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Divisibility of zeta functions of curves in a covering

By

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Abstract. We prove, as an analogy of a conjecture of Artin, that if $Y \longrightarrow X$ is a finite flat morphism between two singular reduced absolutely irreducible projective algebraic curves defined over a finite field, then the numerator of the zeta function of X divides that of Y in $\mathbb{Z}[T]$. Then, we give some interpretations of this result in terms of semi-abelian varieties.

1. Introduction. Let ζ_K be the Dedekind zeta function of a number field K:

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s} \quad (\mathrm{Re}(s) > 1)$$

where the sum ranges over the non zero ideals I of the ring of integers $\mathcal{O}_K$ of K and where $N(I)$ is the norm of the ideal I i.e. the number of elements of the residue class ring $\mathcal{O}_K/I$. It is well-known that it extends to a meromorphic function on $\mathbb{C}$. Emil Artin conjectured that, for any extension of number fields L/K, the ratio

$$\frac{\zeta_L(s)}{\zeta_K(s)}$$

is entire.

We are interested here in a similar question in the following geometric context. Let X be a projective algebraic variety defined over the finite field $\mathbb{F}_q$ and let $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ be the corresponding variety over an algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$. The zeta function of X is defined as

$$Z_X(T) = \exp\left(\sum_{n=1}^{\infty} \sharp X(\mathbb{F}_{q^n}) \frac{T^n}{n}\right)$$

where $\sharp X(\mathbb{F}_{q^n})$ is the number of $\mathbb{F}_{q^n}$-rational points of X.

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We consider the ℓ-adic étale cohomology spaces with compact support $H^i_c(\overline{X}, \mathbb{Q}_\ell)$ of $\overline{X}$ where ℓ is a prime number distinct from the characteristic of $\mathbb{F}_q$.

A crude analogy of Artin's conjecture would state that if $Y \longrightarrow X$ is a surjective morphism between two projective algebraic varieties then the ratio $Z_Y(T)/Z_X(T)$ of their zeta functions is a polynomial in T. It turns to be false, for instance for the blowing up of the projective plane at some rational point, where the ratio equals $\frac{1}{1-qT}$.

However, by Grothendieck-Lefschetz formula, the zeta function of X can be written as

$$Z_X(T) = \prod_{i=0}^{2\dim X} (\det(1 - FT \mid H^i_c(\overline{X}, \mathbb{Q}_\ell)))^{(-1)^{i+1}}$$

where F is the map on cohomology induced by the Frobenius morphism on $\overline{X}$. So, one could ask whether it is true that if there is a surjective morphism $Y \longrightarrow X$ between two projective algebraic varieties Y and X defined over $\mathbb{F}_q$, then the polynomial $\det(1 - FT \mid H^i(\overline{X}, \mathbb{Q}_\ell))$ divides the polynomial $\det(1 - FT \mid H^i(\overline{Y}, \mathbb{Q}_\ell))$ in $\mathbb{Z}[T]$. This is the case for instance for the previous example of the blowing up of the plane at a point.

More generally, the answer is yes provided that X and Y are smooth. Indeed, thanks to the projection formula and Poincaré duality, Kleiman proved that in this case, there is a Galois invariant injection between the cohomology spaces (see [8, prop. 1.2.4]).

Unfortunately, we cannot expect this divisibility in full generality (even for curves) since it does not hold for the desingularization of the nodal cubic curve.

The main result of this paper is:

Theorem 1. *Let $Y \longrightarrow X$ be a flat finite morphism between two reduced absolutely irreducible projective algebraic curves Y and X defined over a finite field. Then, the numerator of the zeta function of X divides Y one in $\mathbb{Z}[T]$.*

We prove this theorem in the following section and we make some remarks in the last one.

2. Proof of the theorem. Let C be an absolutely irreducible and reduced projective algebraic curve defined over the finite field $k = \mathbb{F}_q$ with q elements. It is known that

$$Z_{C,k}(T) = \frac{\det(1 - TF \mid H^i_c(\overline{C}, \mathbb{Q}_\ell))}{(1-T)(1-qT)},$$

where F is the Frobenius morphism on the first group of ℓ-adic cohomology with compact support $H^i_c(\overline{C}, \mathbb{Q}_\ell)$ of $\overline{C}$, and that the eigenvalues of the Frobenius have modulus $\sqrt{q}$ or 1 (see [6]). In fact, the authors have shown in [1] the following result. Denote by $\tilde{C}$ the normalization of C and $\nu_C : \tilde{C} \longrightarrow C$ the normalization map. If P is a closed point of C, we denote by $d_k(P) = [k(P) : k]$ the residual degree of P. Then, the numerator polynomial of the zeta function of C can be writen precisely as (see [1]):

