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SEQUENTIAL COMPLETENESS AND SPACES WITH THE GLIDING HUMPS PROPERTY

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Introduction. In [2] Bennett and Kalton proved the following property of the BK–space ℓ_∞ of bounded (real or complex) sequences: Given any separable FK–space F containing c_o such that $F \cap \ell_\infty$ is dense in ℓ_∞, one actually has $\ell_\infty \subset F$. The method of proof leading to this result is based on a detailed analysis of ℓ_∞ and its subspaces using two–norm convergence.

In the present paper we obtain the following generalization of the Bennett/Kalton result, using a different approach. We prove that for every BK–AB–space E having the so–called *strong gliding humps property*, the following is true: Given any separable FK–space F containing Φ such that $F \cap E$ is dense in E, one has $E \subset F$.

The method of proof we use to establish this result consists in checking the following two properties satisfied by every BK–AB–space E having the strong gli– ding humps property. Firstly, (*i*) every dense subspace D of E necessarily satisfies $D^\beta = E^\beta$, and secondly, (*ii*) the topology $\sigma(E^\beta,E)$ is sequentially complete. Combining these facts yields the sequential completeness of $\sigma(D^\beta,D)$ and therefore permits applying Kalton's closed graph theorem to the inclusion mapping $(D,\tau(D,D^\beta)) \longrightarrow F$, where F is a separable FK–space containing Φ such that $D = F \cap E$ is dense in E.

Both properties (*i*) and (*ii*) are of interest in themselves. Proving the sequential completeness of weak topologies of the form $\sigma(E^\beta,E)$ involves techniques familiar in bounded consistency theory. We refer to [3] for a survey of these techniques. In fact, our present approach derives sequential completeness of the topology $\sigma(E^\beta,E)$ from a weak form of the gliding humps property for the multiplier space $M(E)$, closely related to the corresponding properties of $M(E)$ considered in [3] and [8]. On the other hand, property (*i*) is related to the circle of problems connected with the

Wilansky property, considered in [1], [4], [5,6,7], [10]. Recall that a BK–space E containing Φ is said to have the Wilansky property if every dense FK–subspace F of E satisfying $F^{\beta} = E^{\beta}$ must coincide with E, i.e. $F = E$. So BK–AB–spaces with the strong gliding humps property are far from having the Wilansky property, but surprisingly enough satisfy a separable version of the Wilansky property, namely, for every separable FK–space F containing Φ such that $D = F \cap E$ satisfies $D^{\beta} = E^{\beta}$, one has $E \subset F$. In the final part of our paper we discuss this circle of problems. We end up with various examples concerning the Wilansky property, its separable version, and the gliding humps property.

1. Preliminaries.

In general our terminology is based on the book [11]. The sections of a sequence $x \in \omega$ are noted $P_k x$, $k = 1, 2, \ldots$ In the following we list some of the notions of particular interest in our present investigation.

A sequence (z^n) of vectors $\neq o$ from Φ is called a *block sequence* if there exists a strictly increasing sequence (k_j) of integers such that z^n is of the form

$$z^n = (o , \ldots , o , z^n_{k_{n-1}+1} , \ldots , z^n_{k_n} , o , \ldots) ,$$

$n \in \mathbb{N}$.

Let $\zeta = (z^n)$ be a block sequence. Then $\ell_\infty(\zeta)$ denotes the sequence space

$$\ell_\infty(\zeta) = \left\{ \sum_n \lambda_n z^n : (\lambda_n) \in \ell_\infty \right\},$$

summation being understood in the coordinatewise sense. Analogously we use the notation $c_o(\zeta)$.

Let E be a BK–space containing Φ. E is called *null for block sequences* (see [4]) if, given any block sequence $\zeta = (z^n)$, the relation $c_o(\zeta) \subset E$ implies $z^n \to o$ $(n \to \infty)$ in E.

The following result was proved in [4] and will again be of use in the present paper.

Lemma 1. *Let* X *be a BK-space containing* Φ, *and let* $\mathrm{E} = \mathrm{X}^{\gamma}$ *be endowed with the* γ*-dual norm (see [11, p. 158]). Suppose* E *is separable. Then it is null for block sequences.*◊

Let E be a BK–space containing Φ. Then E is said to have the *strong gliding humps property* if, given any block sequence (z^n) bounded in E, there exists a sequence (n_k) of indices having $\sum_k z^{n_k} \in E$, where summation is understood in the coordinatewise sense.

