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Generic Gâteaux-Differentiability of Convex Functions on Small Sets

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INTRODUCTION

The theory of generic differentiability of convex functions on Banach spaces is by now a well-explored part of infinite-dimensional geometry. All the attempts to solve this kind of problem have in common, as a working hypothesis, one special feature of the finite-dimensional case. Namely, convex functions are always considered to be defined on convex sets with nonempty interior. But typically, a convex set in a Banach space does not have interior points even when it is not contained in a closed hyperplane. So this raises the problem of expounding a theory of differentiability of convex functions defined on small sets, i.e., sets without interior points. In our paper [N2] we have made an attempt to solve this problem, discussing questions of generic Fréchet-differentiability of convex functions on small sets. In the present paper we deal with problems of generic Gâteaux-differentiability in this context.

1. MAXIMAL SUBGRADIENTS

In the following let E be a normed space and let C be a convex subset of E not contained in a closed hyperplane. For fixed $x \in C$ let $K(C, x)$ denote the convex cone with vertex at the origin consisting of all vectors $z \in E$ having $x + tz \in C$ for small $t > 0$.

DEFINITION. Let $\varphi : C \to \mathbb{R}$ be a convex function. φ is called

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(1) Gâteaux-differentiable at $x \in C$ if there exists a continuous linear functional $f_x \in E'$ satisfying

$$\lim_{t \downarrow 0} \frac{1}{t} (\varphi(x+tz) - \varphi(x)) = f_x(z)$$

for all $z \in K(C, x)$;

(2) Fréchet-differentiable at $x \in C$ if there exists $f_x \in E'$ satisfying

$$\lim_{t \downarrow 0} \sup_{\substack{z \in K(C,x) \\ \|z\| \leqslant 1}} \left| \frac{1}{t} (\varphi(x+tz) - \varphi(x)) - f_x(z) \right| = 0.$$

Notice that Fréchet-differentiability of φ at x implies Gâteaux-differentiability. Moreover, if φ is Gâteaux-differentiable at $x \in C$, then its derivative f_x at x is uniquely determined in view of the fact that C is not part of a closed hyperplane. So the notation $\varphi'(x) = f_x$ is justified in both cases.

It is worth noting that our definition generalizes the classical concepts of differentiability of convex functions. This follows from the fact that for an interior point x of C the cone $K(C, x)$ is all of E.

It is well known that a continuous convex function φ is Gâteaux-differentiable at an interior point x of its domain C if and only if its subdifferential $\partial\varphi(x)$ at x contains a unique element. It is easy to see that this characterization does not remain valid for boundary points of C, when our notion of differentiability is used. So what we shall first be interested in is a substitute for the subdifferential characterization of Gâteaux-differentiability.

PROPOSITION 1. *Let E be a normed space and let C be a convex subset of E not contained in a closed hyperplane. Let $\varphi: C \to \mathbb{R}$ be a convex function and let $x \in C$ be fixed. Suppose that either*

(i) *there exists a neighbourhood U of x in C such that $\partial\varphi(y)$ is nonempty on U and $\varphi \mid U$ extends to a convex and lower semi-continuous function $\psi: E \to \mathbb{R} \cup \{\infty\}$, or*

(ii) *there exists a neighbourhood U of x in C and a dense subset D of U such that $\partial\varphi(y) \neq \emptyset$ on D and $\varphi \mid U$ extends to a lower semi-continuous function on E and is continuous on U.*

Then the following statements are equivalent:

(1) *φ is Gâteaux-differentiable at x;*

(2) *$\partial\varphi(x)$ is nonempty and contains a unique maximal element with respect to the order induced by $K(C, x)$.*

Proof. Assume (1). We consider the sublinear operator $\partial\varphi(x;\ )$ defined on $K(C, x)$ with

$$\partial\varphi(x; z) = \inf_{t>0} \frac{1}{t} (\varphi(x+tz) - \varphi(x)).$$

Note that $\partial\varphi(x;\ )$ is well-defined in view of $\partial\varphi(x) \neq \emptyset$. Moreover, the elements of $\partial\varphi(x)$ are precisely the continuous linear support functionals of $\partial\varphi(x;\ )$. But φ is Gâteaux-differentiable at x, so $\partial\varphi(x;\ ) = \varphi'(x)$ holds on $K(C, x)$. This implies $f(z) \leqslant \varphi'(x)(z)$ for all $z \in K(C, x)$ and all $f \in \partial\varphi(x)$, proving that $\varphi'(x)$ is the unique maximal element of $\partial\varphi(x)$ with respect to the order induced by $K(C, x)$.

