Stability Optimization of Hybrid Periodic Systems via a Smooth Criterion

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Abstract

We consider periodic orbits of controlled hybrid dynamic systems and want to find open-loop controls that yield maximally stable limit cycles. Instead of optimizing the spectral or pseudo-spectral radius of the monodromy matrix A, which are nonsmooth criteria, we propose a new approach based on the smoothed spectral radius $\rho_\alpha(A)$, a differentiable criterion favorable for numerical optimization. Like the pseudo-spectral radius, the smoothed spectral radius $\rho_\alpha(A)$ converges from above to the exact spectral radius $\rho(A)$ for $\alpha \to 0$. Its derivatives can be computed efficiently via relaxed Lyapunov equations. We show that our new smooth stability optimization program based on $\rho_\alpha(A)$ has a favourable structure: it leads to a *differentiable* nonlinear optimal control problem with periodicity and matrix constraints, for which tailored boundary value problem methods are available. We demonstrate the numerical viability of our method using the example of a walking robot model with nonlinear dynamics and ground impacts as a complex open-loop stability optimization example.

Index Terms

stability, periodic orbits, Lyapunov equation, eigenvalue optimization, smoothed spectral radius, robotic motion, robustness

I. Introduction

Stability optimization of nonlinear periodic systems with hybrid dynamics is a difficult but very important task. It arises when a technical system is best operated periodically and has to be controlled in such a way that its *cyclic steady state* or *periodic orbit* is stable, robust against

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perturbations, and optimized with respect to certain desirable features. A typical example is human or robotic running, where the periodic motion has to be robustly stable and allow the runner to move as fast as possible. The dynamics of running or hopping are often described by hybrid dynamics due to the ground impacts.

Other examples are periodically operated simulated moving bed (SMB) processes [25], looping kites [15], or iterative feedback tuning with time and frequency domain constraints [3], [7].

Periodic systems can be stabilized in two ways: one based on sensors, actuators, and feedback control, the second based on intrinsically *open-loop* stable orbits [19], [20]. Here we treat the second, where system parameters – for example limb lengths in the case of a running robot – and time varying inputs, or *control functions* – for example periodic torque commands – are simultaneously optimized to yield inherently stable periodic orbits, along with other operational constraints or performance objectives. We note that the first case – feedback control – can be addressed by our approach by including the unknown feedback controller gains among the decision variables.

II. Models of periodic hybrid dynamic systems and sensitivities

Hybrid dynamical systems include both continuous phases and discrete jump events. Each continuous phase is described by its own set of differential equations and each jump is described by its own discrete equations. Writing the system state as $y(t) \in \mathbb{R}^n$, free model parameters as $p \in \mathbb{R}^{n_p}$, time varying control functions, i.e. external inputs, as $u(t) \in \mathbb{R}^{n_u}$, and time as t, the dynamics of all n_{ph} phases can be written as

$$\begin{aligned} \dot{y}(t) &= f_j(t, y(t), u(t), p) \quad \text{for} \quad t \in [\tau_{j-1}, \tau_j], \\ & \qquad j = 1, ..., n_{\mathrm{ph}},\ \tau_0 = 0, \tau_{n_{\mathrm{ph}}} = T \qquad (1) \\ y(\tau_j^+) &= y(\tau_j^-) + J_j(y(\tau_j^-), p) \quad \text{for} \quad j = 1, ..., n_{\mathrm{ph}}, \qquad (2) \end{aligned}$$

Here the right hand sides f_j are class $\mathcal{C}^2$ within phases, but discontinuities J_j may occur at junctions between f_j and f_{j+1}. The J_j are twice continuously differentiable, and τ_j denote the phase boundaries, which are the roots of the switching functions

$$s_j(\tau_j, y(\tau_j), p) = 0. \qquad (3)$$

If T is the overall cycle time, T-periodic systems have to satisfy periodicity constraints of the form

$$y(T) = y(0). \tag{4}$$

Stability is determined by the sensitivity of the solution of the hybrid dynamic system with respect to perturbations in the initial values $Y(t) = \frac{dy(t)}{dy(0)}$, and in particular, its value at the end of the cycle $Y(T)$. This can be computed by solving the variational differential equation

$$\dot{Y}(t) = \frac{\partial f_j}{\partial y}(t, y(t), u(t), p) \cdot Y(t). \tag{5}$$

