Available online at www.sciencedirect.com
ScienceDirect
Automatica 43 (2007) 724–731
automatica
www.elsevier.com/locate/automatica

Brief paper

Nonsmooth optimization for multiband frequency domain control design[☆]

Pierre Apkarian[a,b,*], Dominikus Noll[c]

[a] ONERA-CERT, Centre d'études et de recherche de Toulouse, Control System Department, 2 av. Edouard Belin, 31055 Toulouse, France
[b] Université Paul Sabatier, Institut de Mathématiques, Toulouse, France
[c] Université Paul Sabatier, Institut de Mathématiques, 118, route de Narbonne, 31062 Toulouse, France

Received 7 November 2005; received in revised form 2 March 2006; accepted 25 August 2006

Abstract

Multiband frequency domain synthesis consists in the minimization of a finite family of closed-loop transfer functions on prescribed frequency intervals. This is an algorithmically difficult problem due to its inherent nonsmoothness and nonconvexity. We extend our previous work on nonsmooth H_∞ synthesis to develop a nonsmooth optimization technique to compute local solutions to multiband synthesis problems. The proposed method is shown to perform well on illustrative examples.

Keywords: H_∞-synthesis; Multidisk problems; Structured controller design; Nonsmooth optimization

1. Introduction

We present a new algorithmic approach to multifrequency band feedback control synthesis. We consider simultaneous minimization of a finite family of closed-loop performance functions

$$f(\mathscr{K}) = \max_{i=1,\dots,N} \|T_{w^i \to z^i}(\mathscr{K})\|_{I_i}, \tag{1}$$

where $\mathscr{K}$ is the feedback controller, $T_{w^i \to z^i}(\mathscr{K}, \cdot)$ the ith closed-loop performance channel, and $\|T_{w^i \to z^i}(\mathscr{K})\|_{I_i}$ the peak value of the transfer function maximum singular value norm on a prescribed frequency interval I_i:

$$\|T_{w^i \to z^i}(\mathscr{K})\|_{I_i} = \sup_{\omega \in I_i} \overline{\sigma}(T_{w^i \to z^i}(\mathscr{K}, j\omega)).$$

Typically, each I_i is a closed interval or a finite union of intervals. For a single channel, $i = 1$ and $I_1 = [0, \infty]$, minimizing $f(\mathscr{K})$ reduces to standard H_∞ synthesis.

The present approach to multiband synthesis expands on (Apkarian & Noll, 2006a,b,c; Noll & Apkarian, 2005), where this idea was laid down for standard H_∞ synthesis. It leads to efficient algorithms, because a substantial part of the computations is carried out in the frequency domain, where the plant state dimension only mildly affects cpu times. Our method avoids the difficulties of bilinear matrix inequalities, where the presence of Lyapunov variables, whose number grows quadratically with the state-space dimension, quickly leads to large size optimization programs as systems get sizable. We have identified this as the major source of breakdown for most existing codes.

Multiband control design is of great practical interest mainly for two reasons:

- Very often design criteria are expressed as frequency domain constraints on limited frequency bands.
- In the traditional approach, weighting functions are used to specify frequency bands. But the search for suitable weighting functions is often critical and increases the controller order.

Despite its importance, only few methods for multiband synthesis have been published. In Iwasaki and Hara (2005), the authors develop an extension of the KYP Lemma (Rantzer,

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Faryar Jabbari under the direction of Editor Roberto Tempo.
* Corresponding author. Tel.: +33 5 62 25 27 84; fax: +33 5 62 25 27 64.
E-mail addresses: apkarian@cert.fr (P. Apkarian), noll@mip.ups-tlse.fr (D. Noll).

0005-1098/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2006.08.031

1996) to handle band restricted frequency domain constraints. The resulting problem is nonconvex even in state-feedback.

There exist classical loop-shaping methods, like QFT (Horowitz, 1982), which exploit graphical tools and interfaces, but to work satisfactorily an advanced level of intuition is required. QFT is no longer suited under additional structural constraints on the controller.

Similar comments apply to methods based on the Youla parametrization, which lead to high-order controllers (Boyd & Barratt, 1991). Classical Bode, Nyquist and Nichols plots to design simple controllers such as PID and phase lag (Bode, 1945; Franklin, Powell, & Emami-Naeni, 1986) are limited to SISO systems, even though some generalizations to MIMO systems have been tried (MacFarlane & Postlethwaite, 1977). Altogether we believe that frequency band synthesis warrants a fresh investigation based on recent progress in optimization.

In contrast with H_∞ (Apkarian & Noll, 2006b) and multidisk syntheses (Apkarian & Noll, 2006c), multiband design leads to an additional difficulty. Closed-loop stability with controller $\mathscr{K}$ has to be built into a mathematical programming constraint. Two possibilities to model this constraint will be discussed, for more details see Apkarian and Noll (2006b, c). In the sequel each $T_{w^i \to z^i}$ is a smooth operator defined on the open domain $\mathscr{D} \subset \mathbb{R}^{(m_2+k)\times(p_2+k)}$ of kth order stabilizing controllers

$$\mathscr{K} := \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}, \quad A_K \in \mathbb{R}^{k\times k},$$

with values in the infinite dimensional space RH_∞.

2. Multiband frequency domain design

We consider a plant P in state-space form

$$P(s) : \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{2}$$

together with N concurring performance specifications, represented as plants $P^i(s)$ in state-space form as

$$P^i(s) : \begin{bmatrix} \dot{x}^i \\ z^i \\ y^i \end{bmatrix} = \begin{bmatrix} A^i & B_1^i & B_2^i \\ C_1^i & D_{11}^i & D_{12}^i \\ C_2^i & D_{21}^i & D_{22}^i \end{bmatrix} \begin{bmatrix} x^i \\ w^i \\ u^i \end{bmatrix}, \quad i = 1, \dots, N, \tag{3}$$

where $x^i \in \mathbb{R}^{n^i}$ is the state of P^i, $u^i \in \mathbb{R}^{m_2}$ the control, $w^i \in \mathbb{R}^{m_1^i}$ the vector of exogenous inputs, $y^i \in \mathbb{R}^{p_2}$ the vector of measurements and $z^i \in \mathbb{R}^{p_1^i}$ the ith performance vector. Without loss of generality, it is assumed that $D=0$ and $D_{22}^i=0$ for all i.

Multiband synthesis requires designing an output feedback controller $u^i = K(s)y^i$ for plants (3) with:

- *Internal stability*: The controller $\mathscr{K}$ stabilizes the original plant P in closed loop.
- *Performance*: Among all internally stabilizing controllers, $\mathscr{K}$ minimizes the worst case performance function $f(\mathscr{K}) = \max_{i=1,\dots,N} \|T_{w^i\to z^i}(\mathscr{K})\|_{I_i}$.