Now assume (2). Let us first consider the case where φ is a Lipschitz function, which consequently extends to a convex Lipschitz function on E. But then $\partial\varphi(x;\ )$ is as well Lipschitz and consequently also extends to a convex Lipschitz function on E. Suppose now that $f_x \in \partial\varphi(x)$ is maximal with respect to the order induced by $K(C, x)$. We claim that $\partial\varphi(x;\ ) = f_x$ holds on $K(C, x)$. Clearly we have $f_x \leqslant \partial\varphi(x;\ )$ on $K(C, x)$. Suppose we had $f_x(z) < \partial\varphi(x; z)$ for some $z \in K(C, x)$. By Hörmander's theorem $\partial\varphi(x;\ )$ may be approximated from below by continuous linear support functionals (see [Hö]). So there exists $f \in E'$, $f \leqslant \partial\varphi(x;\ )$ on $K(C, x)$ such that $f(z) > f_x(z)$. But note that $f \in \partial\varphi(x)$, and this contradicts the maximality of f_x in $\partial\varphi(x)$. This implies $f_x = \partial\varphi(x;\ )$ on $K(C, x)$, from which we deduce that φ is Gâteaux-differentiable at x with derivative f_x. This ends the proof in the case where φ is Lipschitz on C.

Let us now consider the more general cases (i) and (ii). For $n \in \mathbb{N}$ let φ_n denote the infimal convolution of φ and $n\|\ \|$, noted $\varphi_n = \varphi \,\square\, n\|\ \|$. Notice that either $\varphi_n \equiv -\infty$ or φ_n is finite everywhere and is Lipschitz with constant n. The latter is the case, e.g., when $\partial\varphi(y) \cap nB' \neq \emptyset$ for some $y \in C$, where B' denotes the dual unit ball (see [HU] for the properties of the functions φ_n). The coincidence set C_r of φ and φ_r is just

$$C_r = \{ y \in C : \partial\varphi(y) \cap rB' \neq \emptyset \}$$

(see [HU]). Now assume that $f_x \in \partial\varphi(x)$ is maximal with respect to the order induced by $K(C, x)$. Let $\|f_x\| \leqslant n$, then we have $\varphi(x) = \varphi_r(x)$ for all $r \geqslant n$. This implies

$$\partial\varphi_r(x) \subset \partial(\varphi_r | C)(x) \subset \partial\varphi(x)$$

for all $r \geqslant n$. Since $f_x \in \partial\varphi_r(x)$, $r \geqslant n$, we deduce that f_x is maximal in $\partial(\varphi_r | C)(x)$ with respect to the order induced by $K(C, x)$ for all $r \geqslant n$. From the first part of the proof of $(2) \to (1)$ we therefore deduce that $f_x = (\varphi_r | C)'(x)$ holds for all $r \geqslant n$.

Let us now assume property (i). Let $z \in K(C, x)$ be fixed and choose $t > 0$ such that $x + tz \in U$. This implies $\partial\varphi(x + tz) \neq \emptyset$. Choose $r \geqslant n$ such that $\partial\varphi(x + tz)$ contains an element of norm not exceeding r. This implies $\varphi(x) = \varphi_r(x)$, $\varphi(x + tz) = \varphi_r(x + tz)$, $r \geqslant n$. Since $\partial\varphi(x) \neq \emptyset$, φ is locally Lipschitz at x along the ray $x + \mathbb{R}_+ z$. Hence, choosing $t > 0$ small enough and r large enough, we may assume that also $\varphi(x + sz) = \varphi_r(x + sz)$ for all $0 \leqslant s \leqslant t$. But this implies $\partial\varphi(x; z) = \partial\varphi_r(x; z)$, hence $\partial\varphi(x; z) = f_x(z)$. Since $z \in K(C, x)$ was chosen arbitrarily, this proves the claim in case (i).

Finally, consider the case where (ii) is satisfied. By assumption there exists a dense subset D of U such that $\partial\varphi(y) \neq \emptyset$ for $y \in D$. Let $D^* = \{z \in E : x + tz \in D \text{ for some } t > 0\}$, then D^* is dense in $K(C, x)$. We claim that $\partial\varphi(x;\ ) = f_x$ is true on D^*. Let $z \in D^*$, $x + tz \in D$ for $t > 0$. This implies $\partial\varphi(x + tz) \neq \emptyset$, so the above reasoning readily implies $\partial\varphi(x; z) = f_x(z)$. Finally, observe that $\partial\varphi(x;\ )$ is continuous on $K(C, x)$ as a consequence of the fact that $\varphi \mid U$ is continuous. Since D^* is dense in $K(C, x)$, we derive $\partial\varphi(x;\ ) = f_x$ on $K(C, x)$. ∎

COROLLARY 1. *Let E be a normed space and let $\varphi : E \to \mathbb{R} \cup \{\infty\}$ be a lower semi-continuous convex function. Let C be the domain of φ and suppose $\varphi \mid C$ is continuous. Let $x \in C$ be fixed. Then the following statements are equivalent:*

(1) *φ is Gâteaux-differentiable at x;*

(2) *$\partial\varphi(x)$ is nonempty and contains a unique maximal element with respect to the order induced by $K(C, x)$.*

Proof. By a result of Ekeland [E] there exists a dense subset D of C such that $\partial\varphi(y)$ is nonempty over D. Therefore the condition (ii) in Proposition 1 is satisfied and the result follows. ∎

Remark. Proposition 1 clearly generalizes the classical statement that a continuous convex function φ is Gâteaux-differentiable at an interior point x of its domain if and only if $\partial\varphi(x)$ contains a unique element.

