On finite element uniqueness studies for Coulomb's frictional contact model

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We are interested in the finite element approximation of Coulomb frictional unilateral contact problem in linear elasticity. Using a mixed finite element method and an appropriate regularization, it becomes possible to prove existence and uniqueness when the friction coefficient is lower than $C\varepsilon^2|\log(h)|^{-1}$ where h and ε denote the discretization and the regularization parameters respectively. This bound converging very slowly towards 0 when h decreases (in comparison with the already known results of the non-regularized case) suggests a minor dependence of the mesh-size on the uniqueness conditions, at least for practical engineering computations. Then we study the solutions of a simple finite element example in the non-regularized case. It can be shown that one, multiple or an infinity of solutions can occur and that, for a given loading, the number of solutions can eventually decrease when the friction coefficient increases.

Keywords : Coulomb friction law, finite elements, mesh-size dependent uniqueness conditions, non-uniqueness example

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1. Introduction and problem set-up

The Coulomb friction model is currently chosen in the numerical approximation of contact problems arising in structural mechanics. From a mathematical point of view, the study of the continuous model in elastostatics using the associated variational formulation obtained in [6] leads to existence results when the friction coefficient is sufficiently small (see [16, 13, 14, 7]). Concerning the associated finite element model, it has been proved in [8, 9] that it admits always a solution and that the solution is unique provided that the friction coefficient is lower than a positive value vanishing as the discretization parameter decreases. Also in reference [8], a convergence result of the finite element model towards the continuous model was established. Besides, in the finite dimensional context, numerous studies and examples of non-uniqueness using truss elements have been exhibited, proving that the problem is generally not well-posed (see in particular the contributions of [12, 15, 2]).

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Our first aim in this paper is to study the influence of a specific regularization (i.e.; the smoothing of the absolute value involved in the friction model) on the uniqueness conditions for the discrete problem. We consider a mixed finite element method in section 2 and, denoting by h and ε the discretization and the regularization parameters respectively, we show in section 3 that the problem admits a unique solution if the friction coefficient is lower than $C\varepsilon^2|\log(h)|^{-1}$, and we notice that a bound of only $Ch^{\frac{1}{2}}$ can be obtained in the case of the exact model (i.e.; when $\varepsilon = 0$). As a consequence, we note that if ε is chosen slowly decreasing towards zero (as h decreases), then the bound of the non-regularized case becomes more satisfactory than the one arising from the exact model.

Our second aim, in section 4, is to choose a particular case of a finite dimensional problem in the non-regularized case: a simple example using finite elements. We study this problem and we show that it can admit one, multiple or an infinity of solutions. Such an example completes and illustrates the already known results using truss elements, especially [15].

Let us now consider an elastic body occupying in the initial configuration a bounded subset $\overline{\Omega}$ of $\mathbb{R}^2$. The boundary $\partial\Omega$ of the domain Ω is supposed to be Lipschitz and consists of three nonoverlapping parts Γ_D, Γ_N and Γ_C. The unit outward normal on $\partial\Omega$ is denoted $\boldsymbol{n} = (n_1, n_2)$ and we set $\boldsymbol{t} = (n_2, -n_1)$. The body is submitted to volume forces $\boldsymbol{f} = (f_1, f_2) \in (L^2(\Omega))^2$ on Ω and to surface forces $\boldsymbol{F} = (F_1, F_2) \in (L^2(\Gamma_N))^2$ on Γ_N. The part Γ_D is embedded and we suppose that the surface measure of Γ_D does not vanish. Initially, the body is in contact with a rigid foundation on the straight line segment Γ_C.

The unilateral contact problem with Coulomb friction consists of finding the displacement field $\boldsymbol{u} = (u_i),\ 1 \le i \le 2$ and the stress tensor field $\boldsymbol{\sigma} = (\sigma_{ij}),\ 1 \le i,j \le 2$, satisfying the following conditions (1.1)–(1.4):

$$\boldsymbol{div}\ \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \boldsymbol{0} \quad \text{in } \Omega, \quad \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{F} \quad \text{on } \Gamma_N, \quad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D, \tag{1.1}$$

where $(\boldsymbol{div}\ \boldsymbol{\sigma}(\boldsymbol{u}))_i = \sigma_{ij,j},\ 1 \le i \le 2$, the notation $_{,j}$ denotes the $j-$th partial derivative and the summation convention of repeated indices is adopted. The stress tensor field is linked to the displacement field by the constitutive law of linear elasticity

$$\sigma_{ij}(\boldsymbol{u}) = \lambda\varepsilon_{kk}(\boldsymbol{u})\delta_{ij} + 2\mu\varepsilon_{ij}(\boldsymbol{u}), \tag{1.2}$$

where λ and μ are the positive Lamé coefficients and where $\varepsilon_{ij}(\boldsymbol{u}) = 1/2(u_{i,j} + u_{j,i})$ denotes the linearized strain tensor field.

On the boundary $\partial\Omega$, we write $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma_n(\boldsymbol{u})\boldsymbol{n} + \sigma_t(\boldsymbol{u})\boldsymbol{t}$ and $\boldsymbol{u} = u_n\boldsymbol{n} + u_t\boldsymbol{t}$. Let $\mathcal{F} > 0$ stand for the friction coefficient on Γ_C. The conditions on the contact zone Γ_C are as follows:

$$u_n \le 0, \quad \sigma_n(\boldsymbol{u}) \le 0, \quad \sigma_n(\boldsymbol{u})\ u_n = 0, \tag{1.3}$$

$$|\sigma_t(\boldsymbol{u})| \le \mathcal{F}|\sigma_n(\boldsymbol{u})|, \quad (|\sigma_t(\boldsymbol{u})| - \mathcal{F}|\sigma_n(\boldsymbol{u})|)u_t = 0, \quad \sigma_t(\boldsymbol{u})\ u_t \le 0. \tag{1.4}$$

The conditions (1.3) express unilateral contact and conditions (1.4) represent Coulomb friction. The closed convex cone $\boldsymbol{K}$ of admissible displacements is the subset in the

Sobolev space $(H^1(\Omega))^2$ of the displacement fields satisfying the embedding and the non-penetration conditions

$$\boldsymbol{K} = \Big\{ \boldsymbol{v} = (v_1, v_2) \in \boldsymbol{V}, \quad v_n \leq 0 \text{ on } \Gamma_C \Big\}, \tag{1.5}$$

where

$$\boldsymbol{V} = \Big\{ \boldsymbol{v} = (v_1, v_2) \in (H^1(\Omega))^2, \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D \Big\}.$$

As in [16], we consider the mapping $\Phi : M \rightarrow M$ with

$$M = \Big\{ \alpha \in H^{-\frac{1}{2}}(\Gamma_C), \quad \alpha \geq 0 \Big\},$$

defined for all $g \in M$ as follows: $\Phi(g) = -\sigma_n(\boldsymbol{u}(g))$, where $\boldsymbol{u}(g) \in \boldsymbol{K}$ is the unique solution of the variational inequality:

$$\begin{aligned} \boldsymbol{u}(g) \in \boldsymbol{K}, \quad & \int_\Omega \sigma_{ij}(\boldsymbol{u}(g))\, \varepsilon_{ij}(\boldsymbol{v} - \boldsymbol{u}(g))\, d\Omega + < \mathcal{F} g, |v_t| - |u_t(g)| >_{\Gamma_C} \\ & \geq \int_\Omega f_i(v_i - u_i(g))\, d\Omega + \int_{\Gamma_N} F_i(v_i - u_i(g))\, d\Gamma, \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \end{aligned} \tag{1.6}$$

where $< .,. >_{\Gamma_C}$ denotes the duality pairing between the fractional Sobolev space $H^{\frac{1}{2}}(\Gamma_C)$ (see [1]) and its dual space $H^{-\frac{1}{2}}(\Gamma_C)$. Following [16, 10], a weak solution of the unilateral contact problem with Coulomb friction is a pair $(\boldsymbol{u}, \gamma)$ where γ is a fixed point of Φ and $\boldsymbol{u}$ is the unique solution of Problem (1.6) with $g = \gamma$.