We consider an interesting class of spaces of this type. Let (k_j) be a strictly increasing sequence of integers. Now let

$$E = \left\{ x \in \omega : \| x \| = \sup_n \| P_{k_n} x - P_{k_{n-1}} x \|_n < \infty \right\},$$

where $\| \ \|_n$ is any monotone norm (cf. [11, p.104]) on the finite dimensional space of all vectors of the form $P_{k_n} x - P_{k_{n-1}} x$. Clearly every such E is a BK–space with the norm $\| \ \|$, and these spaces have the strong gliding humps property. Choosing for (k_j) a proper subsequence of the integers and setting $\| \ \|_n = \| \ \|_\ell$ for instance provides examples where the sequence (n_k) arising in the definition of the strong gliding humps property has to be chosen as a proper subsequence of the integers.

2. Dense Subspaces.

In this section we establish property *(i)* for the class of BK–AB–spaces having the strong gliding humps property.

Theorem 2. *Let* E *be a BK-AB-space with the strong gliding humps property, and let* D *be a dense linear subspace of* E *containing* Φ. *Then* $D^\gamma = E^\gamma$ *, and hence* $D^\beta = E^\beta$, *is satisfied.*

Proof. As D is dense in E , the coincidence of γ–duals implies the coincidence of the β–duals. Indeed, suppose $D^\gamma = E^\gamma$ has been proved. Let $a \in D^\beta$. Define $f_n \in E'$ by $f_n(x) = \sum_{i=1}^{n} a_i x_i$. Then , in view of $a \in E^\gamma$, the sequence (f_n) is pointwise bounded and, because of $a \in D^\beta$, pointwise converges on the dense subspace D of E . The Banach– Steinhaus Theorem therefore implies the convergence of (f_n) on E , which means $a \in E^\beta$. Hence it suffices to establish the first statement.

Let $a \in D^\gamma$. This means that the triangular matrix A whose n–th row is

$$a_1 , a_2 , \ldots\ldots\ldots , a_n , o , o , \ldots\ldots$$

maps the space D to ℓ_∞. It therefore suffices to show that the rows of A are uniformly bounded in the γ–dual norm , i.e. $K = \sup_n \| P_n a \|_\gamma < \infty$ (cf. [11, p.159]). Indeed, if this has been proved, then A turns out to be a continuous operator $D \to \ell_\infty$ with respect to the topology of the space E on D. This follows by considering the estimate

$$\begin{aligned} |(Ax)_n| &= | \langle P_n a , x \rangle | = | \langle P_n a , P_n x \rangle | \\ &\leq \sup_k | \langle P_n a , P_k x \rangle | \\ &\leq \| x \|_E \cdot \| P_n a \|_\gamma \quad (\text{definition of } \| \ \|_\gamma) \end{aligned}$$

$$\leq K \cdot \| x \|_E .$$

But then A extends to a continuous linear operator $E \to \ell_\infty$, and the latter is still represented by the matrix A, which means that A maps E to ℓ_∞, i.e. $a \in E^\gamma$.

Assume on the contrary that $\sup_n \| P_n a \|_\gamma = \infty$, and choose a sequence (n_k) of integers such that $\| P_{n_k} a \|_\gamma \to \infty$ $(k \to \infty)$. Now let B be the infinite matrix whose k–th row is

$$a_1 / \| P_{n_k} a \|_\gamma , \dots , a_{n_k} / \| P_{n_k} a \|_\gamma , o , o , \dots\dots$$

Then B maps D to c_o, since for $x \in D$ we have

$$(Bx)_k = (Ax)_{n_k} / \| P_{n_k} a \|_\gamma .$$

Notice that B is continuous as an operator $D \to c_o$ in view of

$$| (Bx)_k | = | \langle P_{n_k} a,x \rangle | / \| P_{n_k} a \|_\gamma \leq \| x \|_E ,$$

hence extends to a continuous operator $E \to c_o$, and the latter is still represented by the matrix B. So B maps E to c_o.

Let (m_k) be a subsequence of (n_k) chosen in such a way that $\| P_{m_{k-1}} a \|_\gamma / \| P_{m_k} a \|_\gamma \to o$ $(k \to \infty)$. Then the matrix C whose k–th row is

$$o , \dots , o , a_{m_{k-1}+1} / \| P_{m_k} a \|_\gamma , \dots , a_{m_k} / \| P_{m_k} a \|_\gamma , o , o , \dots\dots$$

still maps E to c_o. This follows by considering the equality

$$(Cx)_k = (Bx)_{m_k} - \langle P_{m_{k-1}} a , x \rangle / \| P_{m_k} a \|_\gamma .$$

Here the last term tends to o $(k \to \infty)$ in view of the estimate

$$| \langle P_{m_{k-1}} a , x \rangle | \leq \| P_{m_{k-1}} a \|_\gamma \cdot \| x \|_E .$$

Using the definition of the γ – dual norm (cf. [11, p. 159]), we find vectors $x^{(k)} \in E$ having $\| x^{(k)} \|_E \leq 1$ and

$$| \sum_{i=1}^{r_k} a_i x_i^{(k)} | \geq \frac{1}{2} \cdot \| P_{m_k} a \|_\gamma ,$$

where r_k is an appropriate index satisfying $r_k \leq m_k$. In the cases where we have

