MARKOV MOVES FOR QUASIPOSITIVE BRAIDS

S.Yu. Orevkov

Let B_m denotes the group of braids with m strings (m-braids). Recall that it is defined by $\sigma_1, \dots, \sigma_{m-1}$ and relations $[\sigma_j, \sigma_k] = 1$ for $|k-j| > 1$ and $\sigma_j\sigma_k\sigma_j = \sigma_k\sigma_j\sigma_k$ for $|k-j| = 1$. Denote by $e : B_m \to \mathbf{Z}$ the "exponent sum" homomorphism: $e(\sigma_i) = 1$ for all $i = 1, \dots, m-1$. An m-braid b is called *quasipositive* (see [5]) if $b = \prod_{j=1}^k a_j \sigma_1 a_j^{-1}$ for some braids $a_j \in B_m$ (recall that all standard generators are conjugated).

One says that $b' \in B_{m+1}$ is obtained from $b \in B_m$ by a *Markov move* if $b' = b\sigma_m^\varepsilon$ for $\varepsilon = \pm 1$ (we identify here B_m with the subgroup of B_{m+1} generated by $\sigma_1, \dots, \sigma_{m-1}$). Say that the Markov move is *positive* if $\varepsilon = 1$ and *negative* otherwise. It follows immediately from the definitions that a braid is quasipositive if it is obtained from a quasipositive braid by a positive Markov move. Here we prove the converse.

Theorem 1. *Let $b' \in B_{m+1}$ be obtained from b by a positive Markov move. If b' is quasipositive then b is also quasipositive.*

This is a pure existence theorem. Our proof is based on Gromov's theory of pseudo holomorphic curves [2] and it is absolutely non-constructive: we do not know an algorithm to find a quasipositive presentation of b starting with a quasipositive presentation of b'.

Remark. If b' is obtained from any braid b by a negative Markov move then b' is never quasipositive. This is an immediate consequence from the following result of Burckel [1] and Laver [3] (a proof was given by Wiest [6]; see also [4]): *any conjugate of a quasipositive braid is positive in Dehornoy right-invariant order.*

Recall that a smooth oriented 2-surface F in a smooth symplectic 4-manifold (X, ω) is call *symplectic* if $j^*(\omega)$ is positive on F where j is the imbedding $F \subset X$. Let (z, w), $z = x + iy$, $w = u + iv$, $i = \sqrt{-1}$, be coordinates in $\mathbf{C}^2$. Denote by ω_0 the standard sypmlectic form $\omega_0 = dx \wedge dy + du \wedge dv$ on $\mathbf{C}^2$.

Lemma. *Let $F \in \mathbf{C}^2$ be the graph of a smooth function $w = f(z) = u(z) + iv(z)$ defined in a domain $D \subset \mathbf{C}$. If $|u'_x|, |u'_y|, |v'_x|, |v'_y| < 1/\sqrt{2}$ then F is symplectic.*

Proof. Let $p(z, w) = z$. Then $((p|_F)^{-1})^*(\omega_0) = (1 + u'_x v'_y - u'_y v'_x) dx \wedge dy$. $\square$

We shall consider m-braids as isotopy classes of m-valued functions $f : [0, 1] \to \mathbf{C}$ such that each of $f(0)$ and $f(1)$ is the set of m points with distinct real parts. The generator $\sigma_k \in B_m$ are represented by $t \mapsto \{1, \dots, k-1, k+(1 \pm e^{\pi i t})/2, k+1, \dots, m\}$.

Proof of Theorem 1. Denote by H the conic $w^2 - 2w = z^2$ and let $p = (0, 2) \in H$. Denote by C_∞ the cylinder $\{(z, w) \mid \operatorname{Im} z \geq 0, |z| \leq 3\}$. Let D be the half-disk

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$C_\infty \cap \{w=0\}$ and $C_p = \{(z,w) \mid \operatorname{Im}\big(2z/(2-w)\big) \ge 0,\ |2z/(2-w)| \le 3\}$ the cone over D with the vertex p (the union of all complex lines (pp'), $p' \in D$).