The first existence result for the unilateral contact problem with Coulomb friction in the case of a sufficiently small friction coefficient $\mathcal{F}$ has been proved in [16]. Generalizations and/or improvements are established in [13, 14, 7]. The uniqueness seems to remain an open problem.

2. The discrete problem

We discretize the domain Ω with a family of triangulations $(\mathscr{T}_h)_h$ where the notation $h > 0$ stands for the discretization parameter representing the greatest diameter of a triangle in $\mathscr{T}_h$. The chosen space of finite elements of degree one is:

$$\boldsymbol{V}_h = \Big\{ \boldsymbol{v}_h;\ \boldsymbol{v}_h \in (\mathscr{C}(\overline{\Omega}))^2,\ \boldsymbol{v}_h|_T \in (P_1(T))^2 \ \ \forall T \in \mathscr{T}_h,\ \boldsymbol{v}_h = \boldsymbol{0} \text{ on } \Gamma_D \Big\},$$

where $\mathscr{C}(\overline{\Omega})$ and $P_1(T)$ denote the space of continuous functions on $\overline{\Omega}$ and the space of polynomial functions of degree one on T respectively. We assume that the families of monodimensional traces of triangulations on Γ_C are quasi-uniform in order to use inverse inequalities (see [4]). Let W_h be the range of $\boldsymbol{V}_h$ by the normal trace operator on Γ_C:

$$W_h = \Big\{ \mu_h;\ \mu_h = \boldsymbol{v}_h|_{\Gamma_C}.\boldsymbol{n}, \quad \boldsymbol{v}_h \in \boldsymbol{V}_h \Big\}.$$

Clearly, the space W_h involves continuous and piecewise of degree one functions. We define M_h as the closed convex cone of Lagrange multipliers expressing nonnegativity:

$$M_h = \Big\{ \mu_h \in W_h, \quad \mu_h \geq 0 \Big\}.$$

For any $\boldsymbol{u}$ and $\boldsymbol{v}$ in $(H^1(\Omega))^2$, define

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, d\Omega, \quad L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}.\boldsymbol{v}\, d\Omega + \int_{\Gamma_N} \boldsymbol{F}.\boldsymbol{v}\, d\Gamma.$$

Finally, let us mention that we still keep the notation $\boldsymbol{v}_h = v_{hn}\boldsymbol{n} + v_{ht}\boldsymbol{t}$ on the boundary $\partial\Omega$, for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$.

For approximating Coulomb frictional contact problem, we choose a mixed finite element method with a nonnegative parameter ε regularizing the absolute value (the case $\varepsilon = 0$ corresponds to the non-regularized problem). As in the continuous framework (1.6), the approximated problem requires the introduction of an intermediate setting with a given slip limit $g_h \in M_h$. It consists of *finding* $\boldsymbol{u}_h \in \boldsymbol{V}_h$ *and* $\lambda_h \in M_h$ *such that*

$$\left\{ \begin{array}{l} a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + \displaystyle\int_{\Gamma_C} \lambda_h (v_{hn} - u_{hn})\, d\Gamma \\ \qquad + \displaystyle\int_{\Gamma_C} \mathcal{F} g_h \Big(\sqrt{v_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2} \Big)\, d\Gamma \geq L(\boldsymbol{v}_h - \boldsymbol{u}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ \displaystyle\int_{\Gamma_C} (\mu_h - \lambda_h) u_{hn}\, d\Gamma \leq 0, \qquad \forall \mu_h \in M_h. \end{array} \right. \tag{2.1}$$

Henceforth, problem (2.1) will be denoted $P_\varepsilon(g_h)$.

Remark 2.1 *It can be checked that if* $(\boldsymbol{u}_h, \lambda_h)$ *solves* (2.1) *then* $\boldsymbol{u}_h$ *is also solution of the variational inequality which consists of finding* $\boldsymbol{u}_h \in \boldsymbol{K}_h$ *satisfying*

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + \int_{\Gamma_C} \mathcal{F} g_h \Big(\sqrt{v_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2} \Big)\, d\Gamma \geq L(\boldsymbol{v}_h - \boldsymbol{u}_h),$$

for all $\boldsymbol{v}_h \in \boldsymbol{K}_h$*. Here,* $\boldsymbol{K}_h$ *stands for a finite dimensional approximation of* $\boldsymbol{K}$ *defined in* (1.5)*:*

$$\boldsymbol{K}_h = \Big\{ \boldsymbol{v}_h \in \boldsymbol{V}_h, \quad \int_{\Gamma_C} \mu_h v_{hn}\, d\Gamma \leq 0,\ \forall \mu_h \in M_h \Big\}.$$

Problem $P_\varepsilon(g_h)$ is also equivalent to find a saddle-point $(\boldsymbol{u}_h, \lambda_h) \in \boldsymbol{V}_h \times M_h$ verifying

$$\mathscr{L}(\boldsymbol{u}_h, \mu_h) \leq \mathscr{L}(\boldsymbol{u}_h, \lambda_h) \leq \mathscr{L}(\boldsymbol{v}_h, \lambda_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\ \forall \mu_h \in M_h,$$

where

$$\mathscr{L}(\boldsymbol{v}_h, \mu_h) = \frac{1}{2} a(\boldsymbol{v}_h, \boldsymbol{v}_h) + \int_{\Gamma_C} \mu_h v_{hn}\, d\Gamma + \int_{\Gamma_C} \mathcal{F} g_h \sqrt{v_{ht}^2 + \varepsilon^2}\, d\Gamma - L(\boldsymbol{v}_h).$$

From results concerning saddle-point problems obtained by Haslinger, Hlaváček and Nečas in [10], p.338, the existence of such a saddle-point follows. Moreover, the $\boldsymbol{V}$−ellipticity of $a(.,.)$ implies that the first argument $\boldsymbol{u}_h$ is unique. Besides, if for any $\mu_h \in W_h$, one has:

$$\int_{\Gamma_C} \mu_h v_{hn}\, d\Gamma = 0, \ \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \qquad \Longrightarrow \qquad \mu_h = 0, \tag{2.2}$$

then the second argument λ_h is unique and $P_\varepsilon(g_h)$ admits a unique solution. Note that condition (2.2) is fulfilled because the space W_h coincides with the space obtained from $\boldsymbol{V}_h$ by the normal trace operator on Γ_C.