$m_{k-1} < r_k \leq m_k$ let

$$x^k = (o, \dots, o, x^{(k)}_{m_{k-1}+1}, \dots, x^{(k)}_{r_k}, o, o, \dots),$$

then the sequence (x^k) is bounded in E by the AB property. Notice that

$$\langle P_{r_k} a, x^k \rangle = \langle P_{r_k} a, x^{(k)} \rangle - \langle P_{m_{k-1}} a, x^{(k)} \rangle,$$

so we find

$$| \langle P_{r_k} a, x^k \rangle | \geq \frac{1}{4} \cdot \| P_{m_k} a \|_\gamma$$

for k sufficiently large in view of the fact that

$$\langle P_{m_{k-1}} a, x^{(k)} \rangle / \| P_{m_k} a \|_\gamma \to o \quad (k \to \infty).$$

In particular, the relation $m_{k-1} < r_k \leq m_k$ is valid for k sufficiently large. Let $\rho_k =$ sign $\langle P_{r_k} a, x^k \rangle$. Using the fact that E has the strong gliding humps property, we find a sequence (k_j) of integers having

$$\tilde{x} = \sum_j \rho_{k_j} x^{k_j} \in E,$$

where summation is understood in the coordinatewise sense. We claim that $C\tilde{x} \notin c_o$, the desired contradiction. Indeed, we have

$$(C\tilde{x})_{k_j} = \rho_{k_i} \langle P_{r_{k_i}} a, x^{k_i} \rangle / \| P_{m_{k_i}} a \|_\gamma \geq 1/4$$

This ends the proof of Theorem 2. ◊

We end this section with the following structural result for the class of BK–AB–spaces having the strong gliding humps property.

Proposition 3. *Let* E *be a BK-AB-space having the strong gliding humps property. Then* Φ *is dense in* E^γ *with respect to the* γ*-dual norm. In particular,* E^γ *is null for block sequences.*

Proof. The second part of the statement follows from the first part together with Lemma 1. So it suffices to check the first part of the statement.

Let $y \in E^\gamma$ be fixed. We prove that a certain sequence $(P_{n_k} y)$ of sections of y converges to y in the γ – dual norm. Assume the contrary, i.e.

$$\| P_n y - y \|_\gamma \geq \varepsilon$$

for all n and some $\varepsilon > o$. Using the definition of the γ – dual norm we choose vectors $x^{(n)} \in E$ having $\| x^{(n)} \|_E \leq 1$ and

$$| \sum_{i=n}^{r_n} y_i x_i^{(n)} | \geq \varepsilon/2,$$

where $r_n \geq n$. Inductively define a sequence (n_j) by setting $n_{j+1} = r_{n_j} + 1$.

Let (x^k) be the block sequence defined by

$$x^k = (o , \dots , o , x_{n_k}^{(n_k)} , \dots , x_{n_{k+1}-1}^{(n_k)} , o , o , \dots),$$

then (x^k) is bounded in E and we have

$$| \langle y , x^k \rangle | = | \sum_{i=n_k}^{n_{k+1}-1} y_i x_i^{(n_k)} | \geq \varepsilon/2,$$

$k \in \mathbb{N}$. Let $\rho_k = \text{sign} \langle y , x^k \rangle$. Using the strong gliding humps property we find a sequence (k_j) having

$$\tilde{x} = \sum_j \rho_{k_j} x^{k_j} \in E \quad \text{(pointwise sum)}.$$

Obviously we have $\tilde{x} \cdot y \notin bs$, contradicting $\tilde{x} \in E, y \in E^\gamma$. This proves the result. ◊

Corollary 4. *Let* X *be a BK-space containing* Φ *such that* $E = X^\gamma$ *has the strong gliding humps property. Then* E *is a dual Banach space, namely* $E = F'$, *where* $F = E^\gamma$. ◊

3. Sequential completeness.

In this section we derive a criterion for the sequential completeness of a weak topology of the type $\sigma(E^\beta, E)$. In particular we obtain condition (*ii*) for the class of BK–AB–spaces having the strong gliding humps property. First we need a definition.

Let X be a sequence space containing Φ . Then X is said to have the *weak gliding humps property* if, given any $x \in X$ and any block sequence (x^k) having $x = \sum_i x^i$ (pointwise sum), every sequence (n_k) of integers admits a subsequence (m_k) such that

$$\tilde{x} = \sum_k x^{m_k} \in X \quad \text{(pointwise sum)}.$$

The following Lemma indicates the relation of the weak gliding humps property of a space X to the type of gliding humps properties of the multiplier space $M(X)$ considered in bounded consistency theory (cf. [3, 8]).

