PARAMETRIC EQUATIONS OF PLANE SEXTIC CURVES WITH A MAXIMAL SET OF DOUBLE POINTS

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Abstract. We give explicit parametric equations for all irreducible plane projective sextic curves which have at most double points and whose total Milnor number is maximal (is equal to 19). In each case we find a parametrization over a number field of the minimal possible degree and try to choose coordinates so that the coefficients are as small as we can do.

To the memory of Shreeram Abhyankar

Let C be a sextic curve in $\mathbb{C}P^2$ with simple singularities. The total Milnor number $\mu(C)$ of C (i. e., the sum of Milnor numbers of singular points) is at most 19. Indeed, if X is the double covering of $\mathbb{C}P^2$ branched at C and $\sigma : \tilde{X} \to X$ is the resolution of singularities of X, then $\tilde{X}$ is a K3 surface, hence the negative index of inertia of the intersection form on $H_2(\tilde{X})$ is 19 and the exceptional curves of σ generate a sublattice of rank $\mu(C)$ with a negative definite form.

Using a more detailed analysis of sublattices of $H_2(\tilde{X})$, a complete list of all maximal sets of simple singularities is obtained by Yang [13]. If $\mu(C) = 19$, then C is rational by the genus formula and C is rigid because the K3 surface is rigid. The latter means that there are only finitely many of such curves up to projective equivalence.

As in [4], in this paper we consider **irreducible sextic curves of total Milnor number 19 which have A_n singularities only**. We give explicit parametric equations for all of them. There are 39 equisingularity classes of such curves. They are listed in Degtyarev's papers [3–5] where the number of projectively non-equivalent curves is computed in each case. For the equisingularity classes (sets of singularities), we use the same numbering as in [3–5].

Explicit defining equations are computed in [4, 5] for 33 of these 39 equisingularity classes (some of them were earlier computed in [2, 3, 8, 9]). The computations in [4, 5] are based on the method proposed by Artal, Carmona, and Cogolludo in [2] (the ACC-method).

Here we use a more straightforward method which we applied already in [9]. Namely, we just write a parametrization of C with indeterminate coefficients and solve the simultaneous equations imposed by the singularity types. It happens that this method works better in the cases when the most singular points have one branch and when it is clear how to choose coordinates to cancel the action of $PSL(3)$ and $PSL(2)$. Otherwise, for example, in the case of the A_{19} singularity treated in [2], the ACC-method works better.

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In the cases when the straightforward method does not work well, we computed parametrizations by a "reengineering" from the defining equations from [4, 5].

In all the 39 cases we find a parametrization over a field E of the minimal possible degree. In all cases except four, we have $E = F$ where F is the minimal field of definition of the curve. In the remaining four cases (namely, **1**, **16**, **34**, **36**) E is a quadratic extension of F.

By rephrasing Abhyankar, one could say: *a nice curve must have a nice equation.* For the curves in question I tried to find equations as nice as I could (at least as short as I could).

I am grateful to Alexei Skorobogatov for useful discussions.

§1. The parametric equations

All the parametrizations below are available on the web page [10] in Maple format. The condition $\mu(C) = 19$ implies that exactly one singularity has two local branches and all the others are irreducible. In the parametrizations below, $p(t)$ is always a quadratic polynomial whose roots are mapped to the reducible singularity (the A_n with odd n). In the brackets we list the values of t mapped to the irreducible singularities in the order they appear in the notation "$A_{n_1} + A_{n_2} + \dots$".

1. A_{19}, see [2]. $F = \mathbb{Q}(\sqrt{5})$, $E = F(\sqrt{-3})$, $\omega^2 + \omega + 1 = 0$, $a = 3 \pm \sqrt{5}$, $p = t$,

$$\begin{aligned} x &= \omega t^2 + a t^3 - \omega^2 t^4, \\ y &= 2(t + t^5) - 5a(t^2 - t^4) + (6 + 3a)t^3, \\ z &= (\omega^2 - \omega t^6) + 4a(\omega t + \omega^2 t^5) + (81a/2 - 34)(t^2 - t^4) + (63a - 54)t^3. \end{aligned}$$

A parametrization over F does not exist because the curve does not have smooth real points.

2. $A_{18}+A_1$, see [2]. $F = E = \mathbb{Q}(a)$, $a^3-2a-2 = 0$, $[\infty]$, $p = 3t^2-12t+5a^2+8a+17$,

$$\begin{aligned} x &= (t + 3 - a)(t + 3 + a)\,p, \qquad y = t^2 + 3a^2 + 4a - 1, \\ z &= \big[9t^4 + 72t^3 - (52a^2 + 100a - 232)t^2 - (204a^2 + 528a - 420)t \\ &\quad - 92a^2 - 326a + 587\big]\,p. \end{aligned}$$

3. $(A_{17} + A_2)$, see [2, 8]. $E = F = \mathbb{Q}$, $[\infty]$, $p = t^2 - 3$,

$$x = (t^2 + 9)p^2, \qquad y = tp, \qquad z = t^4 - 12t^2 + 3;$$

the curve is symmetric under $(x : y : z) \mapsto (x : -y : z)$, $t \mapsto -t$.