$$P_{C,k}(T) := (1-T)(1-qT)Z_{C,k}(T) = P_{\tilde{C},k}(T) \prod_{P \in C} L_{C,P,k}(T),$$

where $P_{\tilde{C},k}$ is the numerator of the zeta function $Z_{\tilde{C},k}$ of $\tilde{C}$, and for a closed point $P \in C$

$$L_{C,P,k}(T) := \frac{\prod_{\tilde{P} \in \nu_C^{-1}(P)} (1 - T^{d_k(\tilde{P})})}{1 - T^{d_k(P)}} \in \mathbb{Z}[T].$$

Let us remark that if P is a non singular point on C then $L_{C,P,k}(T) = 1$. Now, consider a finite flat morphism f from Y to X as in the theorem. By Kleiman's theorem quoted in the introduction, the polynomial $P_{\tilde{X},k}$ divides $P_{\tilde{Y},k}$ (alternatively see 3.4 below for a proof in the case of curves). Thus, the theorem follows immediately from the following proposition:

Proposition 2. *If P and Q are closed points respectively on X and Y with $f(Q) = P$, then $L_{X,P,k}$ divides $L_{Y,Q,k}$ in $\mathbb{Z}[T]$.*

Let us begin by two lemmas.

Lemma 3. *Proposition 2 holds if $d_k(Q) = d_k(P)$.*

P r o o f. By flatness of f, the fibred product $Z = Y \times_X \tilde{X}$ is an irreducible curve (see [2]) and thus $\tilde{Y} \longrightarrow Z$ is surjective since it is not constant.

$$\begin{tikzcd}
\tilde{Y} \arrow[dr] \arrow[drr, "\tilde{f}"] \arrow[ddr, "\nu_Y"'] & & \\
& Y \times_X \tilde{X} \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "\nu_X"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{Q} \arrow[drr] \arrow[ddr] & & \\
& & \tilde{P} \arrow[d] \\
& Q \arrow[r] & P
\end{tikzcd}$$

This implies that, for any closed points $\tilde{P}$ over P in $\tilde{X}$, there exists a $\tilde{Q}$ over Q in $\tilde{Y}$ such that $\tilde{f}(\tilde{Q}) = \tilde{P}$. Let

$$\alpha_P = \sharp \nu_X^{-1}(P)$$

be the number of closed points of $\tilde{X}$ above P in the normalization map. By reordering the sets $\nu_X^{-1}(P)$ and $\nu_Y^{-1}(Q)$, we can suppose that for all $1 \leqq i \leqq \alpha_P$, we have $\tilde{f}(\tilde{Q}_i) = \tilde{P}_i$.

Thus the residue field $k(\tilde{Q}_i)$ of $\tilde{Q}_i$ is an extension field of that of $\tilde{P}_i$, so that $d_k(\tilde{P}_i)$ divides $d_k(\tilde{Q}_i)$. Thus,

$$(1 - T^{d_k(P)})L_{X,P,k} = \prod_{i=1}^{\alpha_P}(1 - T^{d_k(\tilde{P}_i)}) \text{ divides } \prod_{i=1}^{\alpha_P}(1 - T^{d_k(\tilde{Q}_i)})$$

which divides himself $\prod_{i=1}^{\alpha_Q}(1 - T^{d_k(\tilde{Q}_i)}) = (1 - T^{d_k(Q)})L_{Y,Q,k}$. Since we have supposed that $d_k(Q) = d_k(P)$, we obtain the desired divisibility. □

Denoting the greatest common divisor of two integers (or two polynomials) d and d' by $\gcd(d, d')$, we can state:

Lemma 4. *If Q is closed point in $f^{-1}(P)$, then we have*:

$$\sum_{\tilde{P}\in\nu_X^{-1}(P)} \gcd(d_k(\tilde{P}), d_k(Q)) \leqq \alpha_Q := \sharp\nu_Y^{-1}(Q).$$