Lemma 5. *Let* X *be a sequence space containing* Φ . *Suppose that either (i) the multiplier space* M(X) *has the gliding humps property in the sense of* [3, 8], *or (ii)* X *is a BK-AB-space having the strong gliding humps property. Then* X *has the weak gliding humps property.*

Proof. First consider the case where statement (*i*) is valid. Let $x = \sum_n x^n$ for a block sequence (x^n). Let (k_j) be the corresponding sequence of integers (cf. section 1). Define the vectors y^n by setting $y_k^n = 1$ for $k_{n-1} < k \leq k_n$, $y_k^n = o$ otherwise. Then $y^n \in M(X)$, $\| y^n \|_{bv} = 2$, $n \in \mathbb{N}$. Let (n_k) be any fixed sequence of integers. Applying the fact that $M(X)$ has the gliding humps property now provides a subsequence (m_k) of (n_k) having

$$\tilde{y} = \sum_k y^{m_k} \in M(X).$$

Since $x \cdot y^{m_k} = x^{m_k}$, we derive $\tilde{x} = \sum_k x^{m_k} \in X$. This proves the result in case (*i*). Case (*ii*) is clear. ◊

Our main interest in the weak gliding humps property lies in the following

Theorem 6. *Let* X *be a sequence space containing* Φ *and having the weak gliding humps property. Then* $\sigma(X^\beta, X)$ *is sequentially complete.*

Proof. Let $(y^{(n)})$ be a Cauchy sequence in $\sigma(X^\beta, X)$. Let y denote its co–ordinatewise limit. We first prove that $y \in X^\gamma$. Assume on the contrary that $(\sum_{i=1}^n x_i y_i)_{n=1}^\infty$ is unbounded for some $x \in X$. We define strictly increasing sequences (n_j) , (k_j) , (r_j) of integers having the following properties:

(α) $r_{j-1} < k_j$ and $| \sum_{i=r_{j-1}}^{k_j} x_i y_i | \geq j + \sum_{t=1}^{j-1} \sum_{i=r_{t-1}}^{k_t} | x_i y_i |$;

(β) $\sum_{i=1}^{k_j} | x_i | \cdot | y_i^{(n_j)} - y_i | < 2^{-j}$;

(γ) $k_j < r_j$ and $| \sum_{i=r}^{s} x_i y_i^{(n)} | < 2^{-j}$ for all $s \geq r \geq r_j$, $n = 1 , \dots , n_j$.

Suppose $n_1 , \dots , n_{j-1}$, $k_1 , \dots , k_{j-1}$, and $r_1 , \dots , r_{j-1}$ have been constructed satisfying (α) – (γ). First we find $k_j > r_{j-1}$ such that

$$\left| \sum_{i=r_{j-1}}^{k_j} x_i y_i \right| \geq j + \sum_{t=1}^{j-1} \sum_{i=r_{t-1}}^{k_t} | x_i y_i |$$

is satisfied. This is possible in view $x \cdot y \notin bs$. Now observe that $y^{(n)} \to y$ pointwise. This permits selecting n_j in accordance with (β). Finally, having regard of the fact that $x \cdot y^{(n)} \in cs$, $(n = 1, \dots, n_j)$, we certainly find an index $r_j > k_j$ such that (γ) is valid.

Let us now define the vectors $x^i \in \Phi$ by setting

$$x^i = (o, \dots, o, x_{r_{i-1}}, \dots, x_{k_i}, o, o, \dots).$$

Then the weak gliding humps property for X provides a sequence (j_s) having

$$\tilde{x} = \sum_s x^{j_s} \in X.$$

We claim that the sequence $\langle \tilde{x}, y^{(n_{j_s})} \rangle$, $s \in \mathbb{N}$, is unbounded, a contradiction with the fact that $(y^{(n)})$ is Cauchy in $\sigma(X^\beta, X)$. Writing $j = j_s$ we have

$$\sum_{i=1}^{\infty} \tilde{x}_i y_i^{(n_j)} = \sum_{i=1}^{k_j} \tilde{x}_i (y_i^{(n_j)} - y_i) + \sum_{i=1}^{k_j} \tilde{x}_i y_i + \sum_{i=k_j+1}^{\infty} \tilde{x}_i y_i^{(n_j)}$$

$$= A_j + B_j + C_j.$$

Here the first term on the right side converges to o $(s \to \infty, j = j_s)$ in view of (β) and

$$| A_j | = \left| \sum_{i=1}^{k_j} \tilde{x}_i (y_i^{(n_j)} - y_i) \right| \leq \sum_{i=1}^{k_j} | x_i | \cdot | y_i^{(n_j)} - y_i | \leq 2^{-j}.$$

