MARKOV MOVES FOR QUASIPOSITIVE BRAIDS

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Let B_m denotes the group of braids with m strings (m-braids). Recall that it is defined by $\sigma_1,\dots,\sigma_{m-1}$ and relations $[\sigma_j,\sigma_k]=1$ for $|k-j|>1$ and $\sigma_j\sigma_k\sigma_j=\sigma_k\sigma_j\sigma_k$ for $|k-j|=1$. Denote by $e: B_m\to\mathbf{Z}$ the "exponent sum" homomorphism: $e(\sigma_i)=1$ for all $i=1,\dots,m-1$. An m-braid b is called *quasipositive* (see [5]) if $b=\prod_{j=1}^k a_j\sigma_1 a_j^{-1}$ for some braids $a_j\in B_m$ (recall that all standard generators are conjugated).

One says that $b'\in B_{m+1}$ is obtained from $b\in B_m$ by a *Markov move* if $b'=b\sigma_m^\varepsilon$ for $\varepsilon=\pm1$ (we identify here B_m with the subgroup of B_{m+1} generated by $\sigma_1,\dots,\sigma_{m-1}$). Say that the Markov move is *positive* if $\varepsilon=1$ and *negative* otherwise. It follows immediately from the definitions that a braid is quasipositive if it is obtained from a quasipositive braid by a positive Markov move. Here we prove the converse.

Theorem 1. *Let $b'\in B_{m+1}$ be obtained from b by a positive Markov move. If b' is quasipositive then b is also quasipositive.*

This is a pure existence theorem. Our proof is based on Gromov's theory of pseudo holomorphic curves [2] and it is absolutely non-constructive: we do not know an algorithm to find a quasipositive presentation of b starting with a quasipositive presentation of b'.

Remark. If b' is obtained from any braid b by a negative Markov move then b' is never quasipositive. This is an immediate consequence from the following result of Burckel [1] and Laver [3] (a proof was given by Wiest [6]; see also [4]): *any conjugate of a quasipositive braid is positive in Dehornoy right-invariant order.*

Recall that a smooth oriented 2-surface F in a smooth symplectic 4-manifold (X,ω) is call *symplectic* if $j^*(\omega)$ is positive on F where j is the imbedding $F\subset X$. Let (z,w), $z=x+iy$, $w=u+iv$, $i=\sqrt{-1}$, be coordinates in $\mathbf{C}^2$. Denote by ω_0 the standard sypmlectic form $\omega_0=dx\wedge dy+du\wedge dv$ on $\mathbf{C}^2$.

Lemma. *Let $F\in\mathbf{C}^2$ be the graph of a smooth function $w=f(z)=u(z)+iv(z)$ defined in a domain $D\subset\mathbf{C}$. If $|u'_x|,|u'_y|,|v'_x|,|v'_y|<1/\sqrt{2}$ then F is symplectic.*

Proof. Let $p(z,w)=z$. Then $((p|_F)^{-1})^*(\omega_0)=(1+u'_xv'_y-u'_yv'_x)dx\wedge dy$. $\square$

We shall consider m-braids as isotopy classes of m-valued functions $f:[0,1]\to\mathbf{C}$ such that each of $f(0)$ and $f(1)$ is the set of m points with distinct real parts. The generator $\sigma_k\in B_m$ are represented by $t\mapsto\{1,\dots,k-1,k+(1\pm e^{\pi it})/2,k+1,\dots,m\}$.

Proof of Theorem 1. Denote by H the conic $w^2-2w=z^2$ and let $p=(0,2)\in H$. Denote by C_∞ the cylinder $\{(z,w)\mid \operatorname{Im} z\ge0,\ |z|\le3\}$. Let D be the half-disk

$C_\infty \cap \{w=0\}$ and $C_p = \{(z,w) \mid \operatorname{Im}\big(2z/(2-w)\big) \ge 0,\ |2z/(2-w)| \le 3\}$ the cone over D with the vertex p (the union of all complex lines (pp'), $p' \in D$).

H is the graph of the 2-valued function $w = 1 \pm \sqrt{1+z^2}$. It has two points of ramification $z = \pm i$. Let h_- and h_+ be the single-valued branches in the band $\{|\operatorname{Im} z| < 1\}$ such that $h_-(0) = 0$ and $h_+(0) = 2$. Denote also the graphs of h_- and h_+ by H_- and H_+.

