Gamma. \qquad (4.21)
\end{aligned}$$

Using again the approximation properties of Π_h^ℓ gives

$$\begin{aligned}
\int_{\Gamma_C} (\lambda - \Pi_h^\ell\lambda)([\boldsymbol{u}.\boldsymbol{n}] - \Pi_h^\ell[\boldsymbol{u}.\boldsymbol{n}])d\Gamma &\le \|\lambda - \Pi_h^\ell\lambda\|_{L^2(\Gamma_C)}\|[\boldsymbol{u}.\boldsymbol{n}] - \Pi_h^\ell[\boldsymbol{u}.\boldsymbol{n}]\|_{L^2(\Gamma_C)} \\
&\le Ch^\nu\|\lambda\|_{H^\nu(\Gamma_C)}h\|[\boldsymbol{u}.\boldsymbol{n}]\|_{H^1(\Gamma_C)}.
\end{aligned}$$

Here, we observe a loss of optimality when approximating the function $[\boldsymbol{u}.\boldsymbol{n}] \in H^{1+\nu}(\Gamma_C)$ with $\Pi_h^\ell[\boldsymbol{u}.\boldsymbol{n}]$. As a consequence

$$b(\boldsymbol{u}_h, \lambda) \le C(\boldsymbol{u})(h^{\frac{1}{2}+\nu}\|\boldsymbol{u} - \boldsymbol{u}_h\| + C(\boldsymbol{u})h^{1+\nu}). \qquad (4.22)$$

Consider estimate (4.21) and suppose now that $\frac{1}{2} < \nu < 1$. Let $N(h)$ represent as in the previous theorem the number of (1D)-segments denoted T_i $(1 \leq i \leq N(h))$, of the triangulation of Ω^ℓ on Γ_C where the change from $[\boldsymbol{u}.\boldsymbol{n}] < 0$ to $[\boldsymbol{u}.\boldsymbol{n}] = 0$ occurs. The integral term in (4.21) is now estimated as follows:

$$\begin{aligned}\int_{\Gamma_C} (\lambda - \Pi_h^\ell \lambda)([\boldsymbol{u}.\boldsymbol{n}] - \Pi_h^\ell[\boldsymbol{u}.\boldsymbol{n}])d\Gamma &= -\int_{\Gamma_C} \lambda\, \Pi_h^\ell[\boldsymbol{u}.\boldsymbol{n}]d\Gamma \\ &\leq \sum_{i=1}^{N(h)} \int_{T_i} |\lambda|\, |\Pi_h^\ell[\boldsymbol{u}.\boldsymbol{n}]|d\Gamma \\ &\leq h_\ell \sum_{i=1}^{N(h)} \|\lambda\|_{L^\infty(T_i)}\, \|\Pi_h^\ell[\boldsymbol{u}.\boldsymbol{n}]\|_{L^\infty(T_i)} \\ &\leq h_\ell \sum_{i=1}^{N(h)} \|\lambda\|_{L^\infty(T_i)}\, \|[\boldsymbol{u}.\boldsymbol{n}]\|_{L^\infty(T_i)}.\end{aligned}$$

The latter term has already been estimated in (4.13). Finally

$$b(\boldsymbol{u}_h, \lambda) \leq C(\boldsymbol{u})(h^{\frac{1}{2}+\nu}\|\boldsymbol{u} - \boldsymbol{u}_h\| + C(\boldsymbol{u})h^{1+2\nu}). \tag{4.23}$$

Step 3. End of the proof.

We combine (4.19) with (4.22) and estimate (4.12) which is still valid. The first convergence result (4.17) of the theorem is then obtained. In the same way, the second bound (4.18) is proved by putting together (4.15) which remains true, (4.23) and (4.19). □

5. Numerical experiments

In this part, we solve numerically two examples of contact problems with quadratic finite elements. Let us mention that we only focus on the numerical convergence rates of the finite element methods in the L^2 and H^1-norms. We compute these rates by considering families of uniform meshes made of triangular or quadrilateral elements. We skip over the study concerning optimized computations obtained with a posteriori error estimators and mesh adaptivity procedures which is beyond the scope of this paper.

