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Sandwich Theorem and Supportability for Partially Defined Convex Operators

DOMINIKUS NOLL

Universität Stuttgart, Mathematisches Institut B, Pfaffenwaldring, 7000 Stuttgart 80, Federal Republic of Germany

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Given a convex operator φ: $E \to F \cup \{\infty\}$ and a concave operator $\psi: E \to F \cup \{-\infty\}$, both defined an a locally convex Fréchet space and having values in an ordered locally convex vector space F with the least upper bound property, and satisfying $\psi \leqslant \varphi$, we ask for conditions ensuring the existence of a *continuous* affine mapping $h: E \to F$ satisfying $\psi \leqslant h \leqslant \varphi$. Besides this continuous sandwich problem we discuss the problem of existence of continuous linear support mappings for partially defined sublinear operators, by reducing it to a sandwich problem.

INTRODUCTION

Let C, D be convex sets in a real vector space E, φ a convex function defined on C, ψ a concave function defined on D, both having values in an ordered real vector space F. Suppose $\psi(x) \leqslant \varphi(x)$ holds for every $x \in C \cap D$. The sandwich problem consists in finding an affine function $h: E \to F$ satisfying

$$h(y) \geqslant \psi(y) \qquad \text{for all} \quad y \in D,$$

$$h(x) \leqslant \varphi(x) \qquad \text{for all} \quad x \in C.$$

In $[Z_2]$, Zowe has investigated this algebraic sandwich problem in the case where F is an ordered vector space with the least upper bound property. The outcome of this investigation is that a sufficient (and also reasonable) condition to ascertain the existence of an affine separator h is that 0 be an algebraic relative interior point of $C - D$, i.e.,

$$0 \in (C - D)^{ri}.$$

Usually, infinite-dimensional vector spaces are endowed with (natural) locally convex topologies. This makes it desirable to find solutions h for the sandwich problem above which are continuous with respect to these

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topologies. In $[Z_1]$, Zowe has established such a continuous sandwich theorem under the assumption that F is a normally ordered locally convex vector space having the least upper bound property. It states that

(1) the continuous sandwich theorem is valid if φ is continuous at some $x_0 \in \operatorname{int} C \cap D$.

This provides an answer in many cases, and the method of proof it relies on has been refined by several authors (see, e.g. [B]). Nevertheless, it is not fully satisfactory in the light of its algebraic counterpart. The existence of a topological interior point in one of the sets C, D is a rather strong restriction. In our paper $[N_2]$ we therefore asked for conditions of a different nature in which a continuous sandwich theorem is valid without this request. We have obtained the following result.

(2) The continuous sandwich theorem is valid if E is a separable Fréchet space, C, D are convex Borel sets in E such that 0 is an algebraic interior point of $C-D$, i.e., $0 \in (C-D)^i$, and there exist weakly Borel measurable $\varphi_0: C \to F$, $\psi_0: D \to F$ satisfying $\varphi \leqslant \varphi_0$ on C, $\psi \geqslant \psi_0$ on D.

This result maintains the symmetry of its algebraic counterpart, and moreover covers all cases of practical relevance in the case of a separable space E, since the measurability assumptions on φ, ψ are quite natural. Of course we would like to avoid separability in statement (2). We do not know whether this is possible in general. In $[N_2]$ we obtained a partial answer.

(3) The continuous sandwich theorem is valid if E is a Fréchet space and there exist closed convex sets $C_0 \subset C$, $D_0 \subset D$ having $0 \in (C_0 - D_0)^i$, and weakly Borel measurable maps $\varphi_0: C_0 \to F$, $\psi_0: D_0 \to F$ satisfying $\varphi \leqslant \varphi_0$ on C_0, $\psi \geqslant \psi_0$ on D_0.

In the present paper we shall, in particular, prove a generalization of statement (3). We introduce the notion of pseudo-complete convex sets and we prove that statement (3) remains valid for such pseudo-complete sets C_0, D_0. Since the class of pseudo-complete convex sets in a Fréchet space is fairly large—in particular it contains as a proper subclass the CS—closed sets in the sense of Jameson and therefore, by a result of Fremlin and Talagrand, all convex G_δ—sets—we obtain a satisfactory version of the continuous sandwich theorem which again covers all cases of a practical relevance.

The sandwich theorem bears some relation to the support problem for partially defined sublinear operators. Recall that the problem of existence of (continuous) linear support mappings for partially defined sublinear operators— in contrast with the case of fully defined operators—does not admit a solution in general. In $[N_2]$ we have outlined a method for producing solutions to the support problem in the partially defined case under some mild measurability assumptions by reducing the support theorem to the sandwich theorem. This method is based on a boundedness result for convex operators with measurability condition. In the present paper we give another application of this method. Roughly speaking we show that lower semi-continuity of convex operators is a one-dimensional property, if a weak measurability assumption is made.

The structure of the paper is as follows. In Section 1 we introduce the concept of pseudo-completeness for convex sets and prove some basic facts. Section 2 presents our main sandwich theorem generalizing statement (3) above. In Section 3 we establish a result on the existence of continuous linear support mappings for partially defined sublinear operators. The final Section 4 presents an application of the support theorem. We prove that a convex function φ defined on a pseudo-complete convex set C in a Fréchet space E which is dominated by a Borel measurable function is actually lower semi-continuous as a mapping $E \to \mathbb{R} \cup \{\infty\}$ if for every one-dimensional linear manifold L in E the restriction $\varphi | L$: $L \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous.

Notations and Preliminaries

We assume that E, F are real locally convex vector spaces and that F is ordered by a normal cone F_+, which means that F has a base of neighborhood of 0 consisting of order-convex sets. F is said to have the least upper bound property (lubp) if every subset of F which is order-bounded below has an infimum.

We adjoin two new elements ∞, $-\infty$ to F, imposing the rules $x + \infty = \infty$, $x + (-\infty) = -\infty$, $-\infty < x < \infty$, $\infty + \infty = \infty$, $(-\infty) + (-\infty) = -\infty$, $0\infty = 0(-\infty) = 0$, $\lambda\infty = \infty$, $\lambda(-\infty) = -\infty$, $(-1)(-\infty) = \infty$ for all $x \in F$, $\lambda > 0$. An operator $\varphi: E \to F \cup \{\infty\}$ is called convex if for x, $y \in E$, $0 \leqslant \lambda \leqslant 1$ we have $\varphi(\lambda x + (1-\lambda) y) \leqslant \lambda\varphi(x) + (1-\lambda)\varphi(y)$, sublinear if it is convex and satisfies $\varphi(\lambda x) = \lambda\varphi(x)$ for all $\lambda \geqslant 0$. φ is called concave if $-\varphi$ is convex, and it is called superlinear if $-\varphi$ is sublinear. The set of $x \in E$ with $\varphi(x) \in F$ is noted $D(\varphi)$. Observe that $D(\varphi)$ is convex when φ is convex or concave, and it is a cone when φ is sub- or superlinear.

