COMPUTATIONAL EXPERIENCE WITH AN INTERIOR POINT ALGORITHM FOR LARGE SCALE CONTACT PROBLEMS

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Abstract

In this paper we present an interior point method for large scale Signorini elastic contact problems. We study the case of an elastic body in frictionless contact with a rigid foundation. Primal and primal-dual algorithms are developed to solve the quadratic optimization problem arising in the variational formulation. Our computational study confirms the efficiency of the interior point methods for this class of optimization problems.

1 Introduction

In this paper we are interested in numerical resolution of contact problems in linear elasticity. Such problems arise in mechanical engineering, when an elastic body is in frictionless contact with a rigid foundation. Due to their importance for applications, there exists a considerable quantity of work dedicated to the numerical resolution of contact problems [3, 4, 12, 14, 1, 24, 23]. The various aspects included approximations by finite elements and the resolution of optimization problem. The use of increasingly finer meshes generates problems with a large number of variables. That is why complex techniques like domain decomposition [23, 18], multigrid methods [17] are widely used in computational mechanics.

The quadratic penalty method and projection method are to date the most popular optimization techniques for contact problems. The augmented Lagrangian method is often used. And even the Uzawa algorithm is still widely used. Domain decomposition techniques allow computations in a parallel environment. Krause and Wohlmuth [18] have tested an algorithm using an iterative Gauss-Seidel solver for

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nonlinear contact problems. Successive Over relaxation (SOR) methods with projection and Gauss-Seidel [24] are also used. Algorithms based on these methods converge very slowly when the mesh gets fine. Penalty methods generate zigzagging which fails convergence. To remedy this phenomenon, one has to take recourse to active constraint strategies. Last but not least, in all these approaches ill-conditioning occurs. This is caused by the choice of the penalty parameter. Preconditioning are proposed to deal with this insufficiency.

The application of optimization techniques in contact mechanics has been the object of many studies [1, 2, 14]. The optimization problem arising in frictionless contact is a convex program with inequality constraints. Interior point methods have proved efficient for this class of problems. Some authors have proven polynomial convergence [19, 10, 22]. Here we propose an interior point method which uses a truncated Newton technique. This approach is particularly suited for large scale problems, arising from very fine meshes. We prove global and local quadratic convergence. One of the first applications of interior point methods in mechanical contact was in shape optimization [2]. The interior approach maintains strict feasibility at each iteration, which is convenient since it guarantees non penetration. Another advantage of interior point methods is that active and inactive constraints need not be distinguished. Despite these advantages, interior point methods are still very little used for applications in mechanical contact. Kloosterman et al. [16] combined barrier methods with the augmented Lagrangian technique. Christensen et al. [6] compare potential reduction interior point method with nonsmooth Newton method in frictional contact.

This article is organized in the following way: In the first section we present the mathematical formulation of contact problem without friction. We give a classical theorem on existence and unicity of solution for the equivalent variational problem. We introduce also a primal and primal-dual method for the resolution of the optimization problem. Section 3 describes the algorithmic resolution by interior point techniques after discretization of the problem. Finally, section 4 provides numerical results.

The following notation is used throughout the paper. The scalar product on $L^2(E)$ is denoted by $\langle \cdot, \cdot \rangle_E$. $\mathbb{R}^p$, $\mathbb{R}^p_-$ denote the p-dimensional Euclidean space and the negative orthant of $\mathbb{R}^p$, respectively. The set of all $p \times p$ matrices with real entries is denoted by $\mathbb{R}^{p \times p}$. The i-th component of a vector $u \in \mathbb{R}^p$ is denoted by u_i. The diagonal matrix corresponding to a vector u is denoted by $diag(u)$ or U and the vector whose i-th component is $1/u_i$ is denoted by u^{-1} or $1/u$. Given u and v in $\mathbb{R}^p$, $u \leq v$ means $u_i \leq v_i$ for every $i = 1, \ldots, p$, uv and u/v denote the vector whose i-th component is $u_i v_i$ and u_i/v_i respectively. For a vector u, the Euclidean norm is denoted by $\| \cdot \|$ and u^T denotes the transpose vector. We denote the vector of all ones by e. Its dimension is always clear from the context.