2. GENERIC GÂTEAUX-DIFFERENTIABILITY

In this section we obtain a generalization of the well-known results on generic Gâteaux-differentiability by Stegall [S1, S2] to the case of convex functions defined on small sets. First let us recall a definition.

A Banach space E is called of class (S) if the following condition is satisfied: Whenever X is a Baire topological space and $\Theta : X \to 2^{E'}$ is a set-valued operator which is upper semi-continuous with respect to $\sigma(E', E)$ and has nonempty convex and $\sigma(E', E)$-compact values, then Θ admits a selector $\vartheta: X \to E'$ which is continuous at the points of a dense G_δ-subset of X.

It has been proved by Stegall [S1] that all weakly compactly generated Banach spaces, and hence in particular all separable Banach spaces, are of class (S). Here we obtain the following generalization of Stegall's result stating that the spaces of class (S) are weak Asplund spaces.

THEOREM 1. *Let E be a Banach space of class (S) and let C be a convex subset of E which is not contained in a closed hyperplane. Suppose that C is a Baire space in the induced topology. Let $\varphi: C \to \mathbb{R}$ be a lower semi-continuous convex function and suppose there exists a dense G_δ-subset G_0 of C such that for every $y \in G_0$ the subdifferential $\partial\varphi(y)$ is nonempty. Then there exists a dense G_δ-subset G of C such that φ is Gâteaux-differentiable at every $x \in G$.*

Proof. Let us first consider the case where φ is Lipschitz on C and therefore extends to a convex Lipschitz function ψ on E. The operator $\partial\psi: C \to 2^{E'}$ is known to be upper semi-continuous with respect to the norm topology on C and the weak star topology $\sigma(E', E)$ on E'. Moreover, $\partial\psi$ has nonempty convex and $\sigma(E', E)$ compact values. By the definition of the spaces of class (S) there exists a selector $x \to f_x$ for $\partial\psi$ which is continuous with respect to the norm topology on C and $\sigma(E', E)$ on E' at the points of a dense G_δ-subset G of C. We claim that $f_x \in \partial\psi(x)$ is the Gâteaux-derivative of $\psi \mid C = \varphi$ at x whenever $x \in G$. Indeed, for fixed $x \in G$, f_x is maximal in $\partial\varphi(x)$ with respect to the order induced by $K(C, x)$. For suppose there exists $f \in \partial\varphi(x)$ having

$$f(z) - f_x(z) =: \varepsilon > 0$$

for some $z \in K(C, x)$. Choose $t_0 > 0$ such that $x + tz \in C$ for $0 \leqslant t \leqslant t_0$. Since $f_{x+tz} \in \partial\psi(x+tz) \subset \partial\varphi(x+tz)$, $0 < t \leqslant t_0$, the monotonicy of the operator $\partial\varphi$ implies

$$(f_{x+tz} - f)(tz) \geqslant 0,$$

so we find $f_{x+tz}(z) - f_x(z) \geqslant \varepsilon$ for all $0 < t \leqslant t_0$, contradicting the continuity of $y \to f_y$ at x. So f_x is actually maximal in $\partial\varphi(x)$. By Proposition 1, case (ii), we derive that $\varphi'(x) = f_x$ holds for $x \in G$.

Let us now consider the general case. Let $\varphi_n = \varphi \,\Box\, n \|\ \|$. For every n let G_n be a dense G_δ-subset of C such that $\varphi_n | C$ is Gâteaux-differentiable at every $x \in G_n$. For $x \in G_n$ let $f_{x,n}$ denote the Gâteaux-derivative $(\varphi_n | C)'(x)$.