Denote by U the cylinder $\{(z,w) \mid \operatorname{Im} z = 0\,, |z| \le 1,\ |w| \le 1\}$. Let us fix a geometric realization $B \subset U$ of the braid $\sigma_1^{-1}\sigma_2^{-1}\dots\sigma_{m-1}^{-1} b\, \sigma_{m-1}^{-1}\dots\sigma_2^{-1}\sigma_1^{-1}$ such that $B \cap \big(\{\pm1\}\times \mathbf{C}\big) = \{\pm 1\} \times (\{-1\}\cup X)$ where X is a finite set $\{x_2,\dots,x_m\} \subset [-1/2, 1/2]$. Thus, the "lower corners" $(-1,-1)$ and $(1,-1)$ belong to B (they correspond to the first string of the braid).

For a real $\varepsilon > 0$, denote by U_ε, and B_ε the images of U, and B under the linear transformation $(z,w) \mapsto \big(\varepsilon z/2\,,\ -h_-(\varepsilon/2)\,w\big)$, and let $X_\varepsilon = \{x_{2,\varepsilon},\dots,x_{m,\varepsilon}\}$ where $x_{j,\varepsilon} = -h_-(\varepsilon/2)x_j$ (so, we place the "lower corners" of U_ε onto the lower branch of H). B_ε is the graph of an m-valued function $w = f(z)$ defined on the segment $[-\varepsilon/2, \varepsilon/2]$. Let us continue f to the rectangle $R = \{|\operatorname{Re} z| \le \varepsilon,\ |\operatorname{Im} z| \le \varepsilon\}$. If $|\operatorname{Re} z| \le \varepsilon/2$, we put $f(z) = f(\operatorname{Re} z)$. If $|\operatorname{Re} z| \ge \varepsilon/2$, we put $f(z) = \{f_1(z), x_{2,\varepsilon},\dots,x_{m,\varepsilon}\}$. where $f_1(z) = h_-(z)$ for $\operatorname{Re} z = \pm\varepsilon$, $f_1(z) = h_-(\varepsilon/2)$ for $\operatorname{Re} z = \pm\varepsilon/2$, and f_1 is linear on each segment $[\pm\varepsilon + yi, \pm\varepsilon/2 + yi]$ with $|y| \le \varepsilon$. For small z, we have $h_-(z) = -z^2 + o(z^2)$. Hence, for $\varepsilon \ll 1$, the branches of f do not meet each other (see Figure 1).

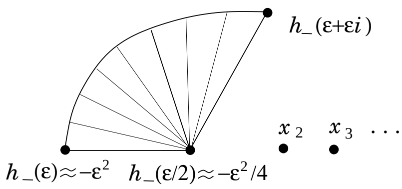

FIGURE 1

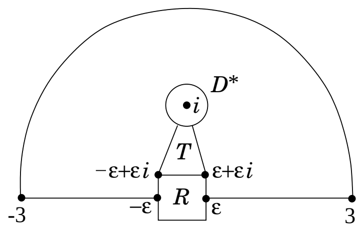

FIGURE 2

When $\varepsilon \to 0$, the height of U_ε decreases faster than the width. Hence, $\max_{z\in R}|f'(z)| \to 0$, and one can choose ε so small that the complexifications of real lines passing throug p meet the graph of f transversally and cut from it a braid B^p_ε which is equivalent to B. For such ε, the graph of the restriction of f onto the upper side of R is contained in C_p.

Put $L = \mathbf{C} \times X_\varepsilon$ and $\bar B_\varepsilon = B_\varepsilon \cup \big(((H \cup L)\cap \partial C_\infty) \setminus U_\varepsilon\big)$. In other words, $\bar B$ is the graph of the $(m+1)$-valued function on the closed curve ∂D. On $[-\varepsilon,\varepsilon]$ it is defined as $\{f, h_+\}$; outside $[-\varepsilon,\varepsilon]$, its graph contains the both branches of H and $m-1$ constant branches L. Easy to check (see Figure 3) that the braid representing by $\bar B$ is conjugated to $b' = b\sigma_m$.

Now, let us construct a symplectic surface F in $C_\infty \cup C_p$ such that $F \cup \partial C_\infty = \bar B$ and $F \cup \partial C_p$ represents b. Define F in $R \times \mathbf{C}$ as the graph of $\{f, h_+\}$. Define F outside $(R\times\mathbf{C}) \cup (C_\infty \cap C_p)$ as $(H\cup L) \setminus \big((C_\infty \cap C_p) \cup (R \times \mathbf{C})\big)$.

The vertical line $\{z = i\}$ is tangent to H at $q = (i,1)$. Since $q \in \operatorname{Int} C_p$ and $\{i\} \times X_\varepsilon \subset \operatorname{Int} C_p$ for $\varepsilon \ll 1$, we may choose a disk $D^* \subset \mathbf{C}$ centered at i such that $(D^* \times \mathbf{C}) \cap (H \cup L) \subset C_p$.