4. $A_{16}+A_3$. see [2, 5]. $E = F = \mathbb{Q}(\sqrt{17})$, $a = (9\pm\sqrt{17})/8$, $[\infty]$, $p = t^2+t-a+2/3$,

$$x = (t^2 - 3\,t - 6 + 11a)\,p, \quad y = t^2 - a, \quad z = (t^2 - 6\,t + 6 + 5a)\,p^2.$$

5. $A_{16} + A_2 + A_1$. see [2, 9]. $E = F = \mathbb{Q}(a)$, $a^3 - a^2 + a - 3 = 0$, $[\infty, 0]$, $p = 2t^2 + (8 + 7a - 7a^2)(2t + 1)$,

$$\begin{aligned} x &= \big[t^2 + (8 + 13a - 13a^2)t - (1672 - 3049a + 1261a^2)/14\big]\,t^2, \\ y &= (2t - 1 + 16a - 13a^2)(2t + 9 + 12a - 15a^2)\,t^2 p, \\ z &= \big(2t^2 + (a - a^2)(12t - 9)\big)\,p. \end{aligned}$$

6. $A_{15}+A_4$, see [2, 5]. $E=F=\mathbb{Q}(i)$. $[\infty]$, $p=15\,t^2-7+6i$,

$$\begin{aligned} x &= (145\,t^2-580i\,t-133-146i)p,\\ y &= (1649\,t+724-465i)(170\,t-7-164i),\\ z &= (5\,t^2+10i\,t+1+2i)\,p^2. \end{aligned}$$

7. $A_{14}+A_4+A_1$, see [2, 5].
$E=F=\mathbb{Q}(a)$, $2a^6-6a^5+10a^4-5a^3-10a^2+4a+8=0$, $[\infty,0]$,
$p=t^2-12\,t-(338a^5-638a^4+458a^3+771a^2-4382a-2592)/187$,

$$\begin{aligned} x &= \bigl(t^2+(4a-12)t-2a^5+6a^4-2a^3-3a^2-14a+24\bigr)\,t^2,\\ y &= \bigl(t^2+4at-(10a^5-22a^4+50a^3-73a^2-38a+16)/11\bigr)\,p,\\ z &= \bigl(t^2+(8a-8)t-(10a^5-22a^4+50a^3-73a^2+50a+16)/11\bigr)\,p\,t^2. \end{aligned}$$

Note that F is a cubic extension of $\mathbb{Q}(\sqrt{-15})$: we have $a=-\frac{6}{11}b^3+b-\frac{2}{11}$ where $b^3=-\frac{5}{4}+\frac{11}{36}\sqrt{-15}$.

8. $(A_{14}+A_2)+A_3$, see [8]. $E=F=\mathbb{Q}$, $[\infty,2]$, $p=t^2+2t-9$,

$$x=t^2-5,\qquad y=(t^2+2t-5)\,p,\qquad z=(t^2+4t-11)\,p^2.$$

9. $(A_{14}+A_2)+A_2+A_1$, see [8]. $E=F=\mathbb{Q}$, $[\infty,0,1]$, $p=t^2+5\,t-5$,

$$x=(2t^2-3)\,p,\qquad y=(t^2+5\,t+1)\,t^2,\qquad z=(t^2+4t-6)\,p\,t^2.$$

10. $A_{13}+A_6$, see [5]. $E=F=\mathbb{Q}(a)$, $a^4=7$, $[\infty]$, $p=3t^2+7a^3-5a^2-13a+39$,

$$\begin{aligned} x &= (3t-2a^3+2a^2-6a+2)(3t+2a^3-a^2-2a+3)\,p,\\ y &= 3t^2+4(a^2-8a+5)\,t+(-1037a^3+2143a^2-1105a-2757)/33,\\ z &= \bigl[3t^2-2(a^2-8a+5)\,t+(31a^3+15a^2-149a+235)/3\bigr]\,p^2. \end{aligned}$$

11. $A_{13}+A_4+A_2$, see [5]. $E=F=\mathbb{Q}(\sqrt{21})$, $a=\pm\sqrt{21}$, $[\infty,3-(3a/2)]$, $p=2t^2+11a-39$, $q=2t+3a-6$,

$$\begin{aligned} x &= (t+a-3)(t-2a-3)\,p,\\ y &= (2t-5a-9)(2t-3a+9)\,q^2,\\ z &= (t-a+3)(t+3)\,p\,q^2. \end{aligned}$$

12. $A_{12}+A_7$, see [5]. $E=F=\mathbb{Q}(i)$, $[\infty]$, $p=3t^3+1$,

$$\begin{aligned} x &= \bigl(13\,t^2+(-8+12i)\,t+7-4i\bigr)\,p,\\ y &= \bigl(13\,t^2+(-20+30i)\,t-11-16i\bigr)\,p^2,\\ z &= (13\,t+3+2i)(13t+1-8i). \end{aligned}$$

13. $A_{12}+A_6+A_1$, see [5]. $E=F=\mathbb{Q}(a)$, $7a^3-16a^2+12a-4=0$, $[\infty,0]$, $p=3t^2+(12a-18)t+28a^2-53a+36$,

$$\begin{aligned} x&=\big(t^2+(4a-2)t+14a^2-15a+4\big)\,t^2,\\ y&=(t^2+4t-7a+4)\,p,\\ z&=\big(t^2+(4a+4)t+13a\big)\,p\,t^2. \end{aligned}$$

14. $A_{12}+A_4+A_3$, see [5]. $E=F=\mathbb{Q}$, $[\infty,0]$, $p=85t^2-442t+507$,

$$x=(5t^2-12t-13)\,t^2,\qquad y=(25t^2-20t-221)\,t^4,\qquad z=t^2-4t+3.$$

15. $A_{12}+A_4+A_2+A_1$. $E=F=\mathbb{Q}(a)$, $15a^3-48a^2+40a-10=0$, $[\infty,0,30a^2-\frac{159}{2}a+\frac{55}{2}]$, $p=t^2+(60a^2-141a+64)t+3828/5\,a^2-1851\,a+772$,

$$\begin{aligned} x&=\big(t^2+9t-24a^2+24a+5\big)t^2,\\ y&=t^2+9at+84a^2-165a+50,\\ z&=\big(t^2+(-9a+18)t-51a^2+24a+41\big)t^4. \end{aligned}$$

