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# Sequence spaces with separable $\gamma$-duals

By

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**Introduction.** Using the sliding hump technique, G. Bennett [2] and W. Stadler [9] independently obtained an affirmative answer to a problem posed by A. Wilansky, who asked whether $c_0$ was the only $FK$-space $E$ densely containing $\Phi$ whose $\beta$-dual is $\ell$. Actually, their approaches give us much more information. Adopting a definition from [2], an $FK$-space $E$ is said to have the Wilansky property (W) provided that every dense $FK$-subspace $F$ of $E$ satisfying $F^\beta = E^\beta$ must coincide with $E$. The Bennett/Stadler result then tells that every $BK$-$AK$-space $E$ whose dual $E'$ is as well a $BK$-$AK$-space has the Wilansky property (W).

In their papers [6, 7, 8], W. Stadler and the author have further investigated this circle of problems. We indicated that not only is the Wilansky property of a theoretical interest in sequence space theory, but also has nice applications in summability. From the summability point of view, however, it would be desirable to have criteria concerning the question when a convergence domain $c_A$ (with respect to an infinite matrix $A$) has the Wilansky property. Unfortunately, the Bennett/Stadler Theorem gives us poor information on that point, since the assumption of sectional convergence, which is a genuine part of the proofs given in [2] and [9], is quite restrictive when imposed on a null domain $(c_0)_A$ of a matrix $A$. The purpose of this paper is to overcome this difficulty by building up a sliding hump argument, which no longer makes use of sectional convergence in $E$ or $E'$. It turns out that the crucial assumption needed to make our argument work is that the $\gamma$-dual space $E^\gamma$ be separable in its intrinsic $BK$-topology.

We obtain criteria for the presence of the Wilansky property in the frame of $BK$-$AD$-spaces. When applied to convergence domains $c_A$, this tells that $c_A$ has the Wilansky property if $A$ is a perfect triangular matrix and $c_A^\beta$ is separable. In particular, this is the case for any triangular matrix $A$ having sectional convergence, so applies to the Cesàro methods $C_\alpha$, $0 \leqq \alpha \leqq 1$, the permanent methods $M_p$ of weighted means $p$, and various other matrix methods. We complete our paper by giving an example of a convergence domain $c_A$, $A$ a permanent lower triangular matrix with diagonal entries $\neq 0$, which lacks the Wilansky property (W). Since $c_A \cong c$, this also proves that the Wilansky type properties are not topological.

**Notations and preliminaries.** Our terminology is mainly based on the books [10] and [11]. In the following we list some of the notions of special interest in our paper.

The sections of a sequence $x \in \omega$ are denoted by $P_n x$, $n \in \mathbb{N}$. We freely use the concepts of $\beta$-, $\gamma$- and $f$-duality as presented in [10]. When $E$ is a $BK$-space containing $\Phi$, then $E^f$ is a quotient of $E'$ under the natural mapping $E' \to E^f$ ([10, p. 105f.]), and $E^f$ is also a $BK$-space. $E^\beta$, $E^\gamma$ are known to be $BK$-spaces under the norms $\|\ \|_\beta$, $\|\ \|_\gamma$ respectively, where

$$\|y\|_\beta = \sup_{x \in E,\, \|x\| \leq 1} \sup_{n \in \mathbb{N}} \left| \sum_{i=1}^{n} x_i y_i \right|,$$

and similarly $\|\ \|_\gamma$. We may iterate this procedure to obtain a norm $\|\ \|_{\beta\beta}$ on the $\beta$-bidual $E^{\beta\beta}$, starting with $\|\ \|_\beta$ on $E^\beta$, and similarly for the $\gamma$-duals. It is important to note here that, although $E^\beta \subset E'$ via the natural identification, the topology generated by $\|\ \|_\beta$ is generally different from (and hence strictly finer than) the topology induced by the dual norm. In the present investigation the spaces $E^\beta$, $E^\gamma$ will always be considered with their intrinsic topologies coming from the norms $\|\ \|_\beta$, $\|\ \|_\gamma$.

**1. Separable $\gamma$-duals.** In this section we introduce a somewhat technical concept which turns out to be quite useful in our investigation to follow.

Let $(z^n)$ be a sequence of vectors from $\Phi$ such that there exists a strictly increasing sequence $(k_j)$ of indices having

$$z^j = (0, \ldots, 0, z^j_{k_{j-1}+1}, \ldots, z^j_{k_j}, 0, 0, \ldots),$$

$j = 1, 2, \ldots$ Then $(z^n)$ will be called a block sequence.

