UDC 517.5

WHEN A CHAIN OF BLOWUPS DEFINES AN AUTOMORPHISM OF $\mathbf{C}^2$.

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1. Introduction. In this note we give a new proof of a theorem of A.G. Vitushkin [1]. The proof is based on a formula for the canonical class of an algebraic compactification of $\mathbf{C}^2$ (formula (5) below). This formula was used in some author's papers (see, e.g., formula (3) in [2]), however, its detailed proof is published here the first time. Similar formulae written in other terms appeared in different authors' papers.

Let L be a line on $\mathbf{C}P^2$. We shall consider it as the infinite line of the affine plane $\mathbf{C}^2$, i.e. $L = \mathbf{C}P^2 \setminus \mathbf{C}^2$. Let $\sigma_1 : V \to \mathbf{C}P^2$ be a birational morphism whose restriction onto $\sigma_1^{-1}(\mathbf{C}^2)$ is an isomorphism, i.e. σ_1 is a composition (a chain) of blowups "at infinity". Let $E_1, \dots, E_n$ be the irreducible components of the curve $E = \sigma_1^{-1}(L)$ where E_1 is the proper transform of L. We say that the chain of blowups σ_1 *defines an automorphism of* $\mathbf{C}^2$ if the last glued curve (denote it by E_2) admets a birational morphism $\sigma_2 : V \to \mathbf{C}P^2$ such that $\sigma_2|_{\sigma_2^{-1}(\mathbf{C}^2)}$ is an isomorphism, $\sigma_2^{-1}(L) = E$, and E_2 is the proper transform of L (in this case, $\sigma_2^{-1}\sigma_1$ is an automorphism of $\mathbf{C}^2$).

Following [1], let us define a *test surface* for the chain of blowups σ_1 as a homology class $S \in H_2(V; \mathbf{Z})$ such that

$$S \cdot E_2 = 1, \qquad S \cdot E_i = 0 \text{ for } i \neq 2 \tag{1}$$

(as above, E_2 is the last glued curve). Since $E_1, \dots, E_n$ is a base in $H_2(V; \mathbf{Z})$, the conditions (1) uniquely define the class S. If a chain of blowups σ_1 defines an automorphism of $\mathbf{C}^2$ then $S = \sigma_2^{-1}(l)$ where l is a generic line on $\mathbf{C}P^2$. Hence, by the adjunction formula, we have

$$S^2 = 1, \qquad S \cdot K_V = -3, \tag{2}$$

where $K_V = -c_1(V)$ is the canonical class of V. A.G. Vitushkin proved that (2) is not only necessary but also a sufficient condition for a chain of blowups σ_1 to define an automorphism $\mathbf{C}^2$:

Theorem. (see [1]) *A chain of blowups σ_1 defines an automorphism of $\mathbf{C}^2$ if and only if the test surface S satisfies (2).*

2. Discriminant of the intersection form. Let $D = D_1 + \dots + D_n$ be a curve on a smooth projective surface V, ($D_1, \dots, D_n$ are its irreducible components). Let $A_D = (D_i \cdot D_j)_{ij}$ be the intersection matrix. Let us define the *discriminant* $d(D)$ of D as $\det(-A_D)$. The sign minus provides the equality $d(\sigma^{-1}(D)) = d(D)$ for a blowup $\sigma : \tilde{V} \to V$. If $D_i \cdot D_j \le 1$ for $i \neq j$ then the *graph of* D is the graph Γ_D whose vertices are $D_1, \dots, D_n$ and whose edges correspond to pairs (D_i, D_j) with $D_i \cdot D_j = 1$.

Lemma 1. (Mumford [3]) *Suppose* Γ_D *has the form of a linear chain* $-\circ-\cdots-\circ-$. *Then*

a). If $D_i^2 \le -2$ *for all* i *then* $d(D) \ge 2$.

b). If A_D *is negatively definite,* $D_i^2 \le -1$ *for all* i, *and* $d(D) = 1$ *then* D *can be blown down to a smooth point.*

3. Proof of Vitushkin's theorem. Let $\sigma_1 : V \to \mathbf{C}P^2$ be a birational morphism and $E = E_1 + \cdots + E_n = \sigma_1^{-1}(L)$. Then $E_1, \ldots, E_n$ is a base of the vector space $H_2(V, \mathbf{Q})$. Let $A = (E_i \cdot E_j)$ be the intersection matrix and let $B = A^{-1}$. The graph of E is a tree. Since $d(E) = -1$, Cramer's rule easily implies

Lemma 2. $b_{ij} = d(E - [ij])$ *where* $[ij]$ *is the minimal connected set of the form* $E_{i_1} \cup \cdots \cup E_{i_k}$ *which contains* $E_i \cup E_j$. $\square$

Lemma 3. *For any* $C_1, C_2 \in H_2(V, \mathbf{Q})$, *one has* $C_1 \cdot C_2 = \sum_{i,j} b_{ij}(C_1 \cdot E_i)(C_2 \cdot E_j)$.