16. $A_{11}+2A_4$, see [5]. $F=\mathbb{Q}(\sqrt{2})$, $E=F(i)$, $a=\pm\sqrt{2}$, $c=4a+34i-27$, $[(3+5i\pm\sqrt{30i}\,)/4]$, $p=t^2+(2ai+a-3i-2)t+a+i-1$,

$$\begin{aligned} x&=\big[17t^4+(34ai-9a-34i-20)t^3+(136ai+49a+34i-148)t^2\\ &\qquad+(-42ai+127a+122i+48)t-37ai-13a-18i+24\big]\,p,\\ y&=c(t+2ai+a-3i-1)(17t-6ai-7a-15i-9)\big(2t^2-(5i+3)t-2\big)^2,\\ z&=(2t-i)\,p^2. \end{aligned}$$

Another parametrization (with $0,\infty\mapsto A_{11}$) is: $E=\mathbb{Q}(b)$, $b^2=-7\pm4\sqrt{2}$, thus $b^4+14b^2+17=0$,

$$\begin{aligned} x&=\big[32t^4-(7b^3-49b^2+17b-135)t^3-(28b^3-44b^2+52b-4)t^2\\ &\qquad+(10b^3+2b^2-26b-50)t+2b^3+4b^2+18b+32\big]\,t,\\ y&=(b^2-2b+9)\big[32t^2+(28b^3+8b^2+28b+16)t+2b^3-b^2+24b+19\big]\\ &\qquad\times\big[16t^2-(b^3-7b^2+7b-17)t+b^3+b^2+15b-1\big]^2,\\ z&=(b^2+2b+9)\big(2t^2-(b^2-1)t-2\big)\,t^2; \end{aligned}$$

A parametrization over F does not exist because the curve does not have smooth real points.

17. $(A_{11}+2A_2)+A_4$, see [8]. $E=F=\mathbb{Q}$, $[\pm i\sqrt{2/3},\infty]$, $p=3t^2+9t+8$,

$$x=(3t^2-10)\,p^2,\qquad y=(3t^2+9t+5)\,p,\qquad z=3t^2+3t+2.$$

18. $A_{10}+A_9$, see [5]. $E=F=\mathbb{Q}(\sqrt{5})$, $a=\pm\sqrt{5}$, $[\infty]$, $p=t^2-11-22/3\,a$,

$$\begin{aligned} x&=(t-3-2a)(t-5+4a)\,p,\\ y&=\big(t^2-(32-8a)t+149+70a\big)\,p^2,\\ z&=t^2+(16-4a)t+161-226/5\,a. \end{aligned}$$

19. $A_{10}+A_8+A_1$, see [5]. $E=F=\mathbb{Q}(a)$, $a^3-4a^2+8a-4=0$, $[\infty,0]$, $p=t^2+(8a-6)t+9a^2-29a+16$,

$$\begin{aligned} x &= \big(t^2+(4a-2)t+15a^2-45a+24\big)\,t^2,\\ y &= \big(t^2-(4a-4)t+5a^2-9a+4\big)\,p,\\ z &= \big(t^2+4t-3a^2+3a\big)\,p\,t^2. \end{aligned}$$

20. $A_{10}+A_7+A_2$, see [5]. $E=F=\mathbb{Q}(\sqrt{3})$, $a=\pm\sqrt{3}$, $[\infty,(3+27a)/22]$, $p=2t^2-6+a$,

$$\begin{aligned} x &= \big(22t^2-(12+20a)t+30+39a\big)\,p,\\ y &= \big(22t^2-(42+26a)t-6+45a\big)\,p^2,\\ z &= 22t^2+(18-14a)t-(300+159a)/11. \end{aligned}$$

21. $A_{10}+A_6+A_3$, see [5]. $E=F=\mathbb{Q}(\sqrt{-7})$, $a=\pm i\sqrt{7}$, $[\infty,0]$, $p=22\,t^2+(27-3a)t+9-3a$,

$$\begin{aligned} x &= (4t+1-a)(t+1)t^2,\\ y &= 22\,t^2+(56-16a)t-3-7a,\\ z &= \big(22\,t^2+(-1+5a)t-5+3a\big)\,t^4. \end{aligned}$$

22. $A_{10}+A_6+A_2+A_1$. $E=F=\mathbb{Q}(a)$, $a^3-a^2+3=0$, $[\infty,0,4a^2-3]$, $p=t^2+(2a^2+6a+3)t-26a^2+78a+123$,

$$\begin{aligned} x &= \big(t^2+9t-22a^2+30a+21\big)\,t^2,\\ y &= t^2-(6a^2-6a+9)t+50a^2-78a+75,\\ z &= \big(t^2+(6a^2-6a+27)t-22a^2+66a-33\big)\,t^4. \end{aligned}$$

23. $A_{10}+A_5+A_4$, see [5]. $E=F=\mathbb{Q}(\sqrt{15})$, $a=\pm\sqrt{15}$, $[\infty,0]$, $p=5t^2+(3a+5)t+(70-24a)/17$,

$$\begin{aligned} x &= (t+a-4)(t+1)\,p,\\ y &= \big(t^2+(4a-12)t-2a+8\big)\,p\,t^2,\\ z &= \big(5t^2+(8a-10)t+(110-2a)/7\big)\,t^2. \end{aligned}$$