We assume that the controller K has the form

$$K(s) = C_K(sI - A_K)^{-1}B_K + D_K, \quad A_K \in \mathbb{R}^{k\times k}, \tag{4}$$

where the case $k = 0$ of a static controller is included. The synthesis problem may then be represented as

$$\begin{aligned} &\text{minimize} \quad f(\mathscr{K}) = \max_{i=1,\dots,N} \|T_{w^i\to z^i}(\mathscr{K})\|_{I_i} \\ &\text{subject to} \quad \mathscr{K} \text{ stabilizes } (A, B, C), \quad \mathscr{K} \in \mathscr{D}. \end{aligned} \tag{5}$$

Note that structural constraints on the controller are easily handled by restricting $\mathscr{K}$ to suitable subspaces.

Remark 1. A difficulty in (5) is that $\mathscr{D} = \{\mathscr{K} : \mathscr{K}$ stabilizes $(A, B, C)\}$ is not a constraint in the sense of mathematical programming. An element $\mathscr{K}$ on the boundary $\partial\mathscr{D}$ is not a valid solution. Since an optimization algorithm for (5) will converge to $\mathscr{K} \in \partial\mathscr{D}$, we have to modify this constraint to avoid numerical failure.

3. Model I: distance to instability

In this section we present a first systematic way to build a constraint which guarantees closed-loop stability. For simplicity we work with static controllers. The case of dynamic controllers simply follows from standard augmentation of the plant (Apkarian & Noll, 2006b).

We start by introducing a stabilizing channel $s \mapsto T_{\text{stab}}(\mathscr{K}, s) := (sI - (A + B\mathscr{K}C))^{-1}$ for (2). Then $\mathscr{K}$ stabilizes P in closed loop iff $T_{\text{stab}}(\mathscr{K})$ is stable. The stability domain $\mathscr{D}$ in (5) is then

$$\mathscr{D} = \{\mathscr{K} \in \mathbb{R}^{m_2\times p_2} : \|(sI - (A + B\mathscr{K}C))^{-1}\|_\infty < +\infty\}.$$

We then replace $\mathscr{D}$ by the smaller closed set

$$\mathscr{D}_b = \{\mathscr{K} \in \mathbb{R}^{m_2\times p_2} : \|(sI - (A + B\mathscr{K}C))^{-1}\|_\infty \leqslant b\},$$

where $b > 0$ is some large constant, and consider the following program:

$$\begin{aligned} &\text{minimize} \quad f(\mathscr{K}) = \max_{i=1,\dots,N} \|T_{w^i\to z^i}(\mathscr{K})\|_{I_i} \\ &\text{subject to} \quad g(\mathscr{K}) := \|T_{\text{stab}}(\mathscr{K})\|_\infty \leqslant b. \end{aligned} \tag{6}$$

The distance to instability of a stable matrix A is defined as

$$\beta(A) = \inf\{\|X\|_F : A + X \text{ instable}\}.$$

It is easy to see that

$$\beta(A) \geqslant \varepsilon \quad \Leftrightarrow \quad \|(sI - A)^{-1}\|_\infty \leqslant 1/\varepsilon.$$

That means, our natural choice is $b = 1/\beta$.

How do we solve program $(6)_b$ numerically? We consider the homotopy program

$$\min_{\mathscr{K}\in\mathbb{R}^{m_2\times p_2}} \max\left\{\max_{i=1,\dots,N} \|T_{w^i\to z^i}(\mathscr{K})\|_{I_i}, \mu\|T_{\text{stab}}(\mathscr{K})\|_\infty\right\}$$
$$= \min_{\mathscr{K}} \max\{f(\mathscr{K}), \mu g(\mathscr{K})\} =: \min_{\mathscr{K}} f_\mu(\mathscr{K}), \tag{7}$$

where $\mu > 0$ is called the homotopy parameter.

Lemma 2. *Let $\mathscr{K}_\mu$ be a local minimum of $(7)_\mu$ which is nondegenerate in the sense that it is neither a critical point of f alone, nor a critical point of g alone. Then $\mathscr{K}_\mu$ is a Karush–Kuhn–Tucker point of program $(6)_{b(\mu)}$ with $b(\mu) = g(\mathscr{K}_\mu) = f(\mathscr{K}_\mu)/\mu$.*

Proof. As $\mathscr{K}_\mu$ is nondegenerate, the necessary optimality conditions for $(7)_\mu$ give $0 < t_\mu < 1$ such that

$$0 \in t_\mu \partial f(\mathscr{K}_\mu) + (1 - t_\mu)\mu\partial g(\mathscr{K}_\mu) \quad \text{and}$$

$$f(\mathscr{K}_\mu) = \mu g(\mathscr{K}_\mu).$$

Let us now write the Karush–Kuhn–Tucker conditions for $(6)_b$: there exists a Lagrange multiplier λ_b such that

$$0 \in \partial f(\mathscr{K}_b) + \lambda_b \partial g(\mathscr{K}_b), \quad g(\mathscr{K}_b) - b \leqslant 0,$$

$$\lambda_b \geqslant 0, \quad \lambda_b(g(\mathscr{K}_b) - b) = 0.$$

We see that the solution $\mathscr{K}_\mu$ of $(7)_\mu$ solves $(6)_b$ if

$$b(\mu) = g(\mathscr{K}_\mu), \quad \lambda_{b(\mu)} = ((1 - t_\mu)\mu)/t_\mu.$$

This proves the claim. □

Lemma 3. *Let $\mathscr{K}_b$ be a local minimum of program $(6)_b$ which is nondegenerate in the sense that it is not a Karush–Kuhn–Tucker point of f alone. Then $\mathscr{K}_b$ is a critical point of program $(7)_{\mu(b)}$ with $\mu(b) = f(\mathscr{K}_b)/g(\mathscr{K}_b)$.*

Proof. We compare the necessary optimality conditions. Reading the formulas backwards, we get

$$\mu(b) = f(\mathscr{K}_b)/g(\mathscr{K}_b).$$

Then reading $\lambda_b = (1 - t_\mu)\mu/t_\mu$ backwards leads to

$$t_{\mu(b)} = \frac{\mu(b)}{\lambda_b + \mu(b)} = \frac{f(\mathscr{K}_b)}{f(\mathscr{K}_b) + \lambda_b g(\mathscr{K}_b)} \in (0, 1). \qquad \square$$

Remark 4. There is a local one-to-one correspondence between $(6)_b$ and $(7)_\mu$ in the sense that $\mathscr{K}_b = \mathscr{K}_{\mu(b)}$ and $\mathscr{K}_\mu = \mathscr{K}_{b(\mu)}$. To find $\mathscr{K}_b$ for $b = \beta^{-1}$ it suffices to find $\mu(b) = \mu(\beta^{-1})$ and solve $(7)_{\mu(\beta^{-1})}$. Using $(7)_\mu$ to solve model $(6)_b$ is basically a homotopy method, because the parameter $b(\mu)$ is gradually driven toward its final value b by adjusting μ. Notice that the problem may become ill-conditioned when b is chosen too large.