Let X be a subset of E. A function $f: X \to F$ is called weakly Borel measurable if $g \circ f$ is Borel measurable on X for every $g \in F'_+$, i.e., for every continuous and positive linear functional g on F. $f: X \to F$ is called weakly Baire if for every $g \in F'_+$ $g \circ f$ is a mapping with the Baire property (see [Ku, p. 399]).

Finally we need a topological notion. Let X be a topological space. A pair (ϑ, T) consisting of a tree $T = (T, \leqslant_T)$ of height $\aleph_0$ and a mapping ϑ

from T to the topology of X is called a web on X if the following conditions (i), (ii) are satisfied:

(i) $\{\vartheta(t): t \in T\}$ is a base for the topology of X;

(ii) For fixed $t \in T$ $\{\vartheta(s): t <_T s \in T\}$ is a base for the topology of $\vartheta(t)$.

Moreover, a web (ϑ, T) on X is called p-complete if the following condition (iii) is satisfied:

(iii) If (t_n) is a cofinal branch in T (i.e., $t_n <_T t_{n+1}$) such that $\vartheta(t_n) \neq \varnothing$ for all n, then $\bigcap \{\vartheta(t_n): n \in \mathbb{N}\} \neq \varnothing$ as well.

Various concepts related with the notion of a web have been discussed. See for instance [CČN] for the concept of a sieve, [DW] for the notion of a sifter. Following [N_1], a topological space X is called p-complete if it admits a p-complete web. Note that the class of p-complete spaces is invariant under products, continuous open images and is G_δ-hereditary. Moreover, every p-complete space is a Baire space.

1. *Pseudo-Complete Convex Sets*

In this section let C be a convex cone with vertex 0 in a separated locally convex vector space E. Let τ denote the topology on C inherited from E. Observe that τ is not invariant under the translations $x \to x + y$, $y \in C$, preserving C. This suggests introducing different topologies on C which are invariant under translations $x \to x + y$, $y \in C$, and at the same time preserve most of the information about the original topology τ. One special topology of this kind may be defined by taking as a base of neighbourhoods of $x \in C$ the sets

$$x + (U \cap C),$$

where U varies over the neighbourhoods of 0 in E. This topology, which we agree to call σ, has been defined by Saint Raymond [SR] in the special case $E = l^1$, $C = l^1_+$, and it has been further investigated in our paper [N_2]. The following properties of σ are easily stated:

(1) The translations $x \to x + y$ with $y \in C$ map (C, σ) homeomorphically onto its open subspace $(C + y, \sigma)$, respectively;

(2) σ is invariant under homotheties $x \to \lambda x$, $\lambda > 0$;

(3) σ is locally convex;

(4) σ is finer than the original topology τ.

In the following, every topology γ on C satisfying conditions (1) to (4) will be called an invariant topology on C. The special invariant topology σ defined above will be called the cone topology of C.

Let us consider an example. Let $E=\mathbb{R}$, $C=\mathbb{R}_+$. Then τ is the euclidean topology on C, while σ is the topology generated by the intervals $[a, b)$, $0 \leqslant a < b$, on C, sometimes called the Sorgenfrey topology. Clearly in this case σ is the only invariant topology on C.

In the general case, suppose there exists $x \in C$ having $-x \notin C$. Then every invariant topology γ on C induces the Sorgenfrey topology on the ray $\{\lambda x : \lambda \geqslant 0\}$, whereas τ naturally induces the euclidean topology. This implies that τ itself is not an invariant topology on C unless C is a vector subspace of E.

Definition 1. Let C be a convex cone with vertex 0 in a separated locally convex vector space E. C is called pseudo-complete if there exists an invariant topology γ on C such that (C, γ) is a p-complete topological space. If, in addition, the topology γ may be chosen to be the cone topology σ, then C is called σ-pseudo-complete.

This definition will be clarified in an instant, when we will become aware of the fact that CS-closed sets are pseudo-complete. Recall that a convex set K in a locally convex vector space E is called CS-closed (see [J]) if every convergent series $\sum_{n=1}^{\infty} \lambda_n x_n$ in E having $x_n \in K$ and $0 \leqslant \lambda_n \leqslant 1$, $\sum_{n=1}^{\infty} \lambda_n = 1$ actually converges to an element of K.

Proposition 1. *Every CS-closed convex cone C with vertex* 0 *in a locally convex Fréchet space E is σ-pseudo-complete. In particular, every convex G_δ-cone with vertex* 0 *in E is σ-pseudo-complete.*

Proof. The second part of the statement follows from the first one together with a result of Fremlin and Talagrand [FT] stating that convex G_δ-sets in Fréchet spaces are CS-closed. So let us prove the first part of the statement.

Since C is a cone, its CS-closedness may be expressed equivalently by saying that every convergent sequence (x_n) of vectors from C having $x_{n+1} - x_n \in C$ in fact converges to an element of C (see [J, Section 2, p. 38]).

Let (U_n) be a base of convex and open neighbourhoods of 0 in E satisfying $U_{n+1} + U_{n+1} \subset U_n$. For $x \in C$ we define $\vartheta(x) = x + (U_1 \cap C)$, which is a σ-open set in C. Suppose now we have already defined σ-open sets $\vartheta(x_1, ..., x_{n-1})$ for all sequences $x_1, ..., x_{n-1}$ of length $n-1$ consisting of elements of C. Then define

$$\vartheta(x_1, ..., x_n) = \begin{cases} x_n + (U_{k_n} \cap C), \text{ if } x_n \in \vartheta(x_1, ..., x_{n-1}) \text{ and where} \\ \quad k_n \text{ is the smallest integer} \geqslant n \text{ having} \\ \quad x_n + ((U_{k_n} + U_{k_n} + U_{k_n}) \cap C) \subset \vartheta(x_1, ..., x_{n-1}); \\ \varnothing, \text{ if } x_n \notin \vartheta(x_1, ..., x_{n-1}). \end{cases}$$

Let T denote the tree of all finite sequences of elements of C ordered in the obvious way. We claim that (ϑ, T) fullfills the conditions (i)–(iii) of a p-complete web on (C, σ). Since conditions (i) and (ii) are clear from the construction, we have the check condition (iii).

Let (x_n) be a sequence in C having $\vartheta(x_1, ..., x_n) \neq \emptyset$ for every n. By construction we find $y_n \in U_{k_n} \cap C$ with $x_{n+1} = x_n + y_n$. In view of $x_{n+1} - x_n \in U_{k_n}$ the sequence (x_n) converges to some $x \in E$. By the above description of CS-closedness we actually have $x \in C$. We conclude by proving $x \in \vartheta(x_1, ..., x_n)$ for every n. So let n be fixed. For $k \geqslant 1$ we have

$$x_{n+k} = x_n + \sum_{i=n}^{n+k-1} y_i.$$

Therefore the sequence $(x_{n+k} - x_n)$ lies in C and converges to some v_n which, in view of

$$(x_{n+k+1} - x_n) - (x_{n+k} - x_n) = x_{n+k+1} - x_{n+k} \in C$$

and the CS-closedness of C must lie in C. This means $v_n = \sum_{i=n}^{\infty} y_i \in C$. But on the other hand we have

$$\sum_{i=n}^{n+j} y_i \in U_{k_n} + \cdots + U_{k_{n+j}} \subset U_{k_n} + U_{k_n},$$

giving

$$\sum_{i=n}^{\infty} y_i \in U_{k_n} + U_{k_n} + U_{k_n}.$$

This readily implies $x = x_n + \sum_{i=n}^{\infty} y_i \in x_n + ((U_{k_n} + U_{k_n} + U_{k_n}) \cap C) \subset \vartheta(x_1, ..., x_n)$. ▮

Proposition 1 tells that the class of pseudo-complete convex cones in a Fréchet space is fairly large. The following example shows that, although the description of pseudo-completeness relies on the concept of Baire category, there exist pseudo-complete sets which are of the first category in themselves.