It becomes then possible to define two maps: the first one denoted $\Psi_{\varepsilon h}$ yielding the first component (i.e.; $\Psi_{\varepsilon h}(g_h) = \boldsymbol{u}_h$) and the second one denoted $\Phi_{\varepsilon h}$ such that

$$\begin{aligned} \Phi_{\varepsilon h} : M_h &\longrightarrow M_h \\ g_h &\longmapsto \lambda_h, \end{aligned}$$

where $(\boldsymbol{u}_h, \lambda_h)$ is the solution of $P_\varepsilon(g_h)$. The introduction of the latter map allows the defining of a solution to Coulomb discrete frictional contact problem.

Definition 2.2 *A solution of Coulomb discrete regularized (resp. non-regularized) frictional contact problem is a solution of $P_\varepsilon(\lambda_h)$ with $\varepsilon > 0$ (resp. $\varepsilon = 0$) where $\lambda_h \in M_h$ is a fixed point of $\Phi_{\varepsilon h}$.*

Set

$$\tilde{\boldsymbol{V}}_h = \Big\{ \boldsymbol{v}_h \in \boldsymbol{V}_h,\ v_{ht} = 0 \text{ on } \Gamma_C \Big\}.$$

It is easy to check that the definition of $\|.\|_{-\frac{1}{2},h}$ given by

$$\|\nu\|_{-\frac{1}{2},h} = \sup_{\boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h} \frac{\displaystyle\int_{\Gamma_C} \nu v_{hn}\, d\Gamma}{\|\boldsymbol{v}_h\|_1}, \tag{2.3}$$

is a norm on W_h (since condition (2.2) holds). The notation $\|.\|_1$ represents the $(H^1(\Omega))^2$-norm.

3. Existence and uniqueness studies

We are now interested in the existence and uniqueness study for the discrete problem. In order to establish existence, it suffices to show that the mapping $\Phi_{\varepsilon h}$ admits a fixed point in M_h by using Brouwer's theorem. Uniqueness is ensured if the mapping is contractive. Such a technique has been already used in the non-regularized case with discontinuous and piecewise constant Lagrange multipliers [8, 9]. Our aim is to study the regularized case (and also the non-regularized case) when using continuous piecewise of degree one Lagrange multipliers.

Theorem 3.1 *Let $\varepsilon > 0$. The following results hold:*

(Existence) For any positive $\mathcal{F}$, there exists a solution to the Coulomb discrete regularized frictional contact problem.

(Uniqueness) Assume that $\overline{\Gamma}_D \cap \overline{\Gamma}_C = \emptyset$. If $\mathcal{F} \le C\varepsilon^2 |\log(h)|^{-1}$ then the problem admits a unique solution. The positive constant C depends neither of h nor of ε.

Proof. Let $(\boldsymbol{u}_h, \lambda_h)$ be the solution of $P_\varepsilon(g_h)$. Taking $\boldsymbol{v}_h = \boldsymbol{0}$ in the equation of (2.1) gives

$$a(\boldsymbol{u}_h, \boldsymbol{u}_h) + \int_{\Gamma_C} \lambda_h u_{hn}\, d\Gamma - \int_{\Gamma_C} \mathcal{F} g_h \Big(\varepsilon - \sqrt{u_{ht}^2 + \varepsilon^2} \Big)\, d\Gamma \le L(\boldsymbol{u}_h). \tag{3.1}$$

Since $g_h \geq 0$, $\varepsilon - \sqrt{u_{ht}^2 + \varepsilon^2} \leq 0$, and according to

$$\int_{\Gamma_C} \lambda_h u_{hn} \, d\Gamma = 0,$$

it follows from (3.1), the $\boldsymbol{V}-$ellipticity of $a(.,.)$ and the continuity of $L(.)$ that:

$$\alpha \|\boldsymbol{u}_h\|_1^2 \leq a(\boldsymbol{u}_h, \boldsymbol{u}_h) \leq L(\boldsymbol{u}_h) \leq C\|\boldsymbol{u}_h\|_1,$$

where α stands for the ellipticity constant of $a(.,.)$. Here, the constant C depends on the external loads $\boldsymbol{f}$ and $\boldsymbol{F}$. Therefore, using the trace theorem yields

$$\|u_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)} \leq C'\|\boldsymbol{u}_h\|_1 \leq \frac{CC'}{\alpha}. \tag{3.2}$$

Besides, the equality in (2.1) implies

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + \int_{\Gamma_C} \lambda_h v_{hn} \, d\Gamma = L(\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h.$$

Denoting by M' the continuity constant of $a(.,.)$ yields

$$\int_{\Gamma_C} \lambda_h v_{hn} \, d\Gamma \leq M'\|\boldsymbol{u}_h\|_1 \|\boldsymbol{v}_h\|_1 + C\|\boldsymbol{v}_h\|_1, \qquad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h.$$

As a consequence

$$\|\lambda_h\|_{-\frac{1}{2},h} \leq M'\|\boldsymbol{u}_h\|_1 + C \leq \Big(\frac{M'}{\alpha} + 1\Big)C.$$

So, we come to the conclusion that

$$\|\Phi_{\varepsilon h}(g_h)\|_{-\frac{1}{2},h} \leq C', \qquad \forall g_h \in M_h, \tag{3.3}$$

where C' only depends on the applied loads $\boldsymbol{f}, \boldsymbol{F}$ and on the continuity and ellipticity constant of $a(.,.)$.

The existence result of Theorem 3.1 consists now to show that the mapping $\Phi_{\varepsilon h}$ is continuous.

Let $(\boldsymbol{u}_h, \lambda_h)$ and $(\overline{\boldsymbol{u}}_h, \overline{\lambda}_h)$ be the solutions of $P_\varepsilon(g_h)$ and $P_\varepsilon(\overline{g}_h)$ respectively (where $g_h \in M_h$ and $\overline{g}_h \in M_h$). From (2.1), we get

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + \int_{\Gamma_C} \lambda_h v_{hn} \, d\Gamma = L(\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h,$$

and

$$a(\overline{\boldsymbol{u}}_h, \boldsymbol{v}_h) + \int_{\Gamma_C} \overline{\lambda}_h v_{hn} \, d\Gamma = L(\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h,$$

which implies by subtraction that

$$\int_{\Gamma_C} (\lambda_h - \overline{\lambda}_h) v_{hn}\, d\Gamma = a(\overline{\boldsymbol{u}}_h - \boldsymbol{u}_h, \boldsymbol{v}_h) \leq M' \|\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h\|_1 \|\boldsymbol{v}_h\|_1, \qquad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h,$$

where the continuity of the bilinear form $a(.,.)$ has been used. So, we get the following estimate

$$\|\lambda_h - \overline{\lambda}_h\|_{-\frac{1}{2},h} \leq M' \|\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h\|_1. \tag{3.4}$$

Next, we show that $\Psi_{\varepsilon h}$ is continuous from M_h into $\boldsymbol{V}_h$. We consider again $(\boldsymbol{u}_h, \lambda_h)$ and $(\overline{\boldsymbol{u}}_h, \overline{\lambda}_h)$, the solutions of $P_\varepsilon(g_h)$ and $P_\varepsilon(\overline{g}_h)$ respectively. We have

$$\begin{aligned} & a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + \int_{\Gamma_C} \lambda_h (v_{hn} - u_{hn})\, d\Gamma \\ & + \int_{\Gamma_C} \mathcal{F} g_h \Big(\sqrt{v_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2} \Big)\, d\Gamma \geq L(\boldsymbol{v}_h - \boldsymbol{u}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \end{aligned}$$

and

$$\begin{aligned} & a(\overline{\boldsymbol{u}}_h, \boldsymbol{v}_h - \overline{\boldsymbol{u}}_h) + \int_{\Gamma_C} \overline{\lambda}_h (v_{hn} - \overline{u}_{hn})\, d\Gamma \\ & + \int_{\Gamma_C} \mathcal{F}\, \overline{g}_h \Big(\sqrt{v_{ht}^2 + \varepsilon^2} - \sqrt{\overline{u}_{ht}^2 + \varepsilon^2} \Big)\, d\Gamma \geq L(\boldsymbol{v}_h - \overline{\boldsymbol{u}}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{aligned}$$