We claim the existence of an open dense subset G_* of C such that φ is locally Lipschitz at every $x \in G_*$. Indeed, let U be a relatively open and nonempty subset of C. By assumption we have $\partial\varphi(y) \neq \varnothing$ on a dense G_δ-subset G_0 of C, which implies $G_0 \subset \bigcup_{n=1}^{\infty} C_n$. Here $C_n = \{z \in C : \varphi(z) = \varphi_n(z)\}$ (see the proof of Proposition 1 and [HU]). We deduce that $\bigcup_{n=1}^{\infty} (C_n \cap U)$ is of the second category in C. This implies $\operatorname{int}(C_n \cap U) \neq \varnothing$ for some n, since C_n is closed in C. Here int refers to the interior relative to C. Choose V_U nonempty and relatively open in C and contained in $C_n \cap U$. Then φ is Lipschitz on V_U with constant n. Setting $G_* = \bigcup \{V_U : U \neq \varnothing \text{ open in } C\}$ finally provides a dense open subset of C such that φ is locally Lipschitz at every x in G_*.

Let $G = G_* \cap \bigcap_{n=1}^{\infty} G_n$, then G is a dense G_δ-subset of C. We prove that G fulfills the requirements of our theorem. Indeed, let $x \in G$ be fixed and choose a neighbourhood U of x in C such that $\varphi = \varphi_n = \varphi_{n+1} = \cdots$ holds on U. This clearly implies $f_{x,n} = f_{x,n+1} = \cdots =: f_x$ on C. Therefore f_x is the Gâteaux-derivative of φ at x. ∎

As we have pointed out already, the Gâteaux-differentiability of a convex function at a boundary point of its domain does not necessarily imply the uniqueness of the subdifferential. The uniqueness of $\partial\varphi(x)$ at a differentiability point x may be obtained, however, if x is a nonsupport point of the domain C.

PROPOSITION 2. *Let E be a Banach space of class (S) and let C be a convex subset of E which has at least one nonsupport point. Suppose C is a Baire space in the induced topology and let $\varphi : C \to \mathbb{R}$ be a convex lower semi-continuous function having nonempty subdifferential on a dense G_δ-subset of C. Then there exists a dense G_δ-subset G of C such that for every $x \in G$, φ is Gâteaux-differentiable at x with unique subdifferential $\partial\varphi(x) = \{\varphi'(x)\}$.*

Proof. By Theorem 1 there exists a dense G_δ-subset G_0 of C such that φ is Gâteaux-differentiable at every $x \in G_0$. Since C has at least one nonsupport point, a result of Phelps [P] tells that the nonsupport points of C form a dense G_δ-subset G_1 of C. Let $G = G_0 \cap G_1$, then G fulfills the requirements of the theorem. Indeed, let $x \in G$, then $\varphi'(x)$ is maximal in $\partial\varphi(x)$ with respect to the order induced by $K(C, x)$. But notice that $K(C, x)$ is dense in E as a consequence of the fact that x is a nonsupport point of C (see [K]). This implies the uniqueness of $\partial\varphi(x)$ since the order induced by $K(C, x)$ reduces to equality. ∎

COROLLARY 2. *Let E be a separable Banach space and let C be a convex subset of E not contained in a closed hyperplane. Suppose C is a Baire space in the induced topology and let $\varphi : C \to \mathbb{R}$ be a lower semi-continuous convex function satisfying $\partial\varphi(x) \neq \varnothing$ on a dense G_δ-subset of C. Then there exists* *a dense G_δ-subset G of C such that φ is Gâteaux-differentiable at every $x \in G$ with unique subdifferential $\partial\varphi(x) = \{\varphi'(x)\}$.*

Proof. E is of class (S) by Stegall's theorem. Moreover, an old result of Klee's [K, H] states that C, being separable, has at least one nonsupport point. ∎

Remark. The result of Ekeland quoted above tells that a lower semicontinuous convex function φ has nonempty subdifferential $\partial\varphi(x)$ on a dense subset of its domain C. In general, however, this set is of the first category in C. Since we are interested in results on generic Gâteaux-differentiability of φ, the theorem of Ekeland is of no help. What we need is a description of the situation when $\partial\varphi(x)$ is nonempty generically. In [N2] we have obtained the answer to this problem. It states that $\partial\varphi(x)$ is nonempty generically on a convex Baire set C if and only if φ is locally Lipschitz at the points of a dense subset of C. Moreover, the reasoning in the proof of Theorem 1 shows that this dense set may in fact be chosen to be open in C.

3. CS-Closed Sets

Our main Theorem 1 gives some information on the differentiability of a convex function defined on a convex subset of a Banach space which need not have interior points but is assumed to be Baire in the induced topology. In the following we show that one may even gain some information on the differentiability of convex functions defined on a fairly larger class of convex sets in Banach spaces, namely, the class of CS-closed sets in the sense of Jameson [J]. Recall that a convex set C in a Banach space E is called CS-closed if every convergent series $\sum_{n=1}^{\infty} \lambda_n x_n$, $x_n \in C$, $0 \leqslant \lambda_n \leqslant 1$, $\sum_{n=1}^{\infty} \lambda_n = 1$, converges to an element of C. Clearly every closed convex subset of E is CS-closed. Fremlin and Talagrand [FT] have proved that every convex G_δ-set in a Fréchet space is CS-closed, so the class of CS-closed sets is fairly large.