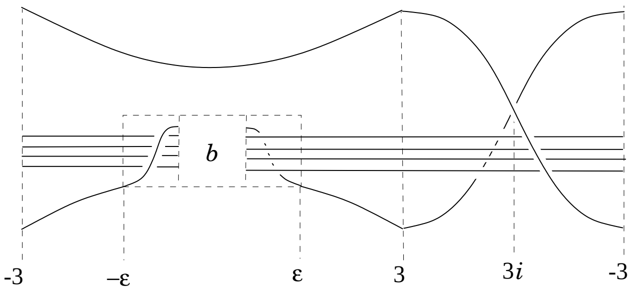

FIGURE 3

Let us extend the surface F into $(D\setminus D^*)\times\mathbf{C}$ so that its projection onto the axis $w=0$ is an unbranched covering over $D\setminus D^*$ and the part of ∂F which is over ∂D^* is contained in C_p. Let $T\subset\mathbf{C}$ be the triangle with vertices $-\varepsilon+\varepsilon i$, $\varepsilon+\varepsilon i$, i (see Figure 2). Define F outside $(R\cup T\cup D^*)\times\mathbf{C}$ as $(H\cup L)\cap\big((D\setminus(R\cup T\cup D^*))\times\mathbf{C}\big)$. It is an unbranched covering over $D\setminus(R\cup T\cup D^*)$ because $L\cap H\subset R\times\mathbf{C}$.

For $z\in\mathbf{C}$, let $C_p(z)=\{w\,|\,(z,w)\in C_p\}$ be the fiber of C_p over z. We have

$$\begin{aligned} C_p(z) &= \{w\,|\ \operatorname{Im}\bar z(w-2)\ge 0,\ |2z|\le 3\cdot|2-w|\}\\ &= \{w\,|\ \operatorname{Arg}(w-2)\in[\theta,\pi+\theta],\ |w-2|\ge 2r/3\}, \qquad z=re^{\theta i}. \end{aligned}$$

In Figure 4, the domain $C_p(z)$ is shaded. Easy to check that for $z\in T$ we have

$$x_{2,\varepsilon}\,,\dots,x_{m,\varepsilon}\,,\operatorname{Re}h_-(z)<1<\operatorname{Re}h_+(z) \qquad \text{and} \qquad h_-(z)\in C_p(z) \tag{1}$$

(the images of T under h_- and h_+ are depicted in Figure 5).

Let $\hat H_+=\{(z,w)\in C_p\,|\,z\in T,\ \operatorname{Arg}(w-h_+(z))=\operatorname{Arg}(z/i)\}$, it is the union of all segments $[(z,h_+(z)),(z,\hat h_+(z))]$ where $z\in T$ is such that $h_+(z)\in C_p$ and $\hat h_+(z)$ is the intersection of $\partial C_p(z)$ with the line orthogonal to the half-plane containing $C_p(z)$ (see Figure 4).

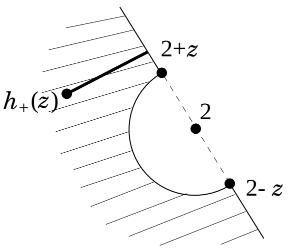

FIGURE 4

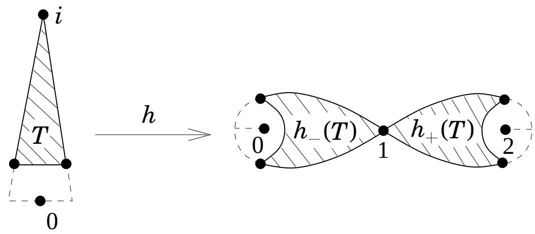

FIGURE 5

Let $E=\operatorname{Int}\big(C_p\cap((T\setminus D^*)\times\mathbf{C})\big)\setminus\hat H_+$ and $\mathrm{pr}:E\to T\setminus D^*$ the projection $(z,w)\mapsto z$. This is a fibration whose fibers are diffeomorphic to $\mathbf{R}^2$. The condition (1) assures that the part of $F\setminus H_+$ which is already constructed, defines pairwise non-intersecting sections of pr over $(\partial T)\setminus D^*$. Since the fibration is trivial, they can be extended to pairwise non-intersecting sections over $T\setminus D^*$. Let us define F in $(T\setminus D^*)\times\mathbf{C}$ as the union of the graphs of these sections and $H_+\cap\big((T\setminus D^*)\times\mathbf{C}\big)$.

