

**SHARPNESS OF RISLER'S UPPER BOUND
FOR THE TOTAL CURVATURE OF AN
AFFINE REAL ALGEBRAIC HYPERSURFACE**

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Let A be a real algebraic hypersurface in \mathbb{R}^{n+1} of degree d . For a point $p \in A$, we denote the curvature of A at p (i.e., the jacobian of the Gauss mapping $\gamma : A \rightarrow S^n$) by $k(p)$. Similarly, for a point p in the complexification $\mathbb{C}A$ of A , we denote the Gaussian curvature of $\mathbb{C}A$ at p by $K(p)$. Using results of Teissier and Langevin, Risler [1] proved that

$$\frac{1}{\sigma_n} \int_A |k| \leq \frac{1}{\sigma_{2n}} \int_{\mathbb{C}A} |K| = \frac{d(d-1)^n}{2} \quad ("><" \text{ instead of "}\leq" \text{ for } d \geq 3) \quad (1)$$

where σ_n is the n -volume of the unit n -sphere. Using Harnack's construction he proved also that this bound is sharp for $n = 1$, i.e., for any d and for any $\varepsilon > 0$ there exists a real algebraic curve A in \mathbb{R}^2 such that $\int_A |k| > \pi d(d-1) - \varepsilon$. The purpose of this note is to show the evidence of the fact that Risler's bound (1) is sharp for any n . Namely,

Proposition. *For any positive integers n, d and for any $\varepsilon > 0$ there exists a real algebraic hypersurface A in \mathbb{R}^{n+1} such that $\int_A |k| > \frac{1}{2}d(d-1)^n \sigma_n - \varepsilon$.*

Proof. Let f be a polynomial in one variable of degree d which has d distinct real roots a_1, \dots, a_d in the segment $[-1, 1]$. For $c \in \mathbb{R}$, let us set

$$F_c(x_0, \dots, x_n) = f(x_0) + c \cdot (f(x_1) + \dots + f(x_n)),$$

and let A_c be the hypersurface in \mathbb{R}^{n+1} defined by the equation $F_c = 0$.

Step 1. *There exists $\delta > 0$ such that $0 < |c| < \delta$ implies that the restriction of x_0 -coordinate to the hypersurface A_c has $d(d-1)^n$ nondegenerate critical points.*

Indeed, critical points of x_0 on A_c are solutions of the simultaneous equations

$$\frac{\partial F_c}{\partial x_1} = \dots = \frac{\partial F_c}{\partial x_n} = F_c = 0 \quad (2)$$

By Rolle's theorem, f' has $d-1$ real roots b_1, \dots, b_{d-1} . Thus, if $c \neq 0$ then the solution of the simultaneous equations $\partial F_c / \partial x_j = 0$, $j = 1, \dots, n$, is the union of $(d-1)^n$ lines $x_j = b_{i_j}$, $1 \leq i_j \leq d-1$. The d hyperlanes $x_0 = a_i$, $i = 1, \dots, d$ are transverse to these lines, so if we include the equation $F_0 = 0$, we obtain $d(d-1)^n$ solutions. It is clear that for $|c| \ll 1$, the hypersurface $F_c = 0$ is C^1 -close to $F_0 = 0$ in the cube $[-1, 1]^{n+1}$, so the perturbed system of equations has the same

number of solutions. Moreover, these solutions are nondegenerate, hence they yield nondegenerate critical points.

Step 2. Let us fix c , $0 < |c| < \delta$, and let us set $A_{c,h} = \{F_{c,h} = 0\}$ where

$$F_{c,h}(x_0, \dots, x_n) = F_c(hx_0, x_1, \dots, x_n).$$

Then one has $\lim_{h \rightarrow 0} \int_{A_{c,h}} |k| \geq \frac{1}{2}d(d-1)^n \sigma_n$.

Indeed, it is clear that $A_{c,h} = s_h(A_c)$ where $s_h : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ is the stretching $(x_0, \dots, x_n) \mapsto (x_0/h, x_1, \dots, x_n)$. Let $p_1, \dots, p_N \in A_c$ (where $N = d(d-1)^n$) be the critical points of $x_0|_{A_c}$. Let us choose disjoint neighbourhoods $U_1, \dots, U_N \subset A_c$ of the points p_1, \dots, p_N . It is enough to prove that $\lim_{h \rightarrow 0} \int_{s_h(U_\nu)} |k| \geq \sigma_n/2$ for any ν . By definition, k is the jacobian of the Gauss map $\gamma_h : A_{c,h} \rightarrow \mathbb{R}\mathbb{P}^n$ with respect to the metric on $\mathbb{R}\mathbb{P}^n$ induced by the standard projection of the unit sphere S^n onto $\mathbb{R}\mathbb{P}^n$. Hence, $\int_{s_h(U_\nu)} |k| \geq \text{vol}_{\mathbb{R}\mathbb{P}^n} \gamma_h(s_h(U_\nu))$. Let us consider the affine chart V_0 on $\mathbb{R}\mathbb{P}^n$ corresponding to the coordinates X_1, \dots, X_n where $X_i = x_i/x_0$. The fact that p_ν is a nondegenerate critical point means that $\gamma_1(U_\nu)$ contains an open ball $B \subset V_0$ centered at the origine. It remains to note that $\gamma_h(s_h(U_\nu)) = S_h(\gamma_1(U_\nu))$ where $S_h : V_0 \rightarrow V_0$ is the homothety $(X_1, \dots, X_n) \mapsto (X_1/h, \dots, X_n/h)$, hence

$$\text{vol}_{\mathbb{R}\mathbb{P}^n} \gamma_h(s_h(U_\nu)) \geq \text{vol}_{\mathbb{R}\mathbb{P}^n} S_h(B) \xrightarrow{h \rightarrow 0} \text{volume}(\mathbb{R}\mathbb{P}^n) = \sigma_n/2. \quad \square$$

Remark. In fact, Risler proved in [1] more than the sharpness of (1) for $n = 1$. He proved that (1) for $n = 1$ remains sharp if one considers only *maximal curves*, i.e. curves which have the maximal possible number $(d-1)(d-2)/2 + d$ of connected components (recall that the curves are affine). Our construction applied for $n = 1$ provides curves which are far from being maximal.

REFERENCES

1. J.-J. Risler, *On the curvature of the real Milnor fiber*, Bull. London Math. Soc. **35** (2003), 445-454.

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