Choosing $\boldsymbol{v}_h = \overline{\boldsymbol{u}}_h$ in the first inequality and $\boldsymbol{v}_h = \boldsymbol{u}_h$ in the second one, we obtain thanks to (2.1):

$$a(\boldsymbol{u}_h, \overline{\boldsymbol{u}}_h - \boldsymbol{u}_h) + \int_{\Gamma_C} \mathcal{F} g_h \Big(\sqrt{\overline{u}_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2} \Big)\, d\Gamma \geq L(\overline{\boldsymbol{u}}_h - \boldsymbol{u}_h),$$

and

$$a(\overline{\boldsymbol{u}}_h, \boldsymbol{u}_h - \overline{\boldsymbol{u}}_h) + \int_{\Gamma_C} \mathcal{F}\, \overline{g}_h \Big(\sqrt{u_{ht}^2 + \varepsilon^2} - \sqrt{\overline{u}_{ht}^2 + \varepsilon^2} \Big)\, d\Gamma \geq L(\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h).$$

Thus

$$a(\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h, \boldsymbol{u}_h - \overline{\boldsymbol{u}}_h) \leq \int_{\Gamma_C} \mathcal{F} (g_h - \overline{g}_h) \Big(\sqrt{\overline{u}_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2} \Big)\, d\Gamma. \tag{3.5}$$

So

$$\alpha \|\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h\|_1^2 \leq \mathcal{F} \|g_h - \overline{g}_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \Big\| \sqrt{\overline{u}_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2} \Big\|_{H^{\frac{1}{2}}(\Gamma_C)}. \tag{3.6}$$

The next step consists of estimating the $H^{\frac{1}{2}}$-norm term in (3.6). To attain our ends, we need to use two lemmas which follow hereafter.

Lemma 3.2 *There exists a positive constant* C *satisfying for all* f *and* g *in* $H^{\frac{1}{2}}(\Gamma_C)\cap L^\infty(\Gamma_C)$*:*

$$\|fg\|_{H^{\frac{1}{2}}(\Gamma_C)} \leq C\Big(\|f\|_{H^{\frac{1}{2}}(\Gamma_C)}\|g\|_{L^\infty(\Gamma_C)} + \|f\|_{L^\infty(\Gamma_C)}\|g\|_{H^{\frac{1}{2}}(\Gamma_C)}\Big). \tag{3.7}$$

Proof. From the definition of the $H^{\frac{1}{2}}(\Gamma_C)-$norm (see [1]), we have

$$\|fg\|^2_{H^{\frac{1}{2}}(\Gamma_C)} = \|fg\|^2_{L^2(\Gamma_C)} + \int_{\Gamma_C}\int_{\Gamma_C} \frac{(f(x)g(x)-f(y)g(y))^2}{(x-y)^2}\, d\Gamma d\Gamma.$$

Let us begin with bounding (roughly) the first term:

$$\|fg\|^2_{L^2(\Gamma_C)} = \int_{\Gamma_C} f^2(x)g^2(x)\, d\Gamma \leq \|f\|^2_{L^2(\Gamma_C)}\|g\|^2_{L^\infty(\Gamma_C)}. \tag{3.8}$$

The second term is handled as follows:

$$\begin{aligned}
&\int_{\Gamma_C}\int_{\Gamma_C} \frac{(f(x)(g(x)-g(y))+g(y)(f(x)-f(y)))^2}{(x-y)^2}\, d\Gamma d\Gamma \\
&\leq 2\int_{\Gamma_C}\int_{\Gamma_C} \frac{f^2(x)(g(x)-g(y))^2}{(x-y)^2} + \frac{g^2(y)(f(x)-f(y))^2}{(x-y)^2}\, d\Gamma d\Gamma \\
&\leq 2\Big(\|f\|^2_{L^\infty(\Gamma_C)}\|g\|^2_{H^{\frac{1}{2}}(\Gamma_C)} + \|f\|^2_{H^{\frac{1}{2}}(\Gamma_C)}\|g\|^2_{L^\infty(\Gamma_C)}\Big).
\end{aligned} \tag{3.9}$$

Putting together (3.8) and (3.9) establishes (3.7). □

Lemma 3.3 *For any real number* $p \in [1,\infty[$*, the following inequality holds:*

$$\|f\|_{L^p(\Gamma_C)} \leq C\sqrt{p}\|f\|_{H^{\frac{1}{2}}(\Gamma_C)}, \quad \forall f \in H^{\frac{1}{2}}(\Gamma_C), \tag{3.10}$$

where C *is independent of* p*.*

Proof. see [3], Lemma A.1. □

Proof of Theorem 3.1. We consider the $H^{\frac{1}{2}}$-norm term in (3.6). Employing estimate (3.7) gives:

$$\begin{aligned}
&\Big\|\sqrt{\overline{u}^2_{ht}+\varepsilon^2} - \sqrt{u^2_{ht}+\varepsilon^2}\Big\|_{H^{\frac{1}{2}}(\Gamma_C)} = \Big\|(u_{ht}-\overline{u}_{ht})\frac{u_{ht}+\overline{u}_{ht}}{\sqrt{\overline{u}^2_{ht}+\varepsilon^2}+\sqrt{u^2_{ht}+\varepsilon^2}}\Big\|_{H^{\frac{1}{2}}(\Gamma_C)} \\
&\leq C\|u_{ht}-\overline{u}_{ht}\|_{L^\infty(\Gamma_C)}\Big\|\frac{u_{ht}+\overline{u}_{ht}}{\sqrt{\overline{u}^2_{ht}+\varepsilon^2}+\sqrt{u^2_{ht}+\varepsilon^2}}\Big\|_{H^{\frac{1}{2}}(\Gamma_C)} \\
&\quad +C\|u_{ht}-\overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)}\Big\|\frac{u_{ht}+\overline{u}_{ht}}{\sqrt{\overline{u}^2_{ht}+\varepsilon^2}+\sqrt{u^2_{ht}+\varepsilon^2}}\Big\|_{L^\infty(\Gamma_C)}.
\end{aligned} \tag{3.11}$$