Let $\zeta = (z^n)$ be a block sequence. We denote by $c_0(\zeta)$ the space

$$c_0(\zeta) = \left\{ \sum_{n=1}^{\infty} \lambda_n z^n : (\lambda_n) \in c_0 \right\},$$

summation being understood in the coordinatewise sense. In the same sense we use the notation $\ell^\infty(\zeta)$.

Definition. A $BK$-space $E$ containing $\Phi$ is called null for block sequences if for every block sequence $\zeta = (z^n)$ in $\Phi$ the relation $c_0(\zeta) \subset E$ implies $\|z^n\| \to 0$ $(n \to \infty)$ in $E$. $\square$

**Proposition 1.** *Let $E$ be a $BK$-space containing $\Phi$. Suppose that $E^\gamma$ is separable. Then it is null for block sequences.*

Proof. Let $\zeta = (z^n)$ be a block sequence having $c_0(\zeta) \subset E^\gamma$. We have to show $\|z^n\|_\gamma \to 0$ $(n \to \infty)$. Clearly it suffices to show that $\inf\{\|z^n\|_\gamma : n \in \mathbb{N}\} = 0$. Assume $\inf\{\|z^n\|_\gamma : n \in \mathbb{N}\} =: \eta > 0$.

Observe that $c_0(\zeta) \subset E^\gamma$ implies

$$\ell^\infty(\zeta) \subset c_0(\zeta)^{\gamma\gamma} \subset E^{\gamma\gamma\gamma} = E^\gamma.$$

Indeed, only the first inclusion needs explanation here, but this follows in much the same way as the classical inclusion $\ell^\infty \subset c_0^{\gamma\gamma}$.

Let us now derive the desired contradiction by showing that the subspace $\ell^\infty(\zeta)$ of $E^\gamma$ is closed and isomorphic with $\ell^\infty$, which means that $E^\gamma$ has a nonseparable subspace.

We define a mapping $\varphi: \ell^\infty(\zeta) \to \ell^\infty$ by setting

$$\varphi\left(\sum_{n=1}^{\infty} \lambda_n z^n\right) = (\lambda_n),$$

the series again being interpreted in the pointwise sense. Clearly $\varphi$ is linear and bijective. We prove that it is continuous. Indeed, we have

$$|\lambda_n| \leqq \eta^{-1} \|\lambda_n z^n\|_\gamma = \eta^{-1} \left\| \sum_{i=1}^{n} \lambda_i z^i - \sum_{i=1}^{n-1} \lambda_i z^i \right\|_\gamma$$
$$= \eta^{-1} \|P_{k_n} z - P_{k_{n-1}} z\|_\gamma \leqq 2\eta^{-1} \|z\|_\gamma,$$

where $z = \sum_{n=1}^{\infty} \lambda_n z^n$, the sequence $(k_j)$ corresponding with $(z^n)$ as in the definition of a block sequence, and where the last inequality is a consequence of the monotonity of the norm $\| \ \|_\gamma$. This actually proves the continuity of $\varphi$.

Let $G$ denote the norm closed hull of $\ell^\infty(\zeta)$ in $E^\gamma$. Then $\varphi$ extends to a continuous linear mapping $\tilde{\varphi}: G \to \ell^\infty$. Since $E^\gamma$ is a $K$-space, the elements $z$ of $G$ may still be represented as $z = \sum_i \lambda_i z^i$, and also from $K$-space reasons we deduce that $\tilde{\varphi}(z) = (\lambda_i)$ holds in this case. So $(\lambda_i)$ is in $\ell^\infty$, proving $z \in \ell^\infty(\zeta)$, which consequently is closed in $E^\gamma$. But now we may apply the open mapping theorem, and this tells us that $\varphi$ is an isomorphism, i.e. $\ell^\infty(\zeta) \cong \ell^\infty$. $\square$

Remarks. 1) Proposition 1 is not valid for $\beta$-duals $E^\beta$. Indeed, the space $cs$ is certainly a separable $\beta$-dual of some $BK$-space $E$ containing $\Phi$ (take for instance $E = \ell \oplus \operatorname{lin}\{e\}$), but it is not null for block sequences. Indeed, let $z^n = e^{2n} - e^{2n+1}$, then $(z^n)$ is a block sequence having $c_0((z^n)) \subset cs$, but $\|z^n\|_{cs} = 1$, $n \in \mathbb{N}$.

