PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 4, April 1989

SLIDING HUMP TECHNIQUE AND SPACES WITH THE WILANSKY PROPERTY

DOMINIKUS NOLL AND WOLFGANG STADLER

(Communicated by John B. Conway)

ABSTRACT. We prove that if E is a BK-AK-space whose dual E' as well is BK-AK, then $\sigma(E', F)$ and $\sigma(E', \overline{F})$ have the same convergent sequences whenever F is a subspace of E'' containing Φ and satisfying $F^\beta = E^\beta$. This extends a result due to Bennett [B_2] and the second author [S]. We provide new examples of BK-spaces having the Wilansky property. We show that the bidual E'' of a solid BK-AK-space E whose dual as well is BK-AK satisfies a separable version of the Wilansky property. This extends a theorem of Bennett and Kalton, who proved that l^∞ has the separable Wilansky property.

INTRODUCTION

G. Bennett [B_2] and the second author [S] have independently obtained a positive answer to the following question of Wilansky: Is c_0 the only FK-space, densely containing Φ, whose β-dual is l^1? Both approaches are essentially based on a characterization of the barrelledness of certain sequence spaces by means of their β-duals. In the present paper we extend the Bennett/Stadler result, providing more examples of BK-spaces having the Wilansky property (in the sense introduced in [B_2]).

Let us explain the situation by considering a typical example. The classical sliding hump argument (Toeplitz/Schur) asserts that $\sigma(l^1, c_0)$-bounded sets are $\| \|_1$-bounded. The Bennett/Stadler result generalizes this to the extent that still $\sigma(l^1, E)$-bounded sets are $\| \|_1$-bounded, when $\Phi \subset E \subset c_0$ and $E^\beta = l^1$. The latter may be expressed equivalently by saying that every subspace E of c_0 containing Φ and having $E^\beta = l^1$ is barrelled. Finally, our present attempt shows that $\sigma(l^1, E)$-bounded sets are $\| \|_1$-bounded when $\Phi \subset E \subset l^\infty$ and $E^\beta = l^1$. Actually, we prove a little more. We show that $\sigma(l^1, E)$ and $\sigma(l^1, \overline{E})$ have the same convergent sequences in case $\Phi \subset E \subset l^\infty$ and $E^\beta = l^1$. This extension requires a modified technique, since both the approaches in [B_2] and

Received by the editors December 16, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 40H05.
Key words and phrases. Sliding hump technique, BK-spaces, spaces with the Wilansky property.

[S] make use of the sectional convergence in E (when $E \subset c_0$), and the latter is no longer available (when $E \subset l^\infty$).

We obtain new classes of BK-spaces having the Wilansky property. For instance, we prove that every BK-AK-space E, such that $S(E')$ is separably complemented in E', has the Wilansky property. Here $S(E')$ denotes the space of all $y \in E'$ which have sectional convergence with respect to the norm.

We prove that the bidual E'' of a solid BK-AK-space E whose dual E' is as well BK-AK has the following separable Wilansky type property: If D is a norm dense subspace of E'' containing Φ and having $D^\beta = E^\beta$ $(= E')$, then every separable FK-space F containing D must actually contain E''. When applied to the case $E = c_0$, this provides a result of Bennett and Kalton [BK_1, W, p. 259].

Notation. The sections of a sequence $x \in \omega$ are denoted by

$$P_n x = \sum_{i=1}^{n} x_i e^i ,$$

where e^i are the unit vectors. If $P_n x \to x$ $(n \to \infty)$, then x is said to have sectional convergence. If E is a BK-space, then $S(E)$ denotes the space of all $x \in E$ having sectional convergence with respect to the norm on E.

Concerning all other notions from sequence space theory, we refer to the book [W].

The main theorem

This section presents our fundamental result.