In the previous estimate, we leave the third term unchanged whereas the last one is bounded by 1. It remains then to bound the first two terms which is performed hereafter. We begin with the first one:

$$\|u_{ht}-\overline{u}_{ht}\|_{L^\infty(\Gamma_C)} \leq Ch^{-\frac{1}{p}}\|u_{ht}-\overline{u}_{ht}\|_{L^p(\Gamma_C)} \leq C\sqrt{p}h^{-\frac{1}{p}}\|u_{ht}-\overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)}, \tag{3.12}$$

for any $p \in [1, \infty[$. In (3.12), we used an easily recoverable inverse inequality (see also [4]) as well as (3.10). The second term of (3.11) is bounded thanks to (3.7):

$$\left\| \frac{u_{ht} + \overline{u}_{ht}}{\sqrt{\overline{u}_{ht}^2 + \varepsilon^2} + \sqrt{u_{ht}^2 + \varepsilon^2}} \right\|_{H^{\frac{1}{2}}(\Gamma_C)} \leq C \|u_{ht} + \overline{u}_{ht}\|_{L^\infty(\Gamma_C)} \left\| \frac{1}{\sqrt{\overline{u}_{ht}^2 + \varepsilon^2} + \sqrt{u_{ht}^2 + \varepsilon^2}} \right\|_{H^{\frac{1}{2}}(\Gamma_C)}$$
$$+ C \|u_{ht} + \overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)} \left\| \frac{1}{\sqrt{\overline{u}_{ht}^2 + \varepsilon^2} + \sqrt{u_{ht}^2 + \varepsilon^2}} \right\|_{L^\infty(\Gamma_C)}$$
$$\leq C\sqrt{p} h^{-\frac{1}{p}} \|u_{ht} + \overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)} \left\| \frac{1}{\sqrt{\overline{u}_{ht}^2 + \varepsilon^2} + \sqrt{u_{ht}^2 + \varepsilon^2}} \right\|_{H^{\frac{1}{2}}(\Gamma_C)} + \frac{1}{2\varepsilon} \|u_{ht} + \overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)}, \tag{3.13}$$

where the first $L^\infty-$norm term is bounded as in (3.12) whereas the other one is roughly bounded by $1/2\varepsilon$. Next, we develop the first $H^{\frac{1}{2}}-$norm term in (3.13)

$$\left\| \frac{1}{\sqrt{\overline{u}_{ht}^2 + \varepsilon^2} + \sqrt{u_{ht}^2 + \varepsilon^2}} \right\|^2_{H^{\frac{1}{2}}(\Gamma_C)} = \left\| \frac{1}{\sqrt{\overline{u}_{ht}^2 + \varepsilon^2} + \sqrt{u_{ht}^2 + \varepsilon^2}} \right\|^2_{L^2(\Gamma_C)}$$
$$+ \int_{\Gamma_C} \int_{\Gamma_C} \frac{1}{(y-x)^2} \left(\frac{1}{\sqrt{\overline{u}_{ht}^2(x) + \varepsilon^2} + \sqrt{u_{ht}^2(x) + \varepsilon^2}} - \frac{1}{\sqrt{\overline{u}_{ht}^2(y) + \varepsilon^2} + \sqrt{u_{ht}^2(y) + \varepsilon^2}} \right)^2 d\Gamma d\Gamma.$$

It is easy to check that the L^2-norm term is lower than $\text{meas}(\Gamma_C)/4\varepsilon^2$. Developing the previous integral, bounding then the denominator and using estimate $(a+b)^2 \leq 2a^2 + 2b^2$ furnishes the following upper bound:

$$\frac{1}{8\varepsilon^4} \int_{\Gamma_C} \int_{\Gamma_C} \frac{\left(\sqrt{\overline{u}_{ht}^2(x) + \varepsilon^2} - \sqrt{\overline{u}_{ht}^2(y) + \varepsilon^2}\right)^2}{(y-x)^2} + \frac{\left(\sqrt{u_{ht}^2(x) + \varepsilon^2} - \sqrt{u_{ht}^2(y) + \varepsilon^2}\right)^2}{(y-x)^2} d\Gamma d\Gamma.$$

We use estimate $|\sqrt{a^2 + \varepsilon^2} - \sqrt{b^2 + \varepsilon^2}| \leq |a - b|$ in the previous expression so that

$$\left\| \frac{1}{\sqrt{\overline{u}_{ht}^2 + \varepsilon^2} + \sqrt{u_{ht}^2 + \varepsilon^2}} \right\|^2_{H^{\frac{1}{2}}(\Gamma_C)} \leq \frac{\text{meas}(\Gamma_C)}{4\varepsilon^2} + \frac{1}{8\varepsilon^4} \Big(\|\overline{u}_{ht}\|^2_{H^{\frac{1}{2}}(\Gamma_C)} + \|u_{ht}\|^2_{H^{\frac{1}{2}}(\Gamma_C)} \Big).$$

Therefore, we deduce from (3.2) that a positive constant C exists verifying

$$\left\| \frac{1}{\sqrt{\overline{u}_{ht}^2 + \varepsilon^2} + \sqrt{u_{ht}^2 + \varepsilon^2}} \right\|_{H^{\frac{1}{2}}(\Gamma_C)} \leq C \Big(\frac{1}{\varepsilon} + \frac{1}{\varepsilon^2} \Big). \tag{3.14}$$

Putting estimate (3.14) in (3.13), using (3.2) and (3.11) gives

$$\left\| \sqrt{\overline{u}_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2} \right\|_{H^{\frac{1}{2}}(\Gamma_C)}$$
$$\leq C \|u_{ht} - \overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)} \left(1 + \sqrt{p} h^{-\frac{1}{p}} \Big(\frac{1}{\varepsilon} + \sqrt{p} h^{-\frac{1}{p}} \Big(\frac{1}{\varepsilon} + \frac{1}{\varepsilon^2} \Big) \Big) \right).$$

Choosing $p = -\log(h)$ (h is assumed sufficiently small) in the previous estimate, we obtain

$$\begin{aligned} &\left\|\sqrt{\overline{u}_{ht}^2+\varepsilon^2}-\sqrt{u_{ht}^2+\varepsilon^2}\right\|_{H^{\frac{1}{2}}(\Gamma_C)} \\ &\qquad\leq C\|u_{ht}-\overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)}\left(1+\frac{\sqrt{-\log h}}{\varepsilon}+\frac{-\log h}{\varepsilon}+\frac{-\log h}{\varepsilon^2}\right). \quad (3.15) \end{aligned}$$

The inequality (3.6) together with (3.15) and the trace theorem becomes:

$$\|\boldsymbol{u}_h-\overline{\boldsymbol{u}}_h\|_1 \leq C\mathcal{F}\|g_h-\overline{g}_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\left(1+\frac{\sqrt{-\log h}}{\varepsilon}+\frac{-\log h}{\varepsilon}+\frac{-\log h}{\varepsilon^2}\right), \quad (3.16)$$

which proves that the mapping $\Psi_{\varepsilon h}$ is continuous. This together with (3.4) implies that $\Phi_{\varepsilon h}$ is continuous. Then, from (3.3) and according to Brouwer fixed point theorem, we conclude to the existence of at least one solution to Coulomb discrete regularized frictional contact problem.

We now consider uniqueness. Under the assumption $\overline{\Gamma}_D \cap \overline{\Gamma}_C = \emptyset$, it has been proved in [5], Proposition 3.2, that there exists a positive constant β (independent of h) satisfying

$$\beta\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq \|\mu_h\|_{-\frac{1}{2},h}, \quad \forall \mu_h \in W_h. \quad (3.17)$$

Assembling this result with (3.16) and (3.4) yields

$$\|\lambda_h-\overline{\lambda}_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq C\mathcal{F}\|g_h-\overline{g}_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\left(1+\frac{\sqrt{-\log h}}{\varepsilon}+\frac{-\log h}{\varepsilon}+\frac{-\log h}{\varepsilon^2}\right).$$

Supposing h and ε small enough, we deduce that the mapping $\Phi_{\varepsilon h}$ is contractive if the friction coefficient $\mathcal{F}$ is lower than $C\varepsilon^2|\log(h)|^{-1}$. That ends the proof of the theorem. □

The non-regularized case (i.e.; $\varepsilon = 0$) is handled in the following proposition.