Example. A CS-closed set which is of the first category in itself. Let $E = l^1$ and let C be the order cone corresponding to the lexicographic order on E, i.e.,

$$C = \{x \in l^1 : x_1 = \cdots = x_{n-1} = 0, x_n \neq 0 \Rightarrow x_n > 0\}.$$

Then C is CS-closed, but it is of the first category in itself, for we have $C = \bigcup_{n,m=1}^{\infty} C_{n,m}$, where

$$C_{n,m} = \{x \in l^1 : x_1 = \cdots = x_{n-1} = 0, x_n \geqslant 1/m\},$$

and the sets $C_{n,m}$ have no interior points relative to C.

Let C be a convex cone with vertex 0 in E. We consider a new topology on C, called the cone topology and noted σ, by taking as a base of neighbourhoods of $x \in C$ with respect to σ the sets

$$V(x, \varepsilon) = \{ y \in C : \| y - x \| < \varepsilon, y - x \in C \},$$

$\varepsilon > 0$. This topology has been introduced by Saint-Raymond [SR] in a special case and has been further investigated in [N1]. Its merits are to be found in the following.

PROPOSITION 3. *Let C be a CS-closed convex cone with vertex 0 in the Banach space E. Then C is a Baire space when endowed with its cone topology σ.*

Proof. Let (G_n) be a sequence of open dense sets in (C, σ) and let U be a nonempty, σ-open set in C. Choose $x_1 \in U \cap G_1$ and some $\varepsilon_1 > 0$ having $V(x_1, \varepsilon_1) \subset U \cap G_1$. Now $V(x_1, \varepsilon_1) \cap G_2 \neq \emptyset$. Let x_2 be chosen in this intersection. Choose $\varepsilon_2 > 0$, $\varepsilon_2 \leqslant \varepsilon_1/2$ such that $V(x_2, \varepsilon_2) \subset V(x_1, \varepsilon_1) \cap G_2$, etc. This yields a sequence (x_n) in C and a sequence (ε_n) with $\varepsilon_n \leqslant \varepsilon_{n-1}/2$ and $V(x_n, \varepsilon_n) \subset V(x_{n-1}, \varepsilon_{n-1}) \cap G_n$, $x_n - x_{n-1} \in C$. Clearly (x_n) converges to some $x \in E$. Since C is CS-closed, we deduce $x \in C$, and moreover $x \in U \cap \bigcap_{n=1}^{\infty} G_n$. This proves the result. ∎

This observation permits us to prove the following theorem on the differentiability of a convex function defined on a CS-closed set in a Banach space.

THEOREM 2. *Let E be a Banach space such that $E \times \mathbb{R}$ is of class (S) and let C be a CS-closed convex subset of E not contained in a closed hyperplane. Let $\varphi : C \to \mathbb{R}$ be a convex function which is locally Lipschitz at the points of a relatively open dense subset of C. Then there exists a dense subset D of C such that φ is Gâteaux-differentiable at every $x \in D$.*

Proof. Let us first consider the case where C is a CS-closed convex cone with vertex 0 in E. Clearly it suffices to prove the statement for every $\varphi \mid U$, where U is a relatively open subset of C such that φ is Lipschitz on U. So let us assume that φ itself is a Lipschitz map. It consequently extends to a convex Lipschitz map ψ on E.

Proposition 4 tells us that C is a Baire space when endowed with its cone topology σ. The subdifferential mapping $\partial\psi$ being upper semi-continuous with respect to the norm topology on C and the weak star topology on E', this is also true with respect to the cone topology σ on C and the weak star topology on E'. Applying the definition of the spaces of class (S), we obtain a dense G_δ-subset D of (C, σ) and a selector $x \to f_x$ for $\partial\psi$ which is continuous with respect to σ on C and $\sigma(E', E)$ on E' at every $x \in D$. We claim that this implies

$$\lim_{t \downarrow 0} \frac{1}{t}(\varphi(x+tz) - \varphi(x)) = f_x(z) \tag{$*$}$$

for every $z \in C$, when $x \in D$. Indeed, suppose we had

$$\frac{1}{t}(\varphi(x+tz) - \varphi(x)) \geqslant f_x(z) + \varepsilon$$

for some $z \in C$, $\varepsilon > 0$, and all $t > 0$. This implies

$$f_{x+tz}(z) \geqslant f_x(z) + \varepsilon$$

for $t > 0$, a contradiction since $x + tz$ converges to x in the cone topology and since $x \in D$. This proves $(*)$.