Theorem 1. *Let E be a BK-AK-space such that $S(E')$ is complemented in E' with separable complement L. Let $M = S(E')^\perp$ be the annihilator of $S(E')$ in E''. Let F be any subspace of $E^{\beta\beta}$ containing Φ and suppose $F^\beta = E^\beta$ $(= E')$. Then $\sigma(E', F + M)$ and $\sigma(E', \overline{F} + M)$ have the same convergent sequences.*

Proof. We need some preparations. We may assume that E has a monotone norm (see [W, p. 104]). Let $p_1 \colon E' \to S(E')$, $p_2 \colon E' \to L$ be the projection operators corresponding with the decomposition $E' = S(E') \oplus L$. Notice that $E'' = S(E')^\perp \oplus L^\perp$, $L' = S(E')^\perp = M$. We define norm continuous linear operators $Q_n \colon E' \to E'$, $n \in \mathbf{N}$, by

$$Q_n y = P_n \circ p_1 y + p_2 y.$$

Then we have

$$\|y - Q_n y\| = \|p_1 y - P_n \circ p_1 y\| \to 0.$$

We have to prove that $\sigma(E', F+M)$-convergent sequences are $\sigma(E', \overline{F}+M)$-convergent. To this end, it suffices to prove that every $\sigma(E', F + M)$-null sequence is bounded in norm. Indeed, suppose this has been proved for a $\sigma(E', F+M)$-null sequence (y^n), $\|y^n\| \le K$, say. Then, for $\overline{x} \in \overline{F}$ fixed and $\varepsilon > 0$ choose $x \in F$ having $\|x - \overline{x}\| < \varepsilon/K$. Then

$$|\langle \overline{x}, y^n \rangle| \le K\|x - \overline{x}\| + |\langle x, y^n \rangle| < \varepsilon$$

for $n \ge n(\varepsilon)$.

Let (y^n) be a $\sigma(E', F+M)$-null sequence and assume it is not bounded in norm, $\|y^n\| \ge n2^n$, say. Let $v^n = y^n/n$.

I. There exist strictly increasing sequences (k_j), (n_j) of integers such that the following conditions (1) and (2) are satisfied:

(1) $\|Q_{k_{j-1}} v^{n_j}\| \le 2^{-j}$, $\quad j = 1, 2, \dots$,

(2) $\|v^{n_j} - Q_{k_j} v^{n_j}\| \le 2^{-j}$, $\quad j = 1, 2, \dots$.

Suppose $k_1, \dots, k_j$ and $n_1, \dots, n_j$ have already been defined in accordance with (1) and (2). We claim that $\|Q_{k_j} v^n\| \to 0$ $(n \to \infty)$. Since (y^n) is $\sigma(E', F+M)$-null, $(p_2 y^n)$ is bounded for $\sigma(L, M)$, hence is norm bounded, hence $\|p_2 v^n\| \to 0$. On the other hand, $y^n = p_1 y^n + p_2 y^n$ implies that $(p_1 y^n)$ is $\sigma(E', F+M)$-bounded, hence $(p_1 v^n)$ is $\sigma(E', F+M)$-null, hence is coordinatewise null in view of $\Phi \subset F$. Clearly this implies $\|P_{k_j} p_1 v^n\| \to 0$, proving our claim. But now it is clear that a choice of $n_{j+1} > n_j$ satisfying (1) is possible.

Next observe that $\|v^{n_{j+1}} - Q_k v^{n_{j+1}}\| \to 0$ $(k \to \infty)$. This permits a choice of $k_{j+1} > k_j$ in accordance with (2).

II. Let $z^j = Q_{k_j} v^{n_j} - Q_{k_{j-1}} v^{n_j} = P_{k_j} p_1 v^{n_j} - P_{k_{j-1}} p_1 v^{n_j}$, and let $\alpha_j = 1/\|z^j\|$. Then (α_j) is an l^1-sequence by (1), (2). Observe that $\alpha_j z^j \to 0$ with respect to $\sigma(E', F+M)$, but $\|\alpha_j z^j\| = 1$. Therefore, a result of Pelczyński [P] guarantees the existence of a basic subsequence $(\alpha_{j_r} z^{j_r})$ of $(\alpha_j z^j)$. To simplify the reasoning in the following, we assume that $(\alpha_j z^j)$ itself is a basic sequence in E'.