EXAMPLE 1. Let $E = l^1$ and let C be the order cone of the lexicographic ordering on E. Then C is CS-closed and hence pseudo-complete. But C is of the first category, since $C = \bigcup_{n,m=1}^{\infty} C_{n,m}$, where $C_{n,m} = \{x \in l^1 : x_1 = \cdots = x_{n-1} = 0,\ x_n \geqslant 1/m\}$, and the sets $C_{n,m}$ have no interior points relative to C.

We continue our investigation with the following invariance properties of the class of pseudo-complete cones.

LEMMA 1. *Let E, E_1 be separated locally convex vector spaces.*

(1) *Let $C \subset E$, $C_1 \subset E_1$ be (σ-) pseudo-complete cones. Then $C \times C_1$ is again (σ-) pseudo-complete;*

(2) *Let $C \subset E$ be a pseudo-complete cone and let $f: E \to E_1$ be a continuous linear operator. Then $f(C)$ is pseudo-complete;*

(3) *Let $C, D \subset E$ be pseudo-complete cones. Then $C + D$ is again pseudo-complete;*

(4) *Suppose E is metrizable and let C, $D \subset E$ be (σ-) pseudo-complete. Then $C \cap D$ is again (σ-) pseudo-complete.*

Proof. Consider (1). Let γ, γ_1 be invariant topologies on C, C_1 such that (C, γ), (C_1, γ_1) are p-complete spaces. Then $\gamma \times \gamma_1$ is invariant and p-complete on $C \times C_1$. This proves the non-bracket part of the statement. Since the product of the cone topologies on C, C_1 is again the cone topology on $C \times C_1$, the bracket part follows as well.

Next consider (2). Let γ on C be invariant and p-complete. We define an invariant topology $\gamma_1 = f(\gamma)$ on $C_1 = f(C)$ by taking as a base of neighborhoods of $y \in C_1$ the sets $y + f(W)$, W varying over the neighborhoods of 0 in (C, γ). This provides in fact an invariant topology on C_1. Since $f \mid C: (C, \gamma) \to (C_1, \gamma_1)$ is by construction a continuous open surjection, we deduce that $\gamma_1 = f(\gamma)$ is a p-complete topology.

Clearly statement (3) follows from (1) and (2). So let us finally consider (4). Let γ, δ be invariant p-complete topologies on C, D, respectively. We define a new invariant topology $\gamma \vee \delta$ on $C \cap D$ by taking as a base of neighborhoods of $x \in C \cap D$ the sets $x + (V \cap W)$, V a neighborhood of 0 in (C, γ), W a neighborhood of 0 in (D, δ). Note that in the case where γ, δ are the cone topologies on C, D the new topology $\gamma \vee \delta$ is again the cone topology on $C \cap D$. It therefore remains to prove that $\gamma \vee \delta$ is p-complete.

Let (ϑ, T) and (ρ, R) be given on (C, γ), (D, δ) in accordance with the definition of p-completeness. Considering property (4) of an invariant topology, we may assume that the sets $\vartheta(t)$, $\rho(r)$ are metrically bounded and that for cofinal branches (t_n) in T, (r_n) in R the relations $\lim_{n \to \infty} \operatorname{diam} \vartheta(t_n) = \lim_{n \to \infty} \operatorname{diam} \rho(r_n) = 0$ are valid.

Let S denote the tree of height $\aleph_0$ consisting of all finite sequences $((t_1, r_1), ..., (t_n, r_n))$ having $t_1 <_T \cdots <_T t_n$, $r_1 <_R \cdots <_R r_n$, and give S the natural order. Define a mapping χ on S by setting

$$\chi((t_1, r_1), ..., (t_n, r_n)) = \vartheta(t_n) \cap \rho(r_n).$$

Then (χ, S) fulfills the requirements of the definition of a p-complete web in the space $(C \cap D, \gamma \vee \delta)$. Since properties (i), (ii) are clear from the definition, we check condition (iii). So let (t_n) and (r_n) be increasing sequences having $\chi((t_1, r_1), ..., (t_n, r_n)) \neq \emptyset$ for every n. This implies $\vartheta(t_n) \neq \emptyset$, $\rho(r_n) \neq \emptyset$ for every n, so there exist $x \in \bigcap \{\vartheta(t_n): n \in \mathbb{N}\}$, $y \in \bigcap \{\rho(r_n): n \in \mathbb{N}\}$. But note that we have $d(x, y) \leqslant \operatorname{diam} \vartheta(t_n) + \operatorname{diam} \rho(t_n) \to 0$, so $x = y$. This proves the result. ∎

EXAMPLE 2. Let $E = l^1$ and let C be the cone consisting of all vectors $x = (x_n)$ in E having $x_1 \geqslant 0$ and $|x_n| \leqslant x_1/n$ for $n \geqslant 2$. It is easy to check that C is CS-closed hence pseudo-complete. Let $f: l^1 \to l^1$ be the left-shift operator. Then $f(C)$ is a pseudo-complete cone in l^1 by Lemma 1. But note that $f(C)$ is no longer CS-closed. This follows from the fact (proved in [J, Cor. 1]) that any dense CS-closed set must be the whole space. Indeed, $f(C)$ is clearly dense in l^1 but it is not all of l^1.

We wish to extend the notion of pseudo-completeness to arbitrary convex sets. This will be achieved by making use of the following standard construction. Given any convex set C in the separated locally convex vector space E, we denote by $\tilde{C}$ the convex cone with vertex $(0, 0)$ in $E \times \mathbb{R}$ generated by the set $C \times \{1\}$, i.e., $\tilde{C} = \mathbb{R}_+(C \times \{1\})$.

DEFINITION 2. Let C be a convex set in the separated locally convex vector space. Then C is called pseudo-complete if the cone $\tilde{C}$ associated with C in $E \times \mathbb{R}$ is pseudo-complete in the sense of Definition 1. Moreover, C is called σ-pseudo-complete if $\tilde{C}$ is a σ-pseudo-complete cone.

Note that Definition 2 makes sense also in the case where C is already a convex cone with vertex 0, for then $\tilde{C} = C \times \mathbb{R}_+$, and thus ($\sigma$-) pseudo-completeness of C in the sense of Definition 1 is equivalent to (σ-) pseudo-completeness of $\tilde{C}$.