H is the graph of the 2-valued function $w = 1 \pm \sqrt{1+z^2}$. It has two points of ramification $z = \pm i$. Let h_- and h_+ be the single-valued branches in the band $\{|\operatorname{Im} z| < 1\}$ such that $h_-(0) = 0$ and $h_+(0) = 2$. Denote also the graphs of h_- and h_+ by H_- and H_+.

Denote by U the cylinder $\{(z,w) \mid \operatorname{Im} z = 0\,, |z| \le 1,\ |w| \le 1\}$. Let us fix a geometric realization $B \subset U$ of the braid $\sigma_1^{-1}\sigma_2^{-1}\dots\sigma_{m-1}^{-1}b\,\sigma_{m-1}^{-1}\dots\sigma_2^{-1}\sigma_1^{-1}$ such that $B \cap \big(\{\pm 1\} \times \mathbf{C}\big) = \{\pm 1\} \times (\{-1\} \cup X)$ where X is a finite set $\{x_2, \dots, x_m\} \subset [-1/2, 1/2]$. Thus, the "lower corners" $(-1,-1)$ and $(1,-1)$ belong to B (they correspond to the first string of the braid).

For a real $\varepsilon > 0$, denote by U_ε, and B_ε the images of U, and B under the linear transformation $(z,w) \mapsto \big(\varepsilon z/2\,,\ -h_-(\varepsilon/2)\,w\big)$, and let $X_\varepsilon = \{x_{2,\varepsilon}, \dots, x_{m,\varepsilon}\}$ where $x_{j,\varepsilon} = -h_-(\varepsilon/2)x_j$ (so, we place the "lower corners" of U_ε onto the lower branch of H). B_ε is the graph of an m-valued function $w = f(z)$ defined on the segment $[-\varepsilon/2, \varepsilon/2]$. Let us continue f to the rectangle $R = \{|\operatorname{Re} z| \le \varepsilon,\ |\operatorname{Im} z| \le \varepsilon\}$. If $|\operatorname{Re} z| \le \varepsilon/2$, we put $f(z) = f(\operatorname{Re} z)$. If $|\operatorname{Re} z| \ge \varepsilon/2$, we put $f(z) = \{f_1(z), x_{2,\varepsilon}, \dots, x_{m,\varepsilon}\}$. where $f_1(z) = h_-(z)$ for $\operatorname{Re} z = \pm\varepsilon$, $f_1(z) = h_-(\varepsilon/2)$ for $\operatorname{Re} z = \pm\varepsilon/2$, and f_1 is linear on each segment $[\pm\varepsilon + yi,\ \pm\varepsilon/2 + yi]$ with $|y| \le \varepsilon$. For small z, we have $h_-(z) = -z^2 + o(z^2)$. Hence, for $\varepsilon \ll 1$, the branches of f do not meet each other (see Figure 1).

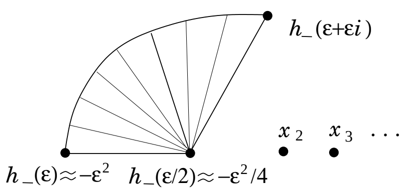

FIGURE 1

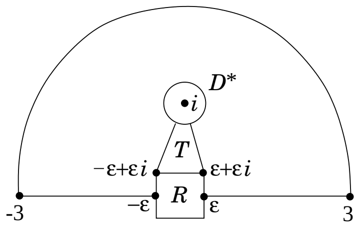

FIGURE 2

When $\varepsilon \to 0$, the height of U_ε decreases faster than the width. Hence, $\max_{z\in R} |f'(z)| \to 0$, and one can choose ε so small that the complexifications of real lines passing throug p meet the graph of f transversally and cut from it a braid B_ε^p which is equivalent to B. For such ε, the graph of the restriction of f onto the upper side of R is contained in C_p.