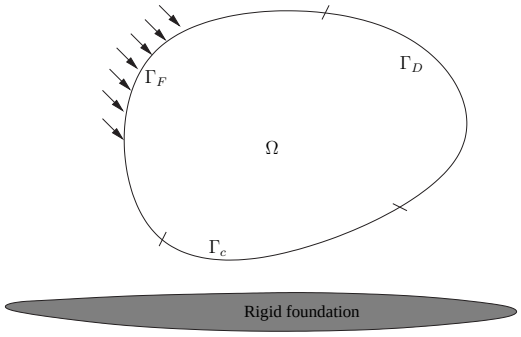


Figure 1: Elastic body coming to contact with a rigid foundation

2 Contact problem and variational formulation

Let Ω be a bounded domain of $\mathbb{R}^d$, $d = 2$ or 3 with boundary Γ. The body is unilaterally supported by a frictionless rigid foundation (see figure 1). The boundary Γ is splited into three disjoint parts, Γ_F, Γ_D and Γ_c. The portion where the body force f and the surface traction p are applied is Γ_F. The body is fixed along Γ_D and the contact surface is Γ_c. We suppose that Γ_D has positive surface measure. A displacement u is admissible if the following condition is satisfied

$$u_n - g \leq 0 \text{ on } \Gamma_c, \tag{1}$$

where g is the initial gap and n denotes the outward unit vector normal to Ω on Γ_c. This is the non penetration condition. We denote by $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^\top u)$ the linearized strain tensor and by $\sigma(u)$ the stress tensor. The mathematical formulation of the contact problem is a free boundary problem also called Signorini's problem: Find a displacement u in Ω, solution of the following system of equalities and inequalities

$$\begin{aligned}
\operatorname{div}\sigma(u) + f &= 0 && \text{in } \Omega, && (2a)\\
\sigma(u)\cdot n &= p && \text{on } \Gamma_F, && (2b)\\
u &= 0 && \text{on } \Gamma_D, && (2c)\\
\sigma_t &= 0 && \text{on } \Gamma_c, && (2d)\\
\langle u_n - g,\ \sigma_n\rangle_{\Gamma_c} &= 0 && && (2e)\\
\left.\begin{aligned} u_n - g &\leq 0\\ \sigma_n &\leq 0 \end{aligned}\right\} && && \text{on } \Gamma_c. && (2f)
\end{aligned}$$

where σ_t, σ_n denote the tangential and normal component of the stress vector on Γ_c respectively. We have $\sigma(u) = \mathcal{A}\varepsilon(u)$, where $\mathcal{A}$ is the Hooke elasticity tensor, having

classical symmetry and coercivity properties. Equation (2a) is the equilibrium condition, (2c) and (2d) are boundary conditions and (2e - f) are the unilateral contact condition. A thorough study of existence and uniqueness of solutions for the Signorini problem is given in Fichera [13]. Kikuchi and Oden [14] use convex optimization theory to solve the problem and describe numerical computations by finite element approximations. A variational formulation of (2) is considered by Kinderlehrer and Stampacchia in their book [15]. In fact, the Stampacchia's minimum theorem may be applied to the Signorini problem. Before recalling these results, let us give some useful notations and definitions. We define the Sobolev space

$$V = \{v \in H^1(\Omega)^d : v = 0 \ \text{ on } \Gamma_D\},$$

with its usual norm $\|\cdot\|_V$. The set of admissible displacements is

$$K = \{v \in V : v_n - g \leq 0 \ \text{ on } \Gamma_c\},$$

which is a closed convex subset of V. Let us denote by a and f the following bilinear and linear forms

$$\begin{aligned} a(u,v) &= \int_\Omega \sigma(u) : \varepsilon(v)\, dx \quad \forall u,\ v \in V, \\ f(v) &= \int_\Omega f v\, dx + \int_{\Gamma_F} p v\, ds \quad \forall v \in V. \end{aligned}$$

Observe that $f : V \to \mathbb{R}$ is a continuous linear functional with respect to $\|\cdot\|_V$. Similarly, the bilinear form $a : V \times V \to \mathbb{R}$ is continuous, symmetric, and V-elliptic provided that Γ_D has non empty interior in $\partial\Omega$ and the usual coercivity property of Hooke tensor due to Korn's inequality. We define the energy functional by

$$J(u) = \frac{1}{2} a(u,u) - f(u). \tag{3}$$

The contact problem is equivalent to the following variational inequality: find $u \in K$ such that

$$a(u, v-u) \geq f(v-u) \text{ for all } v \in K. \tag{4}$$

Now we are in the position to recall Stampacchia's minimum theorem.