PROPOSITION 2. *Every CS-closed convex set in a locally convex Fréchet space is σ-pseudo-complete. In particular, every convex G_δ-set in a Fréchet space is σ-pseudo-complete.*

Proof. This follows from Proposition 1 when we observe that the cone $\tilde{C}$ associated with a CS-closed convex set C is itself CS-closed. ∎

2. *Sandwich Theorem*

Let C, D be convex cones with vertices 0 in a locally convex vector space E. The pair (C, D) is said to induce an open decomposition of E if for every neighborhood U of 0 in E the set

$$\tilde{U} = (U \cap C) - (U \cap D)$$

is again a neighborhood of 0. A classical result of Banach tells that in a

Fréchet space E every pair (C, D) of closed cones C, D satisfying $E = C - D$ induces an open decomposition. This has been generalized to the CS-closed case in [J]. Here we present the following pseudo-complete version.

THEOREM 1. *Let E be a locally convex Fréchet space and let C, D be pseudo-complete convex cones with vertices* 0 *in E having $E = C - D$. Then (C, D) induces an open decomposition of E.*

Proof. Let γ, δ be invariant and p-complete topologies on C, D. Then $C \times D$ is p-complete with the invariant topology $\gamma \times \delta$. Let α be the image of the topology $\gamma \times \delta$ under $(x, y) \to x - y$, i.e., $\alpha = \gamma - \delta$, then α is invariant and p-complete on E, where p-completeness follows from the fact that $(x, y) \to x - y$ is continuous and open with respect to the topologies $\gamma \times \delta$ and α.

Let $-\alpha$ be the image of α under $x \to -x$, then $-\alpha$ is again an invariant and p-complete topology on E. Let $\beta = \alpha \vee -\alpha$ be defined as in part (4) of the proof of Lemma 1, then β turns out to be once more an invariant and p-complete topology on E. But note that β has a base of neighborhoods of 0 consisting of symmetric sets. So β is actually a vector space topology.

Let us consider the identity mapping $i: (E, \beta) \to (E, \tau)$, where τ denotes the original Fréchet topology on E. Then i is continuous and by [Kö, p. 24, (1)] is nearly open. Since (E, β) is p-complete, we may apply our open mapping theorem from [N_1, Theorem 3], and this implies $\beta = \tau$. But note that for every τ-neighborhood U of 0 the set $\tilde{U}$ is a neighborhood of 0 with respect to α and so with respect to β. This finally proves that $\tilde{U}$ is a neighborhood of 0 with respect to τ. ∎

As we shall see next, Theorem 1 is an important tool in order to establish our main Sandwich Theorem.

THEOREM 2. *Let E be a locally convex Fréchet space and let F be a normally ordered locally convex vector space with lubp. Let $\varphi: E \to F \cup \{\infty\}$ be a convex, $\psi: E \to F \cup \{-\infty\}$ a concave operator satisfying $\psi \leqslant \varphi$. Suppose there exist pseudo-complete sets $C \subset D(\varphi)$, $D \subset D(\psi)$ having $0 \in (C - D)^i$ such that φ on C and $-\psi$ on D are majorized, respectively, by weakly Borel measurable maps. Then there exists a continuous affine function $h: E \to F$ such that for all $x \in E$*

$$\psi(x) \leqslant h(x) \leqslant \varphi(x)$$

is satisfied.

Proof. Let us first consider the case where φ, ψ are sublinear, resp. superlinear, and where C, D are pseudo-complete convex cones with vertices at 0 having $E=C-D$ and $C\subset D(\varphi)$, $D\subset D(\psi)$ and such that φ, $-\psi$ are weakly Borel measurably dominated on C, D, respectively.

Define a sublinear operator $\chi: E\to F$ by setting

$$\chi(z)=\inf\{\varphi(x)-\psi(y): z=x-y\}.$$

Note that χ is well-defined in view of the fact that $E=C-D$ and hence $E=D(\varphi)-D(\psi)$ and since F has the lubp. Now the Hahn–Banach theorem provides a linear support mapping f for χ, which consequently satisfies $\psi\leqslant f\leqslant\varphi$. It remains to prove that f is continuous. F being normally ordered , we are led to prove that $g\circ f$ is continuous for every $g\in F'$ with $g\geqslant 0$. So let $g\in F'$ positive be fixed. We prove that $g\circ\chi$ and hence $g\circ f$ is continuous.

Let γ, δ be invariant p-complete topologies on C, D. In particular, this implies that (C,γ), (D,δ) are Baire spaces. This permits us to apply the following

Lemma 2. *Let C be a convex cone with vertex 0 in a separated locally convex vector space E and let γ be an invariant topology on C such that (C,γ) is a Baire space. Let $\varphi: C\to\mathbb{R}$ be a convex function and suppose there exists a Borel measurable function $\psi: C\to\mathbb{R}$ having $\varphi(x)\leqslant\psi(x)$ for all $x\in C$. Then there exists a dense G_δ-subset G of (C,γ) such that $\varphi\,|\,G$: $(G,\gamma)\to\mathbb{R}$ is continuous.*

Proof of the Lemma. This is just Lemma 1 in $[\mathrm{N}_2]$, where the argument has been presented in the case $\gamma=\sigma$. Since only properties (1)–(4) of an invariant topology γ have been used in $[\mathrm{N}_2]$, the present result holds as well. ∎

Let G be a dense G_δ-subset of (C,γ) so that $g\circ\varphi\,|\,G: (G,\gamma)\to\mathbb{R}$ is continuous and let H be a dense G_δ-subset of (D,δ) so that $g\circ\psi\,|\,H$: $(H,\delta)\to\mathbb{R}$ is continuous. Fix $x_0\in G$ and $y_0\in H$ and choose some open neighborhood V of 0 in (C,γ) and some open neighborhood W of 0 in (D,δ) such that

$$g(\varphi((x_0+V)\cap G))\subset g(\varphi(x_0))+[-1,1]$$
$$g(\psi((y_0+W)\cap H))\subset g(\psi(y_0))+[-1,1]$$

are satisfied. Let $U=(x_0+V)-(y_0+W)$, then U is an open neighborhood of x_0-y_0 in E in view of the fact that (C,D) induces an open decomposition of E (Theorem 1) and hence $(x,y)\to x-y$ is continuous and open with respect to the Fréchet topology on E and $\gamma\times\delta$ on $C\times D$.

Since $(x_0+V)\cap G$ is residual in x_0+V and $(y_0+W)\cap H$ is residual in

$y_0 + W$ and since the restriction of $(x, y) \to x - y$ to $(x_0 + V) \times (y_0 + W)$ is again continuous and open onto U, Lemma 4 below tells us that

$$R = ((x_0 + V) \cap G) - ((y_0 + W) \cap H)$$

is a residual subset of U.