24. $A_{10}+2A_4+A_1$, see [5]. $E=F=\mathbb{Q}(a)$, $a^3-a^2-a-1=0$, $[\infty,\pm\sqrt{5a^2-20}]$, $p=t^2+(-12a^2+18a+8)t-39a^2+12a+116$,

$$\begin{aligned} x &= (t-a)(t+4a^2-5a-4)\,p,\\ y &= \big[t^2+(-20a^2+30a+8)t+45a^2+20a-140\big](t^2-5a^2+20)^2,\\ z &= t^2+(4a^2-6a)t+a^2+4a+4. \end{aligned}$$

25. $A_{10}+A_4+A_3+A_2$, see [5]. $E=F=\mathbb{Q}$, $[\infty,0,4]$, $p=20t^2-55t-121$,

$$x=(7t^2-35t+22)t^2,\qquad y=(8t^2-52t+77)t^4,\qquad z=2t^2-7t-7.$$

26. $A_{10}+A_4+2A_2+A_1$. $E=F=\mathbb{Q}(\sqrt{5})$, $a=\pm\sqrt{5}$, $b=a-1$, $[\infty, 0, \text{roots of } t^2+3t+(15-a)/6]$, $p=t^2+(a-4)t+(15-23a)/6$,

$$x=\big(6t^2+6t+9a-25\big)t^2, \quad y=6t^2+6at+5b, \quad z=\big(6t^2+(12-6a)t-11b\big)\,t^4.$$

27. $A_9+A_6+A_4$, see [5]. $E=F=\mathbb{Q}(a)$, $a^3-5a-5=0$, $[0,\infty]$, $p=3t^2-(4a^2+4a-7)t+4a^2+13a+20$,

$$\begin{aligned} x&=\big[3t^2-(4a^2+4a-25)t-(274a^2-365a-1510)/31\big]\,t^2,\\ y&=(t-a^2-a+1)(3t-a^2-a-5)\,t^2p,\\ z&=\big[t^2+6t-2a^2-11a-10\big]\,p. \end{aligned}$$

28. $A_9+2A_4+A_2$, see [5]. $E=F=\mathbb{Q}$, $[\pm i\sqrt{15/2},\infty]$, $p=2t^2-5$,

$$x=pt, \qquad y=4t^4-80t^2+15, \qquad z=(2t^2+75)p^2;$$

the curve is symmetric under $(x:y:z)\mapsto(-x:y:z)$, $t\mapsto -t$.

29. $(2A_8)+A_3$, see [8]. $E=F=\mathbb{Q}$, $[0,\infty]$, $p=t^2+2t-1$,

$$x=3t^2-4t+1, \qquad y=(3t^2+4t-3)t^2, \qquad z=(t^2+4t+3)t^4;$$

the curve is symmetric under $(x:y:z)\mapsto(z:-y:x)$, $t\mapsto -1/t$.

30. $A_8+A_7+A_4$, see [5]. $E=F=\mathbb{Q}(i)$, $[\infty,(i-1)/2]$, $p=34t^2-8+19i$,

$$\begin{aligned} x&=\big(30\,t^2+20(1-i)t-4-3i\big)\,p,\\ y&=\big(10t^2-(2+6i)t+5i\big)\,p^2,\\ z&=(10t+9-3i)(30t+19-13i). \end{aligned}$$

31. $A_8+A_6+A_4+A_1$, see [5]. $E=F=\mathbb{Q}(a)$, $a^3-a^2-a+5=0$, $[\infty,0,1]$, $p=t^2+(a^2-4a+5)t+(43a^2-104a+120)/7$,

$$\begin{aligned} x&=\big[15t^2+(5-20a+5a^2)t-70+64a-23a^2\big]\,t^2,\\ y&=\big[21t^2+(49-28a+7a^2)t-18+24a-11a^2\big]\,(t-1)^2,\\ z&=\big[5t^2+(1-4a-a^2)t-10+5a^2\big]\,t^2\,(t-1)^2. \end{aligned}$$

32. $(A_8+A_5+A_2)+A_4$, see [8]. $E=F=\mathbb{Q}(i)$, $[\infty,0,1]$, $p=t^2+2it-(8+4i)/5$,

$$x=(t^2-4)pt^2, \quad y=(3t^2-2t+2)p, \quad z=(3t^2-(2-6i)t-4-4i)t^2.$$

33. $(A_8+3A_2)+A_4+A_1$, see [8]. $E=F=\mathbb{Q}$, $[\infty, \text{ roots of } t^3-3t-3,\ 0]$, $p=t^2+3t-9$,

$$x=(t^2+t-3)t^2, \qquad y=(t^2+3t+3)t^4, \qquad z=t^2-t-1.$$

34. A_7+2A_6, see [5]. $F=\mathbb{Q}(\sqrt{-7})$, $E=F(\sqrt{-3})$, $a=\pm i\sqrt{7}$, $b=\pm i\sqrt{3}$, $p=t$,

$$\begin{aligned}x&=4(a+1)\big[16t^4-(ab+a+b-23)t^3-(12ab-20a-28b-60)t^2\\&\qquad-(23ab+2a+3b-106)t-10ab+46\big]t,\\y&=(ab+5)\big[8t^2+(2ab-16a-26b-12)t+5ab+15a+23b-23\big]\\&\qquad\times\big[8t^2-(ab+3a+5b-9)t-ab-3a-5b+5\big]^2,\\z&=2(a+1)^2(ab-5)\big(2t^2-(a-3)t+1\big)\,t^2.\end{aligned}$$

The implicit equation has coefficients in F:

$$\begin{aligned}&1/4\,y^4+\big[x^2+(2-6a)x+50+(23/2)a\big]y^3\\&+\big[x^4-(10+26a)x^3+(162-99a)x^2+(1883-265a)x+1799+1428a\big]y^2\\&-(a+1)\big[7x^3+(79-2a)x^2+(182+9a)x+658+336a\big]h(x)y\\&-\big[(15+7a)x^2+(175+43a)x+623-21a\big]h(x)^2,\qquad h=4x^2+4ax-7+7a,\end{aligned}$$

and the A_6 points are on the line $y=0$ at the roots of h.