4. Model II: shifting poles

Let us consider a second possibility to fix a closed subset of $\mathscr{D}$ based on the shifted H_∞ norm, (Boyd & Barratt, 1991, p. 100): $\|H(\cdot)\|_{\infty,\alpha} = \|H(\cdot + \alpha)\|_\infty$. For $\alpha < 0$, condition $\|H\|_{\infty,\alpha} < +\infty$ guarantees that the poles of $H(s)$ lie to the left of $\Re s = \alpha < 0$. This means that for every $\alpha < 0$, the closure $\overline{\mathscr{D}}^\alpha$ of the open domain

$$\mathscr{D}^\alpha = \{\mathscr{K} \in \mathbb{R}^{m_2\times p_2} : \|(sI - (A + B\mathscr{K}C))^{-1}\|_{\infty,\alpha} < +\infty\}$$

is a tractable constraint set, because $\overline{\mathscr{D}}^\alpha \subset \mathscr{D}$. Indeed, elements $\mathscr{K} \in \partial\mathscr{D}^\alpha$ still have $\Re\lambda \leqslant \alpha < 0$ for the poles λ of $A + B\mathscr{K}C$, hence these $\mathscr{K}$ are closed-loop stabilizing. This suggests the optimization program

$$\text{minimize} \quad f(\mathscr{K}) = \max_{i=1,\dots,N} \|T_{w^i\to z^i}(\mathscr{K})\|_{I_i}$$

$$\text{subject to} \quad \mathscr{K} \in \overline{\mathscr{D}}^\alpha. \tag{8}$$

Having prepared its rationale, let us discuss an algorithm for $(8)^\alpha$. The situation is slightly more complicated than for model I, because $\overline{\mathscr{D}}^\alpha$ is not easily represented as a constraint set in the sense of nonlinear programming. What we have, though, is a barrier function for $\mathscr{D}^\alpha$. Putting

$$h_\alpha(\mathscr{K}) = \|T_{\text{stab}}(\mathscr{K})\|_{\infty,\alpha},$$

where $T_{\text{stab}}(s) := (sI - (A + B\mathscr{K}C))^{-1}$ is the stabilizing channel for plant P, we see that

$$\mathscr{D}^\alpha = \{\mathscr{K} \in \mathbb{R}^{m_2\times p_2} : h_\alpha(\mathscr{K}) < +\infty\}.$$

We may then consider the following family of programs:

$$\min_{\mathscr{K}\in\mathbb{R}^{m_2\times p_2}} \max\{f(\mathscr{K}), \mu h_\alpha(\mathscr{K})\} =: \min_{\mathscr{K}\in\mathbb{R}^{m_2\times p_2}} f_{\mu,\alpha}(\mathscr{K}), \tag{9}$$

where $f_{\mu,a}(\mathscr{K})$ is the barrier function. We link (8) to (9).

Lemma 5. *Let $\mathscr{K}^{\mu,\alpha}$ be a local minimum of $(9)^{\mu,\alpha}$ which is nondegenerate in the sense that it is neither a critical point of f alone, nor a critical point of h_α alone. Let $\mathscr{K}^\alpha$ be an accumulation point of the sequence $\mathscr{K}^{\mu,\alpha}$ as $\mu \to 0$. Suppose $\min_{\omega\in\mathbb{R}_+} \underline{\sigma}(\mathrm{j}\omega I - (AB\mathscr{K}^a C - \alpha I))$ is attained on a finite set of frequencies. Then $\mathscr{K}^\alpha$ is a critical point of program $(8)^\alpha$.*

Proof. (1) The KKT conditions for $(8)^\alpha$ at $\mathscr{K}^\alpha$ give a subgradient $G \in \partial f(\mathscr{K}^\alpha)$ such that $-G$ is in the Clarke normal cone $N_{\overline{\mathscr{D}}^\alpha}(\mathscr{K}^\alpha)$ of $\overline{\mathscr{D}}^\alpha$ at $\mathscr{K}^\alpha$; cf. (Bonnans & Shapiro, 2000).

(2) The KKT conditions for $(9)^{\mu,\alpha}$ say that there exists $0 < t_{\mu,\alpha} < 1$ such that $f(\mathscr{K}^{\mu,\alpha}) = \mu h_\alpha(\mathscr{K}^{\mu,\alpha})$ and

$$0 \in t_{\mu,\alpha}\partial f(\mathscr{K}^{\mu,\alpha}) + (1 - t_{\mu,\alpha})\mu\partial h_\alpha(\mathscr{K}^{\mu,\alpha}). \tag{10}$$

We introduce the level sets

$$\mathscr{D}^\alpha(\mu) = \{\mathscr{K} \in \mathbb{R}^{m_2\times p_2} : h_\alpha(\mathscr{K}) \leqslant h_\alpha(\mathscr{K}^{\mu,\alpha})\}.$$

There are now two cases. Either $h_\alpha(\mathscr{K}^{\mu,\alpha}) \to \infty$ as $\mu \to 0$, or there exists a subsequence for which these values are bounded.

In the latter case, $f(\mathscr{K}^{\mu,\alpha}) = \mu h_\alpha(\mathscr{K}^{\mu,\alpha})$ gives $f(\mathscr{K}^{\mu,\alpha}) \to 0$, hence $f(\mathscr{K}^\alpha) = 0$. This case is excluded, because here $\mathscr{K}^\alpha$ is a global minimum of f alone.

(3) Now assume that $h_\alpha(\mathscr{K}^{\mu,\alpha}) \to \infty$, so that the $\mathscr{D}^\alpha(\mu)$ grow as $\mu \to 0$. Then $\cup_{\mu>0}\mathscr{D}^\alpha(\mu) = \mathscr{D}^\alpha$. By (10) there exists a subgradient $G_{\mu,\alpha} \in \partial f(\mathscr{K}^{\mu,\alpha})$ such that

$$-(1-t_{\mu,\alpha})\mu G_{\mu,\alpha}/t_{\mu,\alpha} \in \partial h_\alpha(\mathscr{K}^{\mu,\alpha}).$$

In other words, the negative subgradient $-G_{\mu,\alpha}$ of f at $\mathscr{K}^{\mu,\alpha}$ is a direction in the normal cone $N_{\mathscr{D}^\alpha(\mu)}(\mathscr{K}^{\mu,\alpha})$ to the level set $\mathscr{D}^\alpha(\mu)$ at $\mathscr{K}^{\mu,\alpha}$. Passing to a subsequence, we may assume $G_{\mu,\alpha} \to G_\alpha$. By upper semi-continuity of the Clarke (1983) subdifferential, $G_\alpha \in \partial f(\mathscr{K}^\alpha)$. We now show that G_α is in the normal cone $N_{\overline{\mathscr{D}}^\alpha}(\mathscr{K}^\alpha)$, because then the necessary optimality condition in step (1) is satisfied.

(4) Let us introduce the following function:

$$\phi_\alpha(\mathscr{K}) = \begin{cases} -h_\alpha(\mathscr{K})^{-2} & \text{if } h_\alpha(\mathscr{K}) < \infty, \\ 0 & \text{else.} \end{cases}$$

Then $\mathscr{D}^\alpha = \{\mathscr{K} : \phi_\alpha(\mathscr{K}) < 0\}$, and $\mathscr{D}^\alpha(\mu) = \{\mathscr{K} : \phi_\alpha(\mathscr{K}) \leqslant -1/h_\alpha(\mathscr{K}^{\mu,\alpha})^2\}$. Notice, however, that $\overline{\mathscr{D}}^\alpha \neq \{\mathscr{K} : \phi_\alpha(\mathscr{K}) \leqslant 0\}$. That means, we cannot directly conclude via upper semi-continuity of the Clarke subdifferential of ϕ_α, as we did for ∂f. This complicates this proof.