We claim that $g \circ \chi$ (and hence $g \circ f$) is bounded above by 2 on the set R. Indeed, let $v \in V$, $x_0 + v \in G$, $w \in W$, $y_0 + w \in H$, then we obtain

$$\begin{aligned} g(\chi((x_0 + v) - (y_0 + w))) &\leqslant g(\varphi(x_0 + v)) - g(\psi(y_0 + w)) \\ &\leqslant 1 + 1 = 2. \end{aligned}$$

Let us define a mapping $\xi_g: U \to \mathbb{R}$ by setting $\xi_g(z) = 2$ in case $z \in R$, $\xi_g(z) = g(\chi(z))$ in case $z \in U \setminus R$. R being a residual subset of U, we deduce that $\xi_g: U \to \mathbb{R}$ has the Baire property and, moreover, majorizes $g \circ \chi$ on U. Therefore the following Lemma 3 implies the continuity of $g \circ \chi$, and this ends the proof of Theorem 2 in the sublinear case.

LEMMA 3. *Let E be a locally convex Fréchet space and let $\varphi: E \to \mathbb{R}$ be a convex function. Suppose there exists a nonempty open set U in E and a mapping $\psi: U \to \mathbb{R}$ with the Baire property such that ψ majorizes φ on U. Then φ is continuous on E.*

Proof of Lemma 3. This is essentially Theorem 3 in [N_2]. ∎

Before stating the announced Lemma 4, let us indicate the proof of Theorem 2 in the general case. First of all we may, if necessary, shift the whole situation such that $0 \in C \cap D$. Now let us define a sublinear operator $\tilde{\varphi}: E \times \mathbb{R} \to F \cup \{\infty\}$ by setting $\tilde{\varphi}(\lambda x, \lambda) = \lambda\varphi(x)$ whenever $x \in D(\varphi)$, $\lambda \geqslant 0$, $\tilde{\varphi}(y, \mu) = \infty$ otherwise, and a superlinear operator $\tilde{\psi}: E \times \mathbb{R} \to F \cup \{-\infty\}$ by setting $\tilde{\psi}(\lambda x, \lambda) = \lambda\psi(x)$ in case $x \in D(\psi)$, $\lambda \geqslant 0$, $\tilde{\psi}(y, \mu) = -\infty$ otherwise. Then $D(\tilde{\varphi}) = D(\varphi)^{\sim}$ and $D(\tilde{\psi}) = D(\psi)^{\sim}$ and $\tilde{\psi} \leqslant \tilde{\varphi}$ are satisfied. Moreover, we have $\tilde{C} \subset D(\tilde{\varphi})$, $\tilde{D} \subset D(\tilde{\psi})$, and $\tilde{C}$, $\tilde{D}$ are pseudo-complete cones having $E \times \mathbb{R} = \tilde{C} - \tilde{D}$, the latter in view of $0 \in (C - D)^i$ and $0 \in C \cap D$. Clearly $\tilde{\varphi}$, $-\tilde{\psi}$ are weakly Borel measurably dominated on $\tilde{C}$, $\tilde{D}$, and therefore we may apply the first part of our proof. This provides a continuous linear mapping $f: E \times \mathbb{R} \to F$ satisfying $\tilde{\psi} \leqslant f \leqslant \tilde{\varphi}$. Setting $h = f(\ , 1)$ finally provides the desired continuous affine separator h for φ, ψ. This ends the proof of Theorem 2. ∎

LEMMA 4. *Let X be a p-complete topological space and let f be a continuous and open surjection from X onto a metrizable space Y. Then f maps residual subsets of X onto residual subsets of Y.*

Proof. Let G be a dense G_δ-subset of X, $G = \bigcap \{G_n : n \in \mathbb{N}\}$ for dense open sets $G_{n+1} \subset G_n$ in X. Let (ϑ, T) be given on X according to the definition of p-completeness. We may assume that (ϑ, T) satisfies the following sharpened versions of conditions (i), (ii):

(i') $\{\vartheta(t) : t \in T_0\}$ is a base for X;

(ii') For fixed $t \in T_n \{\vartheta(s) : t <_T s \in T_{n+1}\}$ is a base for $\vartheta(t)$.

Here T_n denotes the set of elements t in T having height n in the tree.

We define a new mapping χ on T by recursion. For $t \in T_0$ let $\chi(t) = \vartheta(t)$ in case $f(\vartheta(t))$ has metric diamater $\leqslant 1$ and $\vartheta(t) \subset G_1$. Otherwise let $\chi(t) = \varnothing$. Suppose $\chi(t)$ has been defined for $t \in T_{n-1}$. Let $s \in T_n$, $t <_T s$. If $\chi(t) \neq \varnothing$ and if $f(\vartheta(s))$ has diameter $\leqslant 1/(n+1)$ and $\vartheta(s)$ is contained in G_{n+1}, then define $\chi(s) = \vartheta(s)$. In all other cases let $\chi(s) = \varnothing$. This defines χ on the level n.

Suppose χ has been defined along these lines. Let $H_0 = \bigcup \{f(\chi(t)) : t \in T_0\}$, then H_0 is open dense in Y. By transfinite induction define a mapping ξ on T_0 such that either $\xi(t) = f(\chi(t))$ or $\xi(t) = \varnothing$ so that $\bigcup \{\xi(t) : t \in T_0\}$ is dense in H_0 and $t, t' \in T_0$, $t \neq t'$ implies $\xi(t) \cap \xi(t') = \varnothing$. Now let $t \in T_0$ be fixed. Define ξ for the immediate successors s of t by setting $\xi(s) = f(\chi(s))$ or $\xi(s) = \varnothing$ in such a way that $\bigcup \{\xi(s) : t <_T s \in T_1\}$ is dense in $\xi(t)$ and, moreover, $t <_T s'$ and $s \neq s'$ implies $\xi(s) \cap \xi(s') = \varnothing$. This defines ξ on the level 1. Continuing in this way, we obtain a nested sequence of disjoint open coverings $\{\xi(t) : t \in T_n\}$ for dense open subsets $O_n = \bigcup \{\xi(t) : t \in T_n\}$ of Y. Since $\bigcap \{O_n : n \in \mathbb{N}\}$ is a dense G_δ-subset of Y contained in $f(G)$, the proof of Lemma 4 is complete. ∎

Remarks. (1) In [N_2] we have obtained Theorem 2 in the case where C, D are closed convex sets. In this situation we were able to apply another method, namely we brought into action the Michael selection theorem. This is no longer possible in the present more general context.

(2) We de not know whether our separable Sandwich Theorem (2) from the introduction carries over to the nonseparable case without imposing stronger conditions on the sets C, D. In [N_3] we therefore presented another nonseparable version of (2) where the sets C, D are chosen to be weakly K-analytic and where the functions φ, ψ are required to satisfy some mild measurability condition involving weak K-analyticity.

We end this section with an application of Theorem 2 to semi-continuous functions φ, ψ. Note that Corollary 1 below is not presented in its possibly strongest form, since we have chosen a version of more practical interest.