III. We claim the existence of a null sequence (λ_j) such that the sequence z, defined by

$$(*) \qquad z_k = \lambda_j \alpha_j z_k^j \quad \text{for } k_{j-1} < k \le k_j,$$

is not an element of $S(E')$.

Let G denote the subspace of E' consisting of all sequences

$$z = \sum_{j=1}^{\infty} \lambda_j \alpha_j z^j,$$

where (λ_j) is in c_0 and the series converges in norm. Define a linear operator $\varphi\colon G \to c_0$ by setting

$$\varphi(z) = \varphi\left(\sum_{j=1}^{\infty} \lambda_j \alpha_j z^j\right) = (\lambda_j).$$

φ is well defined since $(\alpha_j z^j)$ is a basic sequence by assumption. We prove that φ is continuous. Let $z \in G$, $z = \sum \lambda_j \alpha_j z^j$. Then

$$\begin{aligned} |\lambda_j| = \|\lambda_j \alpha_j z^j\| &= \left\| \sum_{i=1}^{j} \lambda_i \alpha_i z^i - \sum_{i=1}^{j-1} \lambda_i \alpha_i z^i \right\| \\ &= \left\| P_{k_j} z - P_{k_{j-1}} z \right\| \leq 2\|z\|, \end{aligned}$$

the latter in view of the monotonicity of the norm on E (and thus on E'). This proves that φ is continuous.

Let $\overline{G}$ be the norm closure of G in E'. Then φ extends to a continuous, linear operator $\overline{\varphi}\colon \overline{G} \to c_0$. Let $z \in \overline{G}$, then $z = \sum \lambda_j \alpha_j z^j$ for some sequence (λ_j), since $(\alpha_j z^j)$ is a basic sequence. But notice that $\overline{\varphi}(z) = (\lambda_j)$ by a K-space argument. So actually (λ_j) is in c_0, hence $z \in G$, proving $G = \overline{G}$.

Notice that φ is a continuous injection. This proves that φ is not surjective. For supposing it were, it would be a homeomorphism by the open mapping theorem, i.e. we would have $G \approx c_0$. But this is absurd, since no separable dual space may contain a copy of c_0. So φ is not surjective. Let (λ_j) be any null sequence which is not in the range of φ. We prove that z, defined by $(*)$, is not in $S(E')$. Indeed, $z \in S(E')$ would imply $\|z - P_{k_j} z\| \to 0$ $(j \to \infty)$. But note that

$$P_{k_j} z = \sum_{i=1}^{j} \lambda_i \alpha_i z^i,$$

hence z would be in G, which was excluded. This ends step III.

IV. We prove that $(P_k z)$ is $\sigma(E', F+M)$-convergent with limit z. Indeed, let $x \in F + M$, $k \in \mathbf{N}$, $k_{j-1} < k \leq k_j$. Then we have

$$\langle x, P_k z\rangle = \sum_{i=1}^{j-1} \lambda_i \alpha_i \langle x, z^i\rangle + \lambda_j \alpha_j \langle x, P_k z^j\rangle.$$

Here the first summand converges $(k \to \infty, k_{j-1} < k \leq k_j)$ since $\langle x, z^i\rangle \to 0$ and $(\alpha_j) \in l^1$. But the second summand converges as well in view of $\lambda_j \to 0$ $(k \to \infty, k_{j-1} < k \leq k_j)$ and

$$|\alpha_j \langle x, P_k z^j\rangle| = |\langle P_k x, \alpha_j z^j\rangle| \leq \|P_k x\| \, \|\alpha_j z^j\| \leq \|x\|.$$

In view of $F^\beta = E'$ this implies $z \in E'$ and so $P_k z \to z$ in $\sigma(E', F+M)$.

