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Generic Fréchet – differentiability of convex functions on small sets

By

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1. Introduction. It is well-known that a continuous convex real-valued function φ defined on a Banach space E is Fréchet-differentiable at each point of a dense G_δ-subset of E provided that E is an Asplund space, which means e.g. that its dual E' has the Radon-Nikodým property (see [1]). In this note we shall prove an analogue of this result in the case where φ is a convex function defined on a small subset C of E, E an Asplund space. Here a set C in a Banach space E is called small when it has no interior points.

Let E be a Banach space, C a convex subset of E, $\varphi: C \to \mathbb{R}$ a convex function. For $x \in C$, the subdifferential $\partial\varphi(x)$ of φ at x is

$$\partial\varphi(x) = \{f \in E' : f(y - x) \leqq \varphi(y) - \varphi(x) \text{ for all } y \in C\}.$$

In contrast with the case where x is an interior point of C, this set may be empty, and – even when it is known to be nonempty – may be unbounded.

Definition. Let E be a Banach space and let C be a convex subset of E not contained in a closed hyperplane. Let $\varphi: C \to \mathbb{R}$ be a convex function. φ is called Fréchet-differentiable at $x \in C$ if there exists $f_x \in \partial\varphi(x)$ satisfying

$$\lim_{\substack{t \to 0 \\ t > 0}} \sup_{\substack{\|z\| \leqq 1 \\ x + tz \in C}} \left| \frac{1}{t}(\varphi(x + tz) - \varphi(x)) - f_x(z) \right| = 0.$$

In this case we use the notation $\varphi'(x) = f_x$. $\square$

Notice that the assumption that C is not contained in a closed hyperplane of E is necessary to ascertain the uniqueness of the Fréchet-derivative $\varphi'(x)$, should it exist.

For $x \in C$ we denote by $K(C, x)$ the cone over C at x, i.e. the set of directions z having $x + tz \in C$ for some $t_0 > 0$ and all $0 < t < t_0$. Then our definition may be rephrased by saying that φ is Fréchet-differentiable at $x \in C$ if $\frac{1}{t}(\varphi(x + tz) - \varphi(x)) - f_x(z)$ tends to 0 $(t \to 0, t > 0)$ uniformly over all $z \in K(C, x)$ having $\|z\| \leqq 1$.

2. Existence of subgradients. It is known that the subdifferential $\partial\varphi(x)$ of a convex function φ is nonempty when x is an interior point of its domain C. If C is small, however, $\partial\varphi(x)$ may be empty throughout C, although it is known (see [3]) that $\partial\varphi(x)$ is nonempty on a dense subset of C in the case where C is closed and φ is lower semi-continuous on C (or rather when $\varphi: E \to \overline{\mathbb{R}}$, extended by $\varphi(x) = \infty$ for all $x \notin C$, is lower semi-continuous in the usual sense). For our present attempt, however, we shall need some information concerning the question when $\partial\varphi(x)$ is nonempty on a large subset of C in the sense of category. This is provided by the following

Proposition. *Let E be a Banach space and let C be a convex subset of E which is a Baire space in its induced topology. Let $\varphi: C \to \mathbb{R}$ be a lower semi-continuous convex function. Then the following statements are equivalent:*

(i) *There exists a dense relatively open subset G of C such that $\partial\varphi(x)$ is nonempty for every $x \in G$;*
(ii) *There exists a dense relative G_δ-subset G_1 of C such that $\partial\varphi(x)$ is nonempty for every $x \in G_1$;*
(iii) *There exists a dense Baire subset G_2 of C such that $\partial\varphi(x)$ is nonempty for every $x \in G_2$;*
(iv) *There exists a dense relatively open subset G of C such that φ is locally Lipschitz at every $x \in G$;*
(v) *There exists a dense subset D of C such that φ is locally Lipschitz at every $x \in D$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear. We prove (iii) $\Rightarrow$ (iv). For $r \in \mathbb{N}$ define a convex function $\varphi_r: E \to \mathbb{R} \cup \{-\infty\}$ by

$$\varphi_r(z) = \inf\{\varphi(x) + r\,\|y\| : x \in C, z = x + y\}. \tag{1}$$

φ_r is called the infimal convolution of φ and $r\,\|\ \|$, noted $\varphi * r\,\|\ \|$. It is known that either $\varphi_r \equiv -\infty$ or φ_r is finite everywhere, and in the latter case is Lipschitz with Lipschitz constant r. The coincidence set C_r of φ and φ_r is

$$C_r = \{x \in C: \partial\varphi(x) \cap rB' \neq \emptyset\}, \tag{2}$$

where B' denotes the dual unit ball. For a detailed discussion of the functions φ_r we refer to [4].