COROLLARY 1. *Let E be a locally convex Fréchet space and let*

$\varphi: E \to \mathbb{R} \cup \{\infty\}$ be convex and lower semi-continuous and let $\psi: E \to \mathbb{R} \cup \{-\infty\}$ be concave and upper semi-continuous. Let $\psi \leqslant \varphi$ be satisfied and suppose $0 \in (D(\varphi) - D(\psi))^i$. Then there exists a continuous affine functional h on E having $\psi \leqslant h \leqslant \varphi$.

Proof. Let $x_0 \in D(\varphi) \cap D(\psi)$ be fixed and define closed convex sets C, D in E by setting

$$C = \{x \in E: \varphi(x) \leqslant |\varphi(x_0)| + 1\},$$

$$D = \{x \in E: \psi(x) \geqslant -|\psi(x_0)| - 1\}.$$

Clearly $C \subset D(\varphi)$, $D \subset D(\psi)$ and φ is dominated by a constant map on C while $-\psi$ is dominated by a constant map on D. In view of Theorem 2 it therefore remains to show that $0 \in (C - D)^i$.

Let $z \in E$ be fixed. Choose $x \in D(\varphi)$, $y \in D(\psi)$ and $\lambda \geqslant 0$ such that $z = \lambda(x - y)$ is satisfied. Let $\mu = \max\{1, |\varphi(x)|, |\psi(y)|\}$. Then we have

$$\frac{1}{\mu} x + \left(1 - \frac{1}{\mu}\right) x_0 \in D(\varphi),$$

$$\frac{1}{\mu} y + \left(1 - \frac{1}{\mu}\right) x_0 \in D(\psi),$$

and the difference of these two vectors is just $(1/\lambda\mu) z$. Moreover, we have

$$\varphi\left(\frac{1}{\mu} x + \left(1 - \frac{1}{\mu}\right) x_0\right) \leqslant \frac{1}{\mu} \varphi(x) + \left(1 - \frac{1}{\mu}\right) \varphi(x_0) \leqslant 1 + |\varphi(x_0)|,$$

$$\psi\left(\frac{1}{\mu} y + \left(1 - \frac{1}{\mu}\right) x_0\right) \geqslant \frac{1}{\mu} \psi(y) + \left(1 - \frac{1}{\mu}\right) \psi(x_0) \geqslant -1 - |\psi(x_0)|.$$

This proves $0 \in (C - D)^i$. ∎

EXAMPLE 3. Let E be an incomplete normed space and let C be a closed convex cone with vertex 0 in E such that $E = C - C$ but C does not induce an open decomposition of E. Then there exists a non-continuous linear functional g on E such that $g | C: C \to \mathbb{R}$ is continuous. Define φ by setting $\varphi | C = g | C$, $\varphi(x) = \infty$ for $x \notin C$ and ψ by setting $\psi | C = g | C$, $\psi(x) = -\infty$ for $x \notin C$. Then φ, ψ fulfill the requirements of Corollary 1, but clearly there does not exist a continuous linear f satisfying $\psi \leqslant f \leqslant \varphi$, for this would imply $g = f$, contradictory to the fact that g is not continuous. This proves that the completeness assumption on E in Corollary 1 is essential.

3. *Support Theorem*

In this section we establish a result on the existence of continuous linear support mappings for partially defined sublinear operators satisfying a weak measurability assumption. First we need the following preparatory proposition which generalizes [N_2, Lemma 3] as well as the result given in [SR].

PROPOSITION 3. *Let E be a locally convex Fréchet space and let F be a normally ordered locally convex vector space with an order-unit. Let C be a σ-pseudo-complete convex cone with vertex 0 in E and let $\varphi: C \to F$ be a sublinear operator which is dominated by a weakly Borel measurable function on C. Then φ is order-bounded below on some neighborhood U of 0 in E.*

Proof. Let (U_n) be a base of neighborhoods of 0 in E. Suppose that φ is not bounded below on any of the sets $U_n \cap C$, $n = 1, 2, \ldots$. Choose vectors $x_n \in U_n \cap C$ with $\varphi(x_n) \not\geqslant -ne$, where e denotes the order-unit on F. There exist continuous positive linear functionals f_n on F satisfying $f_n(e) = 1$ and

$$f_n(\varphi(x_n)) < f_n(-ne) = -n,$$

$n = 1, 2, \ldots$ Setting $\varphi_n(x) = (1/\sqrt{n})\, f_n(\varphi(x))$, $n = 1, 2, \ldots$, we therefore define a sequence (φ_n) of sublinear functionals on C which pointwise converges to 0. It suffices to prove that (φ_n) is uniformly bounded below on a neighborhood of 0, for then $\varphi_n(x_n) < -\sqrt{n}$ provides the desired contradiction.

By assumption, φ is weakly Borel measurably dominated, hence each φ_n is majorized by a Borel measurable function on C. By Lemma 2 there exists a dense G_δ-subset G of (C, σ) such that each $\varphi_n | G: (G, \sigma) \to \mathbb{R}$ is continuous.

For $k \in \mathbb{N}$ let $F_k = \{x \in G: \sup_{n,m \geqslant k} |\varphi_n(x) - \varphi_m(x)| \leqslant \frac{1}{4}\}$, then the F_k are closed in (G, σ). Let $X = \bigcup \{\partial F_k : k \in \mathbb{N}\}$, where the boundary ∂ refers to the space (G, σ), then X is a first category subset of G and $G' = G \backslash X$ is a dense G_δ-set in (C, σ).

Let $x \in G'$ be fixed. Since $\varphi_n(x) \to 0$, there exists k having $x \in F_k$, hence $x \in \operatorname{int}_\sigma F_k$ (in view of $x \notin X$), where $\operatorname{int}_\sigma$ refers to the interior operator in the space (G, σ). Consequently, there exists a neighborhood V of 0 in E such that

$$(x + (V \cap C)) \cap G \subset F_k.$$

Note that in view of the continuity of $\varphi_k | G$, V may in addition be chosen such that $y \in (x + (V \cap C)) \cap G$ implies

$$|\varphi_k(x) - \varphi_k(y)| < \tfrac{1}{4}.$$

We claim that for every $n \geqslant k$ and every $v \in V \cap C$ $\varphi_n(v) \geqslant -2$, which finally provides the desired contradiction. So let $n \geqslant k$ and $v \in V \cap C$ be fixed. Let $G'' = \{z \in C: z + \frac{1}{2}v \in G\}$, then G'' is a dense G_δ-set in (C, σ) since it is the preimage of $G \cap (C + \frac{1}{2}v)$ under the σ-homomorphism $x \to x + \frac{1}{2}v$ mapping (C, σ) onto $(C + \frac{1}{2}v, \sigma)$. Consequently the set

$$(x + (\tfrac{1}{2}V \cap C)) \cap G \cap G''$$

is nonempty and we may choose y within. This means $y \in (x + (\frac{1}{2}V \cap C)) \cap G \subset (x + (V \cap C)) \cap G \subset F_k$, giving

$$|\varphi_n(y) - \varphi_k(y)| \leqslant \tfrac{1}{4}.$$

On the other hand $y + \frac{1}{2}v \in [x + (\frac{1}{2}V \cap C) + (\frac{1}{2}V \cap C)] \cap G \subset (x + (V \cap C)) \cap G$ implies $y + \frac{1}{2}v \in F_k$, hence

$$|\varphi_n(y + \tfrac{1}{2}v) - \varphi_k(y + \tfrac{1}{2}v)| \leqslant \tfrac{1}{4}.$$