For hybrid dynamical systems, the above equation is used for the continuous phases (with $Y(0) = I$), and updates of form

$$\begin{aligned} Y(\tau_j^+) = \Bigg(\bigg(& f_{j+1}(\tau_j^+) - f_j(\tau_j^-) - \frac{\partial J_j}{\partial t} - \frac{\partial J_j}{\partial y} f_j(\tau_j^-) \bigg) \cdot \\ & \frac{1}{\dot{s}_j} \frac{\partial s_j}{\partial y}^T + I + \frac{\partial J_j}{\partial y} \Bigg) Y(\tau_j^-) \quad \text{for} \quad j = 1, ..., n_{\text{ph}} \end{aligned} \tag{6}$$

at phase changes. Eq. (6) takes into account that the τ_j change under a perturbation of the initial values. Despite the non-smoothness of the trajectory (exhibiting jumps in time), both $y(t)$ and the end values $y(T)$ are twice continuosly differentiable with respect to the initial values $y(0)$, and with respect to the controls $u(t)$ and the parameters p. This is true as long as the ordering of the phases is not changed.

III. Stability of hybrid dynamical systems

Stability of periodic solutions of nonlinear periodic systems can be defined according to Lyapuov's first method, based on the eigenvalues of the so-called monodromy matrix

$$A = Y(T) = \left. \frac{dy(t)}{dy(0)} \right|_{t=T}, \tag{7}$$

which is the Jacobian of the Poincaré map of the periodic system; cf. [16]:

Theorem 3.1 (Stability of Nonlinear Periodic Systems): A T-periodic solution of a T-periodic nonlinear system is locally asymptotically stable if the spectral radius of the monodromy matrix A satisfies $\rho(A) < 1$. It is unstable if $\rho(A) > 1$.

Remarks: (a) As shown in [20], this theorem can be generalized to the case of hybrid dynamic systems, if the differentiability assumptions on f_j and J_j stated in section II are satisfied. The computation of A for hybrid systems follows eqns. (5) and (6). (b) If not all entries of the vector

y are periodic, e.g. in walking motion, the non-periodic directions have to be eliminated from the matrix A before application of this criterion. (c) For autonomous systems the above stability criterion is not directly applicable, because A always has an eigenvalue one, and only orbital asymptotic stability can be achieved. However, by eliminating the eigendirections associated with this eigenvalue, i.e., computing the projection A' of A on the complement of the eigenspace, autonomous systems can be treated in a similar way. The analogous condition is then that the eigenvalues of A' have to have modulus < 1.

IV. Optimal control problem involving stability

Optimal control problems for hybrid dynamical systems with a stability related criterion are based on the augmented system dynamics (i.e trajectories $y(t)$ and their sensitivities $Y(t)$), and typically comprises additional path constraints (8b) and pointwise equality and inequality constraints (8c) and (8d), cf. [20].

$$\min_{y(\cdot),Y(\cdot),u(\cdot),\tau,p} \Phi_{\mathbf{stab}}(Y(T)) \tag{8a}$$

$$\begin{aligned}
\text{s.t.} \quad & \text{(1), (2), (5), (6) - augmented hybrid system dynamics} \\
& g_j(t,y(t),u(t),p) \geq 0 \qquad \text{for} \quad t \in [\tau_{j-1},\tau_j] & \text{(8b)} \\
& r_{\text{eq}}(y(0),..,y(T),p) = 0, \ \text{e.g. (3), (4)} & \text{(8c)} \\
& r_{\text{ineq}}(y(0),..,y(T),Y(T),p) \geq 0 & \text{(8d)}
\end{aligned}$$

The stability objective $\Phi_{\mathbf{stab}}$ depends on the monodromy matrix $A = Y(T)$, and will be further discussed below.

A. *Numerical Optimal Control Methods and Resulting Nonlinear Optimization Problem*

If all problem data are twice differentiable, problem (8) can be solved by the direct boundary value problem approach [8] based on multiple shooting. This technique discretizes both state and control variables and transforms the optimal control problem into a nonlinear mathematical program (NLP), which can be written as

$$\min_x \Phi_{\mathbf{stab}}(A(x)) \quad \text{s.t.} \quad g_E(x) = 0, \ g_I(x) \leq 0. \tag{9}$$

Here the vector $x \in \mathbb{R}^{n_x}$ collects all optimization variables of the discretized optimal control problem. The dynamic system equations have n states $y(t)$ plus the $n \times n$ sensitivity states $Y(t)$. If the control and state variables are discretized on a grid with n_{int} intervals, the dimension of x is $n_x = (n + n^2)(n_{\text{int}} + 1) + n_u n_{\text{int}} + n_p + n_{\text{ph}}$. NLP (9) could in principle be solved by any standard NLP software, but since it is sparse and - due to the discretization - very structured, for efficiency reasons it is solved by a tailored structure-exploiting SQP method. For details, see [8], [17], [18].