Let us now consider the general case. Again we may restrict ourselves to the situation where φ is Lipschitz on C. Now let $\tilde{C}$ denote the convex cone with vertex $(0, 0)$ in $E \times \mathbb{R}$ generated by $C \times \{1\}$, i.e., $\tilde{C} = \mathbb{R}_+(C \times \{1\})$. Let $\tilde{\varphi}: \tilde{C} \to \mathbb{R}$ be the sublinear operator defined by $\tilde{\varphi}(\lambda x, \lambda) = \lambda\varphi(x)$, $\lambda \geqslant 0$, $x \in C$. φ being Lipschitz, we deduce that $\tilde{\varphi}$ is locally Lipschitz at points $(\lambda x, \lambda)$, $x \in C$, $\lambda > 0$, in view of the estimate

$$|\tilde{\varphi}(\lambda x, \lambda) - \tilde{\varphi}(\mu y, \mu)| \leqslant \mu \, \|\varphi(x) - \varphi(y)\| + \|\varphi(x)\| \, |\lambda - \mu|.$$

Therefore the first part of our proof may be applied in view of the fact that $E \times \mathbb{R}$ is again of class (S) and C is a CS-closed convex cone. This provides a dense subset $\tilde{D}$ of $\tilde{C}$ such that $\tilde{\varphi}$ satisfies property $(*)$, i.e., for all $(\lambda x, \lambda) \in \tilde{D}$ there exists $f_{(\lambda x, \lambda)} \in (E \times \mathbb{R})'$ satisfying

$$\lim_{t \downarrow 0} \frac{1}{t}(\tilde{\varphi}((\lambda x, \lambda) + t(\mu y, \mu)) - \tilde{\varphi}(\lambda x, \lambda)) = f_{(\lambda x, \lambda)}(\mu y, \mu) \tag{$**$}$$

for $\mu \geqslant 0$, $y \in C$. Since $\tilde{\varphi}$ is sublinear, differentiability in this restricted sense at some $(\lambda x, \lambda) \in \tilde{C}$ clearly implies differentiability at every $(\mu x, \mu)$, $\mu > 0$. So we may and will assume that $\tilde{D}$ is radial, i.e., $(\lambda x, \lambda) \in \tilde{D}$ for some $\lambda > 0$ implies $(\mu x, \mu) \in \tilde{D}$ for all $\mu > 0$.

Let D denote the subset of C consisting of all vectors z such that $(z, 1) \in \tilde{D}$. Then D is dense in C. We prove that φ is Gâteaux-differentiable at every $x \in D$. Let $x \in D$ be fixed. Then $f_{(x,1)} \in (E \times \mathbb{R})'$ admits a representation $f_{(x,1)} = f_x + \gamma_x$ for some $f_x \in E'$ and some scalar γ_x. Notice that $\gamma_x = \varphi(x) - f_x(x)$, which may be seen by inserting $(x, 1)$ in formula $(**)$. We prove that f_x is the Gâteaux-derivative of φ at x.

Let $z \in K(C, x)$ be fixed and choose $t_0 > 0$ having $x + t_0 z \in C$. Now let

$0<t<t_0$. Setting $s=t/(t_0-t)$, we find $t=st_0/(1+s)$. Applying $(**)$ to $(x, 1)$ and $(x+t_0 z, 1)$, we obtain

$$\lim_{t \downarrow 0} \frac{1}{s}\left(\tilde{\varphi}((x, 1)+s(x+t_0 z, 1))-\tilde{\varphi}(x, 1)\right)=f'_{(x, 1)}(x+t_0 z, 1),$$

since s tends to 0 as $t \downarrow 0$. But notice that we have

$$\begin{aligned}
&\frac{1}{t}(\varphi(x+tz)-\varphi(x)) \\
&\quad=\frac{1}{s}\left[(1+s)\, \varphi(x+tz)-\varphi(x)-s\varphi(x+tz)\right] \frac{s}{t} \\
&\quad=\frac{1}{s}\left[\tilde{\varphi}((x, 1)+s(x+t_0 z, 1))-\tilde{\varphi}(x, 1)\right] \frac{s}{t}-\varphi(x+tz) \frac{s}{t},
\end{aligned}$$

so this quotient tends to

$$f'_{(x, 1)}(x+t_0 z, 1) \frac{1}{t_0}-\frac{1}{t_0} \varphi(x)=f'_x(z) \qquad (t \downarrow 0).$$

This ends the proof of Theorem 2. ∎

Again we may ask for conditions under which φ is Gâteaux-differentiable on a dense subset of a CS-closed set such that, in addition, the subdifferential is unique.