Proof. For $k = 1, 2$, let us set $X_k = (x_k^1, \ldots, x_k^n)$, $Y_k = (y_k^1, \ldots, y_k^n)$, where $C_k = \sum x_k^i E_i$ and $y_k^i = C_k \cdot E_i$. Then $Y_k = AX_k$, hence $X_k = BY_k$ and $C_1 \cdot C_2 = \langle X_1, AX_2 \rangle = \langle BY_1, Y_2 \rangle$. $\square$

Let, as in Sect. 1, E_2 be the last glued curve and S be the test surface. It follows from Lemmas 2,3 and (1) that

$$S^2 = b_{22} = d(E - E_2). \tag{3}$$

Let us denote $\nu_i = E_i \cdot (E - E_i)$. The adjunction formula for E_i yeilds $(K_V + E) \cdot E_i = \nu_i - 2$, hence by (1) and Lemma 3, we have

$$(K_V + E) \cdot S = \sum_{i=1}^{n} b_{i2}(\nu_i - 2). \tag{4}$$

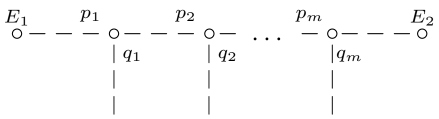

FIG. 1

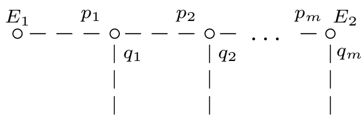

FIG. 2

Suppose that (2) holds. Without loss of generality, we may assume that the curve E is *minimal* in the following sense. There does not exist $j \ge 3$ such that $E_j^2 = -1$ and $\nu_j \le 2$. This means that we blow up each time a point of the exceptional curve of the previouse blowup. Therefore, the induction with respect to the number of blowups easily shows that Γ_E is either as in Fig. 1 or as in Fig. 2 where the dashed lines denote linear chains of vertices. For curves E_j such that $\nu_j = 3$, let us denote the discriminants of the connected components of $E - E_j$ in accordance with Figures 1 and 2. Lemma 1 and the minimality of E imply that $q_j \ge 2$ for all j. Hence, (2) and (3) imply that Fig. 2 is impossible. Applying (4) and Lemma 2 to Fig. 1 and using the fact that $ES = 1$, we obtain

$$K_V \cdot S + 1 = \sum_{\nu_i = 3} b_{i2} - \sum_{\nu_i = 1} b_{i2} = \sum_{j=1}^{m} p_j q_j r_j - r_0 - \sum_{j=1}^{m} p_j r_j - b_{22} = -b_{22} - 1 + s \tag{5}$$

where $r_j = q_{j+1} \ldots q_m$ and $s = \sum (p_j - 1)(q_j - 1) r_j$. Substituting (2) and (3) into (5), we obtain $s = 0$.

Lemma 4. *Let q be the restriction of a nondegenerate quadratic form of the signature $(-,+,\dots,+)$ onto a subspace. If the discriminant of q is positive then q is positively definite.* $\square$

By (2) and (3), we have $d(E - E_2) = S^2 = 1 > 0$. Hence, by Lemma 4, the minus intersection form is positively definite on $E - E_2$. In particular, all q_j and p_j are positive, hence, all the summands in the sum s (see (5)) are non-negative. Using the fact that all $q_j > 1$, the equality $s = 0$ implies $p_1 = \cdots = p_m = 1$. Hence, by Lemma 1b, the leftmost linear branch of Γ_E can be blown down. As the result, we obtain another graph of the same form which has $m - 1$ triple vertices. The discriminant of the leftmost linear branch of the new graph is p_2. Contunuing this process, we blow down all the components of E except E_2. The Theorem is proved.

4. Remark. It is proved in [1] that the conditions (2) are necessary and sufficient for a chain of blowups to be a composition of so-called triangular chains (see the definition in [1]). Explicitely writing down the blowups in coordinates, it is not difficult to see that triangular chains are those and only those which define triangular (i.e. of the form $(x, y) \mapsto (x + f(y), y)$) transformations of $\mathbf{C}^2$ up to linear changes of coordinates. Hence, the arguments from [1] prove more than the above Theorem. They provide also a proof of Jung's theorem which claims that any automorphism of $\mathbf{C}^2$ is decomposable into a product of affine and triangular transformations. The author is grateful to A.G. Vitushkin for useful discussions

References

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