Theorem 2.1 (Stampacchia's minimum theorem [15]). *Let $a(u,v)$ be a continuous and V-elliptic bilinear form, K closed convex and non empty. If $f \in V^*$ then there exists a unique $u \in K$ such that*

$$a(u, v-u) \geq f(v-u) \text{ for all } v \in K.$$

Moreover, if the bilinear form is symmetric, then u is characterized as being the unique minimizer of the energy functional:

$$J(u) = \min_{v \in K} J(v).$$

This theorem shows that a solution to the contact problem exists and is unique. It also establishes a relation between the Signorini problem and convex optimization theory. Thus we consider the following optimization problem :

$$\min_{v \in K} J(v). \tag{5}$$

An approximate solution can be found using suitable optimization algorithm. An approach based on projection method and penalty method is proposed in [23]. Carstensen et al [4] handle inequality constraints with a quadratic penalty function defined by

$$\varphi_\epsilon(v) = \frac{1}{2\epsilon} \int_{\Gamma_c} |v_n^+|^2 \, ds,$$

where ϵ is a penalty parameter and ${v_n}^+ = \max(0,\ v \cdot n_c - g)$. The constrained problem is transformed into an unconstrained penalty problem

$$\min_{v \in V} J_\epsilon(v) = J(v) + \varphi_\epsilon(v). \tag{6}$$

Invoking the optimality conditions for this optimization problem we obtain an equivalent cast: find u_ϵ solution to the following equation

$$a(u, v) + c_\epsilon^+(u, v) = f(v) \text{ for all } v \in V, \tag{7}$$

where $c_\epsilon^+(u, v) = \dfrac{1}{\epsilon} \langle {u_n}^+,\ v_n \rangle_{\Gamma_c}$. A solution to (7) may be computed by a homotopy approach for the family of problems depending on the parameter ϵ tending to 0, and where each individual problem is solved by Newton's method. However, ill conditioning appears in the linear systems when computing the Newton step. In the projection method, the projection on K is computationally very expensive, especially when the problem dimension is large. Other approaches based on the augmented Lagrangian have been proposed, see for instance [5]. This strategy combines the penalty method with Lagrange multipliers method. The algorithm has a main iteration including an inner step, in which the Lagrange multiplier is kept constant or supposed to be known. In the main or external iteration, the multiplier is updated. This method is the well known Uzawa algorithm. Non differentiable methods like generalized Newton are also used [1].

2.1 Barrier methods

The use of barrier methods in contact mechanics is relatively recent [16]. This approach handles the constraints using a logarithmic barrier function. The problem becomes

$$\min_{v \in V} J(v) - \epsilon \int_{\Gamma_c} \log(-c(v))\, ds \tag{8}$$

and a solution u of (8) is characterized by

$$a(u, v) - \langle \lambda(u), v_n \rangle_{\Gamma_c} = f(v) \quad \forall v \in V$$

where $\lambda(u) = \epsilon / c(u)$ is the Lagrange multiplier estimate. This equation is nonlinear and solved by Newton's method. The barrier function is defined as

$$\psi_\epsilon(v) = -\epsilon \int_{\Gamma_c} \log(-c(v))\, ds$$

Our present approach uses this formulation, also called a primal method. There are other proposals for the choice of the barrier function. For example in [16] the authors use

$$\psi_\epsilon(v) = -\epsilon \int_{\Gamma_c} \bar{\lambda} \log\left(1 + \frac{c^+(v)}{\epsilon}\right) ds$$

where $\bar{\lambda}$ is a fixed Lagrange multiplier estimate. The function $J_\epsilon(v) = J(v) + \psi_\epsilon(v)$ is continuous convex and twice differentiable. The solution of (8) converges strongly to the solution of the Signorini problem as ϵ tends to zero. Methods with logarithmic penalty terms require a feasible starting point, which may sometimes pose a problem. The cost to find such a feasible iterate may exceed that of the entire optimization process. To circumvent this difficulty, one may add slack variables and modify the constraints. The optimization problem then becomes

$$\begin{aligned} \min_{v \in V,\, q \in L^2(\Gamma_c)} \quad & J(v) + \varphi(q) \\ & v_n - g + q = 0 \end{aligned} \tag{9}$$

where φ is defined by

$$\varphi(q) = \begin{cases} 0, & \text{if } q \geq 0 \text{ a.e. } \Gamma_c; \\ +\infty, & \text{otherwise.} \end{cases}$$

The new logarithmic barrier function is

$$\psi_\epsilon(q) = -\epsilon \int_{\Gamma_c} \log(q)\, ds$$

Thus the associated barrier problem is

$$\min_{v\in V,\, q\in L^2(\Gamma_c)} J(v)+\psi_\epsilon(q)$$
$$v_n - g + q = 0$$