The stability function $\Phi_{\mathbf{stab}}$ could be chosen in various ways. While any induced matrix norm of A could be used to evaluate contraction of perturbations, the spectral radius is particularly appealing, since it is the most stringent criterion. Unfortunately it has two major drawbacks when it comes to minimizing instability: (a) $\rho(A)$ is typically a non-smooth (and even non-Lipschitz) function of the entries of matrix A [9]. (b) $\rho(A)$ is not robust against parametric uncertainties in the system. For this reason, more robust criteria have been proposed, like the pseudo-spectral radius [11], [26] which however still suffers from non-smoothness and from high computational costs within an optimization procedure, where the objective function needs to be evaluated many times. It is the goal of this paper to discuss an alternative to the spectral radius, which is also suited as objective within optimal control problems for hybrid dynamic systems.

V. The Smoothed Spectral Radius

In this note we propose a criterion for stability, which is both smooth and computationally attractive. At its core is a well-known observation.

Lemma 5.1: Let $\|\cdot\|$ be any matrix norm. Then $A \in \mathbb{R}^{n \times n}$ is stable (i.e. $\rho(A) < 1$) if and only if the series $\sum_{k=1}^{\infty} \|A^k\|^2$ converges.

Proof: If the sum $\sum_{k=1}^{\infty} \|A^k\|^2$ converges, then in particular $\lim_{k\to\infty} A^k = 0$. This implies for any eigenvalue and -vector pair (λ, v) with $v \neq 0$ that $\lim_{k\to\infty} A^k v = \left(\lim_{k\to\infty} \lambda^k\right) v$ which implies that $|\lambda| < 1$, i.e. we have shown that $\rho(A) < 1$. Conversely, if $\rho(A) < 1$ we use the spectral radius formula [14] that states for any matrix norm that $\rho(A) = \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}}$. This implies that for any $\epsilon > 0$ there exists a $k_0 \in \mathbb{N}$ so that $\sum_{k=k_0}^{\infty} \|A^k\| \leq \sum_{k=k_0}^{\infty} (\rho(A) + \epsilon)^k$. By chosing ϵ small enough, the upper bound is a converging sum. $\square$

Inspired by this observation, we now introduce a relaxed and weighted H_2-norm, namely the matrix function $f : \mathbb{R}^{n\times n} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$

$$f(A,s) := \sum_{k=1}^{\infty} s^{-2k} \, \|A^k\|^2_{\mathrm{F},V,W}, \tag{10}$$

where $\|B\|^2_{\mathrm{F},V,W} := \|V^{\frac{1}{2}} B W^{\frac{1}{2}}\|^2_{\mathrm{F}} = \mathrm{Tr}(VBWB^T)$ is a scaled Frobenius norm with symmetric positive definite matrices V, W. We have the following

Lemma 5.2: $f(A,s) < \infty \quad \Leftrightarrow \quad s > \rho(A)$.

Proof: We use Lemma 5.1 by noting that $f(A,s) = \sum_{k=1}^{\infty} \|(As^{-1})^k\|^2_{\mathrm{F},V,W}$ and that $s > \rho(A)$ is equivalent to stability of the matrix As^{-1}. ❑

Lemma 5.3: For every fixed $A \in \mathbb{R}^{n\times n} \setminus \{0\}$ one has for $s > \rho(A)$ that $\frac{\partial f(A,s)}{\partial s} < 0$. Moreover, $\{f(A,s) | s > \rho(A)\} = \mathbb{R}_{++}$.

This result can be shown by direct calculation. It allows us now to introduce the *smoothed spectral radius* $\rho_\alpha(A)$ as the implicit function of the equation $f(A,s) = \alpha^{-1}$ with respect to s.