35. $A_7+A_6+A_4+A_2$, see [5]. $E=F=\mathbb{Q}(\sqrt{21})$, $a=\pm\sqrt{21}$, $[\infty,0,4]$, $p=2t^2-(9-a)t+(2a-78)/5$,

$$\begin{aligned}x&=\big(2t^2-(23-3a)t+116-24a\big)\,t^2p,\\y&=(t^2-5t-2+4/3\,a)t^2,\\z&=(2t^2-(1+a)t+4-8/3\,a)\,p.\end{aligned}$$

36. $A_7+2A_4+2A_2$. $F=\mathbb{Q}$, $E=\mathbb{Q}(\sqrt{-3})$, $\omega^2+\omega+1=0$, $\{\omega,\omega/2\}\to 2A_2$, $p=t$,

$$\begin{aligned}x&=\omega t(2t^2-2\omega t+\omega^2)(4t^2+5t+2),\\y&=(2t^2-t+1)\big(2t^2+(2+\omega)t-\omega^2\big)^2,\\z&=\omega^2t^2(2t^2+2t+1).\end{aligned}$$

The implicit equation has rational coefficients:

$$\begin{aligned}&32y^4+(-16x^2-288x+248)y^3+(2x^4+96x^3+570x^2-2816x-198)y^2\\&\quad+(-6x^3+16x^2+948x+238)hy-(9x^2+105x+49)h^2.\quad h=x^2+11x-1;\end{aligned}$$

the A_4 points are on the line $y=0$ at the roots of h.

37. $3A_6+A_1$, see [5]. $E=F=\mathbb{Q}$, $[0,1,\infty]$, $p=t^2-t+1$,

$$x=(3t^2-3t+1)t^2,\qquad y=(t^2+t+1)(t-1)^2,\qquad z=(t^2-3t+3)t^2(t-1)^2;$$

the curve is invariant under $(x:y:z)\mapsto(y:z:x)$, $t\mapsto 1/(1-t)$.

38. $2A_6+A_4+A_2+A_1$. $E=F=\mathbb{Q}(\sqrt{21})$, $a=\pm\sqrt{21}$, [double roots of $z(t),\infty,0$], $p=2t^2+(3-a)t+6-\frac{4}{3}a$,

$$\begin{aligned}x&=\big(2t^2+(17+a)t-2-4a\big)t^2,\\y&=10t^2+(19-a)t+10,\\z&=\big(2t^2-(1+a)t+8-2a\big)\big(2t^2+6t+1-a\big)^2.\end{aligned}$$

39. $A_6+A_5+2A_4$. $E=F=\mathbb{Q}(\sqrt{7})$, $a=\pm\sqrt{7}$, [∞, roots of q], $p=t^2-21-8a$, $q=t^2+(6+a)t+7+3a$, $b=(7+2a)/3$,

$$x=(t+2+a)(t+4+a)\,p,\quad y=(t^2+12t+15+8a)\,p^2,\quad z=\big(t^2-2at-b\big)\,q^2.$$

§2. The cases when $E \neq F$

In this section we show that a curve C of equisingularity types **1**, **16**, **34**, **36** does not admit any parametrization with coefficients in F. Recall that F is the field of minimal degree such that the curve is defined over F. In cases **1** and **16** this is evident because F is a real field but the curve does not have any real smooth point. In the other two cases we use a birational equivalence over F of C with a plane conic $aX^2 + bY^2 = 1$, $a, b \in F$. Such an equivalence exists due a classical result by Hilbert and Hurwitz [6]. It follows that any rational curve defined by an equation over F admits a parametrization over a quadratic extension of F. Moreover, a parametrization over F exists if and only if the equation $aX^2 + bY^2 = 1$ has a solution in F.

There are known several methods to find explicit formulas for a birational equivalence of a rational curve with a plane conic defined over the same field. The proof in [6] provides a recursive construction: at each step a curve of degree d is mapped birationally to a curve of degree $d-2$ using a generic 3-dimensional system of adjoint curves of degree $d-2$. A more efficient algorithm based on similar ideas can be found in [11].

Probably, the simplest modern proof of the Hilbert-Hurwitz theorem consists in saying that the anticanonical linear system on the normalization of a rational curve is defined over the same field; it has degree 2 and dimension 3, thus it embeds the curve to $\mathbb{P}^2$ as a conic. Of course, theoretically, this proof can be developed up to an algorithm. This was done in practice by van Hoeij in [7].

Here we use a construction similar that in [2, 4]. We consider a pencil of cubic curves passing through 8 double points of C including infinitely near points. It happens that for curves considered here, the 8 points can be chosen so that the pencil is defined over F (note that this is not the case, for example, for a generic rational sextic curve). Each curve of this pencil has two intersections with C outside the basepoints, hence the pencil defines a double covering onto $\mathbb{P}^1$. By choosing in addition a pencil of lines, we obtain a birational equivalence of C with a rational hyperelliptic curve which can be further transformed into a plane conic. This method is illustrated below in the example of the curves of equisingularity types **34** and **36**. The underlying computations can be found in the files `a7a4a4a2a2.mws` and `a7a6a6.mws` on the web page [10].