Let U be any relatively open and nonempty subset of C. By (2) and (iii), the set $\bigcup_{r \geqq 1} (C_r \cap U)$ is of the second category in U. φ being lower semi-continuous on C, the sets $C_r = \{x \in C: \varphi(x) = \varphi_r(x)\}$ are closed in C, hence for some r, $C_r \cap U$ must have nonempty interior in C. Therefore U contains some nonempty relatively open subset V_U such that $\varphi = \varphi_r$ on V_U, so that φ is Lipschitz on V_U with Lipschitz constant r. But now $G_0 = \bigcup \{V_U: U \neq \emptyset$ relatively open in $C\}$ is an open dense subset of C such that φ is locally Lipschitz at every $x \in G_0$. This proves (iv).

Trivially (iv) implies (v). Proving that (v) implies (i) remains. Let D be given as in the statement of (v) and let $x \in D$ be fixed. Let U be a convex and relatively open neighbourhood of x in C such that $\varphi \mid U$ is Lipschitz with constant $r \in \mathbb{N}$, say. Let $\psi = \varphi \mid U$, $\psi_r = \psi * r\,\|\ \|$ the infimal convolution of ψ and $r\,\|\ \|$. We claim that ψ and ψ_r coincide on U. Assume the contrary. Then there exists $z \in U$ having $\psi_r(z) < \psi(z)$. Hence there exists $x \in U$ and $y \in E$ with $z = x + y$ such that $\psi(x) + r\,\|y\| < \psi(z)$, equivalently,

$$\psi(z) - \psi(x) > r\,\|z - x\|. \tag{3}$$

This contradicts the fact that ψ is Lipschitz on U with constant r, so $\psi = \psi_r$ on U is proved.

We have proved that φ and ψ_r coincide on U. This implies $\partial\psi_r(y) \subset \partial\varphi(y)$ for every $y \in U$ since the notion of a subdifferential is a local one. But notice that ψ_r is globally defined and hence satisfies $\partial\psi_r(y) \neq \emptyset$ everywhere. This proves $\partial\varphi(y) \neq \emptyset$ on U. Since $x \in D$ and U were chosen arbitrarily, the proof of (i) is complete. □

Remarks. 1) We do not need any category assumption on the set D in statement (v). But actually we do need a category assumption in statement (iii), i.e. the existence of a dense subset of points x having $\partial\varphi(x) \neq \emptyset$ does not imply local Lipschitz. Indeed, let us consider the following example. Let $C \subset l_2$ be the cube $\prod_{n \in \mathbb{N}} \left[-\frac{1}{n}, \frac{1}{n}\right]$ and define $\varphi : C \to \mathbb{R}$ by

$$\varphi(x) = \sum_{n=1}^{\infty} 2^{-n} \varphi_n(x_n),$$

where φ_n is the convex real function defined by the lower part of the circle with radius $1/n$ and centre 0. Then φ is continuous on C but is nowhere locally Lipschitz, since every nonempty relatively open subset U of C contains a point x such that $|x_n| = 1/n$ for some n. Nevertheless, $\partial\varphi(x)$ is nonempty on a dense subset D of C. In fact, $\partial\varphi(x)$ will be nonempty if the sequence x is eventually "away from $1/n$" so that the derivatives $\varphi_n'(x_n)$ provide an l_2 sequence.

2) Notice that lower semi-continuity of $\varphi : C \to \mathbb{R}$ does not mean that the function $\bar{\varphi} : E \to \mathbb{R} \cup \{\infty\}$ defined by $\bar{\varphi} \mid C = \varphi$, $\bar{\varphi}(x) = \infty$ otherwise, is lower semi-continuous in the usual sense. Both notions coincide if the set C is closed, but in general, lower semi-continuity of $\varphi \mid C$ is a weaker statement.

3. The main result. In this section we prove our main result on the existence of Fréchet-derivatives for convex functions on small sets.