Obviously, $(\boldsymbol{u}_h, \lambda_h) \in \mathbf{V}_h \times M_h$ (where $M_h = Q_h^{\ell,*}$ or $M_h = L_h^{\ell,*}$ with $\ell = 1$ or 2) is the solution of (3.1) if and only if $(\boldsymbol{u}_h, \lambda_h)$ is a saddle-point of the Lagrangian defined by

$$\mathscr{L}(\boldsymbol{v}_h, \mu_h) = \frac{1}{2}a(\boldsymbol{v}_h, \boldsymbol{v}_h) - L(\boldsymbol{v}_h) + \int_{\Gamma_C} \mu_h[\boldsymbol{v}_h.\boldsymbol{n}]d\Gamma,$$

so that problem (3.1) consists of finding $(\boldsymbol{u}_h, \lambda_h) \in \mathbf{V}_h \times M_h$ satisfying

$$\mathscr{L}(\boldsymbol{u}_h, \mu_h) \leq \mathscr{L}(\boldsymbol{u}_h, \lambda_h) \leq \mathscr{L}(\boldsymbol{v}_h, \lambda_h), \qquad \forall \boldsymbol{v}_h \in \mathbf{V}_h,\ \forall \mu_h \in M_h.$$

The solving of the saddle-point problem is achieved with the finite element code CASTEM 2000 developed at the CEA and an SUN UltraSparc computer has been used. For more details concerning the algebraic formulation of the problem, we refer the reader to [15]. In the numerical examples, we choose Hooke's law as a constitutive relation in (2.2):

$$\sigma_{ij}^{\ell} = \frac{E_\ell \nu_\ell}{(1-2\nu_\ell)(1+\nu_\ell)} \delta_{ij} \varepsilon_{kk}(\boldsymbol{u}^\ell) + \frac{E_\ell}{1+\nu_\ell} \varepsilon_{ij}(\boldsymbol{u}^\ell) \qquad \text{in } \Omega^\ell,$$

where E_ℓ denotes Young's modulus, ν_ℓ is Poisson's ratio and δ_{ij} represents the Kronecker symbol.

5.1. *First example*

We consider the problem depicted in Figure 1. The length of the edges of Ω^1 and Ω^2 is 10 mm and plane strain conditions are assumed. Let Ω^1 and Ω^2 be characterized by Poisson's ratios $\nu_1 = 0.4$, $\nu_2 = 0.2$ and Young's modulus $E_1 = 20000$ Mpa, $E_2=$ 15000 Mpa. Both bodies are clamped on $\Gamma_D^\ell, \ell = 1, 2$. On the boundary parts Γ_N^1 and Γ_N^2 the applied loads are of $(1, -10)$ daN.mm^{-2} and $(-1, -10)$ daN.mm^{-2} respectively and body forces are absent.

Unilateral contact problems generally do not admit analytical solutions. In order to obtain error estimates, we must compute a reference solution denoted $\boldsymbol{u}_{ref}$ corresponding to a mesh which is as fine as possible. Practically, a family of nested meshes is built from a coarse mesh by the natural subdivision of a triangle (or a quadrangle) into four triangles (or quadrangles). The most refined mesh furnishes the reference solution $\boldsymbol{u}_{ref}$. To obtain the convergence curve of the error, the approximate solutions $\boldsymbol{u}_h$ are computed on the other meshes (excepted on the both most refined).

We consider here only the case of matching meshes. We compare quadratic finite elements and linear non-interpenetration conditions (i.e. $M_h = L_h^{1,*} = L_h^{2,*}$ corresponding to the theoretical results of Theorem 4.4) with linear finite elements (and linear non-interpenetration conditions corresponding to the theoretical results in [15, 7]). Moreover, the computations are performed on meshes made of triangular elements or rectangular elements.

In the present example, we can numerically observe a separation of the bodies (i.e. $[\boldsymbol{u}_h.\boldsymbol{n}] < 0$) on the right part of the contact zone (see Figure 2). The relative errors for the displacements in the H^1 and in the L^2-norms are reported in Tables 1 and 2. Supposing that these errors behave like Ch^α where h denotes the discretization parameter and C is a constant, we can deduce the convergence rates α shown in Table 3. Finally, Figures 3,4 depict the relative errors as a function of the number of degrees of freedom.