Proposition 3.4 *Let $\varepsilon = 0$. The following results hold:*

(Existence) For any positive $\mathcal{F}$, there exists a solution to the Coulomb discrete frictional contact problem.

(Uniqueness) Assume that $\overline{\Gamma}_D \cap \overline{\Gamma}_C = \emptyset$. If $\mathcal{F} \leq Ch^{\frac{1}{2}}$ then the problem admits a unique solution. The positive constant C is independent of h.

Proof. Estimates (3.3) and (3.4) remain still valid when $\varepsilon = 0$. The starting point of the analysis is (3.5):

$$\begin{aligned} a(\boldsymbol{u}_h-\overline{\boldsymbol{u}}_h,\boldsymbol{u}_h-\overline{\boldsymbol{u}}_h) &\leq \int_{\Gamma_C}\mathcal{F}(g_h-\overline{g}_h)(|\overline{u}_{ht}|-|u_{ht}|)\,d\Gamma \\ &\leq \mathcal{F}\,\|\overline{g}_h-g_h\|_{L^2(\Gamma_C)}\|\,|\overline{u}_{ht}|-|u_{ht}|\,\|_{L^2(\Gamma_C)} \\ &\leq C\mathcal{F}\,h^{-\frac{1}{2}}\|\overline{g}_h-g_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\|\overline{u}_{ht}-u_{ht}\|_{L^2(\Gamma_C)} \\ &\leq C'\mathcal{F}\,h^{-\frac{1}{2}}\|\overline{g}_h-g_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\|\overline{\boldsymbol{u}}_h-\boldsymbol{u}_h\|_1, \end{aligned}$$

where an inverse inequality between $L^2(\Gamma_C)$ and $H^{-\frac{1}{2}}(\Gamma_C)$ has been used. From the latter bound, combined with (3.4) and (3.17), we deduce

$$\|\lambda_h - \overline{\lambda}_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq C\mathcal{F}h^{-\frac{1}{2}}\|g_h - \overline{g}_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}.$$

Hence the result. □

Remark 3.5 *1. In the proof of proposition 3.4, we are not able to remove the mesh dependent uniqueness condition also when avoiding the $L^2(\Gamma_C)$-norms and using only $H^{\frac{1}{2}}(\Gamma_C)$-norms and $H^{-\frac{1}{2}}(\Gamma_C)$-norms. More precisely, there does not exist a positive constant C independent of h such that $\|\,|\overline{g}_h - g_h|\,\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq C\|\overline{g}_h - g_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}$ or $\|\,|\overline{u}_{ht}| - |u_{ht}|\,\|_{H^{\frac{1}{2}}(\Gamma_C)} \leq C\|\overline{u}_{ht} - u_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)}$.*

2. The use of inverse inequalities in the proofs of Theorem 3.1 and Proposition 3.4 implies that it is not possible to generalize the calculus to the continuous problem.

4. The study of a simple finite element example

We consider the triangle Ω of vertexes $A = (0,0)$, $B = (\ell, 0)$ and $C = (0, \ell)$ with $\ell > 0$. We define $\Gamma_D = [B, C]$, $\Gamma_N = [A, C]$, $\Gamma_C = [A, B]$ and $\{X_1, X_2\}$ denotes the canonical orthonormal basis (see Figure 1). We suppose that the volume forces $\boldsymbol{f}$ are absent and that the surface forces denoted $\boldsymbol{F} = F_1 X_1 + F_2 X_2$ are such that F_1 and F_2 are constant on Γ_N.

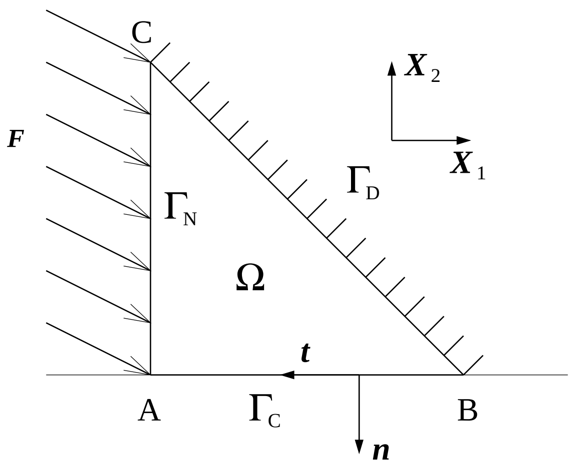

Figure 1: Setting of the problem

We suppose that Ω is discretized with a single finite element of degree one. Consequently, the finite element space becomes:

$$\boldsymbol{V}_h = \Big\{\boldsymbol{v}_h = (v_{h1}, v_{h2}) \in (P_1(\Omega))^2, \quad \boldsymbol{v}_h|_{\Gamma_D} = \boldsymbol{0}\Big\}.$$

In this case

$$M_h = \Big\{g_h \in P_1(\Gamma_C), \quad g_h \geq 0, \quad g_h(B) = 0\Big\}.$$

Clearly, $\boldsymbol{V}_h$ is of dimension two and M_h belongs to the space W_h of linear functions on Γ_C vanishing at B, which is of dimension one. Moreover, since (2.2) or equivalently (2.3) is satisfied, it follows that existence is ensured for all $\varepsilon \geq 0$ according to Theorem 3.1 and Proposition 3.4.

Let $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $\mu_h \in M_h$. Then we denote by (V_T, V_N) the value of $\boldsymbol{v}_h(A)$ corresponding to the tangential and the normal displacements at point A respectively (in our example, we have $V_T = -v_{h1}(A)$ and $V_N = -v_{h2}(A)$). We also denote by Θ the value of μ_h at point A. Then, for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $\mu_h \in M_h$, one obtains

$$\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \frac{1}{2\ell}\begin{pmatrix} 2V_T & V_T + V_N \\ V_T + V_N & 2V_N \end{pmatrix}$$

and

$$\boldsymbol{\sigma}(\boldsymbol{v}_h) = \frac{1}{\ell}\begin{pmatrix} (\lambda + 2\mu)V_T + \lambda V_N & \mu(V_T + V_N) \\ \mu(V_T + V_N) & (\lambda + 2\mu)V_N + \lambda V_T \end{pmatrix}.$$

Therefore

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) = \frac{1}{2}\Big((\lambda + 3\mu)(U_T V_T + U_N V_N) + (\lambda + \mu)(U_T V_N + U_N V_T)\Big)$$

and

$$L(\boldsymbol{v}_h) = -\frac{1}{2}\ell(F_1 V_T + F_2 V_N).$$

Besides

$$\int_{\Gamma_C} \mu_h v_{hn}\, d\Gamma = \frac{\Theta V_N \ell}{3} \quad \text{and} \quad \int_{\Gamma_C} \mathcal{F}\mu_h |v_{ht}|\, d\Gamma = \frac{\mathcal{F}\Theta |V_T| \ell}{3}.$$

Let $(\boldsymbol{u}_h, \lambda_h)$ be a solution of the discrete unilateral contact problem with Coulomb friction and without regularization (i.e.; with $\varepsilon = 0$ in (2.1)). As above-mentioned, the notation (U_T, U_N) stands for the value of $\boldsymbol{u}_h(A)$ ($U_T = -u_{h1}(A)$ and $U_N = -u_{h2}(A)$). We also denote by Λ' the value of λ_h at point A. To simplify the notations and the forthcoming calculations, we set $\Lambda = 2\Lambda'/3$.