Also the third term converges to o $(s \to \infty, j = j_s)$ when we observe that $\tilde{x}_k = o$ holds for $k_j + 1 \leq k \leq r_{j+1} - 1$ $(j = j_s)$, which means that

$$| C_j | = \left| \sum_{i=r_{j+1}}^{\infty} \tilde{x}_i y_i^{(n_j)} \right| \leq \sum_{t=j+1}^{\infty} \left| \sum_{i=r_t}^{k_{t+1}} x_i y_i^{(n_j)} \right| \leq \sum_{t=j+1}^{\infty} 2^{-t} \longrightarrow o$$

in view of property (γ). Finally, the term $| B_j |$ tends to ∞ $(s \to \infty, j = j_s)$ in view of the estimate

$$| B_j | \geq \left| \sum_{i=r_{j-1}}^{k_j} x_i y_i \right| - \sum_{t=1}^{j-1} \sum_{i=r_{t-1}}^{k_t} | x_i y_i | \geq j,$$

$(j = j_s)$, where we use (α) and the fact that $\tilde{x}_k = x_k$ holds for $r_{j-1} \le k \le k_j$, $|\tilde{x}_i| \le |x_i|$ otherwise. This proves our claim $y \in X^\gamma$.

Let us now prove that for fixed $x \in X$ the series $\sum x_i y_i$ converges and that its value is just $\lim_{n\to\infty} \sum x_i y_i^{(n)}$. Assume the contrary. In view of $y \in X^\gamma$ this means that there exists a strictly increasing sequence (m_k) of integers such that

$$a := \lim_{k\to\infty} \sum_{i=1}^{m_k} x_i y_i$$

exists but is different from $b := \lim_n \sum x_i y_i^{(n)}$. Passing to a subsequence of (m_k) if necessary, we may assume that

$$(1) \qquad \Big| \sum_{i=m_k+1}^{m_r} x_i y_i \Big| \le 2^{-k} \quad (r \ge k)$$

is satisfied. Now we define strictly increasing sequences (n_j), (k_j), (r_j) of integers as follows.

Let $k_1 = 1$. Choose n_1 such that $|x_1|\,|y_1^{(n_1)} - y_1| \le 2^{-1}$. Then choose r_1 such that

$$\Big| \sum_{i=m_r+1}^{m_s} x_i y_i^{(n)} \Big| \le 2^{-1}$$

holds for $n = 1, \dots, n_1$ and for all $r \ge r_1$, $s \ge r$.

Suppose $k_1, \dots, k_{j-1}$, $n_1, \dots, n_{j-1}$, $r_1, \dots, r_{j-1}$ have been constructed. Let $k_j = m_{r_{j-1}}$. Then choose n_j in such a way that

$$(2) \qquad \sum_{i=1}^{k_j} |x_i| \cdot |y_i^{(n_j)} - y_i| \le 2^{-j}.$$

Finally choose $r_j > r_{j-1}$ so that

$$(3) \qquad \Big| \sum_{i=m_r+1}^{m_s} x_i y_i^{(n)} \Big| \le 2^{-j}$$

holds for $n = 1, \dots, n_j$, $r \ge r_j$, $s \ge r$.

Suppose the sequences have been defined. For fixed $i \in \mathbb{N}$ let

$$x^i = (o, \dots, o, x_{k_i+1}, \dots, x_{k_{i+1}}, o, o, \dots).$$

Applying the definition of the weak gliding humps property to the sequence of even integers, we obtain a sequence (j_s) such that

$$\tilde{x} = \sum_s x^{2j_s} \in X \qquad (\text{pointwise sum}).$$

We derive the desired contradiction by proving that the sequence $\langle \tilde{x}, y^{(n)} \rangle$ is not convergent.

First we consider the sequence $\langle \tilde{x}, y^{(n_j)} \rangle$, where $j = 2j_s - 1$. Here we have

$$\sum_{i=1}^{\infty} \tilde{x}_i y_i^{(n_j)} = \sum_{i=1}^{k_j} \tilde{x}_i y_i^{(n_j)} + \sum_{i=k_j+1}^{k_{j+1}} \tilde{x}_i y_i^{(n_j)} + \sum_{i=k_{j+1}+1}^{\infty} \tilde{x}_i y_i^{(n_j)}$$

$$= A_j + B_j + C_j .$$

Notice that $\tilde{x}_k = o$ holds for $k_j < k \le k_{j+1}$ $(j = 2j_s - 1)$, so we have $B_j = o$ in this case. Moreover, we have

$$| C_j | \le \sum_{t=j}^{\infty} \Big| \sum_{i=k_{t+1}+1}^{k_{t+2}} x_i y_i^{(n_j)} \Big| \le \sum_{t=j}^{\infty} 2^{-t} \longrightarrow o \ (s \to \infty, j = 2j_s - 1),$$

where we use the fact that on the blocks $k_{t+1}+1, \ldots, k_{t+2}$ the sequence $\tilde{x}$ either agrees with x or is identically o.