The drawback of this approach is that it increases the problem size. Nonetheless, for the contact problems considered here this technique is suited because the number of slack variables equals the number of contact nodes, which is one order of magnitude below the total number of decision variables. As a result, we can say that the addition of slack variables has little incidence on the problem size. Let $\mathcal{L}$ be the Lagrangian of the constrained optimization problem (5),

$$\mathcal{L}(u,\lambda) = J(u) - \langle u_n - g,\ \lambda\rangle_{\Gamma_c}$$

We know that u solves (5) if there exists $\lambda \in L^\infty(\Gamma_c)$ such that

$$a(u,v) - \langle \lambda,\, v_n\rangle_{\Gamma_c} = f(v) \quad \forall v \in V \tag{10a}$$
$$\left.\begin{aligned} \langle u_n - g, \lambda\rangle_{\Gamma_c} &= 0 \\ u_n - g &\le 0 \\ \lambda &\le 0 \end{aligned}\right\} \quad \text{on } \Gamma_c \tag{10b}$$

This system expresses the first order optimal conditions or Karush Kuhn Tucker conditions. We say that (u,λ) is a Karush Kuhn Tucker (KTT) point when it satisfies (10). Since $J(u)$ is strictly convex on K, the KKT conditions are necessary and sufficient for optimality. Thus if (u,λ) is a KKT point, then u satisfies (5). We remark a similarity between (10b) and the last term (2e) in the strong formulation of Signorini's problem . A straightforward result using Green's formula states that λ is the normal component of the stress vector on Γ_c, at least in a week sense. Thus we can say that the present method is a mixed method. The nonlinear system (10) is equivalent to

$$a(u,v) - \langle \lambda,\, v_n\rangle_{\Gamma_c} = f(v) \quad \forall v \in V \tag{11a}$$
$$\left.\begin{aligned} (u_n - g)\lambda &= 0 \\ u_n - g &\le 0 \\ \lambda &\le 0 \end{aligned}\right\} \quad \text{on } \Gamma_c \tag{11b}$$

The primal-dual interior point method consists in solving the following perturbed Karush Kuhn Tucker system

$$a(u,v) - \langle v_n,\ \lambda\rangle_{\Gamma_c} = f(v) \quad \forall v \in V \tag{12a}$$

$$\left.\begin{aligned} (u_n - g)\lambda &= \epsilon \\ u_n - g &\leq 0 \\ \lambda &\leq 0 \end{aligned}\right\} \quad \text{on } \Gamma_c \tag{12b}$$

Using a sequence of decreasing values for ϵ towards zero, the solution of this nonlinear system converges to (u, λ) [26]. Where u is the solution of (4) and λ the optimal Lagrange multiplier.

The success of the interior point methods began in linear programming. Early work proposing extensions to convex programming was published by Nestorov and Nemirovskii [22]. Polynomiality of the interior point algorithm was established in [22, 19]. Our method to solve (2) is based on an interior point algorithm for the discretized contact problem by finite element approximation.

3 Description of the interior point method

We consider a standard finite element method to discretize the convex quadratic optimization problem arising in the variational formulation. Error estimation is not addressed. This question is widely treated in the literature, see for instance [3, 4] and the references there. Let $\{\Sigma_h\}_{h>0}$ be a family of triangulations of Ω, V_h the standard conforming linear finite element space over Σ_h and $I_h : C(\bar{\Omega}_h) \to V_h$ the standard linear Lagrange interpolant. Denote by V_h and K_h the usual finite element subspaces. The approximations of the bilinear, the linear form and the energy functional are respectively given by a_h, f_h and J_h. The discrete problem is formulated as follows

$$\begin{aligned} &\text{minimize } J_h(v) = \frac{1}{2}a_h(v,\ v) - f_h(v) \\ &\text{subject to } v \in K_h \end{aligned} \tag{13}$$

Let $\{p_k,\ k = 1, \dots S\}$ be the nodes of Σ_h and $\{\varphi_k,\ k = 1, \dots S\}$ the nodal basis of V_h. Considering the properties of the nodal basis a straightforward calculation leads to the complete discretized problem

$$\begin{aligned} &\text{minimize } J_h(v) = \frac{1}{2}v^T Q v - b^T v \\ &\text{subject to } c(v) = Av \leq d,\ v \in \mathbb{R}^n \end{aligned} \tag{14}$$

where $J_h : \mathbb{R}^n \longrightarrow \mathbb{R}$, Q is a symmetric positive definite matrix of order n, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$. We always denote by $K = \{v \in \mathbb{R}^n : Av - d \leq 0\}$ the set of admissible displacements. The relative interior of K is denoted by $ri(K)$. Several studies on interior point methods in quadratic programming are available [9, 22, 19].