Now observe that the operators Q_r are $\sigma(E', F+M)$-continuous, so $Q_r(P_k z) \to Q_r z$ $(k \to \infty)$, proving $P_r z = Q_r z$, hence $z \in S(E')$. But this contradicts step III and therefore ends the proof. $\square$

In the case where $S(E') = E'$, i.e. when E' has sectional convergence, the proof may be simplified. Here we have $M = \{0\}$, $Q_n = P_n$. This yields the following.

Corollary 1. *Let E be a BK-AK-space such that E' is as well BK-AK. Let F be a subspace of E'' containing Φ and satisfying $F^\beta = E^\beta$ $(= E')$. Then $\sigma(E', F)$ and $\sigma(E', \overline{F})$ have the same convergent sequences.* $\square$

Spaces with the Wilansky property

An FK-space E is said to have the Wilansky property if every subspace F of E satisfying $F^\beta = E^\beta$ is barrelled in E (see [B$_2$]). In [B$_2$] and [S] it is proved that every BK-AK-space E whose dual E' is as well a BK-AK-space has the Wilansky property. Here we obtain:

Theorem 2. *Let E be a BK-AK-space such that $S(E')$ has a separable complement L in E'. Let G be any FK-space having $E \subset G \subset E^{\beta\beta}$. Then G has the Wilansky property if and only if E is of finite codimension in G.*

Proof. Necessity. Suppose E is of infinite codimension in G. Let (y^n) be a linearly independent sequence in $G \backslash E$. Since $E^\beta = G^\beta$, E is barrelled as a subspace of G, hence is closed in G. But now $F = E + \text{lin}\{y^n\colon n \in \mathbf{N}\}$ is a subspace of G having $F^\beta = G^\beta$ which is not barrelled. Indeed, we may define a sequence (f_n) in G' such that f_n is 0 on $E + \text{lin}\{y^1, \dots, y^{n-1}\}$ and satisfies $f_n(y^n) = n|y^n|$ (for some continuous seminorm $|\cdot|$ on G). Then $f_n \to 0$, $\sigma(G', F)$, but (f_n) is not bounded in G'.

Sufficiency. Let F be a subspace of G with $F^\beta = G^\beta$. We may assume that F contains Φ (see [B$_2$, Theorem 1]).

Let U be a barrel in F. Since $M \cap E = \{0\}$, $M = S(E')^\perp$, the space $F \cap M$ is finite dimensional. Let S be some topological complement of $F \cap M$ in M. Let B denote the unit ball in S. Note that B is $\sigma(E'', E')$-compact, since the unit ball in $M \approx L'$ is weak $*$ compact and S is of finite codimension in M. Now let $V = U + B$. Then V^0, the $\langle E'', E' \rangle$-polar of V, is $\sigma(E', F+M)$-bounded, since V spans $F + M$. By Theorem 1, $\sigma(E', F+M)$-bounded sets are norm bounded in E', so that V^0 is norm bounded in E'. Hence V^{00} is a norm neighbourhood of 0 in E'', hence $V^{00} \cap F$ is a norm neighbourhood of 0 in F, since G (and hence F) must have the topology induced by E''. We end the proof by showing $V^{00} \cap F \subset U$. By the definition of V, we have $V^{00} = \overline{U} + B$, the closure being taken in $\sigma(E'', E')$, since B is $\sigma(E'', E')$-compact. But $V^{00} \cap F = \overline{U} \cap F$ in view of $B \cap F = \{0\}$. Since F has only finitely many dimensions "outside E", we deduce that $\overline{U} \cap F = U$, which ends the proof of Theorem 2. $\square$

Corollary 2. [B_2, § 6]. *c and cs have the Wilansky property.* □

More generally, a BK-AK-space E has the Wilansky property if $S(E')$ is of finite codimension in E', and the same is true for any G having $E \subset G \subset E^{\beta\beta}$ such that E is of finite codimension in G. In a forthcoming paper [NS], we use this fact to prove that for every invertible permanent triangular matrix A whose inverse A^{-1} is a bidiagonal matrix, the convergence domain c_A has the Wilansky property.