Finally, let $c := \lim_{j\to\infty} \sum_{i=1}^{k_j} \tilde{x}_i y_i$, which exists in view of (1) and the fact that $k_j \in \{ m_t : t \in \mathbb{N} \}$. We prove that $A_j \to c$ $(s \to \infty, j = 2j_s - 1)$. Indeed, we have

$$| A_j - c | \le \Big| \sum_{i=1}^{k_j} \tilde{x}_i \cdot (y_i^{(n_j)} - y_i) \Big| + \Big| \sum_{i=k_j+1}^{\infty} \tilde{x}_i y_i \Big| .$$

Here the first term on the right side is $\le 2^{-j}$ in view of (2) and $| \tilde{x}_i | \le | x_i |$, whilst the second term tends to o in view of the convergence of $\sum_{i=1}^{k_j} \tilde{x}_i y_i$.

Let us now consider the subsequence $\langle \tilde{x}, y^{(n_j)} \rangle$, $j = 2j_s$. We prove that it has a limit different from c, from which it readily follows that $\langle \tilde{x}, y^{(n)} \rangle$ is not convergent. Indeed, we have the same decomposition $\langle \tilde{x}, y^{(n_j)} \rangle = A_j + B_j + C_j$, and we find as above that $A_j \to c$, $C_j \to o$ $(s \to \infty, j = 2j_s)$. We prove that B_j converges to a limit different from o. Indeed, observe that we have $\tilde{x}_k = x_k$ for $k_j < k \le k_{j+1}$. This gives

$$B_j = \sum_{i=k_j+1}^{k_{j+1}} x_i y_i^{(n_j)}$$
$$= \sum_{i=1}^{\infty} x_i y_i^{(n_j)} - \sum_{i=1}^{k_j} x_i y_i - \sum_{i=1}^{k_j} x_i (y_i^{(n_j)} - y_i) - \sum_{i=k_{j+1}+1}^{\infty} x_i y_i^{(n_j)} .$$

Here the first term on the right side converges to b , the second term converges to a $(s \to \infty , j = 2j_s)$. The third term converges to o in view of (2) , and so does the fourth term as a consequence of (3) . So $B_j \to b - a \neq o$, which provides the desired contradiction. This completes our argument. ◊

Let us consider the following example , which was communicated to us by Prof. Dr. J. Boos, taken from the thesis of his student Dr. D. Seydel. Let $X = m_o$ be the space of sequences taking only finitely many values. Then m_o clearly has the weak gliding humps property, so $\sigma(\ell, m_o)$ is sequentially complete. But $X = m_o$ does *not* satisfy the following statement (*) considered in [3].

(*) $X \cap W_A \subset c_B$ implies $X \cap W_A \subset W_B$.

To see this we choose for A the matrix $Z_{1/2}$ (cf. [12, p.125]), and we define $B = (b_{nk})$ by setting $b_{nk} = 1$ for $n = k$ or $n = k + 1$ (k even), $b_{nk} = o$ otherwise. Then it is easy to see that $m_o \cap W_A \subset c_B$, but $m_o \cap W_A$ is not contained in W_B.

Statement (*) implies the sequential completeness of $\sigma(X^\beta, X)$ (cf. [3]), and (*) in turn is implied by the conditions imposed on the multiplier space $M(X)$ in [3]. This indicates that these conditions are fairly stronger than the weak gliding humps property considered here .

We consider another example, $X = bs$. Here X does not have the weak gliding humps property, but nevertheless $\sigma(bv_o, bs)$ is sequentially complete. Notice that bs even satisfies statement (*) .

4. The main Lemma.

In this section we prove a technical result, which plays the crucial role towards our result stated in the introduction.

Lemma 7. *Let* E *be a BK-space containing* Φ *, and let* D *be a dense linear subspace of* E *containing* Φ *and satisfying* $D^\beta = E^\beta$. *Suppose that* E^β *is null for block sequences. Then* $\sigma(\Phi, E)$ *and* $\sigma(\Phi, D)$ *have the same null sequences.*

Proof. Let (y^n) be a null sequence in $\sigma(\Phi, D)$. We have to show that (y^n) is bounded in E^β with respect to the β–dual norm. For suppose this has been proved,

$\| y^n \|_\beta \leq M$, say. Then for fixed $x \in E$ and $\varepsilon > o$ we choose $\tilde{x} \in D$ having $\| x - \tilde{x} \|_E < \varepsilon/2M$. Then we find

$$| \langle x , y^n \rangle | \leq \| x - \tilde{x} \|_E \cdot \| y^n \|_\beta + | \langle \tilde{x} , y^n \rangle |$$
$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for n sufficiently large.