We show that ϕ_α is locally Lipschitz. Since h_α is locally Lipschitz, this is true inside $\mathscr{D}^\alpha$. Only points $\mathscr{K} \in \partial D^\alpha$ might cause problems. But

$$\begin{aligned}\phi_\alpha(\mathscr{K}) &= -h_\alpha(\mathscr{K})^{-2} \\ &= -1/\max_{\omega\in\mathbb{R}_+} \overline{\sigma}((\mathrm{j}\omega I - A - B\mathscr{K}C + \alpha I)^{-1})^2 \\ &= -\min_{\omega\in\mathbb{R}_+} \underline{\sigma}(\mathrm{j}\omega I - A - B\mathscr{K}C + \alpha I)^2\end{aligned}$$

and this is locally Lipschitz, because for fixed ω, the minimum eigenvalue of an Hermitian matrix is locally Lipschitz. This also shows that ϕ_α has value 0 outside $\overline{\mathscr{D}}^\alpha$, which is therefore not the level set of ϕ_α at level 0.

Using upper semi-continuity of the Clarke subdifferential $\limsup_{\mu\to 0}\partial\phi_\alpha(\mathscr{K}^{\mu,\alpha}) \subset \partial\phi_\alpha(\mathscr{K}^\alpha)$ implies

$$\limsup_{\mu\to 0} N_{\mathscr{D}^\alpha(\mu)}(\mathscr{K}^{\mu,\alpha}) \subset \Lambda_\alpha(\mathscr{K}^\alpha),$$

where $\Lambda_\alpha(\mathscr{K})$ is the convex cone generated by the compact convex set $\partial\phi_\alpha(\mathscr{K})$, because the normal cone to $\mathscr{D}^\alpha(\mu)$ is generated by the subdifferential of ϕ_α at $\mathscr{K}^{\mu,\alpha}$. Recall the difficulty: our proof is not finished because $\Lambda_\alpha(\mathscr{K}^\alpha)$ is *not* identical with the Clarke normal cone $N_{\overline{\mathscr{D}}^\alpha}(\mathscr{K}^\alpha)$ to $\overline{\mathscr{D}}^\alpha$ at $\mathscr{K}^\alpha$.

Let us show that $\Lambda_\alpha(\mathscr{K}^\alpha)$ is pointed, that is, $\Lambda_\alpha(\mathscr{K}^\alpha) \cap -\Lambda_\alpha(\mathscr{K}^\alpha) = \{0\}$. This follows as soon as we show that $\pm G \in \partial\phi_\alpha(\mathscr{K}^\alpha)$ implies $G = 0$.

By hypothesis, the minimum singular value at $\mathscr{K}^\alpha$ is attained on a finite set of frequencies. This implies that ϕ_α is Clarke regular at $\mathscr{K}^\alpha$. Hence the Clarke directional derivative coincides with the Dini directional derivative:

$$\begin{aligned}\partial\phi_\alpha(\mathscr{K}^\alpha) &= \{G : \forall D \langle G, D\rangle \leqslant \phi'_\alpha(\mathscr{K}^\alpha; D) \\ &= \lim_{t\to 0^+}\inf t^{-1}(\phi_\alpha(\mathscr{K}^\alpha + tD) - \phi_\alpha(\mathscr{K}^\alpha))\}.\end{aligned}$$

But $\phi_\alpha(\mathscr{K}^\alpha) = 0$ so for fixed $\varepsilon > 0$ we can find $t_\varepsilon > 0$ such that $\langle G, D\rangle \leqslant t_\varepsilon^{-1}\phi_\alpha(\mathscr{K}^\alpha + t_\varepsilon D) + \varepsilon \leqslant \varepsilon$, the latter since $\phi_\alpha \leqslant 0$. We have shown $\langle G, D\rangle \leqslant \varepsilon$, and since ε was arbitrary, we have $\langle G, D\rangle \leqslant 0$. Now we use the fact that $-G$ is also a subgradient. Repeating the argument gives $-\langle G, D\rangle \leqslant 0$. Altogether, $\langle G, D\rangle = 0$, and since D was arbitrary, $G = 0$.

(5) Having shown that $\Lambda_\alpha(\mathscr{K}^\alpha)$ is pointed, it follows that the convex hull of $\lim\sup_{\mu\to 0} N_{\mathscr{D}^\alpha(\mu)}(\mathscr{K}^{\mu,\alpha})$ is pointed, because by (4) it is contained in $\Lambda_\alpha(\mathscr{K}^\alpha)$. Now we use Proposition 4.1 and Theorem 2.3 in Cornet and Czarnecky (1999) to deduce that $\lim\sup_{\mu\to 0} N_{\mathscr{D}^\alpha(\mu)}(\mathscr{K}^{\mu,\alpha}) \subset N_{\overline{\mathscr{D}}^\alpha}(\mathscr{K}^\alpha)$. In the terminology of that paper, this is referred to as normal convergence. That completes the proof. □

Remark 6. (1) The above reasoning carries over to dynamic controllers via the augmentation (Apkarian & Noll, 2006b). (2) $f_{\mu,\alpha}$ in (9) and f_μ in (7) have almost identical structure, so the algorithms for both models are similar. (3) Normal convergence defined in Cornet and Czarnecky (1999) is a suitable concept to describe approximation of mathematical programs. If the constraint set is represented as the level set of a locally Lipschitz operator, normal convergence is satisfied. However, in our case, the limiting set $\overline{\mathscr{D}}^\alpha$ is not a level set, which complicates the situation. Academic counterexamples where normal convergence fails can be constructed; see Cornet and Czarnecky (1999).

The method of this section is a barrier method, because α is fixed from start, while μ is driven to 0 to give convergence. So μ plays a role similar to the barrier parameter in interior-point methods. As our experiments show, this requires a final $\mu \ll -\alpha$, so ill-conditioning may occur only when α is chosen too small.

5. Algorithms for multiband control design

In this section we develop tools and present algorithms to solve models I via $(7)_\mu$ and II via $(9)^{\mu,\alpha}$.