The discrete unilateral contact problem with Coulomb friction and without regularization issued from (2.1) and Definition 2.2 consists then of finding $(U_T, U_N, \Lambda) \in \mathbb{R}^3$ such that

$$\begin{cases} (\lambda + 3\mu)(U_T V_T + U_N V_N) + (\lambda + \mu)(U_T V_N + U_N V_T) + \Lambda \ell V_N + \mathcal{F}\Lambda\ell|V_T| \\ \qquad \geq -\ell(F_1 V_T + F_2 V_N), \qquad \forall V_T \in \mathbb{R}, \quad \forall V_N \in \mathbb{R}, \\ (\lambda + 3\mu)(U_T^2 + U_N^2) + 2(\lambda + \mu)(U_T U_N) + \mathcal{F}\Lambda\ell|U_T| = -\ell(F_1 U_T + F_2 U_N), \\ \Lambda \geq 0, \qquad U_N \leq 0, \qquad \Lambda U_N = 0, \end{cases}$$

or equivalently

$$\begin{cases} (\lambda+3\mu)U_N + (\lambda+\mu)U_T + \Lambda\ell = -\ell F_2, \\ (\lambda+\mu)U_N + (\lambda+3\mu)U_T + \mathcal{F}\Lambda\ell \geq -\ell F_1, \\ (\lambda+\mu)U_N + (\lambda+3\mu)U_T - \mathcal{F}\Lambda\ell \leq -\ell F_1, \\ (\lambda+3\mu)(U_T^2+U_N^2) + 2(\lambda+\mu)(U_T U_N) + \mathcal{F}\Lambda\ell|U_T| = -\ell(F_1 U_T + F_2 U_N), \\ \Lambda \geq 0, \qquad U_N \leq 0, \qquad \Lambda U_N = 0. \end{cases} \tag{4.1}$$

Let us now search the solutions to the equations (4.1). Clearly, a solution of (4.1) satisfies either $U_N = 0$ or $\Lambda = 0$.

(i) Case 1: $U_N = 0$. Equations (4.1) become:

$$\begin{cases} (\lambda+\mu)U_T + \Lambda\ell = -\ell F_2, \\ (\lambda+3\mu)U_T + \mathcal{F}\Lambda\ell \geq -\ell F_1, \\ (\lambda+3\mu)U_T - \mathcal{F}\Lambda\ell \leq -\ell F_1, \\ (\lambda+3\mu)U_T^2 + \mathcal{F}\Lambda\ell|U_T| = -\ell F_1 U_T, \\ \Lambda \geq 0. \end{cases}$$

- Suppose that $U_T = 0$. Then

$$\Lambda = -F_2, \qquad F_2 \leq 0, \qquad |F_1| \leq \mathcal{F}|F_2|.$$

- Suppose that $U_T > 0$. Then

$$\begin{cases} (\lambda+\mu)U_T + \Lambda\ell = -\ell F_2, \\ (\lambda+3\mu)U_T + \mathcal{F}\Lambda\ell = -\ell F_1, \\ \Lambda \geq 0. \end{cases}$$

  - Assume that $\mathcal{F} \neq \frac{\lambda+3\mu}{\lambda+\mu}$. Then

$$U_T = \frac{\ell(\mathcal{F}F_2 - F_1)}{(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)} > 0, \qquad \Lambda = \frac{(\lambda+\mu)F_1 - (\lambda+3\mu)F_2}{(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)} \geq 0.$$

  - Assume that $\mathcal{F} = \frac{\lambda+3\mu}{\lambda+\mu}$. Then
    * If $F_1 = \mathcal{F}F_2$, the solutions are

$$(\lambda+\mu)U_T + \Lambda\ell = -\ell F_2, \qquad U_T > 0, \qquad \Lambda \geq 0.$$

    * If $F_1 \neq \mathcal{F}F_2$ then there are no solutions.

- Suppose that $U_T < 0$. Then

$$\begin{cases} (\lambda+\mu)U_T + \Lambda\ell = -\ell F_2, \\ (\lambda+3\mu)U_T - \mathcal{F}\Lambda\ell = -\ell F_1, \\ \Lambda \geq 0, \end{cases}$$

which gives

$$U_T = \frac{\ell(\mathcal{F}F_2 + F_1)}{-(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)} < 0, \qquad \Lambda = \frac{-(\lambda+\mu)F_1 + (\lambda+3\mu)F_2}{-(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)} \geq 0.$$

(ii) Case 2: $\Lambda = 0$.

$$\begin{cases} (\lambda+3\mu)U_N + (\lambda+\mu)U_T = -\ell F_2, \\ (\lambda+\mu)U_N + (\lambda+3\mu)U_T = -\ell F_1, \\ U_N \leq 0, \end{cases}$$

so that

$$U_T = \frac{\ell((\lambda+\mu)F_2 - (\lambda+3\mu)F_1)}{4\mu(\lambda+2\mu)}, \qquad U_N = \frac{\ell((\lambda+\mu)F_1 - (\lambda+3\mu)F_2)}{4\mu(\lambda+2\mu)} \leq 0.$$

All the results are reported in the following proposition. There are three cases which consist of comparing the friction coefficient $\mathcal{F}$ with the critical value $\frac{\lambda+3\mu}{\lambda+\mu} = 3 - 4\nu$ (ν denotes Poisson ratio with $0 < \nu < 1/2$). The results are also depicted in Figures 2, 3 and 4.

Proposition 4.1 *1. If* $\mathcal{F} < \frac{\lambda+3\mu}{\lambda+\mu}$ *then the problem* (4.1) *admits a unique solution:*
(Separation) If $F_2 > \frac{\lambda+\mu}{\lambda+3\mu}F_1$*, then*

$$U_T = \frac{\ell((\lambda+\mu)F_2 - (\lambda+3\mu)F_1)}{4\mu(\lambda+2\mu)},\ U_N = \frac{\ell((\lambda+\mu)F_1 - (\lambda+3\mu)F_2)}{4\mu(\lambda+2\mu)},\ \Lambda = 0. \quad (4.2)$$

(Stick) If $|F_1| \leq \mathcal{F}|F_2|$ *and* $F_2 \leq 0$ *then*

$$U_T = 0,\ U_N = 0,\ \Lambda = -F_2. \quad (4.3)$$

(Right slip) If $F_2 \leq \frac{\lambda+\mu}{\lambda+3\mu}F_1$, $\mathcal{F}F_2 + F_1 > 0$ *then*

$$U_T = \frac{\ell(\mathcal{F}F_2 + F_1)}{-(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)},\ U_N = 0,\ \Lambda = \frac{-(\lambda+\mu)F_1 + (\lambda+3\mu)F_2}{-(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)}. \quad (4.4)$$