P r o o f. The point Q of degree $d = d_k(Q)$ over $\mathbb{F}_q$ is sum of d $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$-conjugate points of degree 1 over $\mathbb{F}_{q^d}$:

$$Q = Q_1 + \cdots + Q_d.$$

Working with $X_d = X \times_{\mathbb{F}_q} \mathbb{F}_{q^d}$ over $\mathbb{F}_{q^d}$ and with Q_1, we have by the preceding lemma that $L_{Y_d,Q_1,\mathbb{F}_{q^d}}$ is divisible by $L_{X_d,P,\mathbb{F}_{q^d}}$ in $\mathbb{Z}[T]$. But, we have on the one hand

$$L_{Y_d,Q_1,\mathbb{F}_{q^d}} = \frac{\prod\limits_{\tilde{Q}_1\to Q_1} (1 - T^{d_{\mathbb{F}_{q^d}}(\tilde{Q}_1)})}{1 - T}.$$

On the other hand, it is easy to see that a point of degree $\tilde{d}$ over $\mathbb{F}_q$ gives $\gcd(\tilde{d}, d)$ points of degree $\frac{\tilde{d}}{\gcd(\tilde{d},d)}$ over $\mathbb{F}_{q^d}$. Thus, we have

$$\begin{aligned} L_{X_d,P,\mathbb{F}_{q^d}} &= \frac{\prod\limits_{\tilde{P}\in\nu_{X_d}^{-1}(P)} (1 - T^{d_{\mathbb{F}_{q^d}}(\tilde{P})})}{1 - T} \\ &= \frac{\prod\limits_{\tilde{P}\to P} (1 - T^{d_{\mathbb{F}_q}(\tilde{P})/\gcd(d_{\mathbb{F}_q}(\tilde{P}),d)})^{\gcd(d_{\mathbb{F}_q}(\tilde{P}),d)}}{1 - T}. \end{aligned}$$

The relation follows from the comparison between their $(1 - T)$-adic valuations. □

We can now prove Proposition 2.

Since $\tilde{Y} \to \tilde{X}$ is surjective, we can reorder, as in Lemma 3, the points $\tilde{P}_i$ and $\tilde{Q}_i$ so that $d_k(\tilde{P}_i)$ divides $d_k(\tilde{Q}_i)$ for $1 \leqq i \leqq \alpha_P$. Thus, we have:

$$\prod_{i=1}^{\alpha_P}(1 - T^{d_k(\tilde{P}_i)}) \text{ divides } \prod_{i=1}^{\alpha_P}(1 - T^{d_k(\tilde{Q}_i)}).$$

If $\alpha_Q \geqq \alpha_P + 1$, the divisibility of $L_{Y,Q,\mathbb{F}_q}$ by $L_{X,P,\mathbb{F}_q}$ is obvious. Since Lemma 4 implies $\alpha_P \leqq \alpha_Q$, we are left to the case $\alpha_Q = \alpha_P$.

By Lemma 4 again, we get $\gcd(d_k(\tilde{P}), d_k(Q)) = 1$ for all $\tilde{P}_i \in \nu_X^{-1}(P)$, in particular for $\tilde{P}_1$. Thus, assuming without lost of generality that $d_P = 1$ (otherwise we can set $U = T^{d_P}$), we obtain:

(i) $(1 - T^{d_k(\tilde{P}_1)})$ divides $(1 - T^{d_k(\tilde{Q}_1)})$,
(ii) $\frac{1-T^{d_k(Q)}}{1-T}$ divides $(1 - T^{d_k(\tilde{Q}_1)})$,
(iii) $\gcd(1 - T^{d_k(\tilde{P}_1)}, \frac{1-T^{d_k(Q)}}{1-T}) = 1$.

Hence

$$\frac{1 - T^{d_k(Q)}}{1 - T}(1 - T^{d_k(\tilde{P}_1)}) \text{ divides } (1 - T^{d_k(\tilde{Q}_1)})$$

which implies that $L_{X,P,k}$ divides $L_{Y,Q,k}$ in $\mathbb{Z}[T]$ and this concludes the proof. $\square$

3. Remarks.

3.1. About the flatness hypothesis. The theorem is false without the flatness hypothesis. In the case of the desingularization of the nodal cubic curve $y^2z = x^2(x + z)$, one has $P_{X,k}(T) = T - 1$ and $P_{Y,k}(T) = 1$. The proof fails in Lemma 3. In this case, $Z = \tilde{X} \times_X \tilde{X}$ is not irreducible: it is the disjoint union of $\tilde{X}$ and of two other points. Hence, the map from $\tilde{Y} = \tilde{X}$ to Z is not surjective.