Theorem. *Let E be an Asplund space and let C be a convex G_δ-subset of E not contained in a closed hyperplane. Let $\varphi : C \to \mathbb{R}$ be a lower semi-continuous convex function such that φ is locally Lipschitz on a dense subset of C. Then there exists a dense G_δ-subset G of C such that*

(i) *φ has a Fréchet-derivative $\varphi'(x) = f_x$ at every $x \in G$;*
(ii) *Whenever $x \in G$, $\varphi'(x) = f_x$, then f_x is a maximal subgradient of φ at x in the sense that given any further $f \in \partial\varphi(x)$, $f(z) \leqq f_x(z)$ holds for all $z \in K(C, x)$.*

Proof. Let r be any integer such that $\varphi_r = \varphi * r \| \|$ is finite. Since φ_r is a continuous, convex function defined on E, the subdifferential mapping $x \to \partial\varphi_r(x)$ is known to be a set-valued, monotone operator having nonempty, convex and $\sigma(E', E)$-compact values in E' which is upper semi-continuous with respect to the norm topology on E and the topology $\sigma(E', E)$ on E' (see [1] or [2] for definitions). Since C is a Baire space with the relative topology, it follows from a result of Christensen and Kenderov [2, Theorem 1.3] that there exists a dense G_δ-subset G_r of C such that for every $x \in G_r$ there exists $f_{x,r} \in \partial\varphi_r(x)$ such that

$$(4)\qquad \begin{cases} \text{for every } \varepsilon > 0 \text{ there exists a neighbourhood } U \text{ of } x \text{ in } C \text{ such that} \\ \text{for every } y \in U \text{ we have} \\ \inf\{\|f_{x,r} - g\| : g \in \partial\varphi_r(y)\} \leqq \varepsilon. \end{cases}$$

Let G_0 be an open dense subset of C on which $\partial\varphi(x) \neq \emptyset$ and let $G = G_0 \cap \bigcap_{r \geqq 1} G_r$. We claim that G fulfills the requirements of the theorem.

Let us first prove that given $x \in G_r$, $f_{x,r}$ is the Fréchet-derivative of $\varphi_r | C$ in the sense of our definition. Suppose this is not true and find vectors $z_n \in E$, $\|z_n\| \leqq 1$, and $t_n > 0$, $t_n \to 0$ having $x + t_n z_n \in C$ such that

$$(5)\qquad \left|\frac{1}{t_n}(\varphi_r(x+t_n z_n) - \varphi_r(x)) - f_{x,r}(z_n)\right| \geqq \varepsilon$$

holds for some $\varepsilon > 0$. $f_{x,r}$ being a subgradient of φ_r at x, this actually implies

$$(6)\qquad \frac{1}{t_n}(\varphi_r(x + t_n z_n) - \varphi_r(x)) - f_{x,r}(z_n) \geqq \varepsilon.$$

Choose $f_n \in \partial\varphi_r(x + t_n z_n)$, then we find

$$f_n(z_n) - f_{x,r}(z_n) \geqq \frac{1}{t_n}(\varphi_r(x + t_n z_n) - \varphi_r(x)) - f_{x,r}(z_n) \geqq \varepsilon,$$

which gives us $\|f_n - f_{x,r}\| \geqq \varepsilon$. Since $x + t_n z_n \to x$, this contradicts Property (4) of the functional $f_{x,r}$. Hence $f_{x,r}$ is in fact the Fréchet-derivative of φ_r at $x \in G_r$.

Next observe that for fixed $x \in G$, there exists a neighbourhood U of x in C such that $\varphi = \varphi_r = \varphi_{r+1} = \ldots$ holds on U. Since C is not contained in a hyperplane, this implies $f_{x,r} = f_{x,r+1} = \ldots =: f_x$. Since $\varphi = \varphi_r$ on U and $\varphi_r'(x) = f_x$, this implies the desired relation $\varphi'(x) = f_x$, the definition of the Fréchet-derivative being a local one. Hence (i) is proved.

In order to prove (ii), it will again be sufficient to show that for fixed $x \in G_r$, $f(z) \leqq f_{x,r}(z)$ will hold for all $z \in K(C, x)$. Indeed, taking into account the formula

$$(7)\qquad \partial\varphi_r(x) = \partial\varphi(x) \cap rB'$$

(see [4]) and the fact that the sequence $f_{x,r}$, $r = 1, 2, \ldots$ is eventually constant for $x \in G$, it is clear that f_x will be maximal in the sense of statement (ii), once the corresponding maximality of $f_{x,r}$ in $\partial\varphi_r(x)$ is proved.