3.1 Primal algorithm

The main idea of interior point methods is to change the constrained problem into a succession of unconstrained problems using a logarithmic barrier penalty function. This was initially introduced by Fiacco and McCormick [10]. Let μ be a positive parameter. We define the barrier problem as follows

$$\min_{v\in\mathbb{R}} \psi(v,\mu) = J_h(v) - \mu \sum_{i=1}^{m} \log((d - Av)_i) \tag{15}$$

We define $\Lambda = diag(\lambda_i,\ i = 1,\cdots,m)$, $C = diag(c_i(x),\ i = 1,\cdots,m)$ and $e = (1,\ldots 1)$ a vector of $\mathbb{R}^m$ which has all is components equals to one. The gradient and the Hessian of the barrier objectif function are given by

$$\nabla\psi(v,\mu) = Qv - b - A^T\lambda, \quad \lambda = \mu/c(v) \tag{16}$$

$$H = \nabla^2\psi(v,\mu) = Q + A^T\Lambda C^{-1}A \tag{17}$$

Let $v(\mu)$ be an optimal solution . We define $\lambda(\mu) \in \mathbb{R}^n$ as

$$\lambda(\mu) = \mu/c(v(\mu)) \tag{18}$$

and observe that $(v(\mu), \lambda(\mu))$ satisfies the perturbed Karush-Kuhn-Tucker system

$$Qv - b - A^T\lambda = 0, \tag{19a}$$

$$\lambda_i(Av - d)_i = \mu, \quad i = 1,\cdots,m \tag{19b}$$

$$Av - d,\ \lambda \leq 0. \tag{19c}$$

In fact, as $\mu \to 0^+$, we expect $(v(\mu), \lambda(\mu))$ to converge to a Karush-Kuhn-Tucker point (v^*, λ^*) for (14). Indeed, (v^*, λ^*) is expected to satisfy the first order optimality condition. This is the primal approach with $\psi(v,\mu)$ as merit function. In the primal-dual approach, v and λ are treated as independent variables, and Newton's method is applied to the perturbed system with merit function $\phi(v,\lambda,\mu)$ given by

$$\phi(v,\lambda,\mu) = \frac{1}{2}\|F(v,\lambda,\mu)\|^2$$

where $F : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^{n+m}$ is defined by

$$F(v, \lambda, \mu) = \begin{pmatrix} Qv - b - A^T\lambda \\ C\Lambda e - \mu e \end{pmatrix} \tag{20}$$

We denote by $(v(\mu), \lambda(\mu))$ the solution of the primal-dual problem (19), $(v(\mu))_{\mu>0}$ defines the primal central path and $(v(\mu), \lambda(\mu))_{\mu>0}$ the primal dual central path. By reducing μ gradually the central path leads us towards the solution of the primal problem or a KKT point, solution of the primal dual problem.

Algorithm 3.1 (Primal interior point algorithm).

Constants ε^*, $\zeta^* > 0$, $\delta \in (0,1)$, $\gamma \in (0,1)$ *and* $\tau \in (0,1)$ *are given*
Choose $v_o \in ri(K)$ *and* ζ_o, $\mu_o > 0$

Loop. *Put counter* $k = 0$.
Initialize inner loop with $\tilde{v}_0 = v_k$.

Inner loop. *Put counter* $\ell = 0$.
Solve $H\Delta\tilde{v}_\ell = -\nabla\psi(\tilde{v}_\ell, \mu_k)$.
Use backtracking linesearch to compute $\sigma \in (0,1)$ *such that*

$\tilde{v}_\ell + \sigma\Delta\tilde{v}_\ell \in ri(K)$ *and* $\psi(\tilde{v}_\ell + \sigma\Delta\tilde{v}_\ell, \mu_k) - \psi(\tilde{v}_\ell, \mu_k) \leq \tau\sigma\Delta\tilde{v}_\ell^T\nabla\psi(\tilde{v}_\ell, \mu_k)$

Update $\tilde{v}_{\ell+1} = \tilde{v}_\ell + \sigma\Delta\tilde{v}_\ell$
if*(*$\|\nabla\psi(\tilde{v}_{\ell+1}, \mu_k)\| < \zeta_k$*)* **then stop inner loop**

continue inner loop *increase counter* ℓ.

if *(*$\|\nabla\psi(\tilde{v}_{\ell+1}, \mu_k)\| < \zeta^*$ **and** $\mu_k < \varepsilon^*$*)* **then stop**

Update $v_{k+1} = \tilde{v}_{\ell+1}$, $\mu_{k+1} = \gamma\mu_k$ *and* $\zeta_{k+1} = \delta\zeta_k$

Continue loop *increase counter* k.

The major computational load of the algorithm lies in the resolution of a linear system. The solution of this system is the Newton step denoted by Δv. We have

$$H\Delta v = -\nabla\psi(v, \mu) \tag{21}$$

From the theoretical point of view one needs to solve this system exactly in order to have a convergent algorithm. In practice one is satisfied with an inexact solution such that $\|H\Delta v + g\| \leq \eta\|g\|$, with η a positive constant selected rather small and g is the gradient of the barrier function. The constant η determines the size of the residual. Iterative methods of the Krylov family are well adapted to compute this

type of truncated solutions. For example the conjugated gradient (CG) or Lanczos method are often used. R. Felkel [9] proposes a interior points method using Lanczos with partial re-orthonormalization. Nash and Sofer [21] try an approach using the CG.