36. $A_7{+}2A_4{+}2A_2$. We start with the homogeneous defining equation $f(x, y, z) = 0$ over $\mathbb{Q}$ given in §1. The curve C has an A_7 singularity at $(0:1:0)$ tangent to the axis $z = 0$. It has two A_4 singularities on the axis $y = 0$ at the roots of $h(x) = 0$ and two A_2 singularities on the line $4x - y = 0$.

Let $g_\lambda(x, y, z) = 0$ be the pencil of cubic curves where $g_\lambda(x, y, 1)$ is

$$8y^2 - \big(2x^2 + (4\lambda + 24)x - 2\lambda + 32\big)y + (\lambda x + 7)h(x).$$

It passes through all the four infinitely near points of A_7 and through each of the other four singular points.

Let us explain how we have found this expression for g_λ. We consider a curve $g = 0$ where $g = 8y^2 - (2x^2 + c_1x + c_2)y + (c_3x + c_4)h(x)$ with indeterminate coefficients c_i. Such a curve passes already through the both A_4 and through three infinitely near points at A_7. Let $\gamma(t) = (t : 1 : \gamma_2t^2 + \gamma_3t^3)$ where $\gamma_2 = 1/4$ and $\gamma_3 = c_1/32 - c_3/8$ are found from the condition $\operatorname{ord}_t(g(\gamma(t))) = 4$. Then γ defines a germ of a smooth curve at $(0:1:0)$ passing thought four infinitely near points of the curve $g = 0$. Then the condition that g passes through the fourth double point at A_7 reads as $\operatorname{ord}_t f(\gamma(t)) = 8$. Due to the chosen form of f and g, this is equivalent to the vanishing of the coefficient of t^6 in $f(\gamma(t))$ which is $(c_1 - 4c_3)^2/24$.

The two A_2 points are on the line $4x - y = 0$, hence the polynomial $f_L(t) = f(t, 4t, 1)$ has a factor $q(t)^2$, $\deg q = 2$ (we have $q(t) = \gcd(f_L, f_L') = t^2 - 5t + 7$; we can compute it because f is known). So, we compute the remainder of the division of $g(t, 4t, 1)$ by $q(t)$ and we equate its coefficients to zero. By eliminating all variables but c_3 from these simultaneous equations, we obtain the above expression for g_λ.

From now on we pass to the affine chart $z \neq 0$. So, we set $z = 1$ and we write $f(x, y)$ and $g_\lambda(x, y)$ instead of $f(x, y, 1)$ and $g_\lambda(x, y, 1)$. Let

$$P(x, \lambda) = \operatorname{Res}_y(f(x, y), g_\lambda(x, y)).$$

It has the form $P = P_1(x, \lambda)P_2(x)$ where the factor $P_2(x)$ corresponds to the base points of the pencil. The curve $P_1(x, \lambda) = 0$ is birationally equivalent to the curve $f(x, y) = 0$. The equivalence is given by the mapping which sends a generic point (x, y) to (x, λ) where λ is found from the condition $g_\lambda(x, y) = 0$ which is a linear equation with respect to λ (of course, the coordinate y should be chosen generically enough to avoid a two-folded covering of $f = 0$ onto $P_1 = 0$). In our case we have

$$\begin{aligned} P_1 = (\lambda^4 - 15\lambda^3 + 63\lambda^2 - 486)x^2 + (4\lambda^4 - 78\lambda^3 + 234\lambda^2 + 2646\lambda - 12636)x \\ - 2\lambda^4 + 156\lambda^3 - 3375\lambda^2 + 28053\lambda - 80109, \qquad P_2 = 256\, q(x)^2 h(x)^2. \end{aligned}$$

Let $D(\lambda) = \operatorname{Discr}_x P_1(\lambda, x)$. Since the curve $P_1(x, \lambda) = 0$ is rational, its projection onto the λ-axis has exactly two branch points. They correspond to roots of D of odd multiplicity. The multiplicity of all other roots is even. This means that $D = D_1 D_2^2$ with $\deg_\lambda D_1 = 2$. It is easy to see that $P_1(x, \lambda)$ (and hence $f(x, y)$ as well) is birationally equivalent to the conic curve $D_1(\lambda) - u^2 = 0$ which in its turn is equivalent to $Q(u, v) = 0$ where $Q(u, v) = \operatorname{Discr}_\lambda(D_1(\lambda) - u^2) - v^2$. In our case we have

$$D_1 = 6(\lambda^2 - 9\lambda + 9), \quad D_2 = 2(\lambda - 15)(\lambda^2 - 9\lambda + 9), \quad Q = 24u^2 + 1620 - v^2.$$

Thus, we reduced the existence of a rational parametrization of the initial curve to the existence of rational solutions to the equation $6u^2 + 5w^2 = 1$ (note that $24 = 6 \times 2^2$ and $1620 = 5 \times 18^2$). The Hilbert symbol $(6, 5)_3$ is equal to -1, see [12; Th. III-1], hence there are no integer solutions.