Finally, $y + \frac{1}{2}v \in (x + (V \cap C)) \cap G$ yields

$$|\varphi_k(x) - \varphi_k(y + \tfrac{1}{2}v)| \leqslant \tfrac{1}{4},$$

hence in view of $|\varphi_k(x) - \varphi_k(y)| \leqslant \frac{1}{4}$ we deduce

$$|\varphi_k(y) - \varphi_k(y + \tfrac{1}{2}v)| \leqslant \tfrac{1}{2}.$$

This proves the result in view of

$$-1 \leqslant \varphi_n(y + \tfrac{1}{2}v) - \varphi_n(y) \leqslant \tfrac{1}{2}\varphi_n(v). \quad \blacksquare$$

THEOREM 3. *Let E be a locally convex vector space and let F be a normally ordered locally convex vector space with lubp and having an order-unit. Let $\varphi: E \to F \cup \{\infty\}$ be a sublinear operator. Suppose that either*

(i) *$D(\varphi)$ is σ-pseudo-complete and φ is weakly Borel measurably dominated on $D(\varphi)$, or*

(ii) *There exist σ-pseudo-complete subcones C, D of $D(\varphi)$ having $E = C - D$ such that φ is weakly Borel measurably dominated on C, D, respectively.*

Then φ admits a continuous linear support function $f: E \to F$.

Proof. First consider case (i). By Proposition 3 φ is order-bounded below on a closed convex neighborhood U of 0 in E, $\varphi \geqslant -e$, say for some order-unit e on F. This implies $\varphi \geqslant -qe$, when q denotes the Minkowski functional of U. Indeed, let $x \in E$ be fixed. If $q(x) > 0$, then $q(x)^{-1}x \in U$, hence $\varphi(q(x)^{-1}x) \geqslant -e$, giving $\varphi(x) \geqslant -q(x)e$. On the other hand suppose we have $q(x) = 0$. This implies $\mathbb{R}_+ x \subset U$, so $\varphi(\lambda x) \geqslant -e$ for all $\lambda > 0$. We claim that this implies $\varphi(x) \geqslant 0$, hence $\varphi(x) \geqslant -q(x)e$ as well. Assume $\varphi(x) \ngeqslant 0$. Then there exists $g \in F'_+$ having $g(e) = 1$ such that $g(\varphi(x)) < 0$. This implies $g(\varphi(\lambda x)) \to -\infty$, $\lambda \to \infty$, a contradiction since $g(\varphi(\lambda x)) \geqslant g(-e) = -1$. So $\varphi(x) \geqslant 0$ is proved.

We may now apply the Sandwich Theorem of Zowe (see (1) in the Introduction) to φ and the concave $\psi = -qe$. This provides us with a continuous linear operator f satisfying $f \leqslant \varphi$.

Next consider case (ii). Here we obtain a closed convex neighborhood U of 0 in E such that φ is bounded below on $U \cap D$, $\varphi \geqslant -e$ say for an order-unit e on F. Let p denote the Minkowski functional of $U \cap D$. Then p is sublinear with domain the cone D. Let ψ be the concave operator $-pe$. We claim that $\varphi \geqslant \psi$. Indeed, this may be established as above.

To conclude we observe that p is Borel measurable and hence ψ is weakly Borel measurable. Since $E = C - D$, we may apply our Sandwich Theorem. This provides a continuous linear operator f with $\psi \leqslant f \leqslant \varphi$. ∎

Remark. The existence of an order-unit in F is essential in Theorem 3. This was indicated in [N_2, Example 1].

4. *Lower semi-continuity*

In this section we prove a somewhat surprising result stating that lower semi-continuity of partially defined convex functions satisfying a mild measurability assumption is essentially a one-dimensional property.

THEOREM 4. *Let E be a locally convex Fréchet space and let $\varphi : E \to \mathbb{R} \cup \{\infty\}$ be a convex function. Suppose there exist σ-pseudo-complete subsets C, D of $D(\varphi)$ with $0 \in (C-D)^i$ such that φ is Borel measurably dominated on C, D, respectively. Suppose that for every one-dimensional linear submanifold L of E the map $\varphi | L : L \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous. Then φ itself is lower semi-continuous.*

Proof. First we treat the case where φ is sublinear and where C, D are σ-pseudo-complete cones contained in $D(\varphi)$ and having $E = C - D$ such that φ is Borel measurably dominated on C, D.

Let $x \in E$ be fixed. We prove that φ may be approximated from below at x by continuous linear support functionals. First we consider the case where $\varphi(x) = -\varphi(-x)$. This means that φ is linear on the line $\mathbb{R}x$. By Theorem 3 part (ii) there exist a continuous linear support functional f for φ. Clearly this must satisfy the equality $f(x) = \varphi(x)$. So let us now consider the difficult case where $\varphi(x) \neq -\varphi(-x)$. Let $\beta < \varphi(x)$. We have to find a continuous linear $f \leqslant \varphi$ with $f(x) > \beta$. First of all we choose an α having $\beta < \alpha < \varphi(x)$ such that $-\alpha < \varphi(-x)$. This is possible since φ is not linear on the line $\mathbb{R}x$.

We define a mapping $\psi : E \to \mathbb{R} \cup \{-\infty, \infty\}$ by setting

$$\psi(z) = \inf\{\varphi(z + \lambda x) - \lambda\alpha : \lambda \in \mathbb{R}\}.$$

It suffices to prove that ψ does not actually assume the value $-\infty$.

For then ψ is a sublinear functional whose domain is the cone $D(\psi) = D(\varphi) + \mathbb{R}x$. Since $\psi \leqslant \varphi$ and since φ is Borel measurably dominated on C, D, we may then apply the Support Theorem (Theorem 3) to ψ, and this provides a continuous linear functional $f \leqslant \psi$. The latter implies $f \leqslant \varphi$ and

$$f(-x) \leqslant \psi(-x) \leqslant \varphi(-x+x) - \alpha,$$

hence f fulfills the requirements.

Let $z \in E$ be fixed. We check $\psi(z) > -\infty$. Let us define a function $\varphi^*: \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$ by setting

$$\varphi^*(\lambda, \mu) = \varphi(\mu z + \lambda x) - \lambda\alpha.$$

φ^* is sublinear, and consequently there exists a neighborhood of $(0, 0)$ in $\mathbb{R}^2$ on which φ^* is bounded below, i.e., there exists $\delta \in (0, 1)$ and $\gamma > 0$ such that $|\lambda|$, $|\mu| \leqslant \delta$ implies $\varphi^*(\lambda, \mu) \geqslant -\gamma$.