3.2 Primal-dual algorithm

The primal dual technique is more efficient than purely primal methods. We apply Newton's method to solve the nonlinear system (19) with (v, λ) as unknown variables. Given $\mu > 0$ we solve the following equivalent non-linear system

$$F(v, \lambda, \mu) = 0 \tag{22}$$

We denote by $(\Delta v, \Delta\lambda)$ the Newton step computed at each iterate. We have

$$\begin{pmatrix} \nabla^2 Q & -A^T \\ \Lambda A & C \end{pmatrix} \begin{pmatrix} \Delta v \\ \Delta\lambda \end{pmatrix} = - \begin{pmatrix} Qv - b - A^T\lambda \\ C\Lambda e - \mu e \end{pmatrix} \tag{23}$$

We can solve this system by means of direct or iterative method. In the large-scale case it may be prohibitive to use a direct solver. Iterative method like GMRES or CG can be used. In the last case the system must be symmetric. A symmetric form may be obtained by premultiplying the $(2, 1)$ block by $-\Lambda^{-1}$. In our method we applied the iterative method to the reduced system

$$H\Delta v = -Qv + b + \mu A^T C^{-1} e \tag{24a}$$

$$\Delta\lambda = -\lambda - C^{-1}\Lambda A\Delta v + \mu C^{-1} e \tag{24b}$$

where $H = Q + A^T\Lambda C^{-1}A$ is the condensed primal-dual matrix. This matrix is symmetric and positive definite. Here we give a summary of the main algorithm.

Algorithm 3.2 (Primal-dual interior point algorithm).

Constants $\varepsilon^* > 0,\ \eta,\ \tau \in (0,\ 1)$
Choose $v_0 \in ri(K),\ \lambda_0 \in \mathbb{R}^m_-,\ w_0 = (v_0, \lambda_0)$ *and* $\mu_0 > 0$

Loop . *Put counter* $k = 0$.

If $\phi(w_k, \mu_k) < \varepsilon^*$ **then stop**

Compute the Newton direction $\Delta w_k = (\Delta v_k, \Delta\lambda_k)$ *from* (24)

Use backtracking to find $\sigma \in (0, 1)$ *such that*

$$\phi(w_k + \sigma\Delta w_k, \mu_k) - \phi(w_k, \mu_k) \leq \tau\sigma\Delta w_k^T \nabla\phi(w_k, \mu_k)$$

and

$$w_k + \sigma \Delta w_k \in ri(K) \times \mathbb{R}^m_-$$

Update $w_{k+1} = w_k + \sigma \Delta v_k,\ \mu_{k+1} = \eta \mu_k$

Continue loop. *Increase counter* k.

The iterative method we use is the Lanczos method with partial reorthogonalization [11]. So we compute an approximated solution following the idea of the truncated Newton method. Let η be a positive real in (0, 1) the primal step satisfies

$$\|H\Delta v + \bar{g}\| \leq \eta \|\bar{g}\|, \quad \text{with } \bar{g} = \nabla J(v) - \mu A^T C^{-1} e \tag{25}$$

Contrary to the primal approach we do not have an inner iteration. One can thus hope to have a faster algorithm. In both cases the Newton step is computed by an iterative method like CG or Lanczos. The feasible initial point is not obvious in both cases. The easiest way to deal with this problem consists in relaxing the feasible constraints, see for instance [9]. We get the following set:

$$\mathcal{S}_\varsigma = \{v \in \mathbb{R}^n : Av - d + \varsigma \geq 0, \ \text{ with } \varsigma \geq 0\}$$

Here ς is a parameter which we decrease towards 0 along with μ. A radical solution to make available a feasible initial point is the slack variables technique.

3.3 Inexact Newton step

In this section we focus on the resolution of the barrier subproblem by iterative methods. At each iteration the barrier subproblem is solved approximately using the truncated Newton method. This is a powerful tool for solving large scale unconstrained optimization problem. An interesting reference on this subject is the paper of R. S. Dembo and T. Steihaug [8] and the recent survey [20]. The particularity of their method is the choice of iterative method for solving the primal Newton system (21) or the primal dual reduced system (24). The conjugate gradient technique is almost always used with preconditioning to speed up the convergence rate. Convergence of the method was studied by R. S. Dembo et al in [7]. Here we are interested in solving the linear system