5.1. Subdifferential of the barrier function

Computation of the Clarke subdifferential of f_μ and $f_{\mu,\alpha}$ is central for our approach. The foundations for the results here are given in Apkarian and Noll (2006b) for the H_∞ norm and in Apkarian and Noll (2006c) for multidisk synthesis. In order to unify the presentation, we introduce a common terminology for both cases. For f_μ at fixed $\mu > 0$, we introduce a new closed-loop transfer channel:

$$T_{w^{N+1}\to z^{N+1}}(\mathscr{K}) = \mu T_{\text{stab}}(\mathscr{K}),$$

so that $f_\mu(\mathscr{K}) = \max_{i=1,\ldots,N+1}\|T_{w^i\to z^i}(\mathscr{K})\|_{I_i}$ when we set $I_{N+1} = [0, \infty]$. Similarly, for $f_{\mu,\alpha}$ at fixed $\mu > 0$, $\alpha < 0$, we introduce the $(N+1)$st channel in the form

$$T_{w^{N+1}\to z^{N+1}}(\mathscr{K},s)=\mu T_{\text{stab}}(\mathscr{K},s+\alpha),$$

so that again $f_{\mu,\alpha}(\mathscr{K})=\max_{i=1,\dots,N+1}\|T_{w^i\to z^i}(\mathscr{K})\|_{I_i}$ with $I_{N+1}=[0,\infty]$.

As formulas for the Clarke subdifferential of $f_{\mu,\alpha}$ are easily inferred from those of f_μ, we will restrict the discussion to f_μ. We introduce the simplifying closed-loop notation in state space

$$\mathscr{A}^i(\mathscr{K}):=A^i+B_2^i\mathscr{K}C_2^i,\quad \mathscr{B}^i(\mathscr{K}):=B_1^i+B_2^i\mathscr{K}D_{21}^i,$$
$$\mathscr{C}^i(\mathscr{K}):=C_1^i+D_{12}^i\mathscr{K}C_2^i,\quad \mathscr{D}^i(\mathscr{K}):=D_{11}^i+D_{12}^i\mathscr{K}D_{21}^i, \tag{11}$$

and in frequency domain

$$\begin{bmatrix} T_{w^i\to z^i}(\mathscr{K},s) & G_{12}^i(\mathscr{K},s)\\ G_{21}^i(\mathscr{K},s) & \star\end{bmatrix} := \begin{bmatrix}\mathscr{C}^i(\mathscr{K})\\ C_2^i\end{bmatrix}(sI-\mathscr{A}^i(\mathscr{K}))^{-1}[\mathscr{B}^i(\mathscr{K})\ B_2^i] + \begin{bmatrix}\mathscr{D}^i(\mathscr{K}) & D_{12}^i\\ D_{21}^i & \star\end{bmatrix}.$$

Here, for $i=N+1$, we define the plant

$$P^{N+1}(s):\begin{bmatrix}\dot{x}^{N+1}\\ z^{N+1}\\ y^{N+1}\end{bmatrix}=\begin{bmatrix}A & I & B\\ I & 0 & 0\\ C & 0 & 0\end{bmatrix}\begin{bmatrix}x^{N+1}\\ w^{N+1}\\ u^{N+1}\end{bmatrix}, \tag{12}$$

where $x^{N+1}\in\mathbb{R}^n$, n is the dimension of A, $u^{N+1}\in\mathbb{R}^{m_2}$, $w^{N+1}\in\mathbb{R}^n$, $y^{N+1}\in\mathbb{R}^{p_2}$, and $z^{N+1}\in\mathbb{R}^n$.

Let us introduce the notion of active frequencies. For a given controller $\mathscr{K}$, active channels or specifications are obtained through the index set $I_\mu(\mathscr{K})$

$$\{i\in\{1,\dots,N+1\}:\|T_{w^i\to z^i}(\mathscr{K})\|_{I_i}=f_\mu(\mathscr{K})\}. \tag{13}$$

Moreover, for each $i\in I_\mu(\mathscr{K})$, we consider the set of active frequencies

$$\Omega_\mu^i(\mathscr{K})=\{\omega\in I_i:\overline{\sigma}(T_{w^i\to z^i}(\mathscr{K},\mathrm{j}\omega))=f_\mu(\mathscr{K})\}.$$

We assume that $\Omega_\mu^i(\mathscr{K})$ is a finite set, indexed as

$$\Omega_\mu^i(\mathscr{K})=\{\omega_v^i:v=1,\dots,p^i\},\quad i\in I_\mu(\mathscr{K}). \tag{14}$$

The set of all active frequencies is $\Omega_\mu(\mathscr{K})$. Now:

Theorem 7. *Assume $\mathscr{K}$ stabilizes P^{N+1} in* (12), *i.e., $\mathscr{K}\in\mathscr{D}$. With* (13) *and* (14), *let the columns of Q_v^i form an orthonormal basis of the eigenspace of $T_{w^i\to z^i}(\mathscr{K},j\omega_v^i)T_{w^i\to z^i}(\mathscr{K},j\omega_v^i)^{\mathrm{H}}$ associated with the largest eigenvalue $\overline{\sigma}(T_{w^i\to z^i}(\mathscr{K},j\omega_v^i))^2$. Then, the Clarke subdifferential of f_μ at K is the compact and convex set*

$$\partial f_\mu(\mathscr{K})=\{\Phi_Y:Y:=(Y_1^1,\dots,Y_{p^1}^1,\dots,Y_1^q,\dots,Y_{p^q}^q)\in\mathbb{B}^p\},$$

where

$$\mathbb{B}^p=\left\{(Y_i):Y_i=Y_i^{\mathrm{H}},Y_i\succcurlyeq 0,\sum_i \mathrm{Tr}(Y_i)=1\right\},$$

and $p:=\sum_{i\in I_\mu(\mathscr{K})}p^i$, q the number of elements in $I_\mu(\mathscr{K})$,

$$\Phi_Y=f_\mu(\mathscr{K})^{-1}\sum_{i\in I_\mu(\mathscr{K})}\sum_{v=1,\dots,p^i}\Re\{G_{21}^i(\mathscr{K},j\omega_v^i)\times T_{w^i\to z^i}(\mathscr{K},j\omega_v^i)^{\mathrm{H}}Q_v^iY_v^i(Q_v^i)^{\mathrm{H}}G_{12}^i(\mathscr{K},j\omega_v^i)\}^{\mathrm{T}}. \tag{15}$$

The formula also applies to $f_{\mu,\alpha}$ when suitably adapted.

Proof. The proof is based on the representation of the Clarke subdifferential of finite maximum functions (Clarke, 1983), and is omitted for brevity. The reader is referred to Apkarian and Noll (2006b, 2006c) and Noll and Apkarian (2005) for related cases. □

5.2. Solving the subproblem

We describe an extension of the nonsmooth technique developed in Apkarian and Noll, 2006a,b for H_∞ synthesis, and in Apkarian and Noll (2006c) for multidisk problems. The method is convergent and has been tested on a variety of sizable problems.