Definition 5.1 (Smoothed Spectral Radius): The smoothed spectral radius is the map $\rho_\alpha : \mathbb{R}_{++} \times \mathbb{R}^{n\times n} \to \mathbb{R}$, $(\alpha, A) \mapsto \rho_\alpha(A)$, that for any $A \neq 0$ uniquely solves the equation

$$f(A, \rho_\alpha(A)) = \alpha^{-1}. \tag{11}$$

For $A = 0$ and any $\alpha > 0$ we define $\rho_\alpha(A) = 0$. As a consequence of Lemma (5.3), $\rho_\alpha(A)$ is well defined on the whole domain, i.e., for any $\alpha > 0$, and any matrix $A \in \mathbb{R}^{n\times n}$.

Theorem 5.4 (Properties of smoothed spectral radius): The smoothed spectral radius $\rho_\alpha(A)$ satisfies $\rho_\alpha(A) > \rho(A)$, and $\lim_{\alpha\to 0} \rho_\alpha(A) = \rho(A)$. Moreover, $\rho_\alpha(A)$ is analytic in both its arguments on the whole domain $\alpha > 0$, $A \in \mathbb{R}^{n\times n}$, and satisfies $\frac{\partial \rho_\alpha(A)}{\partial \alpha} > 0$.

Proof: The first two properties follow from the fact that $f(A, \rho_\alpha(A))$ is finite, but tends to infinity for $\alpha \to 0$. The differentiability properties follow from the implicit function theorem and the fact that $f(A,s)$ is analytic in both its arguments and finite for $s = \rho_\alpha(A) > \rho(A)$, and from the fact that $\frac{\partial f(A,s)}{\partial s} < 0$. ❑

For a visualization see Fig. 1. From the above properties, and from Theorem 3.1, we easily obtain the following

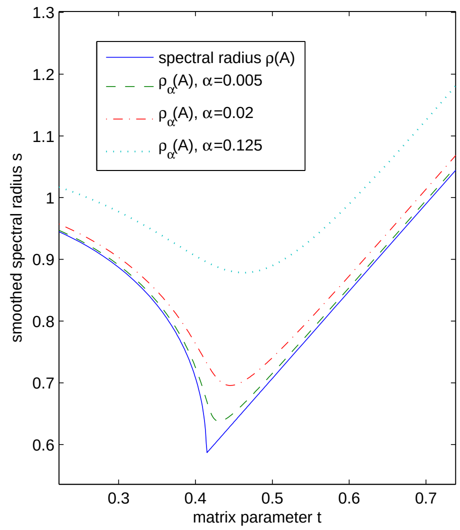

Fig. 1. The smoothed spectral radius $\rho_\alpha(A)$ for three different values of α, at the example of a parameter dependent matrix $A(t) = \big((1,-t)^T,(t,t^2)^T\big)$. The solid (nonsmooth) line is the spectral radius $\rho(A)$, which is a lower bound for $\rho_\alpha(A)$, cf. Theorem 5.4.

Corollary 5.5: If $\rho_\alpha(A(x)) \le 1$ for some $\alpha > 0$, then the nonlinear periodic system is locally asymptotically stable. If $\lim_{\alpha\to 0}\rho_\alpha(A(x)) > 1$, it is unstable.

A. Computing the Smoothed Spectral Radius

When it comes to algorithmic optimization, the fact that the smoothed spectral radius is differentiable allows us to use derivative based methods without any restriction, a major advantage. However, in order to exploit this fact, we will also have to ascertain that computing $\rho_\alpha(A)$ is sufficiently easy, a fact which is not straightforward, given the implicit definition of ρ_α. Fortunately, it turns out that the smoothed spectral radius can be computed by solving a relaxed Lyapunov equation.

Theorem 5.6 (Smoothed Spectral Radius Computation): For any $\alpha > 0$ and $s > 0$, the following statements are equivalent:

(a) $\rho_\alpha(A) \le s$

(b) $f(A,s) \le \alpha^{-1}$

(c) $\exists P_1 \succeq 0 :\ s^2 P_1 = A(W+P_1)A^T, \mathrm{Tr}(VP_1) \le \alpha^{-1}$

(d) $\exists P_2 \succeq 0 :\ s^2 P_2 = A^T(V+P_2)A, \mathrm{Tr}(WP_2) \le \alpha^{-1}$.