Thus, F is constructed over $D \setminus D^*$. Let us fill D^*. Since $F \cap \big((D \setminus D^*) \times \mathbf{C}\big)$ is unbranched over $D \setminus D^*$, its boundary braid over ∂D^* is b'. Let $\prod_{j=1}^k a_j \sigma_1 a_j^{-1}$ be a quasipositive presentation of b' (with respect to some base point z_0). Choose inside D^* distinct points $z_1, \dots, z_k$ and paths α_j connecting z_0 to some $z_j' \in \partial D_j$ where D_j is a small disk centered at z_j. Define $F_j = F \cap (D_j \times \mathbf{C})$ as the graph of $(m+1)$-valued function $z \mapsto \{\pm\sqrt{z - z_j}\,,\ 2\delta_j, \dots, m\delta_j\}$ where δ_j and the radius of D_j are chosen so small that $F_j \subset C_p$. Note that each $F_j \cap \partial D_j$ represents the braid $\sigma_1 \in B_{m+1}$.

Let us define the sets $A_j = F \cup (\alpha_j \times \mathbf{C})$ so that they geometrically represent the braids a_j and define the part of F over $D^* \setminus \bigcup (D_j \cup \alpha_j)$ as the grapf of an isotopy between b' and $\prod_{j=1}^k a_j \sigma_1 a_j^{-1}$. Since the both braids are inside C_p, and $\mathrm{pr} : C_p \cap (D^* \times \mathbf{C}) \to D^*$ is a trivial fibration by disks (see Figure 4), the isotopy can be choosen also inside C_p.

Let $z_* = 2z/(2-w)$, $w_* = w/(2-w)$ be the affine coordinates in $\mathbf{CP}^2$ where C_p is the cylinder $\{(z_*, w_*) \mid \operatorname{Im} z_* \ge 0,\ |z_*| \le 3\}$ over D. Then F defines an m-valued function $w_*(z_*)$ on ∂D. The corresponding braid is b. Indeed, the branches of $F \cap \partial C_p$ close to ∂D are isotopic to those of $F \cap \partial C_\infty$, and H becomes a parabola $w_* = -z_*^2/4$, hence, the braid looks as in Figure 6.

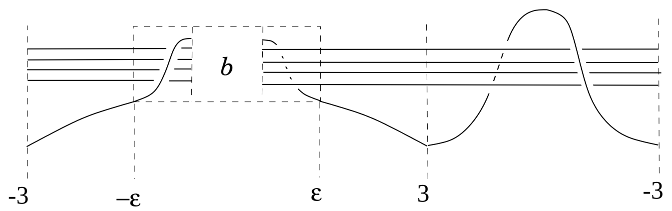

FIGURE 6

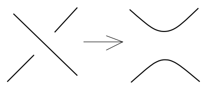

FIGURE 7

Let us describe what happens with the braid when the projection point p_t moves continuousely from $(0:1:0)$ to p along the segment $z = \operatorname{Re} w = 0$, $\operatorname{Im} w \ge 2$ (the segment $z_* = \operatorname{Re} w_* = 0$, $\operatorname{Im} z_* \le 1$ in coordinates $(z_*\,, w_*)$). The two branches of H over the point $q_0 = (3i, 0)$ approaches to each other and at the moment when the line (p_t, q_0) is tangent to H, they bifurcate as in Figure 7. After the bifurcation, σ_m disappears and the $(m+1)$-th string turns into a circle non-linked with the rest of the braid. Then this circle mounts and goes away to infinity.

Let us smooth the constructed surface F in small neighbourhoods of its non-smooth points. The surface $F \subset (C_\infty \cap C_p)$ contains a complex analytic part F_{an} such that $F \setminus F_{an}$ is compact and unbranched with respect to the projection $(z, w) \mapsto z$. Hence, $A(F)$ is symplectic by Lemma where $A(z, w) = (z, aw)$, $0 < a \ll 1$, and $B(F)$ is symplectic with respect to Fubini-Studi symplectic form ω_{FS} on $\mathbf{CP}^2$ where $B(z, w) = (bz, aw)$, $0 < a \ll b \ll 1$ (we suppose that $\mathbf{C}^2$ is embedded to $\mathbf{CP}^2$ by $(z, w) \mapsto (z : w : 1)$). Thus, the closure of F in $\mathbf{CP}^2$ is symplectic with respect to $\omega = B^*(\omega_{FS})$.

Let us choose an almost complex structure J, tamed by ω (see [2]), such that F and all lines (pp') for $p' \in \partial D$ are J-holomorphic. This is possible because all these lines meet F transversally and the intersections are positive. By the results of Gromov [2], $C_p \setminus \{p\}$ is fibered over D by J-holomorphic lines passing through

p. Since F is J-holomorphic, its projection onto D along the fibers is a branched covering which has only positive ramifications. Hence, b is quasipositive. Theorem 1 is proven.

References

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