2) The above example also shows that $\ell^\infty(\zeta) \not\subset c_0(\zeta)^{\beta\beta}$ in general. Indeed, we have $c_0((z^n))^\beta = bv$ in the above case, proving that $c_0((z^n))^{\beta\beta} = cs \not\supset \ell^\infty((z^n))$.

3) Notice that nullity for block sequences may be guaranteed by any condition ensuring that $c_0$ does not embed into the space under consideration. We refer to [5] for various conditions of this kind.

**2. The main result.** In this section we present our main theorem. It is the outcome of a detailed analysis of the original proof given in [9], motivated by the necessity of avoiding the use of sectional convergence. See also [7] for a related argument.

**Theorem 1.** *Let $E$ be a $BK$-space containing $\Phi$ whose $\beta$-dual $E^\beta$ (resp. $\gamma$-dual $E^\gamma$) is null for block sequences. Let $F$ be a dense subspace of $E$ containing $\Phi$ and and satisfying $F^\beta = E^\beta$ (resp. $F^\beta \subset E^\gamma$). Then $(\Phi, \sigma(\Phi, F))$ and $(\Phi, \sigma(\Phi, E))$ have the same convergent sequences.*

Proof. Let us first treat the non-bracket part of the statement. It suffices to prove that every null sequence $(y^n)$ in $(\Phi, \sigma(\Phi, F))$ is $\| \|_\beta$-bounded. Indeed, suppose this has been shown for a $\sigma(\Phi, F)$-null sequence $(y^n)$. Then $(y^n)$ is as well bounded with respect to the dual norm $\| \|$, where $\Phi$ is now considered as a subspace of $E'$. Let us say $\|y^n\| \leqq M, n \in \mathbb{N}$. Now let $x \in E$ and $\varepsilon > 0$ be fixed. Choose $x^1 \in F$ satisfying $\|x - x^1\| < \varepsilon/2M$. Then we have

$$\begin{aligned} |\langle x, y^n\rangle| &\leqq |\langle x - x^1, y^n\rangle| + |\langle x^1, y^n\rangle| \\ &\leqq \|x - x^1\| \, \|y^n\| + |\langle x^1, y^n\rangle| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for $n \geqq n_0$.

Assume that $(y^n)$ is unbounded in norm, $\|y^n\|_\beta \geqq 2^n$, say. Now observe that, in view of $\Phi \subset F$, the sequence $(y^n)$ is coordinatewise null. This permits us to select strictly increasing sequences $(k_j)$ and $(n_j)$ of indices satisfying

$$\text{(1)} \qquad y^{n_j} = P_{k_j} y^{n_j}, \quad j = 1, 2, \ldots,$$

$$\text{(2)} \qquad \|P_{k_{j-1}} y^{n_j}\|_\beta \leqq 2^{-j}, \quad j = 1, 2, \ldots .$$

Indeed, suppose $k_1, \ldots, k_j$ and $n_1, \ldots, n_j$ have already been chosen with (1) and (2). Since $(y^n)$ is coordinatewise null, we deduce that $P_{k_j} y^n \to 0 \; (n \to \infty)$ in norm $\| \|_\beta$. This permits selecting $n_{j+1} > n_j$ satisfying condition (2). But $y^{n_{j+1}} \in \Phi$, so certainly $k_{j+1} > k_j$ in accordance with (1) exists.

Let us define the vectors $v^j = y^{n_j} - P_{k_{j-1}} y^{n_j}, j = 1, 2, \ldots$ and the scalars $\alpha_j = 1/\|v^j\|_\beta$. Clearly $(\alpha_j)$ is an $\ell$-sequence by (2) and $\|y^n\|_\beta \geqq 2^n$.