Proposition 4. *Let E be a Banach space of class (S) and let C be a CS-closed convex subset of E having at least one nonsupport point. Let $\varphi: C \rightarrow \mathbb{R}$ be a convex function which is locally Lipschitz at the points of a relatively open dense subset of C. Then there exists a dense subset D of C such that φ is Gâteaux-differentiable at every $x \in D$ with unique subdifferential.*

Proof. Suppose x is a nonsupport point of C. Then $(x, 1)$ is a nonsupport point of $\tilde{C}$, the convex cone with vertex $(0,0)$ in $E \times \mathbb{R}$ generated by $C \times \{1\}$. The result of Phelps quoted in Section 2 tells that the nonsupport points of $\tilde{C}$ therefore form a dense G_δ-subset of $\tilde{C}$ with respect to the norm topology. But note that the set of nonsupport points is even dense in $\tilde{C}$ with respect to the cone topology σ. Indeed, if $(x, 1)$ is a nonsupport point of $\tilde{C}$, then so is every point of the form $(\lambda y, \lambda)+t(x, 1)$, $t>0$, hence every point $(\lambda y, \lambda) \in \tilde{C}$ can be approximated by nonsupport points in the sense of the cone topology. Therefore the set of nonsupport points of $\tilde{C}$ is a dense G_δ-subset of $\tilde{C}$ with respect to the cone topology.

Passing through the proof of Theorem 2 once more, we may now start with a dense subset $\tilde{D}$ of $\tilde{C}$ consisting of nonsupport points of $\tilde{C}$. Notice that this is not destroyed even when we replace $\tilde{D}$ by its radial hull. Letting D the set of $x \in C$ for which $(x, 1) \in \tilde{D}$, we finally construct a dense subset of C consisting of nonsupport points of C, for which the statement of Theorem 2 is satisfied. But now the argument used in Proposition 2 shows that for every $x \in D$, the subdifferential $\partial\varphi(x)$ is singleton. ∎

Remark. It is not clear from our approach whether Theorem 2 is still valid in the Fréchet case. Clearly a Fréchet version of $(*)$ may be proved, but the "uniform nature" of the argument is destroyed in the second part of our proof.

4. Differentiability at the Boundary

So far we have been dealing with problems of generic differentiability of convex functions on sets without interior points. When applied to the classical case of a function whose domain actually has interior points, we just obtain the classical results on generic differentiability. So our approach does not give us any new information concerning the problem of existence of points of differentiability at the boundary of a convex set C when C *has* interior points. The following result shows that some information may be obtained concerning this question when a stronger continuity assumption is imposed on the function φ. First, however, we need a definition.

A Banach space E is said to have Čech complete ball if its unit ball B is a Čech complete topological space when endowed with the weak topology (see [EW]). For a survey on this and related properties we must refer to [EW].

Proposition 5. *Let E be Banach space with Čech complete ball and let C be a bounded convex body in E. Let $\varphi: C \to \mathbb{R}$ be a convex function such that locally φ extends to a convex function ψ on E such that ψ is continuous with respect to some compatible topology on E strictly weaker than the norm topology. Then there exists at least one boundary point x of C such that φ is Fréchet-differentiable at x.*

Proof. We may assume that φ itself extends to a convex function ψ on E which is continuous with respect to the compatible topology τ, $\sigma(E, E') \leqslant \tau < \|\ \|$. Since C is bounded, it is contained in some multiple of B, hence C is Čech complete with $\sigma(E, E')|C$. We claim that $(C, \tau | C)$ therefore is a Baire space.

Since $(C, \sigma(E, E')| C)$ is Čech complete, there exists a sequence $(\mathfrak{U}_n)$ of weakly relatively open coverings of C such that $\mathfrak{U}_{n+1}$ refines $\mathfrak{U}_n$ and such that every filter $\mathfrak{F}$ on C having $\mathfrak{F} \cap \mathfrak{U}_n \neq \emptyset$ for every n has a cluster point in C with respect to $w=\sigma(E,E')|C$. Now let (G_n) be a sequence of dense open subsets of (C,τ) and let U be nonempty and τ-open in C. Since τ is F-linked to w, i.e., every $x\in C$ has a τ-neighbourhood base consisting of w-closed sets, there exists a nonempty τ-open set U_1 such that U_1 is contained in some element of $\mathfrak{U}_1$ and such that the w-closure of U_1 is contained in $U\cap G_1$. Similarly, there exists a nonempty τ-open set U_2 contained in some element of $\mathfrak{U}_2$ such that the w-closure of U_2 is contained in $U_1\cap G_2$, etc. This provides a sequence (U_n) of τ-open sets such that U_i is contained in some element of $\mathfrak{U}_i$ and such that U_i is nonempty and its w-closure is contained in $U_{i-1}\cap G_i$. Let $\mathfrak{F}$ be the filter generated by (U_n). Then $\mathfrak{F}\cap\mathfrak{U}_n$ is nonempty for every n, so $\mathfrak{F}$ has a w-cluster point $x\in C$. But now we must have $x\in U_n$ for every n, since x is in the w-closure of each U_n. This proves $U\cap\bigcap_{n=1}^{\infty}G_n\neq\emptyset$.