As before, we consider minimization of f_μ for fixed μ, and minimization of $f_{\mu,\alpha}$ for fixed μ, α. We define

$$f_\mu(\mathscr{K},\omega):=\max_{i=1,\dots,N+1}\{\overline{\sigma}(T_{w^i\to z^i}(\mathscr{K},\mathrm{j}\omega)):\omega\in I_i\},$$

so that $f_\mu(\mathscr{K})=\max_{\omega\in[0,\infty]}f_\mu(\mathscr{K},\omega)$. Minimizing f_μ is then a semi-infinite program for the family $f_\mu(\cdot,\omega)$. Clearly, $f_\mu(\mathscr{K},\omega)\leqslant f_\mu(\mathscr{K})$ for $\omega\in[0,\infty]$ and $f_\mu(\mathscr{K},\omega)=f_\mu(\mathscr{K})$ for $\omega\in\Omega_\mu(\mathscr{K})$, the set of active frequencies. By Theorem 7, the subdifferential of $f_\mu(\mathscr{K},\omega)$ is the set of subgradients

$$\Phi_{Y,\omega}:=f_\mu(\mathscr{K},\omega)^{-1}\sum_{i\in I_\omega(\mathscr{K})}\Re\{G_{21}^i(\mathscr{K},\mathrm{j}\omega)\times T_{w^i\to z^i}(\mathscr{K},\mathrm{j}\omega)^{\mathrm{H}}Q_\omega^iY_\omega^i(Q_\omega^i)^{\mathrm{H}}G_{12}^i(\mathscr{K},\mathrm{j}\omega)\}^{\mathrm{T}},$$

where $I_\omega(\mathscr{K})$ is the index set of active models at frequency ω:

$$\{i\in\{1,\dots,N+1\}:\omega\in I_i,\ \overline{\sigma}(T_{w^i\to z^i}(\mathscr{K},\mathrm{j}\omega))=f_\mu(\mathscr{K},\omega)\}.$$

Here the columns of the matrix Q_ω^i form an orthonormal basis of the eigenspace of $T_{w^i\to z^i}(\mathscr{K},\mathrm{j}\omega)T_{w^i\to z^i}(\mathscr{K},\mathrm{j}\omega)^{\mathrm{H}}$ associated with its largest eigenvalue, and

$$\sum_{i\in I_\omega(\mathscr{K})}\mathrm{Tr}\,Y_\omega^i=1,\quad Y_\omega^i=(Y_\omega^i)^{\mathrm{H}}\succcurlyeq 0.$$

An important feature of our technique is to allow finite extensions of the set of active frequencies: $\Omega_{\mathrm{e},\mu}(\mathscr{K})\supseteq\Omega_\mu(\mathscr{K})$. In Section 5.3 we show how $\Omega_{\mathrm{e},\mu}(\mathscr{K})$ is constructed. The idea is as follows: at the current $\mathscr{K}$ only a finite set of

$f_\mu(\cdot,\omega)$, $\omega \in \Omega_\mu(\mathscr{K})$ is active. Therefore, minimizing f_μ in a neighborhood of $\mathscr{K}$ is reduced to minimizing this finite family. The subgradients of f at $\mathscr{K}$ only depend on these active $f_\mu(\cdot,\omega)$, $\omega \in \Omega_\mu(\mathscr{K})$. As we move away from the current $\mathscr{K}$ to a nearby $\mathscr{K}'$, other functions $f_\mu(\mathscr{K}',\omega')$, $\omega' \notin \Omega_\mu(\mathscr{K})$, will become active, of course. If this happens too early, the descent step proposed by the local model will be poor. By choosing an enlarged set $\Omega_{e,\mu}(\mathscr{K})$, including some frequencies ω' outside $\Omega_\mu(\mathscr{K})$, we render the step from $\mathscr{K}$ to the new $\mathscr{K}'$ more robust.

For any such finite extension $\Omega_{e,\mu}(\mathscr{K})$, and for fixed $\delta > 0$, we introduce a corresponding optimality function

$$\begin{aligned}\theta_{e,\mu}(\mathscr{K}) := \inf_{H\in\mathbb{R}^{m_2\times p_2}} \sup_{\omega\in\Omega_{e,\mu}(\mathscr{K})} \sup_{\sum_{i\in I_\omega(\mathscr{K})} \operatorname{Tr} Y_\omega^i=1,\ Y_\omega^i\succcurlyeq 0} & -f_\mu(\mathscr{K}) + f_\mu(\mathscr{K},\omega) + \langle \Phi_{Y,\omega}, H\rangle \\ & + \tfrac{1}{2}\delta\|H\|_F^2. \end{aligned} \tag{16}$$

When $\Omega_{e,\mu}(\mathscr{K}) = \Omega_\mu(\mathscr{K})$, we write $\theta_\mu(\mathscr{K})$. Since $\Omega_\mu(\mathscr{K}) \subset \Omega_{e,\mu}(\mathscr{K})$, we have $\theta_\mu(\mathscr{K}) \leqslant \theta_{e,\mu}(\mathscr{K})$ for any extensions. $\theta_\mu(\mathscr{K})$ and $\theta_{e,\mu}(\mathscr{K})$ are called optimality functions because they share the following property: $\theta_{e,\mu}(\mathscr{K}) \leqslant 0$ for all $\mathscr{K}$, and $\theta_{e,\mu}(\mathscr{K}) = 0$ implies that $\mathscr{K}$ is a critical point of f_μ (Apkarian & Noll, 2006b). Similar optimality functions have been used in the work of Polak (1987, 1997) and Polak and Wardi (1982). They can be used to generate descent steps. In order to do this, we show that optimality function (16) has a tractable dual form.

Proposition 8. *The dual formula for $\theta_{e,\mu}(\mathscr{K})$ is*

$$\begin{aligned}\theta_{e,\mu}(\mathscr{K}) = \sup_{\sum_{\omega\in\Omega_{e,\mu}(\mathscr{K})}\tau_\omega=1,\ \tau_\omega\geqslant 0} \sup_{\sum_{i\in I_\omega(\mathscr{K})} \operatorname{Tr} Y_\omega^i=1,\ Y_\omega^i\succcurlyeq 0} & \times \sum_{\omega\in\Omega_{e,\mu}(\mathscr{K})} \tau_\omega (f_\mu(\mathscr{K},\omega) - f_\mu(\mathscr{K})) \\ & - \frac{1}{2\delta}\left\| \sum_{\omega\in\Omega_{e,\mu}(\mathscr{K})} \tau_\omega \Phi_{Y,\omega}\right\|_F^2. \end{aligned} \tag{17}$$

The associated optimal descent direction in the controller space is given as

$$H(\mathscr{K}) := -\frac{1}{\delta}\sum_{\omega\in\Omega_{e,\mu}(\mathscr{K})} \tau_\omega \Phi_{Y,\omega}. \tag{18}$$

Proof. The proof is essentially covered by the results in Apkarian and Noll (2006c) and is omitted for brevity. □

Remark 9. The appealing feature of the dual program (17) is that it is a small size SDP, or even a convex QP when singular values are simple. It is worth noticing that band restricted norms $\|\cdot\|_{I_i}$ and peak frequencies $\omega \in \Omega_\mu(\mathscr{K})$ are easily computed via an extension of the bisection algorithm in Boyd, Balakrishnan, and Kabamba (1989).

Proposition 8 suggests the following descent scheme for the subproblems for given $\mathscr{K}$ and μ, respectively, μ, α.

Nonsmooth descent algorithm for the subproblem:

- Fix $\delta > 0$, $0 < \vartheta < 1$, $0 < \rho < 1$.