Suppose (y^n) is not bounded in norm. Passing to a subsequence if necessary, we may assume that $\| y^n \|_\beta \geq 2^n$. As $\Phi \subset D$, we have $y^n \to o$ coordinatewise. This permits selecting strictly increasing sequences (n_j) , (k_j) of integers such that

$$(i) \quad \| P_{k_{j-1}} y^{n_j} \|_\beta \leq 2^{-j} ;$$

$$(ii) \quad y^{n_j} \text{ has length } \leq k_j .$$

Setting $v^i = y^{n_i} - P_{k_{i-1}} y^{n_i}$ therefore provides a block sequence (v^i) which is still $\sigma(\Phi,D)$ null, but has $\| v^i \|_\beta \geq 2^{n_i} - 2^i$. We may assume that $n_i \geq 2i$. Let $\alpha_i = 1/\| v^i \|_\beta$, $z^i = \alpha_i \cdot v^i$. Then $\| z^i \|_\beta = 1$. Since E^β is null for block sequences, there exists a sequence $(\lambda_i) \in c_o$ such that

$$z = \sum_i \lambda_i \cdot z^i \quad \text{(pointwise sum)}$$

is *not* an element of E^β . We achieve a contradiction by proving $z \in D^\beta$. So let $x \in D$ be fixed. Let $k \in \mathbb{N}$, and find j having $k_{j-1} < k \leq k_j$, where (k_i) is the sequence of integers corresponding with the block sequence (z^i) . Then we have

$$\sum_{i=1}^{k} x_i z_i = \sum_{i=1}^{j-1} \lambda_i \alpha_i \sum_{s=k_{i-1}+1}^{k_i} x_s v_s^i + \sum_{s=k_{j-1}+1}^{k} \lambda_j \alpha_j x_s v_s^j$$
$$= \sum_{i=1}^{j-1} \lambda_i \alpha_i \langle x , v^i \rangle + \lambda_j \langle P_k x - P_{k_{j-1}} x , \alpha_j \cdot v^j \rangle .$$

Here the first term on the right side converges $(k \to \infty , k_{j-1} < k \leq k_j)$ in view of $(\lambda_j \alpha_j) \in \ell$, $\langle x , v^i \rangle \to o$. The second term converges as well in view of $\lambda_j \to o$ and the estimate

$$| \langle P_k x - P_{k_{j-1}} x , \alpha_j \cdot v^j \rangle | \leq 2 \| x \|_{\beta\beta} \| \alpha_j \cdot v^j \|_\beta = 2 \| x \|_{\beta\beta} ,$$

where we use the fact that the norm $\| \; \|_{\beta\beta}$ is monotone (cf. [11, p.159]) . This ends the proof of the Lemma. ◊

5. Consequences.

In this sections we obtain applications of our main Lemma.

Theorem 8. *Let* E *be a BK-AB-space having the strong gliding humps property. Let* F *be a separable FK-space containing* Φ . *Suppose* $F \cap E$ *is dense in* E. *Then* $E \subset F$.

Proof. Let $D = F \cap E$, then $D^{\beta} = E^{\beta}$ holds by Theorem 2. Also Φ is dense in E^{γ} by Proposition 3, and E^{γ} is null for block sequences by Lemma 1. But clearly we must have $E^{\beta} = E^{\gamma}$ here, so E^{β} is null for block sequences.

Since Φ is norm dense in E^{β} , it follows from our main Lemma that the topologies $\sigma(E^{\beta},D)$ and $\sigma(E^{\beta},E)$ have the same null sequences. Consequently, they also have the same Cauchy sequences. But Theorem 6 tells that $\sigma(E^{\beta},E)$ is sequentially complete, hence the same must be true for $\sigma(E^{\beta},D)$.

Sequential completeness of $\sigma(D^{\beta},D) = \sigma(E^{\beta},D)$ permits applying Kalton's closed graph theorem to the inclusion function $\iota: (D, \tau(D,D^{\beta})) \longrightarrow F$ (see [2] or [11, p.251]), and this implies the continuity of ι.

We claim that $\tau(E,E^{\beta})|D = \tau(D,D^{\beta})$. Indeed, this follows since $\sigma(E^{\beta},D)$ and $\sigma(E^{\beta},E)$ have the same convergent sequences, hence also have the same compact sets (see [11, p.252]). But now ι extends to a continuous linear operator

$$\tilde{\iota}: (E, \tau(E,E^{\beta})) \longrightarrow F .$$

From K–space reasons it is clear that $\tilde{\iota}$ must again be the inclusion mapping, which means $E \subset F$, as desired. This ends the proof of Theorem 8. ◊

Theorem 8 generalizes the Bennett/Kalton result stated in the introduction, since ℓ_{∞} clearly has the strong gliding humps property. We mention another generalization of their result obtained by Snyder [9].