Since ψ is τ-continuous by assumption, the subdifferential mapping $\partial\psi: C\to 2^{E'}$ is upper semi-continuous with respect to $\tau|C$ and $\sigma(E',E)$ on E' and, moreover, has nonempty convex and $\sigma(E',E)$ compact values. Therefore by the result of Christensen and Kenderov in [CK] there exist a τ-dense G_δ-subset G of C and a selection $x\to f_x$ for $\partial\psi$ which is continuous with respect to τ on C and the norm topology on E' at the points x of G. Here we have made use of the fact that E, having Čech complete ball, is an Asplund space (see [EW]). Clearly this implies that $\varphi=\psi|C$ is Fréchet-differentiable at every $x\in G$. It therefore remains to prove that G contains at least one norm-boundary point of C.

Observe that ∂C, the norm boundary of C, is a G_δ-set in (C,w). Indeed, let $0\in C$ and let q be the Minkowski functional of C, then q is weakly lower semi-continuous and we have $\partial C=\{x\in E: q(x)=1\}=\bigcap\{\{x\in C: q(x)>1-1/n\}: n\in\mathbb{N}\}$. Consequently, ∂C is as well a G_δ-set in (C,τ). Suppose now we had $\partial C\cap G=\emptyset$. So $C\backslash\partial C$ is a second category F_σ-set in (C,τ). This implies that it has nonempty interior in (C,τ). Since τ is a locally convex topology, there exists a nonempty convex and τ-open set V in E such that $V\cap C\subset C\backslash\partial C$ is nonempty. But this implies $V\subset C$, for otherwise the set V, being convex, would contain some point of ∂C. So C has nonempty τ-interior. Since C is norm-bounded by assumption, we derive that $\tau=\|\ \|$. But this was excluded, so we obtain the desired contradiction. So G intersects ∂C. ∎

5. Rademacher's Theorem—Category Version

In this final section we prove a category analogue of the classical Rademacher theorem in infinite-dimensional vector spaces.

Let C be a convex body in a normed space E and let $\varphi: \partial C\to\mathbb{R}$ be any function. Let $x \in \partial C$ be fixed. φ is called Fréchet-differentiable at x if there exists $f \in E'$ satisfying

$$\lim_{\substack{\|z\| \to 0 \\ x+z \in \partial C}} \frac{1}{\|z\|} (\varphi(x+z) - \varphi(x) - f(z)) = 0.$$

The classical Rademacher theorem tells in particular that if φ is a Lipschitz mapping defined on a Lipschitzian manifold M in $\mathbb{R}^n$ of dimension $n-1$, then φ is differentiable in the above sense at the points $x \in M \setminus N$, where N is a subset of M having $(n-1)$-dimensional Hausdorff measure 0. Dealing with convex functions, we shall be concerned only with Lipschitzian manifolds M of a very special kind, namely, with manifolds $M = \partial C$, where C is a convex body in a Banach space. We obtain the following.

PROPOSITION 6. *Let E be an Asplund space and let C be a convex body in E. Let $\varphi: C \to \mathbb{R}$ be a convex function satisfying $\partial\varphi(x) \neq \emptyset$ for every $x \in C$. Then there exists a first category subset P of ∂C such that $\varphi | \partial C$ is Fréchet-differentiable at every $x \in \partial C \setminus P$.*

Proof. Using our standard reduction argument, we may restrict our considerations to the case where φ is a Lipschitz function on C and therefore extends to a convex Lipschitz function ψ on the whole space E.

We consider the operator $\partial\psi: \partial C \to 2^{E'}$. Using the result of Christensen and Kenderov [CK], we find a first category subset P of ∂C and a selection $x \to f_x$ for $\partial\psi$ on ∂C such that $x \to f_x$ is continuous at every $x \in \partial C \setminus P$ with respect to the norm topologies on $\partial C \setminus P$ and E', respectively. We claim that for fixed $x \in \partial C \setminus P$ f_x is a Fréchet-derivative for $\varphi | \partial C$ at x.

Assume the contrary. Then there exists a sequence (z_n) with $\|z_n\| \to 0$, $x + z_n \in \partial C$, such that

$$\varphi(x+z_n) - \varphi(x) - f_x(z_n) \geqslant \varepsilon \|z_n\|$$

holds for some $\varepsilon > 0$. Having regard of the inequality

$$f_{x+z_n}(z_n) \geqslant \varphi(x+z_n) - \varphi(x),$$

we find that $\|f_{x+z_n} - f_x\| \geqslant \varepsilon$, and this contradicts the continuity of the mapping $y \to f_y$ at x. This proves the result. ∎

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