Proof: The equivalence of (a) and (b) follows from the definition of the smoothed spectral radius, Eq. (11), and the monotonicity of $f(A,s)$ in s, Lemma 5.3. Equivalence with (c) follows from the observations that (i) $f(A,s)$ is finite if and only if As^{-1} is stable, and (ii) that As^{-1} is stable if and only if a positive definite matrix $\tilde{P}_1$ exists that uniquely solves the Lyapunov equation $\tilde{P}_1 = W + (As^{-1})\tilde{P}_1(As^{-1})^T$. This matrix satisfies $\tilde{P}_1 = \sum_{k=0}^{\infty} s^{-2k} A^k W (A^k)^T$, and $P_1 := \tilde{P}_1 - W = \sum_{k=1}^{\infty} s^{-2k} A^k W (A^k)^T$ is then positive semidefinite and the unique solution of the equation $s^2 P_1 = A(W + P_1)A^T$. Therefore,

$$f(A,s) = \mathrm{Tr}\left(V \sum_{k=1}^{\infty} A^k W (A^k)^T\right) = \mathrm{Tr}(VP_1).$$

The equivalence with (d) follows with a similar argument, now with a positive semidefinite $P_2 = \tilde{P}_2 - V$, where $\tilde{P}_2$ satisfies the Lyapunov equation $\tilde{P}_2 = V + (As^{-1})^T \tilde{P}_2 (As^{-1})$. □

From the above proof we also obtain, using the intermediate matrices $\tilde{P}_1$ and $\tilde{P}_2$, the following corollary.

Corollary 5.7: For any $\alpha > 0$ and $s > 0$, $\rho_\alpha(A) \le s$ is equivalent to the existence of a unique positive definite, symmetric matrix $\tilde{P}_1$, respectively $\tilde{P}_2$, satisfying

$$\tilde{P}_1 = W + s^{-2} A \tilde{P}_1 A^T, \mathrm{Tr}(V\tilde{P}_1) \le \mathrm{Tr}(VW) + \alpha^{-1} \tag{12}$$

$$\tilde{P}_2 = V + s^{-2} A^T \tilde{P}_2 A, \mathrm{Tr}(W\tilde{P}_2) \le \mathrm{Tr}(VW) + \alpha^{-1} \tag{13}$$

VI. Robust Stability Optimization

The differentiable dependence of the smoothed spectral radius on A makes it attractive as an objective for the stability optimal control problem (9). We will now discuss two variants on how to use employ this criterion algorithmically.

A. *Optimization of Smoothed Spectral Radius*

The first variant that comes to mind is to simply choose an $\alpha > 0$ and then to solve a stability optimization problem of the form (9), namely

$$\min_x \; \rho_\alpha(A(x)) \quad \text{s.t.} \quad g_E(x) = 0,\; g_I(x) \le 0, \tag{14}$$

or the equivalent formulation involving a Lyapunov matrix, cf. Corollary 5.7,

$$\begin{aligned}
\text{minimize} \quad & s \\
\text{subject to} \quad & P = V + s^{-2} A(x)^T P A(x), && \text{(15a)} \\
& \mathrm{Tr}(WP) = \mathrm{Tr}(VW) + \alpha^{-1}, && \text{(15b)} \\
& P \succeq 0 && \text{(15c)} \\
& g_E(x) = 0,\ g_I(x) \le 0. && \text{(15d)}
\end{aligned}$$

Note that the relaxed Lyapunov equation (15a) and the trace condition (15b) are both linear equations in the matrix variable P, and that (15a) will always have a unique positive definite solution $P(x,s)$ if $s > \rho(A(x))$. Non-positive definite shadow solutions (which even exist for $s \le \rho(A(x))$) are excluded by the LMI constraint (15c). This constraint, however, is not active at the unique solution $P(x,s)$ for $s > \rho(A(x))$. If s is initialized suitably, it is therefore often possible to ignore the constraint (15c), and for given α and $A(x)$, Eqs. (15a) and (15b) have a unique solution (s,P) with $s = \rho_\alpha(A(x)) > \rho(A(x))$ and P positive definite (possible elimination of these variables would lead again to problem (14)). Finally, if we find a solution with $s = \rho_\alpha(A) \le 1$, then – by virtue of Corollary 5.5 – we have found a stable solution.