34. $A_7 + 2A_6$. The proof is the same as in case **36**. We find a pencil g_λ passing through 4 double points at A_7 and through 2 double points at each A_6. Since we know a parametrization of the curve, the simplest way to do it is to substitute the parametrization into $y^2 + (2x^2 + c_1 + c_2 + x^2)y + (c_3 x + c_4)h(x)$. The result of the substitution is of the form $e(t)q(t)^2t^3$ where $q(t)^2$ is the last factor of $y(t)$, see §1-**34**. Let $r(t) = e_0 + e_1 t$ be the remainder of the division of $e(t)$ by $q(t)$. Then $e(0) = e_0 = e_1 = 0$ is the condition that g_λ passes through the required double points. So, we obtain

$$\begin{aligned} g_\lambda(x, y) = y^2 + \big[2x^2 + 2(\lambda + a + 9)x + (19a - 7)/8\,\lambda + 49(a - 1)/2\big]y \\ + \big[\lambda x + (7 - a)(\lambda + 6)/2\big]\big(4x^2 + 4ax + 7a - 7\big). \end{aligned}$$

Further, we compute: $P(x,\lambda) = \mathrm{Res}_y(f, g_\lambda)/h^4$; $\mathrm{Discr}_x P = D_1 D_2^2$ where

$$D_1 = (a-1)d(\lambda)/14, \qquad D_2 = 3^7(3a+1)(7\lambda+37a-35)d(\lambda)/1024$$
$$\text{and} \quad d(\lambda) = \lambda^2 + (11a-1)\lambda - 46a - 54;$$
$$Q = \mathrm{Discr}_\lambda(D_1(\lambda) - u^2) - v^2 = (2u/a)^2\pi + \big(6(a-3)/7\big)^2 a - v^2$$

where $\pi = (1-a)/2$ (recall that $a = \sqrt{-7}$). Thus we obtain the equation

$$\pi X^2 + aY^2 = 1. \tag{1}$$

It does not have a solution in F, since $(\pi, a)_\pi = -1$. Since the ring of integers of F is Euclidean, Hilbert symbols in F are computed in the same way as in $\mathbb{Q}$. This means that the fact that (1) has no solution in F admits the following "high school algebra proof".

Let $\mathcal{O}_F$ be the ring of integers of F. If there is a solution to (1) in F, then there exist $X, Y, Z \in \mathcal{O}_F$ such that $\pi X^2 + aY^2 = Z^2$ and there is no common non-unit divisor of X, Y, Z in $\mathcal{O}_F$. Let us consider our equation reduced mod π^3. The additive group $\mathcal{O}_F$ is generated by 1 and π, hence the ideal (π^3) is generated by π^3 and π^4. Since $\pi^4 - 3\pi^3 = 8$, we can chose π^3 and 8 as a base of (π^3). Hence $\mathcal{O}_F/(\pi^3) \cong \mathbb{Z}/8\mathbb{Z}$. We have $\pi = -2-\pi^3 \equiv 6 \mod \pi^3$ and $a = 1-2\pi \equiv 1-2\times 6 \equiv 5$. Thus (X, Y, Z) should be a solution to the congruence $6X^2 + 5Y^2 \equiv Z^2 \mod 8$. Since the only squares mod 8 are 0, 1, and 4, we conclude that X, Y, Z represent even classes in $\mathbb{Z}/8\mathbb{Z}$. Since π divides 2 (indeed, $2 = \pi - \pi^2$), it follows that X, Y, Z are all divisible by π in $\mathcal{O}_F$.

3. How the parametrizations were found

We used different approaches for different curves. Each approach is discussed in a separate subsection of this section. We used `maple`, `pari/gp`, `Singular` and `sage` software for computations.

3.1. Parametric equations with indeterminate coefficients. This approach works well when the most portion of $\mu(C)$ is contributed by irreducible singularities. In all cases when we used this approach (except one case), the computations were as easy as in [9] and they took few seconds (sometimes minutes) of CPU time.

The most difficult case was $A_7 + 2A_4 + 2A_2$ (no. **36**). In this cases certain heuristic tricks were applied to fasten the computations and despite them, many hours of parallel work of many processors were needed to achieve the result. An interesting coincidence is that Yang writes in [13; p. 225] that:

> *"The largest discriminant is* 3600*, the sextic curve is irreducible and its singularities correspond to* $A_7 + 2A_4 + 2A_2$*. This is the only sextic curve whose discriminant reaches* 3600*."*

I tried several ways to write a system of simultaneous equations for the coefficients of the parametrization but every time the system, happened to be so huge that I was not able to solve it on the available computers. The system that I finally managed to solve is as follows. We write a parametrization of a curve with indeterminate coefficients in the form

$$x = (1 + a_1 t + a_2 t^2)t(t-1)^2, \quad y = (1 + b_1 t + b_2 t^2)t^2, \quad z = (1 + c_1 t + c_2 t^2)(t-1)^4.$$

A generic curve C of this form has an A_3 point at $(0:0:1)$ tangent to the line $y=0$ (corresponds to $t=0$ and $t=\infty$) and an A_4 point at $(0:1:0)$ tangent to $z=0$ (corresponds to $t=1$).

Let $e(x,y,z)=z^2y-x^2z-\gamma x^3$ where $\gamma=c_1+b_1-2a_1$ is found from the condition that $\operatorname{ord}_t e=4$. Then the curve $e=0$ is smooth at $(0:0:1)$ and it has tangency of order 4 with the branch of C at $t=0$. We denote the coefficients of t^{16} and t^{15} in $e(x(t),y(t),z(t))$ by e_1 and e_2. We have $e_1=c_2(b_2c_2-a_2^2)$ and $e_2=(2a_1-b_1-c_1)a_2^3+(8c_2-c_1)a_2^2+2(b_2c_1-a_1a_2)c_2+(b_1-8b_2)c_2^2$. Then the condition $e_1=e_2=0$ implies that the curve has an A_k point at $(0:1:0)$ with an odd $k\ge 7$ unless it is a multiple curve of a smaller degree.