1. *Initialization*: Find a controller $\mathscr{K}$ which stabilizes the plant P in (2).
2. *Generate frequencies*: Given the current $\mathscr{K}$, compute $f_\mu(\mathscr{K})$ and obtain active frequencies $\Omega_\mu(\mathscr{K})$. Select a finite enriched set of frequencies $\Omega_{e,\mu}(\mathscr{K})$ containing $\Omega_\mu(\mathscr{K})$.
3. *Descent direction*: Compute $\theta_{e,\mu}(\mathscr{K})$ and the solution (τ, Y) of SDP or convex QP (17). If $\theta_{e,\mu}(\mathscr{K}) = 0$, stop, because $0 \in \partial f_\mu(\mathscr{K})$. Otherwise compute descent direction $H(\mathscr{K})$ given in (18).
4. *Line search*: Find largest $t = \vartheta^k$ such that $f_\mu(\mathscr{K} + tH(\mathscr{K})) \leqslant f_\mu(\mathscr{K}) + t\rho\theta_\mu(\mathscr{K})$ and such that $\mathscr{K} + tH(\mathscr{K})$ remains stabilizing.
5. *Step*: Replace $\mathscr{K}$ by $\mathscr{K} + tH(\mathscr{K})$, increase iteration counter by one, and go back to step 2.

Remark 10. Results in Apkarian and Noll, 2006b, c can be used to prove convergence to a critical point $0 \in \partial f_\mu(\mathscr{K}_\mu)$ for fixed μ, starting from an arbitrary $\mathscr{K} \in \mathscr{D}$. Convergence of the overall scheme follows when we combine this with Lemmas 2, 3 and 5. The subproblems become ill-conditioned when μ gets too small, shown by a large number of iterations or even failure to reach criticality. This can be avoided by choosing β (in model I) and α (in model II) moderately small.

5.3. Enriched sets of frequencies

Choosing an extended set of frequencies $\Omega_{e,\mu}$ in step 2 is a key ingredient for the success of our technique and is beneficial mainly for two reasons:

- It renders the algorithm less dependent on the accuracy within which peak frequencies in Ω_μ are computed. A consequence is that the computed search direction behaves more smoothly.
- It captures more information on the frequency responses $\omega \mapsto \overline{\sigma}\left(T_{w^i\to z^i}(\mathscr{K}, \mathrm{j}\omega)\right)$ on their associated intervals I_i. This leads to better step lengths.

5.4. Combined algorithm

We assemble the elements of the previous sections into an algorithm. Here a difference between models (7) and (9) occurs. In $(7)_b$ we have to drive μ to the specific value $b(\mu) = \beta^{-1}$, where $\beta > 0$ is our prior threshold for the distance to instability (homotopy method). In model $(9)^\alpha$, we fix threshold $\alpha < 0$ for the poles λ in closed loop, that is $\Re\lambda \leqslant \alpha < 0$, but drive μ to 0 (barrier method). In both cases we start with a moderate size μ to solve $(7)_\mu$, respectively, $(9)^{\mu,\alpha}$. Then we update μ to μ^+ and use the solution $\mathscr{K}^{\mu,\alpha}$ as initial for the next subproblem. Different strategies to steer the parameter μ are discussed in the experimental section. We also discuss to what precision the early subproblems need to be solved, and how a successive refinement should be organized.

6. Numerical experiments

We consider the double integrator $G(s)=s^{-2}$, one of the most fundamental plants in control. Multiband design specifications are borrowed from George (2004) and involve sensitivity $S := (I + GK)^{-1}$ and complementary sensitivity $T := GK(I + GK)^{-1}$. Multiband constraints are

- disturbance rejection and tracking

  $$|S(\mathrm{j}\omega)| \leqslant 0.85 \quad \text{for } \omega \in I_1 := [0,\ 0.5],\ \text{rad/s},$$

- gain-phase margins

  $$|S(\mathrm{j}\omega)| \leqslant 1.30 \quad \text{for } \omega \in I_2 := [0.5,\ 2],\ \text{rad/s},$$

- bandwidth

  $$|T(\mathrm{j}\omega)| \leqslant 0.707 \quad \text{for } \omega \in I_3 := [2,\ 4],\ \text{rad/s},$$

- roll-off

  $$|w(\mathrm{j}\omega)\, T(j\omega)| \leqslant 1.0 \quad \text{for } \omega \in I_4 := [4,\ \infty],\ \text{rad/s},$$

  where $w(s)$ is the weighing function

  $$w(s) := \frac{0.2634s^2 + 1.659s + 5.333}{0.0001s^2 + 0.014s + 1}.$$

This problem is cast as a multiband H_∞ synthesis problem in form (5):

$\min\{f(\mathscr{K}) : \ \mathscr{K} \text{ stabilizes } G(s)\}$,

with the definition

$f(\mathscr{K}) := \max\{\frac{1}{0.85}\|S\|_{I_1}, \frac{1}{1.30}\|S\|_{I_2}, \frac{1}{0.707}\|T\|_{I_3}, \|w(s)T\|_{I_4}\}.$

As explained in Section 2, the stability constraint could be represented either as distance to instability constraint, using the homotopy function:

minimize $f_\mu(\mathscr{K}) := \max\{f(\mathscr{K}), \mu\|T_{\mathrm{stab}}(\mathscr{K})\|_\infty\}$

(model I) where μ is the homotopy parameter, and where $T_{\mathrm{stab}}(\mathscr{K}, s) = (sI - (A + B\mathscr{K}C))^{-1}$ is the stabilizing channel for the plant, or as a barrier approach (model II), where

minimize $f_{\mu,\alpha}(\mathscr{K}) := \max\{f(\mathscr{K}), \mu\|T_{\mathrm{stab}}(\mathscr{K})\|_{\infty,\alpha}\}$

for a threshold $\alpha < 0$, restricting poles λ of the closed-loop system to $\Re\lambda \leqslant \alpha < 0$, and for the barrier parameter $\mu > 0$. In particular, it will be interesting to see the relationship between β and $-\alpha$.

6.1. Model I: numerical difficulties with a single solve

To emphasize numerical difficulties with small homotopy parameters we report experiments for various values of β, assuming that the corresponding μ-values are known. All experiments are started from the same stabilizing $\mathscr{K}$ of order $k = 1$.

Table 1
Numerical difficulties when solving directly for μ_β

β	μ	Multiband performance				Iter
0.68	10	0.09	0.42	1.43	<u>1.44</u>	26
0.35	1	0.17	1.06	1.07	<u>2.84</u>	32[a]
0.07	0.01	<u>1.44</u>	1.17	0.25	<u>1.44</u>	> 200
1.3e − 4	1e − 3	0.20	0.51	1.13	<u>7.57</u>	5[a]

[a]Failure to achieve descent.

Table 2
Designs with algorithm models I and II

	(α, β)	Multiband performance			
Init	(−0.76, 0.26)	0.091	0.28	2.41	<u>42.37</u>
Mod. I	(−6.322e − 5, 1e − 5)	<u>0.84</u>	<u>0.84</u>	0.31	<u>0.84</u>
Mod. II	(−1.02e − 5, 7.32e − 8)	<u>0.84</u>	<u>0.84</u>	0.31	<u>0.84</u>

In model I the value $\beta = 10^{-5}$ is fixed, in model II the value $\Re\lambda = -\alpha = 10^{-5}$ is imposed.