3.2. The étale case. We can show easily (for simplicity in the case where all points have degree 1) the divisibility for an étale morphism (that is an unramified and flat morphism). Indeed, we have, for a sufficiently large base field (i.e. when all particular points are rational),

$$Z_X(T) = Z_{\tilde{X}}(T) \prod_{P \in \mathrm{Sing}(X)} (1 - T)^{\alpha_P - 1},$$

where $\alpha_P = \sharp \nu_X^{-1}(P)$. So, it suffices to prove that:

$$(\sharp) \qquad \alpha_P - 1 \leqq \sum_{Q \in f^{-1}(P)} (\alpha_Q - 1)$$

for any $P \in X$. Note that the inequality $\alpha_P \leqq \sum_{Q \in f^{-1}(P)} \alpha_Q$ is trivial since there is a finite morphism $\tilde{f}$ between $\tilde{Y}$ and $\tilde{X}$ which send the points of $\nu_Y^{-1}(Q)$ on $\nu_X^{-1}(P)$ for all $Q \in f^{-1}(P)$. But this is not sufficient to prove ($\sharp$).

The étale hypothesis gives an isomorphism

$$\hat{\mathcal{O}}_{X,P} \otimes_{k(P)} k(Q) \cong \hat{\mathcal{O}}_{Y,Q}$$

between the completions of the local rings at $Q \in Y$ and $P = f(Q) \in X$. This implies that $\alpha_Q = \alpha_P$ for all $Q \in f^{-1}(P)$. So, the result follows.

3.3. Inequality for the numbers of rational points. Let us remark that Theorem 1 implies the following inequality which holds whenever we have a finite flat morphism $Y \longrightarrow X$ between two reduced absolutely irreducible projective algebraic curves Y and X defined over $\mathbb{F}_q$ (this result was proved by the authors in [2]):

$$|Y(\mathbb{F}_q) - X(\mathbb{F}_q)| \leqq 2(g_Y - g_X)\sqrt{q} + \Delta_Y - \Delta_X \leqq 2(\pi_Y - \pi_X)\sqrt{q}$$

where g_X is the geometric genus of X, π_X its arithmetic genus and Δ_X is the difference between the number of points of X and its normalization (same notations for Y). This inequality contains the Weil bound for smooth curves, its generalization for singular plane curves proved in [9] and for general singular curves proved in [1] (see also [3]).

3.4. Covering of smooth curves. During the proof of our theorem, we used the following proposition which is a particular case of a proposition of Kleiman quoted in the introduction. We give here a proof in the special case of smooth curves which may be well known to the experts.

Proposition 5. *Let $f : Y \longrightarrow X$ be a finite morphism between two reduced absolutely irreducible smooth projective algebraic curves Y and X defined over a finite field k. Then, the numerator of the zeta function of X divides that of Y in $\mathbb{Z}[T]$.*

Proof. For any prime number ℓ distinct from the characteristic of $\mathbb{F}_k$, consider the $\mathbb{Q}_\ell$-vector space $T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ of dimension $2g_X$, where $T_\ell(J_X)$ is the Tate module of the Jacobian J_X of X and g_X is the (geometric) genus of X. The numerator $P_X(T)$ of the zeta function of X is the reciprocal polynomial of the characteristic polynomial of the Frobenius endomorphism on $T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. The map

$$f^* : J_X \longrightarrow J_Y$$

induced by f on the Jacobians has finite kernel and sends the ℓ^n-torsion points of J_X on those of J_Y. Then, tensorising by $\mathbb{Q}_\ell$, we get an injective morphism of $\mathbb{Q}_\ell$-vector spaces

$$T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{f^* \otimes 1} T_\ell(J_Y) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

The Frobenius morphism on $T_\ell(J_Y) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ leaves fixed the subspace $T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Hence the characteristic polynomial of the latter divides the characteristic polynomial of the former in $\mathbb{Q}_\ell[T]$, hence in $\mathbb{Z}[T]$ since both $P_X, P_Y \in \mathbb{Z}[T]$ have constant term equals to 1. Thus, we have that $P_X(T)$ divides $P_Y(T)$. □

3.5. The weight-zero part. For a reduced absolutely irreducible projective algebraic curve X over $k = \mathbb{F}_q$, we have seen in Section 2 that the numerator of its zeta function can be written as:

$$P_X = P_{\tilde{X}} \times P_{X/\tilde{X}}$$

where $P_{\tilde{X}} = P_{\tilde{X},k}$ is the numerator of the zeta function of the normalization $\tilde{X}$ of X and $P_{X/\tilde{X}} = \prod_{P \in X} L_{X,P,k}$ is a polynomial with roots of modulus one, i.e. of weight zero as in the terminology of Deligne (see [6]).