Remark. In Theorems 1,2, the assumption that E has separable dual may be replaced by any condition ensuring that c_0 does not embed into E'. See for instance [Kw].

Separable Wilansky property

It is clear from Theorem 2 that the bidual E'' of a BK-AK-space E whose dual E' is as well BK-AK does not have the Wilansky property unless E has finite codimension in E''. Nevertheless, the bidual space E'' satisfies some weaker Wilansky type property, which might be called the separable Wilansky property.

Theorem 3. *Let E be a solid BK-AK-space whose dual E' is as well BK-AK. Let D be a norm dense subspace of E'' containing Φ and satisfying $D^\beta = E^\beta$ $(= E')$. Then every separable FK-space F which contains D, actually contains E''.*

Proof. Let $x \in E''$ be fixed. Since D is a norm dense in E'', it is also $\tau(E'', E')$-sequentially dense in E'', i.e. there exists a sequence (x^n) in D which converges to x in $\tau(E'', E')$. We claim that $\tau(E'', E')|D = \tau(D, E')$.

Indeed, by Theorem 1, $\sigma(E', D)$ and $\sigma(E', E'')$ have the same convergent sequences, hence the same compact sets [W, p. 252]. This implies $\tau(E'', E')|D = \tau(D, E')$.

Consequently, the sequence (x^n) is Cauchy in $(D, \tau(D, E'))$. We prove that the inclusion mapping i: $(D, \tau(D, E')) \to F$ is continuous. This is a consequence of Kalton's closed graph theorem (see [BK_2, Theorem 5]), for $\sigma(E', D)$ is sequentially complete. Indeed, since $\sigma(E', D)$ and $\sigma(E', E'')$ have the same convergent sequences, they also have the same Cauchy sequences. But $\sigma(E', E'')$ is sequentially complete as a consequence of the fact that E, and hence $E' = E^\alpha$, is solid. This proves that $\sigma(E', D)$ is sequentially complete.

Since i: $(D, \tau(D, D^\beta)) \to F$ is continuous, the sequence (x^n) is Cauchy in F, and hence converges to some $\overline{x} \in F$. From K-space reasons, we have $x = \overline{x}$, proving $x \in F$. □

Certainly, in Theorem 3, the solidity of the space E may be replaced by the condition that $\sigma(E', E'')$ is sequentially complete.

Corollary 3. (*Compare* [BK_1, *Theorem 3*].) *Let F be a separable FK-space containing Φ and suppose $F \cap l^\infty$ is norm dense in l^∞. Then $l^\infty \subset F$.*

Proof. This follows from Theorem 3 and the fact that every norm dense subspace D of l^∞ satisfies $D^\beta = l^1$ (see [W, Lemma 16.3.3]). □

The result of Bennett and Kalton has been generalized by Snyder [Sn] to a nonseparable version. He proves that every FK-space F containing Φ and satisfying $F + c_0 = l^\infty$ must have $F = l^\infty$.

Scarce copies

The concept of scarce copies of sequence spaces has been introduced by Bennett [B_1]. He proves that every scarce copy of ω and l^1 is barrelled, but that all other standard sequence spaces do not have this property. For instance, c_0 does not have any barrelled scarce copy at all (see [B_1] for details). Here we obtain another negative result on the barrelledness of scarce copies.

Theorem 4. *Let E be a FK-AB-space contained in l^∞ such that $E^\gamma \subset bs$. Then E does not have any barrelled scarce copy.*

Proof. Suppose $\Sigma(E, r)$ is a barrelled scarce copy of E. This implies $\Sigma(E, r)^\beta \subset E^f = E^\gamma$, the latter since E has AB (see [W, p. 167]). Therefore $\Sigma(E, r)^\beta \subset bs$.