Let $z^j = \alpha_j v^j, j = 1, 2, \ldots$, then $(z^j)$ is a block sequence with $\|z^j\|_\beta = 1, j = 1, 2, \ldots$. Since $E^\beta$ is assumed to be null for block sequences, we deduce that $c_0((z^n))$ is not contained in $E^\beta$. So let us select a null sequence $(\lambda_n)$ such that $z = \sum_{n=1}^{\infty} \lambda_n z^n$, the series being understood in the coordinatewise sense, is not an element of $E^\beta$. We obtain the desired contradiction by proving that $z \in F^\beta$. So let $x \in F$ be fixed and let $k \in \mathbb{N}$. Choose $j$ satisfying $k_{j-1} < k \leqq k_j$. Then we have

$$\sum_{i=1}^{k} x_i z_i = \sum_{i=1}^{j-1} \sum_{r=k_{i-1}+1}^{k_i} \lambda_i \alpha_i x_r v_r^i + \sum_{r=k_{j-1}+1}^{k} \lambda_j \alpha_j x_r v_r^j .$$

Here the first term on the right side converges $(k \to \infty, k_{j-1} < k \leqq k_j)$ in view of the fact that $(\alpha_j) \in \ell$ and

$$\left| \sum_{r=k_{i-1}+1}^{k_i} x_r v_r^i \right| = |\langle x, y^{n_i} - P_{k_{i-1}} y^{n_i}\rangle| \leqq |\langle x, y^{n_i}\rangle| + \|x\|_{\beta\beta} 2^{-i} = o(1).$$

But the second term converges as well in view of the fact that $\lambda_j \to 0\,(k \to \infty, k_{j-1} < k \leqq k_j)$ and the estimate

$$\begin{aligned} \left| \sum_{r=k_{j-1}+1}^{k} \alpha_j x_r v_r^j \right| &= |\langle P_k x - P_{k_{j-1}} x, \alpha_j v^j \rangle| \\ &\leqq \| P_k x - P_{k_{j-1}} x \|_{\beta\beta} \| \alpha_j v^j \|_\beta \\ &\leqq 2\, \| x \|_{\beta\beta}, \end{aligned}$$

which uses the fact that the norm $\| \ \|_{\beta\beta}$ is monotone. Clearly this ends the proof of the non-bracket part of the statement.

Let us now consider the bracket part. Here the above proof may be adapted with minor changes. Replacing $\| \ \|_\beta$ by $\| \ \|_\gamma$, one has to show that every $\sigma(\Phi, F)$-null-sequence $(y^n)$ is $\| \ \|_\gamma$-bounded. This leads again to a sequence $z = \sum_i \lambda_i z^i$ with $z \notin E^\gamma$. But the above argument shows $z \in F^\beta$, hence the desired contradiction appears once more. This ends the proof of Theorem 1. □

Theorem 1 may be recast to obtain the following perhaps more convenient form, which in particular clarifies the relation of our argument to the proof given in [9].

**Corollary 1.** *Let $E$ be a BK-space containing $\Phi$ such that $E^\beta$ (resp. $E^\gamma$) is null for block sequences. Let $F$ be a dense linear subspace of $E$ containing $\Phi$. Then the following statements are equivalent:*

(1) $F^\beta = E^\beta$ *(resp. $F^\gamma = E^\gamma$).*
(2) *Whenever $A$ is a lower triangular matrix having $F \subset c_A$, then $E \subset c_A$.*

Proof. The implication (1) ⇒ (2) is just a reformulation of Theorem 1. Conversely, assume (2) and let $z \in F^\beta$. This means that the lower triangular matrix $A$,

$$A = \begin{pmatrix} z_1 & & & \\ z_1 & z_2 & & 0 \\ z_1 & z_2 & z_3 & \\ \cdots & & & \end{pmatrix}$$

has $F \subset c_A$. So $E \subset c_A$, proving $z \in E^\beta$. Since equality of the $\beta$-duals always implies equality of the $\gamma$-duals, the bracket part of the implication follows as well. □

Remark. Let $E = \ell \oplus \operatorname{lin}\{e\}$, then $E^\beta = cs$ is separable, but the result of Corollary 1 is not true for $E$. This may be seen by using examples from [2] or [9], given to show that $\ell$ does not have the Wilansky property.

Notice that, in various cases, Corollary 1 gives a criterion for the question when $F^\gamma = E^\gamma$ implies $F^\beta = E^\beta$.

**3. Wilansky property.** We recall Bennett's definition given in [2]. An $FK$-space $E$ is said to have the Wilansky property (W) if every dense $FK$-subspace $F$ of $E$ satisfying $F^\beta = E^\beta$

must coincide with $E$, i.e. $F = E$. We are going to prove that a $BK$-space $E$ whose $\beta$-dual $E^\beta$ is null for block sequences has the Wilansky property (W) provided that $\Phi$ is dense in $E'$. Since $E'$ is in general not a sequence space, the latter phrase needs explanation.