Let $p(t)=t^3+at^2+bt+c$ where a,b,c are indeterminates. Let $e_3+e_4t+e_5t^2$ and $e_6+e_7t+e_8t^2$ be the remainders of the division of $(xy'-yx')/(t^2(t-1))$ and $(zy'-yz')/(t(t-1)^3)$ by $p(t)$. Then the condition $e_3=\dots=e_8$ implies that the curve has an A_k point with an even $k\ge 2$ at each root of p. Thus a generic solution to $e_1=\dots=e_8=0$ is a curve with $A_7+A_4+3A_2+A_1$. It is hard but possible to eliminate all the variables except b and c from these simultaneous equations. So we obtain an equation that we denote by $f[p]$ (in this notation we assume that f is a polynomial in the coefficients of p, not a polynomial in p).

Let $t\mapsto\varphi(t)$ be a parametrization of a $A_7+2A_4+2A_2$ curve such that $0,\infty\mapsto A_7$ and $1\mapsto A_4$. Let $q(t)=t^2+ut+v$ be the polynomial whose roots are mapped by φ to the A_2 points and let $\varphi(w)$ be the other A_4. By changing the parameter of φ, we may obtain three more parametrizations of the same curve such that $0,\infty\mapsto A_7$ and $1\mapsto A_4$, namely,

$$
\begin{array}{llll}
t\mapsto\varphi(wt) & \text{maps: } 0,\infty\to A_7, & 1,w^{-1}\mapsto 2A_4, & \text{roots of } q(wt)\mapsto 2A_2;\\
t\mapsto\varphi(t^{-1}) & \text{maps: } 0,\infty\to A_7, & 1,w^{-1}\mapsto 2A_4, & \text{roots of } q(t^{-1})\mapsto 2A_2;\\
t\mapsto\varphi(wt^{-1}) & \text{maps: } 0,\infty\to A_7, & 1,w\mapsto 2A_4, & \text{roots of } q(wt^{-1})\mapsto 2A_2.
\end{array}
$$

Thus, u,v,w satisfy the equations

$$
\begin{aligned}
0&=f\big[(t-w)(t^2+ut+v)\big]=f\big[(t-w^{-1})(t^2+uw^{-1}t+vw^{-2})\big]\\
&=f\big[(t-w^{-1})(t^2+uw^{-1}t+w^{-1})\big]=f\big[(t-1)(t^2+uwv^{-1}t+w^2v^{-1})\big].
\end{aligned}
$$

By solving these simultaneous equations, we find the desired parametrization.

3.2. Dual curve. The degree of the dual curve $\check{C}$ is equal to $30-19-k$, $k=\#\operatorname{Sing}(C)$. For the set of singularities $(A_8+3A_2)+A_4+A_1$ (no. **33**) it is equal to 5 (in all the other cases it is ≥ 6) and the set singularities of $\check{C}$ is A_8+A_4. So, we find a parametrization $(\check{x}(t):\check{y}(t):\check{z}(t))$ of $\check{C}$ and we set

$$x=\check{y}'\check{z}-\check{z}'\check{y},\quad y=\check{z}'\check{x}-\check{x}'\check{z},\quad z=\check{x}'\check{y}-\check{y}'\check{x}.$$

Remark. One can check that the curves of equisingularity types with $k=5$, i. e., **26** $A_{10}+A_4+2A_2+A_1$, **36** $A_7+2A_4+2A_2$, **38** $2A_6+A_4+A_2+A_1$ are autodual.

3.3. From an implicit equation to a parametrization. In many cases we used the defining equations given in [2] (for no. **1**) and in [5]. The problem of finding a parametrization of a rational curve defined by an implicit equation is classical. In computer era it found new motivations in geometric modeling and computer graphic. It worth to mention here that Abhyankar and Bajaj [1] gave an excellent presentation of a solution to this problem (based on adjoint curves) accessible to people without any algebro-geometric background. Their algorithm, however, is not appropiate for our goals because the coefficients of the obtained parametrization do not belong to a field extension of minimal degree.

We applied the method explained in §2. The only non-evident step is to find a solution in F to the equation $Q = 0$ (see §2). This is an equation of the form $aX^2 + bY^2 = 1$, $a, b \in F$. In the most cases it was possible to choose the pencil g_λ so that the polynomial $D_1(\lambda)$ factorized into a product of two linear factors. In the remaining cases, fortunately, the coefficients of Q had the form $a = a_1 a_2^2$, $b = b_1 b_2^2$, $a_i, b_i \in F$, with a_1 and b_1 small enough that it was possible to find a solution just by trying all pairs (X, Y) one by one.

The most difficult case was $A_{10} + 2A_4 + A_1$ (no. **24**). Starting with the implicit equation from [5], performing the computations as in §2 (with the pencil of cubics through A_1, $2A_4$, and 5 infinitely near points at A_{10}), and changing the field generator to α, $\alpha^3 - \alpha^2 - \alpha - 1 = 0$ (found by `pari`'s command `polred`), we obtain very big expressions for a and b. However, `sage`'s commands

```
K.<a>=NumberField(x^3-x^2-x-1)
K.ideal( the coefficient ).factor()
```

give a good hint how to extract quadratic factors of a and b and we reduce our equation to $(2-a)X^2 - 5a(a+2)Y^2 - Z^2 = 0$. Then we try all pairs (X, Y) of the form $X = x_0 + x_1\alpha + x_2\alpha^2$, $Y = y_0 + y_1\alpha + y_2\alpha^2$, $x_i, y_i = 0, \pm1, \pm2, \dots$ until the left hand side of the equation factorizes over F into linear factors. Thus we obtain a solution $X = 2 + 2a^2$, $Y = 1 + a - a^2$, $Z = 2 - a^2$. The complete computation for $A_{10} + 2A_4 + A_1$ (from the equation in [5] to the final formulas in §2) with detailed comments can be found in the file `a10a4a4a1.mws` on the web page [10].

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