We prove that $\Sigma(c_0, r)$ is a barrelled scarce copy of c_0, thus obtaining a contradiction, since c_0 has no barrelled scarce copies. Since c_0 has the Wilansky property, barrelledness of $\Sigma(c_0, r)$ will be a consequence of $\Sigma(c_0, r)^\beta \subset l^1$. So let $y \notin l^1$. Since $c_0^\gamma = l^1$, there exists $x \in c_0$ such that $xy \notin bs$, hence $xy \notin \Sigma(E, r)^\beta$. Let $z \in \Sigma(E, r)$ be chosen with $xyz \notin cs$. By the definition of $\Sigma(E, r)$, there exist $z^1, \dots, z^n \in \sigma(E, r)$ having $z = z^1 + \cdots + z^n$. This implies $xyz^i \notin cs$ for some i. We claim that $xz^i \in \sigma(c_0, r) \subset \Sigma(c_0, r)$. Since $z^i \in E \subset l^\infty$, we have $xz^i \in c_0$. On the other hand,

$$c_n(xz^i) \le c_n(z^i) \le r_n$$

for every n implies $xz^i \in \sigma(c_0, r)$. This proves $y \notin \Sigma(c_0, r)^\beta$. □

References

[**B₁**] G. Bennett, *Some inclusion theorems for sequence spaces*, Pacific J. Math. **46** (1973), 17–30.

[**B₂**] ——, *Sequence spaces with small β-duals*, Math. Z. **194** (1987), 321–329.

[**BK₁**] G. Bennett and N. J. Kalton, *Addendum to "FK-spaces containing c_0"*, Duke Math. J. **39** (1972), 819–821.

[**BK₂**] ——, *Inclusion theorems for K-spaces*, Canad. J.Math. **25** (1973), 511–524.

[**Kw**] S. Kwapien, *On Banach spaces containing c_0*, Studia Math. **52** (1974), 187–188.

[**NS**] D. Noll and W. Stadler, *Zerlegung von Wachstumsbereichen und Wirkfeldern für die Verfahren bewichteter Mittel*, Manuscripta Math. **60** (1988), 197–209.

[**P**] A. Pelczyński, *A note on the paper of I. Singer: "Basic sequences and reflexivity in Banach spaces"*, Studia Math. **21** (1962), 371–374.

[**Sn**] A. K. Snyder, *A property of the embedding of c_0 in l_∞*, Proc. Amer. Math. Soc. **97** (1986), 59–60.

[S] W. Stadler, *Zu einer Frage von Wilansky*, Arch. Math. **48** (1987), 149–152.

[W] A. Wilansky, *Summability through functional analysis*, North-Holland, New York, 1984.

UNIVERSITÄT STUTTGART, MATHEMATISCHES INSTITUT B, PFAFFENWALDRING, 7000 STUTTGART 80, BRD

UNIVERSITÄT STUTTGART, MATHEMATISCHES INSTITUT A, PFAFFENWALDRING, 7000 STUTTGART 80, BRD

LINKED CITATIONS

You have printed the following article:

Sliding Hump Technique and Spaces with the Wilansky Property
Dominikus Noll; Wolfgang Stadler
Proceedings of the American Mathematical Society, Vol. 105, No. 4. (Apr., 1989), pp. 903-910.
Stable URL:
http://links.jstor.org/sici?sici=0002-9939%28198904%29105%3A4%3C903%3ASHTASW%3E2.0.CO%3B2-0

This article references the following linked citations. If you are trying to access articles from an off-campus location, you may be required to first logon via your library web site to access JSTOR. Please visit your library's website or contact a librarian to learn about options for remote access to JSTOR.

References

Sn **A Property of the Embedding of c0 in l #**
A. K. Snyder
Proceedings of the American Mathematical Society, Vol. 97, No. 1. (May, 1986), pp. 59-60.
Stable URL:
http://links.jstor.org/sici?sici=0002-9939%28198605%2997%3A1%3C59%3AAPOTEO%3E2.0.CO%3B2-H

NOTE: *The reference numbering from the original has been maintained in this citation list.*