The experiment confirms that it is not a good idea to solve program (7) directly for the “correct” value μ_β giving $b(\mu_\beta) = \beta^{-1} = b$, because numerical difficulties arise.

Column 2 in Table 1 gives those values $\mu = \mu_\beta$ needed to achieve the distance to instability β in column 1. Column 3 gives the achieved multiband performances. Column 4 gives the number of inner iterations to reach convergence. Underlined values give the final (max) multiband performance showing that adequate performance was not achieved. The conclusion of this first experiment is that a homotopy search in the parameter μ is required. Steering μ directly or too quickly to the correct value μ_β causes failure.

6.2. Design with algorithms I and II

The above difficulties can be avoided by decreasing μ gradually. In Table 2 we used the update $\mu \leftarrow \mu/3$. Our stopping test for the subproblems uses the criticality measure $\theta_{\mathrm{e},\mu}(\mathscr{K}) \leqslant 0$ in (17) and is defined as $\theta_{\mathrm{e},\mu}(\mathscr{K}) > -\varepsilon_s$ with the updating rule $\varepsilon_s \leftarrow \max(1\mathrm{e}-4, \varepsilon_s/2)$ and the initialization $\varepsilon_s = 10$. In this form, we require less computations in the early iterations, while accuracy is gradually increased as we get closer to a local solution.

Design with algorithm I: According to Section 3, we have set b to a large value, $b = 10^5$, which corresponds to the distance to instability $\beta = 10^{-5}$. The parameter μ is decreased as long as $\|(sI - (A + B\mathscr{K}C))^{-1}\|_\infty < b$. Results are given in Table 2.

Design with algorithm II: Here the strategy is different as we require a minimum stability degree using the shifted H_∞ norm in Section 4. Based on the idea $-\alpha \approx \beta$, we set $\alpha = -1\mathrm{e} - 5$. The barrier parameter μ is driven to zero with the same updating rule as long as $\mu > 1\mathrm{e} - 8$. Both algorithms I and II are initialized with the same stabilizing controller, see Table 2. “(α, β)” gives initial and final closed-loop spectral abscissa and distance to instability. The last column shows the achieved multiband performances. "μ" gives the final values of the homotopy, respectively, barrier parameter, "iter" the total number of inner iterations to meet our termination criterion. Controllers obtained with algorithms I and II both meet all design requirements since all band restricted performances are below unity. This represents 20% improvements over the results in George (2004). It is also instructive to see both techniques terminate at a nonsmooth local minimum where three among the four band restricted performances coincide.

7. Conclusion

Multiband H_∞ synthesis is a practically important problem for which convincing approaches are lacking. We have presented a new approach to this difficult problem using methods from nonsmooth optimization.

References

Apkarian, P., & Noll, D. (2006a). Controller design via nonsmooth multidirectional search. *SIAM Journal on Control and Optimization*, *44*(6), 1923–1949.

Apkarian, P., & Noll, D. (2006b). Nonsmooth H_∞ synthesis. *IEEE Transactions on Automatic Control*, *51*(1), 71–86.

Apkarian, P., & Noll, D. (2006c). Nonsmooth optimization for multidisk H_∞ synthesis. *European Journal of Control*, *12*(3), 229–244.

Bode, H. W. (1945). *Network analysis and feedback amplifier design*. New York: Van Nostrand.

Bonnans, J. F., & Shapiro, A. (2000). *Perturbation analysis of optimization problems*. Springer Series in Operations Research.

Boyd, S., Balakrishnan, V., & Kabamba, P. (1989). A bisection method for computing the H_∞ norm of a transfer matrix and related problems. *Mathematics of Control, Signals, and Systems*, *2*(3), 207–219.

Boyd, S., & Barratt, C. (1991). *Linear controller design: Limits of performance*. Englewood Chiffs, NJ: Prentice-Hall.

Clarke, F. H. (1983). *Optimization and nonsmooth analysis*. Canadian Mathematical Society Series. New York: Wiley.

Cornet, B., & Czarnecky, M.-O. (1999). Smooth normal approximations of epi-Lipschitzian subsets of $\mathbb{R}^n$. *SIAM Journal Control and Optimization*, *37*(3), 710–730.

Franklin, G. F., Powell, J. D., & Emami-Naeni, A. (1986). *Feedback control of dynamic systems*. Reading, MA: Addison-Wesley.

George, V. I. (2004). Optimum design of robust controller: Frequency domain approach using MATLAB. *Journal IE(I)*, 78–82.

Horowitz, I. (1982). Quantitative feedback theory. *IEE Proceedings*, *129-D*(6), 215–226.

Iwasaki, T., & Hara, S. (2005). Generalized KYP lemma: Unified frequency domain inequalities with design applications. *Transactions on Automatic Control*, *50*(1), 41–59.

MacFarlane, A. G. J., & Postlethwaite, I. (1977). The generalized Nyquist stability criterion and multivariable root loci. *International Journal of Control*, *25*, 81–127.

Noll, D., & Apkarian, P. (2005). Spectral bundle methods for nonconvex maximum eigenvalue functions: First-order methods. *Mathematical Programming Series B*, *104*(2), 701–727.

Polak, E. (1987). On the mathematical foundations of nondifferentiable optimization in engineering design. *SIAM Review*, *29*, 21–89.

Polak, E. (1997). Optimization: Algorithms and consistent approximations. *Applied Mathematical Sciences*, New York: Springer Verlag.

Polak, E., & Wardi, Y. (1982). A nondifferentiable optimization algorithm for the design of control systems subject to singular value inequalities over a frequency range. *Automatica*, *18*(3), 267–283.

Rantzer, A. (1996). On the Kalman–Yacubovich–Popov lemma. *Systems & Control Letters*, *28*(1), 7–10.

Pierre Apkarian received the Ph.D. degree in control engineering from the "Ecole Nationale Supérieure de l'Aéronautique et de l'Espace" (ENSAE) in 1988. He was qualified as a professor from the Université Paul Sabatier in both control engineering and applied mathematics in 1999 and 2001, respectively.
Since 1988, he is a research scientist at ONERA-CERT and associate Professor in the Maths Department of Université Paul Sabatier in Toulouse. His research interests include robust and gain-scheduling control theory, LMI and nonsmooth optimization techniques and their application in aeronautics. Pierre Apkarian has served as an associate editor for the IEEE Transactions on Automatic Control.

Dominikus Noll received his Ph.D. in 1983 and his habilitation in 1989 in mathematics from the Universität Stuttgart (Germany). Since then he has held temporary positions at Dalhousie University and the University of Waterloo (Canada), and since 1995 is professor of applied mathematics at Université Paul Sabatier in Toulouse (France). D. Noll's research interests are in continuous optimization, medical imaging, and automatic control. He is an associated editor of the journal Convex Analysis.