But, if X is a reduced connected scheme of dimension 1 of finite type over $\mathrm{Spec}(\overline{k})$, we can define the Picard scheme Pic_X of X which is a smooth group scheme over $\overline{k}$. We have a group isomorphism

$$\mathrm{Pic}_X(\overline{k}) \cong \mathrm{Pic}(X)$$

with the group Pic X of isomorphism classes of inversible sheaves on X.

Denote by J_X the identity component of Pic_X. This a group scheme called the Jacobian of X.

We have the following exact sequence of smooth connected commutative group schemes over $\overline{k}$ (see [4]):

$$(*) \qquad 0 \longrightarrow L_X \longrightarrow J_X \longrightarrow J_{\tilde{X}} \longrightarrow 0$$

where L_X is a smooth connected linear algebraic group which can be writen $L_X = U_X \times T_X$ with U_X a unipotent group and T_X a torus. Since $\tilde{X}$ is smooth and proper over $\overline{k}$, $J_{\tilde{X}}$ is an abelian variety and thus the Jacobian J_X is a semi-abelian variety i.e. an extension of an abelian variety by a linear group.

Proposition 6. *For any reduced absolutely irreducible projective algebraic curve X defined over $\mathbb{F}_q$, we have, for any ℓ distinct from the characteristic of $\mathbb{F}_q$:*

$$P_X(T) = \det(1 - TF \mid T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$$

where F is the Frobenius endomorphism.

Proof. We have

$$P_X(T) = \det(1 - TF \mid H_c^1(\overline{X}, \mathbb{Q}_\ell)).$$

But Deligne has proved in [5, p. 71] that

$$H_c^1(\overline{X}, \mathbb{Z}_\ell) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(J_X), \mathbb{Z}_\ell)$$

which enable us to conclude. □

Then, we have:

Corollary 7.

$$P_{X/\tilde{X}}(T) = \det(1 - TF \mid T_\ell(T_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$$

where T_X is the toric part of the Jacobian of X.

Proof. By the exact sequence $(*)$, we get

$$T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong (T_\ell(J_{\tilde{X}}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \times (T_\ell(L_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell).$$

The contribution in the Tate module of the linear part is exclusively given by the toric part. Then, the result follows from the identity

$$P_X(T) = P_{\tilde{X}}(T) P_{X/\tilde{X}}(T)$$

and the previous proposition. $\square$

3.6. About the Jacobians. The main theorem admits the following corollary on semi-abelian variety. Note that this corollary is false without the flatness assumption as shown by the desingularization of the nodal cubic curve X: the Jacobian of X is the multiplicatif group $\mathbb{G}_m$ and the Jacobian of $\tilde{X}$ is a point.

Proposition 8. *If*

$$f : Y \longrightarrow X$$

is a flat finite morphism between two reduced absolutely irreducible projective algebraic curves over a finite field k, then the jacobian J_X of X is k-isogenous to a semi-abelian subvariety of the Jacobian J_Y of Y defined over k.

Proof. An extension of an abelian variety by the multiplicatif group $\mathbb{G}_m$ is parametrized by a point of the dual of the abelian variety (see [10]). Over a finite field, such a point is a torsion point, thus the extension is isogenous to the trivial extension. Hence, for an extension J_X of $J_{\tilde{X}}$ by a torus T_X, there is an isogeny between J_X and $J_{\tilde{X}} \times T_X$ which induces a Galois-equivariant isomorphism between $T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and

$$T_\ell(J_{\tilde{X}} \times T_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong (T_\ell(J_{\tilde{X}}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \times (T_\ell(T_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell).$$

Since the Frobenius endomorphism acts semi-simply on abelian varieties so as on torus, we deduce that it acts semi-simply on semi-abelian variety too, thus on $T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and $T_\ell(J_Y) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Furthermore, by Proposition 6, their characteristic polynomials are P_X and P_Y. By Theorem 1, P_X divides P_Y, thus we deduce that $T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is $\mathrm{Gal}(\bar{k}/k)$-isomorphic to a $\mathrm{Gal}(\bar{k}/k)$-subspace of $T_\ell(J_Y) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Furthermore, the theorem of Tate on abelian varieties (see [11]) remains true for semi-abelian varieties: Jannsen in [7] has proved that for any semi-abelian variety A defined over a finite field k, we have:

$$\mathrm{End}_k(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \mathrm{End}_{\mathrm{Gal}(\bar{k}/k)}(T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell).$$

Imiting the proof of Tate in [11], we get the desired result. $\square$

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