Let $x \in G_r$ be fixed. Let $f \in \partial\varphi_r(x)$ and suppose there exists $z \in K(C, x)$ satisfying

$$(8)\qquad f(z) - f_{x,r}(z) =: \varepsilon > 0.$$

Recall that the set-valued operator $\partial\varphi_r$ is locally bounded (see [7] or [1]). Hence there exists a neighbourhood U of x in C such that $\partial\varphi_r(U)$ is contained in some closed ball B in E' with centre 0. Choose $t_0 > 0$ such that $x + tz \in U$ holds for all $0 < t < t_0$. Let $g_t \in \partial\varphi_r(x + tz)$. Using the monotonicity of the subdifferential mapping $\partial\varphi_r$, we find that

$$(9)\qquad g_t(z) - f(z) \geqq 0.$$

This gives us

$$\begin{aligned} g_t(z) - f_{x,r}(z) &= g_t(z) - f(z) + f(z) - f_{x,r}(z) \\ &\geqq \quad 0 \quad + \quad \varepsilon = \varepsilon. \end{aligned}$$

This proves that $\partial\varphi_r(x + tz) \subset \{h \in E' : h(z) \geqq \varepsilon + f_{x,r}(z)\} =: K$ for $0 < t < t_0$. But $\partial\varphi_r(x + tz)$ is as well contained in B for $0 < t < t_0$ by the choice of U. Now $K \cap B$ is convex and $\sigma(E', E)$ – compact and does not contain $f_{x,r}$. Consequently, the separation theorem gives us some $y \in E$, $\delta > 0$ having

$$f_{x,r}(y) > \delta \geqq g(y), g \in K \cap B. \tag{10}$$

Consequently, $f_{x,r}(y) > \delta \geqq g_t(y)$, $0 < t < t_0$, hence $\|y\| \, \|f_{x,r} - g_t\| > \delta$, hence $\|f_{x,r} - g_t\| > \delta/\|y\|$, a contradiction with (4). This proves the claimed maximality of $f_{x,r}$ in $\partial\varphi_r(x)$. □

4. Uniqueness of subgradients. Dealing with convex functions on small sets, we may not expect that existence of the Fréchet-derivative of φ at $x \in C$ in the sense of our definition implies the uniqueness of the subdifferential $\partial\varphi(x)$ of φ at x, as it naturally does in the case where x is an interior point of C. Nevertheless, statement (ii) of our theorem tells that $\partial\varphi(x)$ has a unique maximal subgradient on a dense G_δ-subset of C, where maximality refers to the order induced by $K(C, x)$. The question as to whether a generic subset of C may be found on which $\partial\varphi(x)$ is singleton, depends, as it turns out, rather on the set C than on the function φ defined on C. Indeed, suppose for $x \in C$ there exists $f \in E'$, $f \neq 0$ satisfying $f(y) \leqq f(x)$ for all $y \in C$. Then no convex function φ defined on C will have a unique subgradient at x, for we may, given any subgradient g for φ at x, produce a new one by taking $g + f$. Clearly, this phenomenon cannot occur in the case where x is a non-support point for the set C. This raises the question whether convex sets have sufficiently many non-support points. In the case where E is a separable Banach space, the answer to this question is in the positive. Klee [5] proves that every separable convex set C which is not contained in a closed hyperplane has non-support points and that the set of non-support points is a dense G_δ in C. In the non-separable case, there exist closed convex sets not sited in a closed hyperplane but having no non-support points. Nevertheless, it has been proved by Phelps [6] that once the set of non-support points is known to be nonempty, it is always a dense G_δ in C. This permits us to state the following

Corollary. *Let E be an Asplund space and let C be a convex G_δ-subset of E having at least one non-support point. Let $\varphi: C \to \mathbb{R}$ be a lower semi-continuous convex function which is locally Lipschitz on a dense subset of C. Then there exists a dense G_δ-subset G of C such that for every $x \in G$, $\partial\varphi(x)$ contains a unique element f_x which is the Fréchet-derivative of φ at x.*

Proof. Since C has a non-support point, it may not be contained in a closed hyperplane. Consequently, by the theorem, there exists a dense G_δ-subset G_0 of C such that conditions (i) and (ii) from the theorem are satisfied. By the result of Phelps, we may find a dense G_δ-subset G of G_0 consisting of non-support points. We claim that $\partial\varphi(x) = \{f_x\}$ for all $x \in G$. Indeed, let $x \in G$, $f \in \partial\varphi(x)$, then condition (ii) implies

$f(z) \leqq f_x(z)$ for all $z \in K(C, x)$. But notice that, x being a non-support point of C, the cone $K(C, x)$ is dense in E (see [5]). Clearly this implies $f \leqq f_x$ on E, hence $f = f_x$. □

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