On this example, the convergence rates are a bit greater when quadratic finite elements are used than for linear finite elements. Note also that for a given number of degrees of freedom, we observe (in Figures 3 and 4) that the error in the L^2 and H^1-norms when using quadratic rectangles is about the half of the error obtained with linear rectangles. When using triangles, the error in the quadratic case is roughly the third of the linear case.

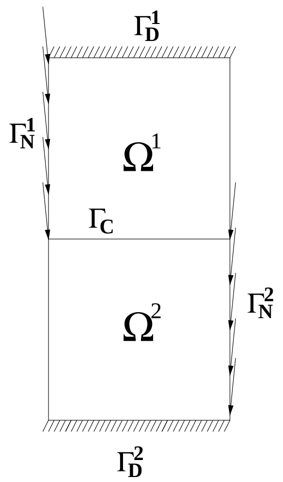

Figure 1: Problem set-up

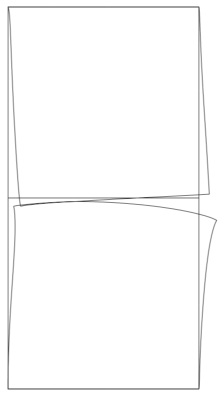

Figure 2: The reference solution with quadratic triangles (deformation is amplified)

Number of elements on Γ_C	1	2	4	8	16	32
6-node triangles	34 , 029	17 , 150	8 , 5397	4 , 7753	2 , 6939	
8-node quadrangles	28 , 289	14 , 283	7 , 4549	4 , 2336	2 , 3959	
3-node triangles	61 , 872	45 , 662	28 , 448	16 , 192	9 , 0573	4 , 9451
4-node quadrangles	56 , 307	33 , 822	19 , 086	10 , 708	6 , 0111	3 , 2980

Table 1: The relative error $\frac{\|\boldsymbol{u}_{ref}-\boldsymbol{u}_h\|}{\|\boldsymbol{u}_{ref}+\boldsymbol{u}_h\|}$ (in %)

Number of elements on Γ_C	1	2	4	8	16	32
6-node triangles	$23,178$	$6,6656$	$2,0487$	$0,72263$	$0,25569$	
8-node quadrangles	$16,700$	$5,2758$	$1,7360$	$0,62892$	$0,22302$	
3-node triangles	$55,966$	$33,644$	$16,594$	$6,3599$	$2,2158$	$0,73018$
4-node quadrangles	$43,848$	$20,740$	$7,4853$	$2,6125$	$0,90513$	$0,30437$

Table 2: The relative error $\frac{\|\boldsymbol{u}_{ref}-\boldsymbol{u}_h\|_{(L^2(\Omega^1\cup\Omega^2))^2}}{\|\boldsymbol{u}_{ref}+\boldsymbol{u}_h\|_{(L^2(\Omega^1\cup\Omega^2))^2}}$ (in %)

Norm	H^1	L^2
6-node triangles	$0,9147$	$1,6255$
8-node quadrangles	$0,8623$	$1,5566$
3-node triangles	$0,7290$	$1,2520$
4-node quadrangles	$0,8187$	$1,4341$