By the choice of α we have $\varphi^*(0, 1) > 0$ and $\varphi^*(0, -1) > 0$. By assumption φ is lower semi-continuous on the lines $L^+ = x + \mathbb{R}z$ and $L^- = -x + \mathbb{R}z$. Consequently, the functions $\lambda \to \varphi^*(\lambda, 1)$ and $\lambda \to \varphi^*(\lambda, -1)$ are lower semi-continuous as well. Hence there exists $\varepsilon \in (0, \delta)$ such that $|\lambda| \leqslant \varepsilon$ implies $\varphi^*(\lambda, \pm 1) > 0$.

We claim that

$$\psi(z) = \inf_{\mu \in \mathbb{R}} \varphi^*(1, \mu) \geqslant -\gamma/\delta\varepsilon.$$

Let $\mu \in \mathbb{R}$ be fixed. We consider two cases, $|\mu| \geqslant 1/\varepsilon$ and $|\mu| < 1/\varepsilon$. First assume $|\mu| \geqslant 1/\varepsilon$. Then we have

$$\varphi^*(1, \mu) = \varphi^*\left(|\mu| \cdot \left(\frac{1}{|\mu|}, \frac{\mu}{|\mu|}\right)\right) = |\mu|\, \varphi^*\left(\frac{1}{|\mu|}, \pm 1\right) > 0.$$

Now let $|\mu| < 1/\varepsilon$, then we have

$$\varphi^*(1, \mu) = \varphi^*\left(\frac{1}{\delta\varepsilon}(\delta\varepsilon, \delta\varepsilon\mu)\right) = \frac{1}{\delta\varepsilon}\varphi^*(\delta\varepsilon, \delta\varepsilon\mu) \geqslant -\frac{\gamma}{\delta\varepsilon},$$

the latter in view of $|\delta\varepsilon|$, $|\delta\varepsilon\mu| \leqslant \delta$. This ends the proof of our claim and therefore of part 1.

(2) Let us now prove the convex result under the additional assumption that E is a Banach space and $D(\varphi)$ is a bounded set and $C, D \subset D(\varphi)$ are (bounded) σ-pseudo-complete sets satisfying $0 \in (C-D)^i$ and $0 \in C \cap D$ such that φ is Borel measurably dominated on C, D. Now let $\tilde{\varphi}$ be the sublinear function associated with φ on $E \times \mathbb{R}$ and let $\tilde{C}$, $\tilde{D}$ be the convex cones associated with C, D. Then we have $E \times \mathbb{R} = \tilde{C} - \tilde{D}$, $\tilde{C}$, $\tilde{D}$ are

σ-pseudo-complete and $\tilde{\varphi}$ is Borel measurably dominated on $\tilde{C}$, $\tilde{D}$. It remains to prove that $\tilde{\varphi}|L$ is lower semi-continuous for every line L in $E\times\mathbb{R}$. For suppose this has been established. Then $\tilde{\varphi}$ is globally lower semicontinuous by part 1 of our proof, and consequently the same is true for $\varphi=\tilde{\varphi}(\ ,1)$.

Let L be a line in $E\times\mathbb{R}$. Then $D(\tilde{\varphi})\cap L$ is either all of L or a ray, an interval, or a single point. The first and the last case are clear, so it remains to consider the two other cases. Clearly here problems may arise only at a boundary point of $L\cap D(\tilde{\varphi})$. So let (x,λ) be such a boundary point and let $(x,\lambda)+\rho_n(y,\mu)$ be a sequence in L tending to (x,λ), i.e., $\rho_n\to 0$. The case where $(x,\lambda)+\rho_n(y,\mu)\notin D(\tilde{\varphi})$ bears no difficulty, so let us assume $(x,\lambda)+\rho_n(y,\mu)\in D(\tilde{\varphi})$. By the definition of $\tilde{\varphi}$ this means that $(\lambda+\rho_n\mu)^{-1}(x+\rho_n y)\in D(\varphi)$. By assumption the set $D(\varphi)$ is bounded and therefore we must have $\lambda\neq 0$. This implies $(\lambda+\rho_n\mu)^{-1}(x+\rho_n y)\to\lambda^{-1}x$. By assumption φ is lower semi-continuous on every line, so we obtain

$$\liminf_{n\to\infty}\varphi((\lambda+\rho_n\mu)^{-1}(x+\rho_n y))\geqslant\varphi(\lambda^{-1}x). \tag{$*$}$$

If $\lambda<0$ then $\lambda+\rho_n\mu<0$ eventually, so that $(x,\lambda)+\rho_n(y,\mu)$ is not an element of $D(\tilde{\varphi})$. Consequently, we must have $\lambda>0$, giving $\lambda+\rho_n\mu>0$. But now we may multiply the inequality $(*)$ with the factors $\lambda+\rho_n\mu$, and this gives $\liminf_{n\to\infty}\tilde{\varphi}((x,\lambda)+\rho_n(y,\mu))\geqslant\tilde{\varphi}(x,\lambda)$. This ends the proof in case (2).

(3) Let us finally consider the general case. Let $x\in E$ be fixed and let (x_n) be a sequence in E converging to x. We have to prove $\liminf_{n\to\infty}\varphi(x_n)\geqslant\varphi(x)$. We choose a closed circeled convex set B in E such that the x_n and x are contained in B and such that (x_n) converges to x in the Banach space E_B generated by B (see [Kö, p. 71] for the possibility of finding such B). We claim that for sufficiently large $n\in\mathbb{N}$ the set $(C\cap nB)-(D\cap nB)$ is absorbing in E_B. Indeed , let $x_0\in C\cap D$ be fixed and choose n with $x_0\in\operatorname{int}(nB)$. Then $0\in((C\cap nB)-(D\cap nB))^i$.

We claim that the assumptions of part 2 of our proof are now satisfied for $\psi: E_B\to\mathbb{R}\cup\{\infty\}$ defined by $\psi(x)=\varphi(x)$ in case $x\in nB$, $\psi(x)=\infty$ otherwise. Clearly ψ is convex and lower semi-continuous on every line, since B is a closed convex set. Moreover, ψ has a bounded domain in the Banach space E_B and is majorized by Borel measurable functions on the sets $C\cap nB$ and $D\cap nB$. Here we make use of the fact that the Banach space topology on E_B is finer than the original topology, and hence Borel measurability with respect to the topology of E implies Borel measurability with respect to E_B. Now part 2 of the proof applies, yielding the result. ∎

EXAMPLE 4. Define $\varphi:\mathbb{R}^2\to\mathbb{R}\cup\{\infty\}$ by $\varphi(\lambda,\mu)=\lambda$ for $\mu>0$, $\varphi(0,0)=0$, $\varphi(\lambda,\mu)=\infty$ otherwise. Then φ is sublinear. Moreover, on each one-dimensional linear subspace of $\mathbb{R}^2$ φ is lower semi-continuous. Since φ itself is not lower semi-continuous, it follows that we may not weaken the assumption in Theorem 4 to the extent that φ be lower semi-continuous on every line passing through 0.

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