$$\nabla^2 \psi(v, \mu)\pi = -\nabla \psi(v, \mu) \tag{26}$$

where $\pi \in \mathbb{R}^n$. The truncated Newton method computes a descent direction Δv solution of (26) that satisfies

$$\|\nabla^2 \psi(v, \mu)\Delta v + \nabla \psi(v, \mu)\| \leq \eta \|\nabla \psi(v, \mu)\|, \tag{27}$$

for a positive parameter η called forcing parameter. Then we compute $v^+ = v + \alpha\Delta v$ the new estimate of the solution. If the stopping test is verified we stop, otherwise we repeat the previous operations. The update formula for v does not guarantee feasibility. To reach this goal we compute an initial step size

$$\alpha_0 = \min(1,\ \sigma\bar{\alpha}), \quad 0 < \sigma < 1$$

where

$$\bar{\alpha} = \min\left\{ \frac{(d - Av)_i}{(A\Delta v)_i} : (A\Delta v)_i > 0 \right\} \tag{28}$$

Then the algorithm generates a sequence of strictly feasible primal points. The algorithm for the barrier subproblem stops when we have found a primal dual point $(\tilde{v},\ \tilde{y})$ such that

$$\|\nabla\psi(\tilde{v}, \mu)\| \leq \zeta \tag{29}$$

where ζ is a fixed tolerance parameter. After computing the new barrier parameter μ, the next barrier subproblem is solved in the same way as described above with $\tilde{v}$ as initial guess. The same approach can be applied to the primal-dual problem on the reduced primal-dual system (24a).

4 Numerical experiments

This section provides numerical results for a small set of test problems. We implemented in Fortran 90 the primal algorithm with the following parameters: $\varepsilon^* = 10^{-9}$, $\zeta^* = 10^{-9}$, $\tau = 0.01$, $\delta = 0.5$ and $\gamma = 0.5$. The initial values were $\zeta_0 = 1500$, $\mu_0 = 1$. If the number of iterations in the truncated Newton method is less than 3 in three successive steps of the main iteration we reduce γ by the rule

$$\gamma_{k+1} = \max\{0.54, 0.99\gamma_k\}$$

If the number of inner iterations (truncated Newton method) per main iteration exceeds 3

$$\gamma_{k+1} = \min\{0.99, 1.05\gamma_k\}$$

and we set

$$\mu_{k+1} = \gamma_{k+1}\mu_k$$

The problems were generated by the finite element tools GETFEM++ [25] for the computation of linearly elastic structures in contact with a rigid foundation. Physical parameters are given by d the dimension of the problem. The discretized parameters are given by κ the degree of Lagrange finite element and n_x the number of spatial steps. We generate the problems for different values of d, κ and n_x. These

Problems	d	κ	n_x	n	m
A	2	1	4	50	5
B	3	1	4	375	25
C	3	2	4	1995	81
D	3	2	8	13203	289
E	3	1	10	3993	121
F	3	3	10	24783	441
G	2	1	20	882	21
H	2	2	20	3362	41
I	2	3	20	7742	61
J	2	1	100	20402	101
K	2	1	50	5202	51
L	3	1	20	27783	441
M	2	1	10	242	11

Table 1: Set of problems

problems sorted in the size from 50 to 27783 variables and from 5 to 441 constraints (see Table 1).

In table 2, ngr, $nfct$ and it are respectively the number of gradient evaluations, the number of fonction evaluations and the total number of main iterations. Generally, the number of function evaluations is equal to the number of gradient evaluations. In our tests the number of main iterations did not exceed 19. The largest part of cpu time is used to solve the barrier subproblem. At the beginning, one iteration is usually enough to solve this subproblem, but the number of iterations increases gradually and reaches a peak, then decreases when a neighborhood of attraction of the solution is approached. An illustration of this phenomenon for problems J, K, G and M is given by Figure 2. The results obtained for the total time of resolution shows rather disparate characteristics (see Table 2). The time obtained for problem L, which has 27783 variables, is less than that necessary to solve the problem J with 20402 variables. It is also noted that the algorithm is slower on problem I. One makes the same remark with problem D. These instabilities are related to the conditioning of the stiffness matrix. The condition number of the stiffness matrix is classically of order $\mathcal{O}(n_x^2)$. Problems L and J were solved for the same values of κ ($\kappa = 1$), but the condition number of L is better than that of J. Conditioning also depends on the value of the degree of Lagrange finite elements. If κ increases, the conditioning becomes worst. This should explains the result observed for I and D.