Given a $BK$-space $E$, we denote by $R_E$, or just by $R$ if no confusion may occur, the norm closed linear hull of the continuous linear functionals $x \to x_n$, $n \in \mathbb{N}$ in $E'$.

**Theorem 2.** *Let $E$ be a BK-space containing $\Phi$ such that $E^\beta$ is null for block sequences. Suppose $R$ is complemented in $E'$. Then $E$ has the Wilansky property* (W).

Proof. The assumption on $R$ means that $E' = R \oplus Q$ holds for some norm closed linear subspace $Q$ of $E'$.

Adopting the reasoning given in the proof of Theorem 2 in [2], it suffices to prove that every dense linear subspace $F$ of $E$ satisfying $F^\beta = E^\beta$ is barrelled in $E$. So let $F$ be of this type. We may assume that $F$ contains $\Phi$ (see the reasoning in the proof of Theorem 1 in [2]). Let $U$ be a barrel in $F$. We have to check that $U$ is a neighbourhood of 0.

Let $R^\perp$ be the annihilator of $R$ calculated in the dual pairing $\langle E'', E' \rangle$. Then $R^\perp \cap E = \{0\}$ in view of the fact that $R$ separates the points of $E$. Notice that $R^\perp \cong Q'$, the dual being calculated with respect to the dual norm on $Q$. Let $B$ be the dual unit ball in $R^\perp$, i.e. the polar of the unit ball from $Q$ in the pairing $\langle R^\perp, Q \rangle$. So $B$ is $\sigma(R^\perp, Q)$-compact, hence is $\sigma(E'', E')$-compact, the latter in view of the fact that $\sigma(E'', E')|_{R^\perp} = \sigma(R^\perp, Q)$.

Let $V = U + B$, then $V$ spans $F + R^\perp$, so $V^0$, its polar calculated in $\langle E'', E' \rangle$, is bounded with respect to the topology $\sigma(E', F + R^\perp)$ on $E'$. We claim that $V^0$ is actually bounded in the dual norm on $E'$.

Let $(y^n)$ be any sequence chosen from $V^0$. We have to prove that $(y^n)$ is bounded in norm. Since $E' = R \oplus Q$, we find sequences $(r^n)$ in $R$, $(q^n)$ in $Q$ having $y^n = r^n + q^n$. But notice that $(q^n)$ is $\sigma(Q, R^\perp)$-bounded. For let $\psi \in R^\perp$ be fixed, then we have

$$\langle \psi, q^n \rangle = \langle \psi, q^n \rangle + \langle \psi, r^n \rangle = \langle \psi, y^n \rangle = O(1).$$

Since $\sigma(Q, R^\perp) = \sigma(Q, Q')$ is the weak topology corresponding with the norm topology on $Q$, we deduce that $(q^n)$ is bounded in norm. It therefore remains proving that $(r^n)$ is also norm bounded.

Recall that $R$ is the closed linear hull of $\Phi$, when the latter is considered a subspace of $E'$ via the natural identification of $e^n \in \Phi$ with the functional $x \to x_n$. Consequently there exists a sequence $(p^n)$ in $\Phi$ having $\| p^n - r^n \| \leqq 1$ for all $n$. Since $(r^n)$ is $\sigma(E', F)$-bounded, we deduce that $(p^n)$ is bounded in $\sigma(\Phi, F)$. But now Theorem 1 comes into action. Since the topologies $\sigma(\Phi, F)$ and $\sigma(\Phi, E)$ have the same convergent sequences, they also have the same bounded sequences, so $(p^n)$ is bounded in $\sigma(\Phi, E)$, and consequently $(r^n)$ is $\sigma(E', E)$-bounded. But now the Banach-Steinhaus Theorem asserts that $(r^n)$ is bounded in norm. This proves our claim.