Put $L = \mathbf{C} \times X_\varepsilon$ and $\bar B_\varepsilon = B_\varepsilon \cup \big(((H \cup L) \cap \partial C_\infty) \setminus U_\varepsilon\big)$. In other words, $\bar B$ is the graph of the $(m+1)$-valued function on the closed curve ∂D. On $[-\varepsilon, \varepsilon]$ it is defined as $\{f, h_+\}$; outside $[-\varepsilon, \varepsilon]$, its graph contains the both branches of H and $m-1$ constant branches L. Easy to check (see Figure 3) that the braid representing by $\bar B$ is conjugated to $b' = b\sigma_m$.

Now, let us construct a symplectic surface F in $C_\infty \cup C_p$ such that $F \cup \partial C_\infty = \bar B$ and $F \cup \partial C_p$ represents b. Define F in $R \times \mathbf{C}$ as the graph of $\{f, h_+\}$. Define F outside $(R \times \mathbf{C}) \cup (C_\infty \cap C_p)$ as $(H \cup L) \setminus \big((C_\infty \cap C_p) \cup (R \times \mathbf{C})\big)$.

The vertical line $\{z = i\}$ is tangent to H at $q = (i, 1)$. Since $q \in \operatorname{Int} C_p$ and $\{i\} \times X_\varepsilon \subset \operatorname{Int} C_p$ for $\varepsilon \ll 1$, we may choose a disk $D^* \subset \mathbf{C}$ centered at i such that $(D^* \times \mathbf{C}) \cap (H \cup L) \subset C_p$.

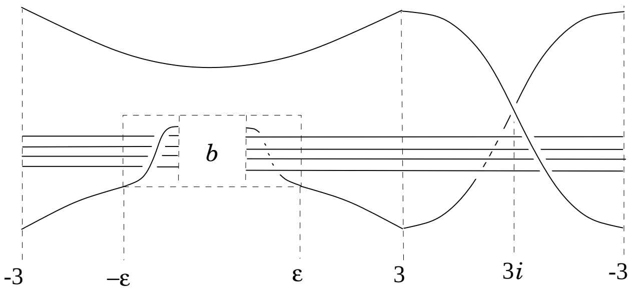

FIGURE 3

Let us extend the surface F into $(D \setminus D^*) \times \mathbf{C}$ so that its projection onto the axis $w = 0$ is an unbranched covering over $D \setminus D^*$ and the part of ∂F which is over ∂D^* is contained in C_p. Let $T \subset \mathbf{C}$ be the triangle with vertices $-\varepsilon + \varepsilon i$, $\varepsilon + \varepsilon i$, i (see Figure 2). Define F outside $(R \cup T \cup D^*) \times \mathbf{C}$ as $(H \cup L) \cap \big((D \setminus (R \cup T \cup D^*)) \times \mathbf{C}\big)$. It is an unbranched covering over $D \setminus (R \cup T \cup D^*)$ because $L \cap H \subset R \times \mathbf{C}$.

For $z \in \mathbf{C}$, let $C_p(z) = \{w \mid (z, w) \in C_p\}$ be the fiber of C_p over z. We have

$$\begin{aligned} C_p(z) &= \{w \mid \ \operatorname{Im} \bar{z}(w-2) \ge 0,\ |2z| \le 3 \cdot |2-w|\} \\ &= \{w \mid \ \operatorname{Arg}(w-2) \in [\theta, \pi+\theta],\ |w-2| \ge 2r/3\}, \qquad z = re^{\theta i}. \end{aligned}$$

In Figure 4, the domain $C_p(z)$ is shaded. Easy to check that for $z \in T$ we have

$$x_{2,\varepsilon}\,, \dots, x_{m,\varepsilon}\,, \operatorname{Re} h_-(z) < 1 < \operatorname{Re} h_+(z) \qquad \text{and} \qquad h_-(z) \in C_p(z) \qquad (1)$$

(the images of T under h_- and h_+ are depicted in Figure 5).