Problems	n	m	ngr	$nfct$	it	optimal value	cpu times
A	50	5	45	84	19	-0.1407047	0.09
B	375	25	59	62	19	-0.0597285072	0.39
C	1995	81	42	42	14	-0.372374397	11.14
D	13203	289	47	47	15	-0.753245045	177.11
E	3993	121	49	49	16	-0.350213401	15.17
F	24783	441	*	*	*	*	*
G	882	21	82	82	18	-0.755060513	1.58
H	3362	41	171	171	16	-1.89351901	38.0199
I	7442	61	330	330	15	-3.05307295	448.19
J	20402	101	224	224	14	-4.15773494	490.660
K	5202	51	134	134	15	-2.03452994	56.889
L	27783	441	49	49	16	-0.819527298	167.669
M	242	11	67	67	18	-0.339076629	0.30

Table 2: Numerical results

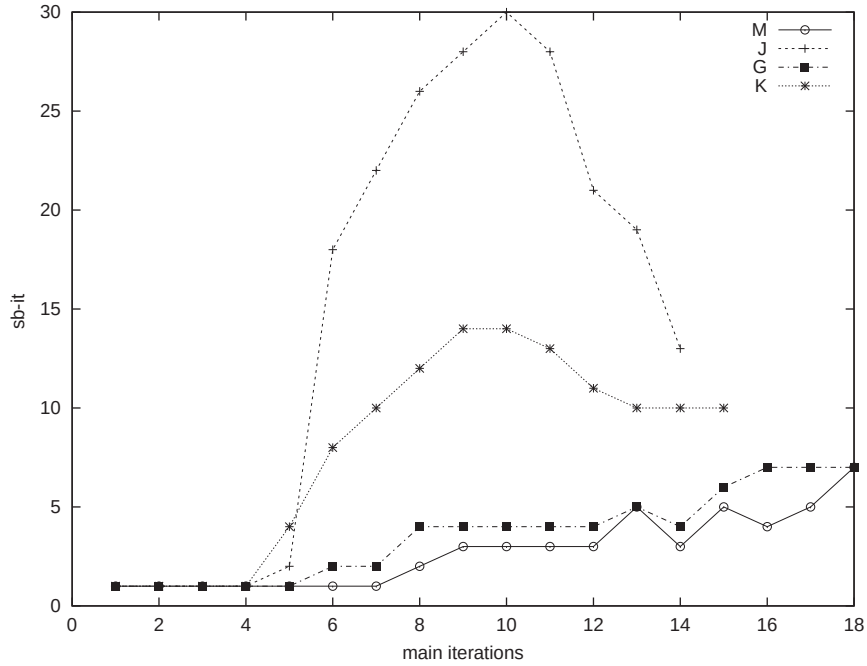
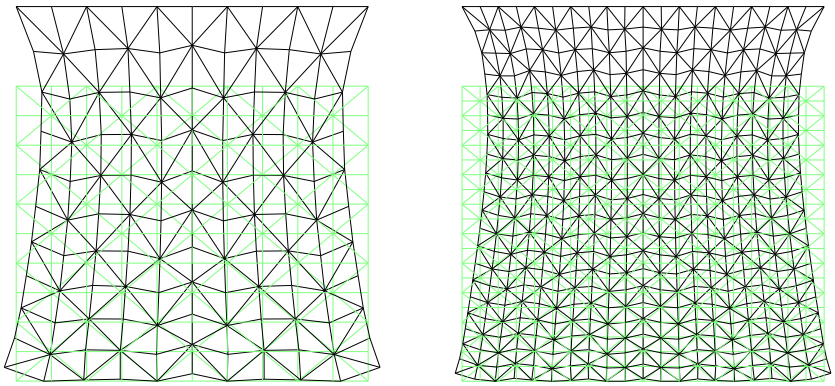


Figure 2: Number of inner iterations for different main iterations



(a) $n_x = 10$ (b) $n_x = 20$

(c) $n_x = 50$ (d) $n_x = 100$

Figure 3: Deformed configurations : An elastic solid is submitted to an artificial non constant volumic force in order to have a serie of contact zone. The problem's dimension and the degree of Lagrange finit element are fixed ($d = 2$ and $\kappa = 1$). In (a) deformation for problem M ($n_x = 10$), (b) deformation for problem G ($n_x = 20$), (c) deformation for problem K ($n_x = 50$) and (d) deformation for problem J ($n_x = 100$).

5 Conclusion

We have proposed an interior point algorithm for the contact problem. The non-linear optimization program arising from the logarithmic barrier function is twice differentiable and solved via a truncated Newton method. The method we proposed is particularly suited for contact problems with a large number of degrees of freedom. Indeed, the truncated Newton technique makes it possible to avoid spending too much time in the resolution of the barrier subproblem. Moreover, storage of the stiffness matrix is not required. Only a result of a matrix vector product is needed.

Numerical results were presented on some examples and confirmed the fast convergence of our algorithm for large scale problems.

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