We have shown that $V^0$ is bounded in norm. Consequently, $V^{00}$ is a norm neighbourhood of 0 in $E''$, and therefore $V^{00} \cap F$ is a norm neighbourhood of 0 in $F$. But

$V^{00} = \bar{V} = \overline{U + B}$, where the closure refers to the topology $\sigma(E'', E')$. Since $B$ is $\sigma(E'', E')$-compact, we deduce $\bar{V} = \bar{U} + B$, hence $V^{00} \cap F = \bar{U} \cap F$. The latter, however, is just the $\sigma(F, E')$-closure of $U$ in $F$, and since $U$ was chosen weakly closed, we finally have proved $V^{00} \cap F = U$. This ends the proof of Theorem 2. $\square$

Remark. One might ask whether Theorem 2 remains valid for a $BK$-space $E$ containing $\Phi$ whose $\beta$-dual $E^\beta$ is null for block sequences and whose dual $E'$ is separable. We give an example showing that this is not the case. Let $q$ denote the space of almost periodic sequences, i.e. the closure of the space $p$ of periodic sequences in $\ell^\infty$. Then $q$ has separable dual (see [1, pp. 68–72]) and its $\beta$-dual is $\ell$. But $\ell$ is also a $\gamma$-dual, and, being separable, it is null for block sequences. Consequently, the space $E = c_0 + q$ has separable dual and its $\beta$-dual $\ell$ is null for block sequences. But $E$ does not have the Wilansky property (see [7]). For details concerning the space $q$ we refer to reference [14A] of [10].

We end this paragraph with the $\gamma$-duality version of Theorem 2. This gives a partial answer to the corresponding question posed by Bennett in [2, §7].

**Theorem 3.** *Let E be a BK-space containing $\Phi$ whose $\gamma$-dual $E^\gamma$ is null for block sequences. Suppose that $R_E$ is complemented in $E'$. Then E has the property* ($\gamma$ – W), *i.e. every dense FK-subspace F of E satisfying $F^\gamma = E^\gamma$ (or just $F^\beta \subset E^\gamma$) must coincide with E.*

Proof. The same as for Theorem 2, but the bracket part of Theorem 1 has to be used now. $\square$

**4. Consequences.** In this paragraph we present some applications of Theorem 2. We obtain conditions for $BK$-$AB$- and $BK$-$AD$-spaces ensuring the validity of the Wilansky property (W).

**Proposition 2.** *Let E be a BK-AB-space such that R is complemented in $E'$ with separable complement Q. Then E has the Wilansky property* (W).

Proof. The assumption on $R$ and $Q$ means that $E' = R \oplus Q$ is itself separable. But notice that, as a consequence of a result of W. L. C. Sargent (see [10, 10.3.11]), the $\gamma$-dual $E^\gamma$, and hence also the $\beta$-dual $E^\beta$, are closed subspaces of $E'$. This means that $E^\beta$ is null for block sequences since it does not contain any copy of $c_0$. Indeed, $c_0 \hookrightarrow E^\beta$ would imply $c_0 \hookrightarrow E'$ by the above observation. It is known, however, that a separable dual space does not contain any copy of $c_0$ (see [5]).

We have checked the assumptions made in Theorem 2. The latter therefore applies and yields the result. $\square$

Our next result presents the announced criterion for the Wilansky property in the class of $BK$-$AD$-spaces. We shall find it of importance again when dealing with perfect convergence domains.

**Proposition 3.** *Let E be a BK-AD-space whose dual $E' = E^f$ is as well a BK-AD-space. Suppose $E^\beta$ is separable. Then E has the Wilansky property.*

Proof. First notice that $AD$ implies that $E'$ may be identified with $E^f$. The $AD$ condition on $E^f$ therefore means that $R_E = E'$. So the first part of the assumptions from Theorem 2 is satisfied.

Next observe that $AD$ also implies $E^\beta = E^\gamma$ (see [10, 7.2.7]). So Proposition 1 applies and shows that $E^\beta$ is null for block sequences. Applying Theorem 2 finally gives the result. $\square$

Remark. It was shown in [2] and [9] that every $BK$-$AK$-space $E$ whose dual $E' = E^\beta$ is as well a $BK$-$AK$-space is unique among those $FK$-$AD$-spaces $F$ satisfying $F^\beta = E^\beta$. One may ask whether the corresponding result is also valid for a $BK$-$AD$-space $E$ whose dual $E' = E^f$ is also $BK$-$AD$ and whose $\beta$-dual $E^\beta$ is separable. If $E$ is not $AK$, then the answer is in the negative, for there always exists a $BK$-$AK$-space $F$ having the same $\beta$-dual $F^\beta = E^\beta$. $F$ is obtained by taking the closure of $\Phi$ in $E^{\beta\beta}$ (see [10, 10.3.23]). Clearly $F$ is the largest $FK$-$AD$-space whose $\beta$-dual is $E^\beta$. One may conjecture that $E$ is the smallest space of this kind, and Proposition 3 above tells that $E$ is certainly a minimal space of this kind, i.e. every $FK$-$AD$-space $G$ contained in $E$ and satisfying $G^\beta = E^\beta$ must have $G = E$. Moreover, Proposition 3 tells that $E$ is unique among $BK$-$AD$-spaces having $BK$-$AD$-dual and with $\beta$-dual $E^\beta$.