B. Optimization of a Heuristic Robustness Measure

When minimizing the smoothed spectral radius, the choice of α is somewhat arbitrary. As seen in Fig. 1 and indicated by Theorem 5.4, $\rho_\alpha(A)$ becomes smoother – and therefore presumably a more robust measure for stability – with increasing values for $\alpha > 0$. Having found a parameter value $\alpha > 0$ with $\rho_\alpha(A(x)) < 1$, we might in consequence search for the largest α such that the stability certificate $\rho_\alpha(A) \le 1$ remains valid. This leads to the optimization program

$$\max_{x,\alpha} \ \alpha \quad \text{s.t.} \quad \rho_\alpha(A(x)) \le 1,\ g_E(x) = 0,\ g_I(x) \le 0. \tag{16}$$

Interestingly, this program can be nicely interpreted as a scaled H_2-norm optimization:

Corollary 6.1 (Equivalence with H_2-norm minimization): Any solution (x^*, α^*) of program (16) also solves

$$\min_x \ \sum_{k=1}^{\infty} \|A(x)^k\|^2_{\mathrm{F},V,W} \quad \text{s.t.} \quad g_E(x) = 0,\ g_I(x) \le 0, \tag{17}$$

with optimal value $(\alpha^*)^{-1}$, and vice-versa.

Proof: From Theorem 5.6 we conclude that problem (16) is equivalent to

$$\max_{x,\alpha} \ \alpha \quad \text{s.t.} \quad f(A(x),1) \leq \alpha^{-1},\ g_E(x) = 0,\ g_I(x) \leq 0. \tag{18}$$

Given the fact that α^{-1} decreases monotonically in α, the constraint $f(A(x),1) \leq \alpha^{-1}$ will always be active, and the problem is equivalent to

$$\min_{x} \ f(A(x),1) \quad \text{s.t.} \quad g_E(x) = 0,\ g_I(x) \leq 0. \tag{19}$$

□

Corollary 5.7 suggests the following computationally more appealing equivalent cast of the stability optimization program:

$$\min_{x,P} \ \mathrm{Tr}(WP) \quad \text{s.t.} \quad P = V + A(x)^T P A(x), \tag{20a}$$

$$P \succeq 0 \tag{20b}$$

$$g_E(x) = 0,\ g_I(x) \leq 0. \tag{20c}$$

Like program (15), this formulation can be addressed by nonlinear programming algorithms that require second or higher order differentiability of the problem functions, if the positivity constraint (20b) is taken care of in the same way as in (15).

Remark 1: The matrix constraint function in (20a) has only $n(n+1)/2$ independent components, due to symmetry. These determine uniquely the $n(n+1)/2$ components of the symmetric matrix variable $P \in \mathbb{R}^{n\times n}$, if $\rho(A(x)) < 1$.

Remark 2: If one wishes to force (20b) to avoid shadow solutions, one could use a nonlinear semi-definite programming (nSDP) solver [22], or one might think of working with a factorization $P = LL^T$ within an NLP solver, where L could e.g. be a Cholesky factor (which, however, introduces additional non-convexity into the problem). In our numerical computations, we did not explicitly enforce the positive definiteness constraint and just solved an NLP without the constraint (20b), which is inactive in the solution.

Remark 3: The positive definite weighting matrices V and W allow us, as usual in H_2-norm optimization, to weight the expected input disturbances by W and the output errors by V.

Remark 4: The number of Lyapunov variables grows quadratically with state dimension n of the system. If the presence of P leads to large size programs it may be preferable to work directly with the cast (17), where the Lyapunov matrix only appears implicitly in function and gradient computations.

C. *Finding Stable Initializations for H_2-Norm Optimization*

During the optimization process, and before a stable solution x has been found, we will have $\rho(A(x)) \geq 1$, so that the Lyapunov constraint (20b) may be infeasible, or meaningless shadow solutions may appear. In order to find an initially stable solution, where $P \succ 0$ can be assured, we propose to use one of the following two homotopy methods:

(i) We start by using a formulation of type (15), with some fixed $\alpha > 0$, in the hope to find a solution with $\rho_\alpha(A(x^*)) \leq 1$. If our local optimization stops at a local minimum with a value $\rho_\alpha(A(x^*)) > 1$, we decrease α and rerun the optimizer until $\rho_\alpha(A(x^*)) \leq 1$ is found. If we do not find any such solution even for arbitrarily small $\alpha > 0$, we have strong evidence that no stable solution exists (at least locally).

(ii) Alternatively, we might choose a fixed scalar $s > \rho(A(x))$, and relax the Lyapunov constraint (20a) to $P = V + s^{-2}A(x)^T PA(x)$, so that it has a unique solution at the initial guess for x. Solution of this relaxed problem yields a matrix $A(x)$ with $\rho(A(x)) < s$, and by decreasing s and solving the problem again at each s, we push the spectral radius of the solution down, until a solution with $s = 1$ is found (if possible). This is then already the desired solution of problem (20) respectively (16).