(Left slip) If $F_2 \leq \frac{\lambda+\mu}{\lambda+3\mu}F_1$, $\mathcal{F}F_2 - F_1 > 0$ *then*

$$U_T = \frac{\ell(\mathcal{F}F_2 - F_1)}{(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)},\ U_N = 0,\ \Lambda = \frac{(\lambda+\mu)F_1 - (\lambda+3\mu)F_2}{(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)}. \quad (4.5)$$

2. If $\mathcal{F} = \frac{\lambda+3\mu}{\lambda+\mu}$ *then, depending on the loadings, the problem* (4.1) *admits either a unique solution or an infinity of solutions:*
(Separation) If $F_2 > \frac{\lambda+\mu}{\lambda+3\mu}F_1$*, then the solution is given by* (4.2)*.*
(Stick) If $(-\mathcal{F}|F_2| < F_1 \leq \mathcal{F}|F_2|$ *and* $F_2 \leq 0)$ *or* $F_1 = F_2 = 0$ *then the solution is given by* (4.3)*.*
(Right slip) If $F_2 \leq \frac{\lambda+\mu}{\lambda+3\mu}F_1$*,* $\mathcal{F}F_2 + F_1 > 0$ *then the solution is given by* (4.4)*.*
(From stick to left slip) If $F_1 = \mathcal{F}F_2$ *and* $F_2 < 0$*, then there exists an infinity of solutions:*

$$U_T = \frac{-\ell(F_2+\beta)}{\lambda+\mu},\ U_N = 0,\ \Lambda = \beta,\ \ \textit{for all}\ \ 0 \leq \beta \leq -F_2.$$

3. If $\mathcal{F} > \frac{\lambda+3\mu}{\lambda+\mu}$ *then, depending on the loadings, the problem* (4.1) *admits one, two or three solutions:*
(Separation) If $F_2 > \frac{\lambda+\mu}{\lambda+3\mu}F_1$ *and* $\mathcal{F}F_2 - F_1 > 0$ *then the solution is given by* (4.2)*.*
(Stick) If $(-\frac{\lambda+3\mu}{\lambda+\mu}|F_2| < F_1 \leq \mathcal{F}|F_2|$ *and* $F_2 \leq 0)$ *or* $F_1 = F_2 = 0$ *then the solution is given by* (4.3)*.*
(Right slip) If $F_2 \leq \frac{\lambda+\mu}{\lambda+3\mu}F_1$*,* $\mathcal{F}F_2 + F_1 > 0$ *then the solution is given by* (4.4)*.*
(Separation and stick) If $F_1 = \mathcal{F}F_2$ *and* $F_2 < 0$*, then there are two solutions given by* (4.2) *and* (4.3)*.*
(Stick and left slip) If $F_1 = \frac{\lambda+3\mu}{\lambda+\mu}F_2$ *and* $F_2 < 0$*, then there are two solutions given by* (4.3) *and* (4.5)*.*
(Separation, stick and left slip) If $-\mathcal{F}|F_2| < F_1 < -\frac{\lambda+3\mu}{\lambda+\mu}|F_2|$ *and* $F_2 \leq 0$ *then there are three solutions given by* (4.2)*,* (4.3) *and* (4.5)*.*

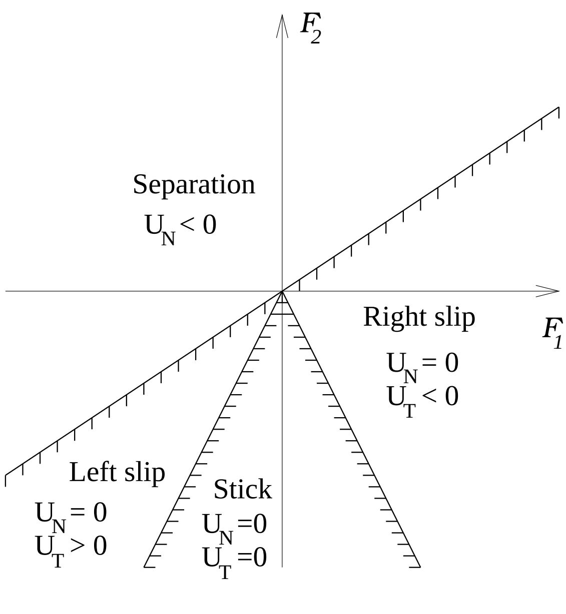

Figure 2: Case $\mathcal{F} < \frac{\lambda+3\mu}{\lambda+\mu} = 3 - 4\nu$. Problem (4.1) admits a unique solution.

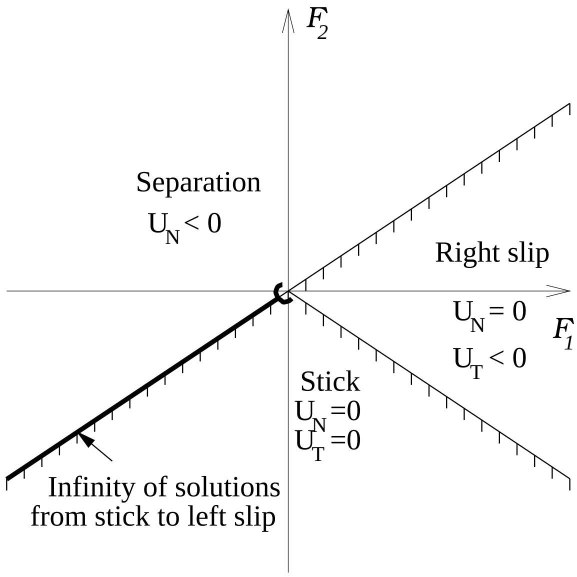

Figure 3: Case $\mathcal{F} = \frac{\lambda+3\mu}{\lambda+\mu} = 3 - 4\nu$. Problem (4.1) admits either a unique or an infinity of solutions.

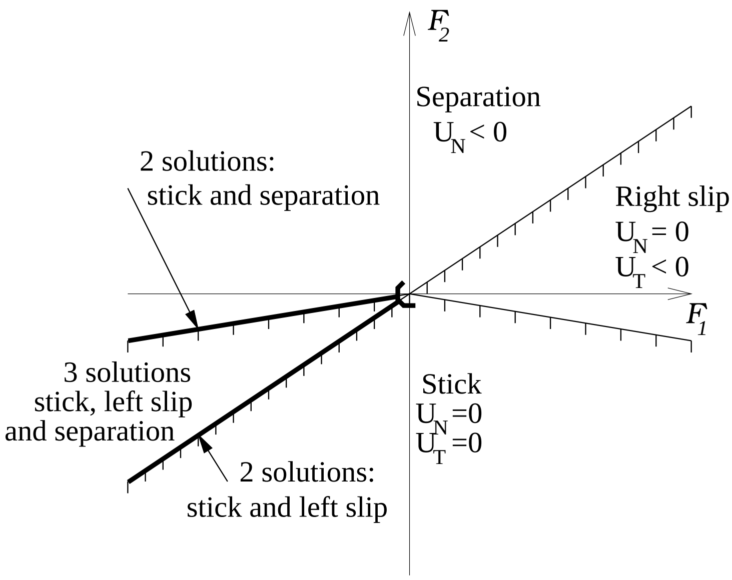

Figure 4: Case $\mathcal{F} > \frac{\lambda+3\mu}{\lambda+\mu} = 3 - 4\nu$. Problem (4.1) admits a unique, two or three solutions.

The study of sufficient conditions of non-uniqueness for Coulomb frictional contact problem in the continuous framework is actually under consideration in [11].

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