**5. Convergence domains.** In this section we discuss the Wilansky property in the framework of convergence domains $c_A$. We assume throughout that $A$ is a lower triangular matrix with diagonal entries $\neq 0$ and with columns in $c_0$. This means that $c_A$ is a $BK$-space containing $\Phi$, in fact $\Phi \subset (c_0)_A$.

Our first step is to clarify the size of the space $R$ in $c'_A$.

**Proposition 4.** *Let $A$ be a perfect lower triangular matrix with diagonal entries $\neq 0$ and with columns in $c_0$. Then $R = R_{c_A}$ has codimension one in $c'_A$.*

Proof. Recall that perfectness of $A$ means that $\Phi$ is dense in $(c_0)_A$, and consequently $\Phi + \operatorname{lin}\{u\}$ is dense in $c_A$ whenever $u$ is a sequence in $c_A$ satisfying $\lim_A u = 1$.

It is well-known that the dual $c'_A$ admits a representation as $\ell \oplus \mathbb{C}$ via the formula

$$\langle x, y + \eta \rangle = \sum_{i=1}^{\infty} (Ax)_i \, y_i + \eta \cdot \lim_A x,$$

$x \in c_A$, $y \in \ell$, $\eta \in \mathbb{C}$. Consequently, each of the projection functionals $x \to x_n$ is represented by some $y^n + \eta_n$ in $\ell \oplus \mathbb{C}$.

We prove that the linear hull of the vectors $y^n$, $n \in \mathbb{N}$ is just $\Phi$. First consider the vector $z^1 = a_{11}^{-1} \cdot e^1$. Then we find

$$\langle e^1, z^1 \rangle = 1, \quad \langle e^k, z^1 \rangle = 0 \quad \text{for } k = 2, 3, \ldots.$$

Therefore $y^1$ and $z^1$ agree on $\Phi$, and perfectness of $A$ implies $y^1 = z^1$. So $\operatorname{lin}\{y^1\} = \operatorname{lin}\{e^1\}$.

Next define $z^2 = a_{22}^{-1} \cdot e^2 - a_{11}^{-1} a_{22}^{-1} a_{21} \cdot e^1$, then again

$$\langle e^2, z^2 \rangle = 1, \quad \langle e^k, z^2 \rangle = 0 \quad \text{for } k \neq 2$$

obtains, proving $y^2 = z^2$ by the perfectness of $A$. Obviously, we have $\operatorname{lin}\{y^1, y^2\} = \operatorname{lin}\{e^1, e^2\}$.

Proceeding in this way, we see that the vectors $y^n$ all have length $n$ and that $\operatorname{lin}\{y^1, \ldots, y^n\} = \{e^1, \ldots, e^n\}$. Therefore the linear hull of the $y^n$ is actually $\Phi$.

Let $Q$ denote the linear hull of the $y^n + \eta_n$ in $\ell \oplus \mathbb{C}$. So $R = \bar{Q}$ via identification. Now it suffices to observe that $Q + \operatorname{lin}\{1\} = Q + \mathbb{C}$ is dense in $\ell \oplus \mathbb{C}$. But clearly $Q + \mathbb{C} = \Phi + \mathbb{C}$ as a consequence of our above observation. Since $\Phi$ is dense in $\ell$, the proof is complete. $\square$

Combining Proposition 4 with Theorem 2 yields the following criterion for the presence of the Wilansky property in the framework of convergence domains.