VII. Discussion

It is well known that computing stabilizing *linear* controllers for *linear* systems can be reduced to a convex optimization problem if the order of the controller is the same as the order of the system. In this case the problem can be addressed via algebraic Riccati equations (AREs) [28], or linear matrix inequalities (LMIs) [13], and is therefore quasi-polynomial. As soon as the order of the controller is smaller than the system order, the problem is in general NP-hard [5], [23] and not accessible to convexity methods. However, non-convex local optimization approaches for stability optimization work fairly well in practice and have been discussed e.g. in [1], [2], [6], [10]–[12].

In this paper, the situation is even more general, as the unknown parameter x may include feedback elements, but may just as well encode other decision parameters that enter the matrix

$A(x)$. For instance, in the stable walking experiment in Section VIII, there is no feedback and x regroups design parameters of the open-loop system. In this case it is particularly advantageous to have a smooth criterion for stability that is compatible with state-of-the-art nonlinear optimal control methods as e.g. the tool MUSCOD-II [18] used for the computations in this paper.

Interestingly, the equivalence between the smoothed spectral radius ρ_α and the H_2-norm, namely

$$\rho_\alpha(A) < 1 \quad \Leftrightarrow \quad f(A,1) < \frac{1}{\alpha} \tag{21}$$

is analogous to a very similar relationship for the pseudo-spectral abscissa $\alpha_\epsilon(A)$ of a linear continuous time system with transfer function $H(z) = (A - z\mathbb{I})^{-1}$, which is related to the H_∞-norm by

$$\alpha_\epsilon(A) < 0 \quad \Leftrightarrow \quad \|H\|_\infty < \frac{1}{\epsilon} \tag{22}$$

as nicely described and proven in [11]. Notice that (21) may be understood as a rigourous form for the intuitive statement that ρ_α is increasingly robust with increasing α. Indeed, similar to the case of the spectral abscissa, where the statement on the right of (22) can be interpreted as the distance to instability of the continuous dynamic system $\dot{x} = Ax$, we can interprete the right hand side of (21) as a statement on the distance to instability of the discrete system $x_{k+1} = Ax_k$.

VIII. Example: open-loop stable walking

In this section we will show how the method can be used to stabilize the hybrid dynamical system of a biped walking robot. The robot has two stiff legs and is powered by torques at the hip and at the ankle, the latter replacing the action of an actuated foot. Its walking motion is shown in Fig. 2. The periodic cycle consists of 2 steps, but the periodic problem considered in the optimal control problem formulation includes one step and the touchdown discontinuity combined with a leg shift.

The model of this robot involves smooth continuous swing phases as well as discrete events at heelstrike in the form of velocity discontinuities. A detailed description of the robot is given in [21]. Stability is very easy to picture in this case. A stable robot persists in its periodic gait, while an unstable robot falls to the ground after a very short time. The task of stability optimization is to determine robot parameters (such as geometry and mass distributions) and actuation pattern

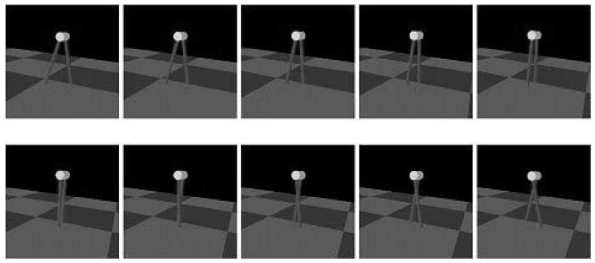

Fig. 2. One step of the symmetric walking cycle of the stiff-legged robot example

that lead to the best possible open-loop stable gaits, not requiring any feedback, which would substantially complicate the set-up due to the need of appropriate sensors and actuators.

For the solution of this problem, we use formulation (20) with weighting matrices $V = W = I$. The dimensions of this problem are as follows: The state space of this robot is of dimension $n = 4$, so that all matrices A, P, V, W are in $\mathbb{R}^{4\times4}$. The sensitivities $Y(t)$ live also in $\mathbb{R}^{4\times4}$ so that the state space of the optimal control problem has dimension $n+n^2 = 20$. There are 8 unknown mechanical parameters in the robot model, meaning dim$(p) = 8$, and, as explained above, there are an additional $n(n+1)/2 = 10$ unknown entries for the Lyapunov matrix P, leading to a total of 18 parameters. To this we add the control u with dimension $n_u = 2$, which is discretized in time on 50 intervals, leading to another 100 degrees of freedom for optimization. Similarly, the state vector is discretized into $(n+n^2) * 50 = 1000$ variables. Finally, this example features exactly two phases, so that dim$(\tau) = 2$, which leads to a total of 1120 variables.