Let $\hat{H}_+ = \{(z, w) \in C_p \mid z \in T,\ \operatorname{Arg}(w - h_+(z)) = \operatorname{Arg}(z/i)\}$, it is the union of all segments $[(z, h_+(z)), (z, \hat{h}_+(z))]$ where $z \in T$ is such that $h_+(z) \in C_p$ and $\hat{h}_+(z)$ is the intersection of $\partial C_p(z)$ with the line orthogonal to the half-plane containing $C_p(z)$ (see Figure 4).

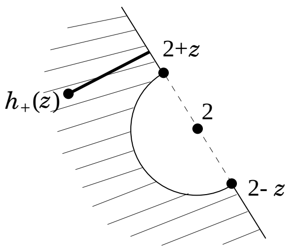

FIGURE 4

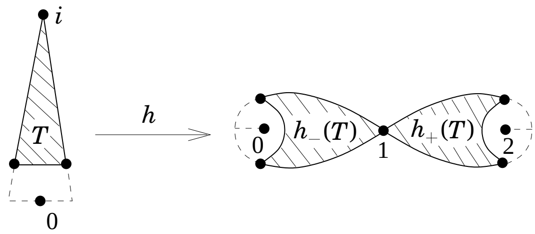

FIGURE 5

Let $E = \operatorname{Int}\big(C_p \cap ((T \setminus D^*) \times \mathbf{C})\big) \setminus \hat{H}_+$ and $\mathrm{pr} : E \to T \setminus D^*$ the projection $(z, w) \mapsto z$. This is a fibration whose fibers are diffeomorphic to $\mathbf{R}^2$. The condition (1) assures that the part of $F \setminus H_+$ which is already constructed, defines pairwise non-intersecting sections of pr over $(\partial T) \setminus D^*$. Since the fibration is trivial, they can be extended to pairwise non-intersecting sections over $T \setminus D^*$. Let us define F in $(T \setminus D^*) \times \mathbf{C}$ as the union of the graphs of these sections and $H_+ \cap \big((T \setminus D^*) \times \mathbf{C}\big)$.

Thus, F is constructed over $D \setminus D^*$. Let us fill D^*. Since $F \cap \big((D \setminus D^*) \times \mathbf{C}\big)$ is unbranched over $D \setminus D^*$, its boundary braid over ∂D^* is b'. Let $\prod_{j=1}^{k} a_j \sigma_1 a_j^{-1}$ be a quasipositive presentation of b' (with respect to some base point z_0). Choose inside D^* distinct points $z_1, \dots, z_k$ and paths α_j connecting z_0 to some $z_j' \in \partial D_j$ where D_j is a small disk centered at z_j. Define $F_j = F \cap (D_j \times \mathbf{C})$ as the graph of $(m+1)$-valued function $z \mapsto \{\pm\sqrt{z - z_j}\,,\ 2\delta_j, \dots, m\delta_j\}$ where δ_j and the radius of D_j are chosen so small that $F_j \subset C_p$. Note that each $F_j \cap \partial D_j$ represents the braid $\sigma_1 \in B_{m+1}$.

Let us define the sets $A_j = F \cup (\alpha_j \times \mathbf{C})$ so that they geometrically represent the braids a_j and define the part of F over $D^* \setminus \bigcup (D_j \cup \alpha_j)$ as the grapf of an isotopy between b' and $\prod_{j=1}^{k} a_j \sigma_1 a_j^{-1}$. Since the both braids are inside C_p, and $\mathrm{pr} : C_p \cap (D^* \times \mathbf{C}) \to D^*$ is a trivial fibration by disks (see Figure 4), the isotopy can be choosen also inside C_p.