Table 3: The convergence rates of the error

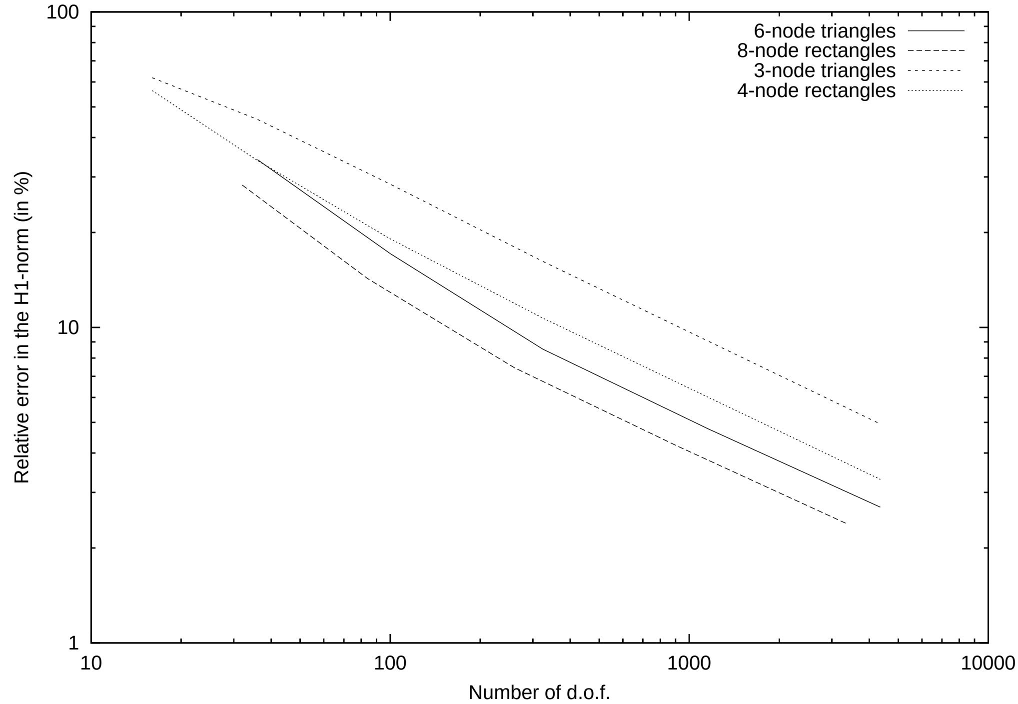

Figure 3: The relative error $\frac{\|\boldsymbol{u}_{ref}-\boldsymbol{u}_h\|}{\|\boldsymbol{u}_{ref}+\boldsymbol{u}_h\|}$ (in %)

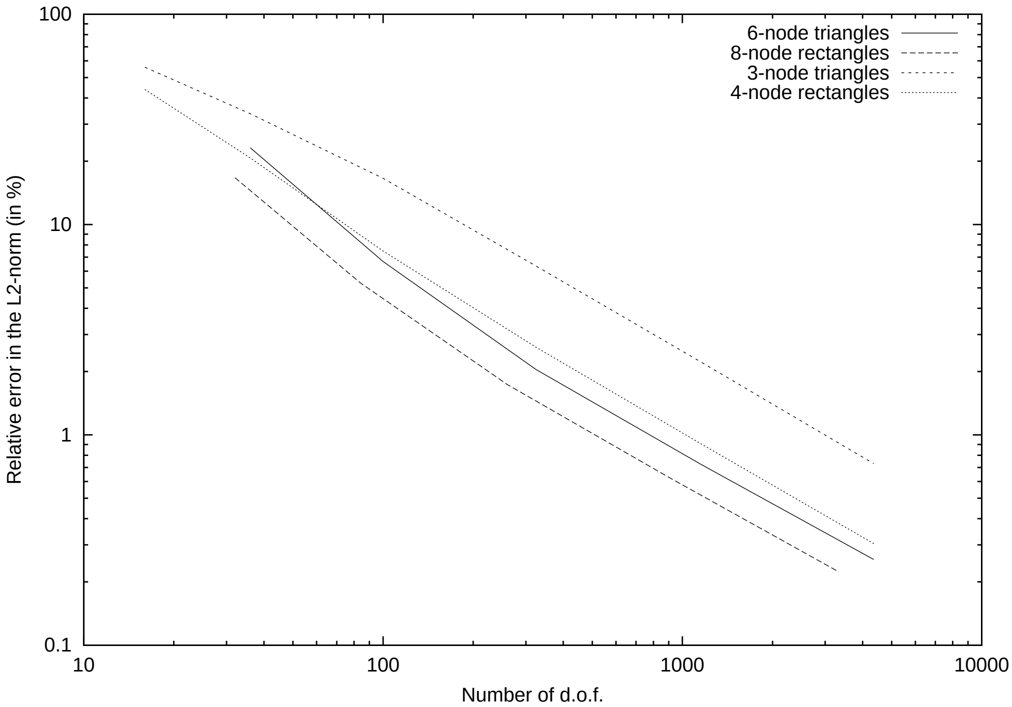

Figure 4: The relative error $\frac{\|\boldsymbol{u}_{ref}-\boldsymbol{u}_h\|_{(L^2(\Omega^1\cup\Omega^2))^2}}{\|\boldsymbol{u}_{ref}+\boldsymbol{u}_h\|_{(L^2(\Omega^1\cup\Omega^2))^2}}$ (in %)

5.2. *Second example*

The method described in the theoretical part can also be used in the more simple case of a single deformable body in contact with a rigid foundation as suggested in Figure 5. Note that this problem does not involve Dirichlet conditions but only symmetry conditions in order to remove singularities coming from adjacent Dirichlet and Neumann conditions. This implies that the problem is not **V**-elliptic but only **K**-elliptic according to [12]. The length of an edge of the square is 10 mm and the elastic characteristics are $\nu = 0.3$, $E = 20000$ Mpa. The applied loads on both parts of the boundary represent 1 daN.mm^{-2} and no body forces are assumed.