**Theorem 4.** *Let $A$ be a perfect lower triangular matrix with diagonal entries $\neq 0$ and with columns in $c_0$. Suppose $c_A^\beta$ is null for block sequences. Then $c_A$ has the Wilansky property. In particular, this is the case when $c_A^\beta$ is separable.*

Proof. For the first part of the statement see Theorem 2. The second part follows by observing that $c_A^\beta = (c_0)_A^\beta$ is also a $\gamma$-dual in view of the fact that $(c_0)_A$ has $AD$. $\square$

**Corollary 2.** *Let $A$ be a lower triangular matrix with diagonal entries $\neq 0$ and with columns in $c_0$. Suppose $(c_0)_A$ is an $AK$-space. Then $c_A$ has the Wilansky property.*

Proof. This is immediate from Theorem 4 since $AK$ means $(c_0)_A^\beta = (c_0)_A'$, and the latter space is known to be separable. $\square$

As a consequence of Corollary 2 we deduce that $c_A$ has the Wilansky property when $A$ is a Cesàro method $C_\alpha$, $0 \leqq \alpha \leqq 1$, a permanent method of weighted means, a Hölder method $H_\alpha$, $0 \leqq \alpha \leqq 1$ or a discrete Riesz method of index $0 \leqq \alpha \leqq 1$ (see [11]). We just mention that the Wilansky property for $C_1$ has already been established in [8], and for the methods of weighted means in [6]. In both cases the original Bennett/Stadler result was used.

We do not know whether the Cesàro methods $C_\alpha$, $\alpha > 1$ have the Wilansky property. Theorem 4 cannot be applied in this case since the $\beta$-dual spaces $c_{C_\alpha}^\beta$ are not separable for $\alpha > 1$. Also we do not know whether the summability domain $c_{C_1 S}$ of the Cesàro method $C_1$ has the Wilansky property.

**6. Concluding remarks.** We present an example of a convergence domain $c_A$ with respect to a lower triangular matrix with diagonal entries $\neq 0$ such that $c_A$ does not have the Wilansky property (W) and such that $c_A \cong c$. This proves in particular that the Wilansky property is not topological.

Let $x$ denote the sequence having $x_n = n^2$. Since $x$ is unbounded, there exists a permanent lower triangular matrix $A$ with diagonal entries $\neq 0$ whose convergence domain $c_A$ is just $c + \text{lin}\{x\}$ (see [11, S. 48]). We show that $c_A$ does not have the Wilansky property (W).

Observe that $c_A^\beta = \{y \in \omega : (n^2 y_n) \in cs\}$. Therefore, the dense $BK$-subspace $F = bv + \text{lin}\{x\}$ of $c_A$ satisfies $F^\beta = c_A^\beta$, but clearly $F \neq c_A$, proving that $c_A$ does not have the Wilansky property (W). Note that $c_A$ fulfills all the requirements of Theorem 4 except perfectness, which is therefore seen to be crucial in Theorem 4.

The above construction may be modified to obtain the following amusing result. Let $x$ be a sequence which is not convergent. Then $c + \text{lin}\{x\}$ has the Wilansky property (W) if and only if $x$ is bounded. Indeed, the necessity of this condition follows from the above reasoning, while its sufficiency has been established in [7] (one may also use Proposition 2).

One may ask for properties of the Wilansky type involving other kinds of duality. For instance, from a practical point of view, Toeplitz $\beta$-duality is of interest, for there exist various $BK$-spaces – in particular convergence domains – which have Toeplitz sectional convergence, but lack ordinary sectional convergence (see [3] and [4] for these concepts). This is the case e.g. for $c_{C_2}$ and $c_{C_1 S}$, which both have $C_1$-sectional convergence, and whose duals consequently can be identified with the factor spaces $(c_{C_2} \to c_{C_1 S})$ and $(c_{C_1 S} \to c_{C_1 S})$ correspondingly. It turns out that $c_{C_2}$ and $c_{C_1 S}$ in fact have the $(\beta_{C_1} - \text{W})$ property, the notation being adopted from [2, §7]. More generally, the following generalization of Theorem 2 is valid. Given a $BK$-space $E$ containing $\Phi$ such that $R_E$ is complemented in $E'$ and such that the Toeplitz $\gamma$-dual space $E^{\gamma_T}$ with respect to a certain Toeplitz matrix $T$ is separable. Then $E$ has the $(\beta_T - \text{W})$ property, which means that every dense $FK$-subspace $F$ of $E$ satisfying $F^{\beta_T} = E^{\beta_T}$ must coincide with $E$. This provides a positive answer to a question of Prof. G. Goes, who asked whether the Bennett/Stadler result carries over to the case of Toeplitz sectional convergence. We have been informed by Prof. Goes that a student of his, U. Böttcher, has as well obtained a positive answer to his question using a different technique.

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