Let $z_* = 2z/(2-w)$, $w_* = w/(2-w)$ be the affine coordinates in $\mathbf{CP}^2$ where C_p is the cylinder $\{(z_*, w_*) \mid \operatorname{Im} z_* \ge 0,\ |z_*| \le 3\}$ over D. Then F defines an m-valued function $w_*(z_*)$ on ∂D. The corresponding braid is b. Indeed, the branches of $F \cap \partial C_p$ close to ∂D are isotopic to those of $F \cap \partial C_\infty$, and H becomes a parabola $w_* = -z_*^2/4$, hence, the braid looks as in Figure 6.

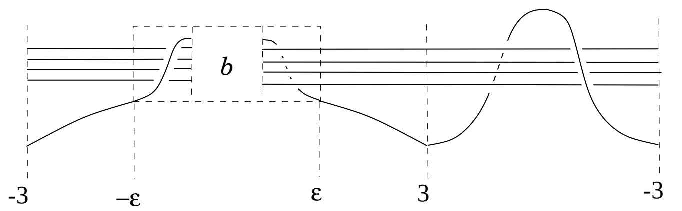

Figure 6

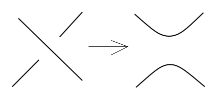

Figure 7

Let us describe what happens with the braid when the projection point p_t moves continuousely from $(0:1:0)$ to p along the segment $z = \operatorname{Re} w = 0$, $\operatorname{Im} w \ge 2$ (the segment $z_* = \operatorname{Re} w_* = 0$, $\operatorname{Im} z_* \le 1$ in coordinates $(z_*\,, w_*)$). The two branches of H over the point $q_0 = (3i, 0)$ approaches to each other and at the moment when the line (p_t, q_0) is tangent to H, they bifurcate as in Figure 7. After the bifurcation, σ_m disappears and the $(m+1)$-th string turns into a circle non-linked with the rest of the braid. Then this circle mounts and goes away to infinity.

Let us smooth the constructed surface F in small neighbourhoods of its non-smooth points. The surface $F \subset (C_\infty \cap C_p)$ contains a complex analytic part F_{an} such that $F \setminus F_{an}$ is compact and unbranched with respect to the projection $(z, w) \mapsto z$. Hence, $A(F)$ is symplectic by Lemma where $A(z, w) = (z, aw)$, $0 < a \ll 1$, and $B(F)$ is symplectic with respect to Fubini-Studi symplectic form ω_{FS} on $\mathbf{CP}^2$ where $B(z, w) = (bz, aw)$, $0 < a \ll b \ll 1$ (we suppose that $\mathbf{C}^2$ is embedded to $\mathbf{CP}^2$ by $(z, w) \mapsto (z : w : 1)$). Thus, the closure of F in $\mathbf{CP}^2$ is symplectic with respect to $\omega = B^*(\omega_{FS})$.

Let us choose an almost complex structure J, tamed by ω (see [2]), such that F and all lines (pp') for $p' \in \partial D$ are J-holomorphic. This is possible because all these lines meet F transversally and the intersections are positive. By the results of Gromov [2], $C_p \setminus \{p\}$ is fibered over D by J-holomorphic lines passing through

p. Since F is J-holomorphic, its projection onto D along the fibers is a branched covering which has only positive ramifications. Hence, b is quasipositive. Theorem 1 is proven.

References

1. S. Burckel, *The wellordering on positive braids*, J. Pure Appl. Algebra **120** (1997), 1–17.
2. M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.
3. R. Laver, *Braid group actions on left distributive structures and well-orderings in the braid groups*, J. Pure Appl. Algebra **108** (1996), 81–98.
5. L. Rudolph, *Algebraic functions and closed braids*, Topology **22** (1983), 191-202.
6. B. Wiest, *Dehornoy's ordering of the braid groups extends the subword ordering*, Pacific J. Math. (to appear); http://protis.univ-mrs.fr/~bertw.

Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina 8, Moscow, 117966, Russia

E-mail address: `orevkov@mi.ras.ru`

Laboratoire E. Picard, UFR MIG, Université Paul Sabatier, 118 route de Narbonne, 31062, Toulouse, France

E-mail address: `orevkov@picard.ups-tlse.fr`