The computation shows a separation of the body on the left part of Γ_C (see Figure 6). The same convergence studies as in the previous example are achieved and reported in Tables 4-6. The convergence rates obtained are a little greater than in the previous example and the use of quadratic finite elements remains also somewhat more attractive than linear finite elements.

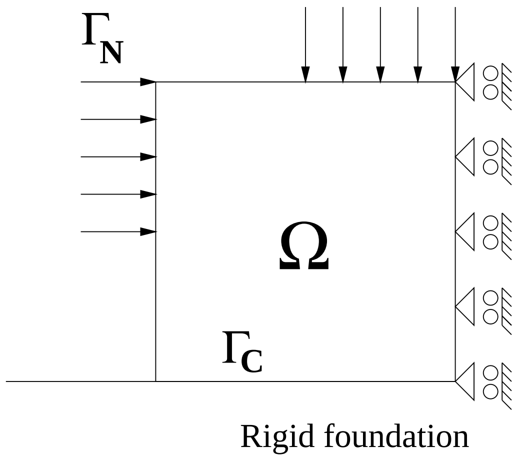

Figure 5: Problem set-up

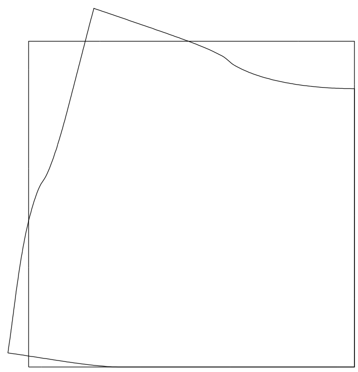

Figure 6: The reference solution with quadratic triangles (deformation is amplified)

Number of elements on Γ_C	2	4	8	16	32
6-node triangles	9 , 0641	4 , 9046	2 , 5025	1 , 2076	
8-node quadrangles	7 , 4772	3 , 6397	1 , 8045	0 , 8678	
3-node triangles	33 , 332	19 , 021	10 , 505	5 , 7683	3 , 0515
4-node quadrangles	20 , 327	11 , 457	6 , 4669	3 , 5615	1 , 8886

Table 4: The relative error $\frac{\|\boldsymbol{u}_{ref}-\boldsymbol{u}_h\|}{\|\boldsymbol{u}_{ref}+\boldsymbol{u}_h\|}$ (in %)

Number of elements on Γ_C	2	4	8	16	32
6-node triangles	3 , 0676	0 , 58016	0 , 20992	0 , 038634	
8-node quadrangles	2 , 9787	0 , 49349	0 , 12561	0 , 025955	
3-node triangles	27 , 446	8 , 6411	2 , 4757	0 , 71111	0 , 18505
4-node quadrangles	8 , 6897	3 , 0529	0 , 72766	0 , 22282	0 , 060311

Table 5: The relative error $\frac{\|\boldsymbol{u}_{ref}-\boldsymbol{u}_h\|_{(L^2(\Omega^1\cup\Omega^2))^2}}{\|\boldsymbol{u}_{ref}+\boldsymbol{u}_h\|_{(L^2(\Omega^1\cup\Omega^2))^2}}$ (in %)

Norm	H^1	L^2
6-node triangles	$0,9693$	$2,104$
8-node quadrangles	$1,036$	$2,280$
3-node triangles	$0,8623$	$1,803$
4-node quadrangles	$0,8570$	$1,793$

Table 6: The convergence rates of the error

6. Conclusion

This paper is a contribution to the numerical analysis and the implementation of quadratic finite elements for unilateral contact problems. We have proposed and studied two mixed quadratic finite element methods (in which the non-interpenetration conditions are either of linear or of quadratic type) and proved that they can lead to optimal convergence rates under reasonable hypotheses. From the numerical examples, it seems that the quadratic finite element approach gives a little better results than linear finite elements.

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