Following [1], a BK–space E containing Φ is said to have the *Wilansky property* if every dense FK–subspace F of E satisfying $F^{\beta} = E^{\beta}$ must coincide with E, i.e. $F = E$. We refer to [1, 4, 5, 6, 7, 10] for information concerning this notion.

Let E be a BK–AB–space having the strong gliding humps property. Suppose that, in addition, E has the Wilansky property. Then *every* dense FK–subspace F of E automatically satisfies $F^{\beta} = E^{\beta}$ as a consequence of Theorem 2. Hence the Wilansky property implies the equality $F = E$ for *every* dense FK–subspace F of E , which means that E has no proper dense FK–subspaces at all. Consequently, BK–spaces E having both, the strong gliding humps property and the Wilansky property are quite peculiar. Actually we do not even know of any BK–space E without proper dense FK–subspaces. Notice, however, that Theorem 8 tells that every BK–AB–space E having the strong gliding humps property satisfies the following separable version of the Wilansky property, which we state as a definition.

A BK–space E containing Φ is said to have the *separable Wilansky property* if, given any separable FK–space F containing Φ such that $D = F \cap E$ is dense in E, the relation $D^{\beta} = E^{\beta}$ implies that $E \subset F$ (cf. [7]).

The following result may be obtained by slightly modifying the proof of Theorem 8 above.

Theorem 9. *Let* E *be a BK-space containing* Φ *such that* $\sigma(E^{\beta},E)$ *is sequentially complete. Suppose* Φ *is norm dense in* E^{γ}. *Then* E *has the separable Wilansky property.*

Proof. Let F be a separable FK–space containing Φ such that $D = F \cap E$ is dense in E and $D^{\beta} = E^{\beta}$ is satisfied. We have to prove $E \subset F$.

Applying the main Lemma shows that $\sigma(\Phi,D)$ and $\sigma(\Phi,E)$ have the same null sequences. Since Φ is norm dense in E^{γ}, we must have $E^{\beta} = E^{\gamma}$, which means that $\sigma(E^{\beta},D)$ and $\sigma(E^{\beta},E)$ have the same null sequences. So $\sigma(E^{\beta},D) = \sigma(D^{\beta},D)$ is sequentially complete. But now we proceed as in the proof of Theorem 8, which finally gives us $E \subset F$. ◊

Remarks. 1) Modifying an example given in [1] shows that ℓ does not even have the separable Wilansky property. Setting

$$F = \left\{ x \in \ell : \lim_{n\to\infty} n \sum_{k\geq 2n} x_k \text{ exists} \right\}$$

provides a proper dense separable FK–subspace of ℓ containing Φ and satisfying $F^{\beta} = \ell^{\beta} = \ell_{\infty}$.

2) Also ω does not have the separable Wilansky property. Here we choose

$$F = \left\{ x \in \omega : \lim_{n\to\infty} (x_{2n} - x_{2n-1}) \text{ exists} \right\}.$$

Then F is a proper dense separable FK–subspace of ω containing Φ and satisfying $F^{\beta} = \Phi$.

3) Let f_o be the space of all allmost null sequences (cf. [3, 8, 12]), then $M(f_o)$ has the gliding humps property, so $\sigma(f_o^{\beta},f_o) = \sigma(\ell,f_o)$ is sequentially complete (cf. [3], [8, §4]). Consequently, by Theorem 9, f_o has the separable Wilansky property, since Φ is dense in ℓ. But f_o does not have the Wilansky property (cf. [7, Theorem 2]). Also notice that f_o does not have the strong gliding humps property, for bs is dense in f_o (cf. [8, §4]), but has β–dual $bs^{\beta} = bv_o$ ($\neq \ell$).

4) Consider the space bs. Theorem 9 above implies that bs has the separable Wilansky property, since $\sigma(bv_o,bs)$ is sequentially complete. Clearly bs does not even have the weak gliding humps property, but nevertheless every dense

FK–subspace F of bs satisfies $F^{\beta} = bv_o$. This may be deduced from the corresponding property of ℓ_∞ using the method of [1, §6].

5) It would be interesting to have an example of a separable BK–space having the separable Wilansky property, but failing the general Wilansky property.

6) In Theorem 9, instead of claiming Φ to be norm dense in E^{γ}, it would be sufficient to make the assumption that Φ is norm dense in E^{β} , and that the latter is null for block sequences. We do not know, however, whether the assumption of norm denseness of Φ in E^{β} alone would be sufficient to obtain the statement of the Theorem, since norm denseness of Φ in E^{β} does *not* imply that E^{β} is null for block sequences. This may be seen by taking $E = bv$, $E^{\beta} = cs$.

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