We started from a previously determined stable solution x_0 with spectral radius $\rho = 0.53$, with the intention to further increase its stability. We could therefore use formulation (20) directly without starting the outlined homotopy method described in Section VI-C. The control variables u are related to the energy input of the robot. Since in the case of robots not only stability but also energy consuption is important, we added a small regularization term $\int_0^T u(t)^2 \; dt$ to the stability objective function.

We applied the direct optimal control technique described in Section IV-A, via control and multiple shooting state discretization and SQP solution of the NLP. Our method finds a locally optimal solution with spectral radius $\rho = 0.157$, which is well below the critical boundary of 1, and much lower than the starting value. The corresponding smoothed spectral radius is 1 with the value $\alpha = 14.08$, while the trace of the optimized P is 4.07109; (see (15b)). Optimal position

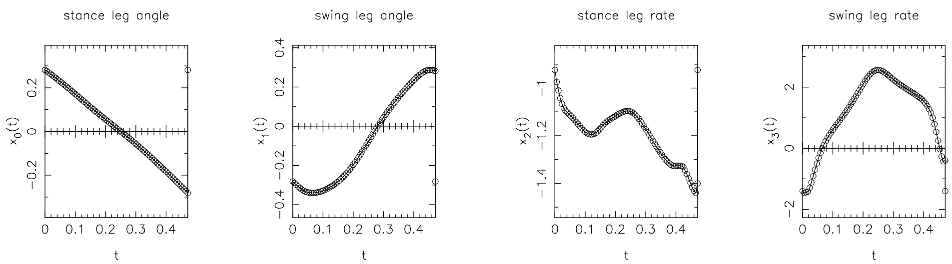

Fig. 3. Trajectories of the most stable solution (joint angles and velocities)

and velocity histories are shown in Fig. 3. [1]

IX. Conclusions

We have presented a new approach to optimize the stability of hybrid periodic systems. In contrast with previous techniques based on minimizing the spectral radius or other non-smooth criteria, we use a smooth problem formulation based on the *smoothed spectral radius*, which allows a solution of the resulting nonlinear programming problem with standard NLP techniques, such as SQP methods. Several smooth stability problem formulations have been presented. The last one was used to robustly stabilize the hybrid dynamic system of a biped walking robot performing periodic symmetric gaits. Numerical experience shows that the proposed method works very well for this challenging application and is able to considerably improve the system stability. Future work aims at investigating the robustness properties of the smoothed spectral radius and applying the approach to further applications in robotic walking and areas like chemical and power engineering (e.g. to simulated moving bed processes [25] or power generating kites [15]). Also, it might be interesting to think of generalizations of the approach to the joint spectral radius of a set of matrices [4], [24]. We note that our approach was already generalized to continuous time systems yielding the “smoothed spectral abscissa” [27].

[1] Parameters of the optimal solution are: leg mass $m = 1\mathrm{kg}$, leg length $l = 0.2\mathrm{m}$, relative center of mass location $c = 0.25$, hip spring and damper constants $k_1 = 0\mathrm{Nm}$ (i.e. no spring), $b_1 = 0.046\mathrm{Nms}$, ankle spring constant $k_2 = 4.51\mathrm{Nm}$, spring offset $\Delta_2 = -0.11$, and damper constant $b_2 = 0.331\mathrm{Nms}$. The initial values are $x^T = (0.346, -0.346, -0.751, -1.24)$, and the cycle time is $T = 0.578\mathrm{s}$.

ACKNOWLEDGEMENTS

The first author thanks B. Houska and W. Michiels for fruitful discussions, and acknowledges financial support by the Research Council KUL (Optimization in Engineering Center OPTEC), the FWO via G.0320.08 and G.0558.08, and the Belgian Federal Science Policy Office: IUAP P6/04 (DYSCO). The second author was supported by the State of Baden-Württemberg within the Margarete von Wrangell program. The third author was supported by grants from Agence Nationale de la Recherche (ANR) under contract *Guidage*, by Fondation d'entreprise EADS under contract *Solving challenging problems in feedback control*, and by Agence Nationale de la Recherche (ANR) under contract *Controvert*.

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