MINIMIZATION OF QUADRATIC FUNCTIONS ON CONVEX SETS WITHOUT ASYMPTOTES

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Abstract. The classical Frank and Wolfe theorem states that a quadratic function which is bounded below on a convex polyhedron P attains its infimum on P. We investigate whether more general classes of convex sets F can be identified which have this Frank-and-Wolfe property. We show that the intrinsic characterizations of Frank-and-Wolfe sets hinge on asymptotic properties of these sets.

Keywords. Quadratic optimization problem · asymptotes · conic asymptotes · Motzkin decomposition · Frank and Wolfe theorem · complementarity problem

1. Introduction

The classical Frank and Wolfe theorem [5, 4] states that a quadratic function q which is bounded below on a convex polyhedron P attains its infimum on P. It is known that this result has consequences with regard to the existence of solutions to linear complementarity problems [6]. Here we investigate ways in which the Frank and Wolfe theorem can be extended.

A first line is to go beyond polyhedra and ask whether there are more general classes of *Frank-and-Wolfe sets*, that is, convex sets F with the property that every quadratic function q which is bounded below on F attains its infimum on F. What one would like to obtain is an internal characterization of Frank-and-Wolfe sets via geometric properties, or likewise, verifiable sufficient conditions for the Frank-and-Wolfe property. In response we will characterize Frank-and-Wolfe sets as those convex sets which do not admit conic asymptotes in a sense to be made precise here.

A variant of the same question concerns the larger class of convex sets F with the property that every quadratic function q which is bounded below on F, and which is in addition convex or quasiconvex on F, attains its infimum on F. It turns out that this class has a nice internal characterization. It consists of those convex sets that do not have affine asymptotes in the sense of Klee [9].

A second idea to extend the Frank and Wolfe theorem would be to go beyond quadratics and look for more general classes of functions f attaining their finite infimum on polyhedra P. For instance, do higher degree polynomials f have this property? It turns out that without convexity this line has little hope for success, as shown by the quartic function $f(x) = x_1^2 + (1 - x_1 x_2)^2$, which has infimum 0 on the plane, but does not attain its infimum there. Positive results can at best be expected for convex polynomial functions f. For instance, Rockafellar [12, Cor. 27.3.1] shows that a convex polynomial f which is bounded below on a polyhedron P attains its infimum on P. Other variations of this theme are for instance Perold [11], Hirsch and Hoffman [7], or Belousov and Klatte [2].

The structure of the paper is as follows. In Section 2 we define Frank-and-Wolfe sets and variants and obtain first basic properties. Section 3 establishes the link between the

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Frank-and-Wolfe property and the absence of f-asymptotes in the sense of Klee. In Section 2 we consider the Frank-and-Wolfe property within the class of Motzkin decomposable sets, where one expects key information to be provided by the recession cone. Section 5 characterizes Frank-and-Wolfe sets by the absence of q-asymptotes, a geometric notion we define in Section 5. In the final Section 6 we obtain an application to generalized complementarity problems.

Notations

We generally follow Rockafellar's book [12]. The closure of a set F is $\overline{F}$. The Euclidean norm in $\mathbb{R}^n$ is $\|\cdot\|$, and the Euclidean distance is $\operatorname{dist}(x,y) = \|x-y\|$. For subsets M, N of $\mathbb{R}^n$ we write $\operatorname{dist}(M,N) = \inf\{\|x-y\| : x \in M, y \in N\}$. A direction d with $x + td \in F$ for every $x \in F$ and every $t \geq 0$ is called a direction of recession of F, and the cone of all directions of recession is denoted as 0^+F.

A function $q(x) = \frac{1}{2}x^\top Ax + b^\top x + c$ with $A = A^\top \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ is called quadratic. The quadratic $q : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex on a convex set $F \subset \mathbb{R}^n$ if the sublevel sets of $q \restriction F : F \to \mathbb{R}$ are convex. Similarly, q is convex on the set F if $q \restriction F$ is convex.

2. Frank and Wolfe sets

We call a convex set F in $\mathbb{R}^n$ *Frank-and-Wolfe* if every quadratic function $q : \mathbb{R}^n \to \mathbb{R}$ which is bounded below on F attains its infimum on F. For short we say that F is a FW-set. In the same vein we call the convex set F *quasi-Frank-and-Wolfe* if the property holds for every quadratic q which is in addition quasiconvex on F. For short, such sets are called qFW-sets.

Formally we may also consider convex sets F where the property holds for every quadratic q which is convex on F. We temporarily call those cFW-sets. Ultimately this class will turn out equivalent to quasi-Frank-and-Wolfe sets, i.e., $cFW = qFW$.

Clearly every bounded closed convex set is Frank-and-Wolfe, so the disquisition is only useful in studying unbounded convex sets. Trivially FW-sets are qFW, and qFW-sets are cFW. The classical theorem of Frank and Wolfe [5] says that every convex polyhedron P is a FW-set. Our first observation is the following.

Lemma 1. *Every cFW-set is closed, hence so are qFW- and FW-sets.*

Proof: Consider $x \in \overline{F}$, then $q(\cdot) = \|\cdot - x\|^2$ is quadratic convex and its infimum on F is 0. Since by hypothesis this infimum is attained, we must have $x \in F$. $\square$

Another useful property of Frank-and-Wolfe sets is the following.

Proposition 1. *Affine images of cFW-sets are cFW-sets. Similarly, affine images of qFW-sets are qFW, and affine images of FW-sets are FW. In particular, affine images of cFW-sets, qFW-sets and FW-sets are closed.*

Proof: Closedness of the affine image of a cFW-set F under an affine image follows from the first part of the statement in tandem with Lemma 1. To prove the first part let F be a cFW-set and $T : \mathbb{R}^n \to \mathbb{R}^m$ an affine operator. Let $\widetilde{F} = T(F)$. We show that $\widetilde{F}$ is cFW. Let $\widetilde{q} : \mathbb{R}^m \to \mathbb{R}$ be quadratic and convex on $\widetilde{F}$, and suppose it is bounded below on $\widetilde{F}$ with infimum γ. Then $q = \widetilde{q} \circ T$ is quadratic and convex on F, and bounded below on F with the same infimum γ. By the hypothesis on F the infimum γ of q is attained at $x_0 \in F$, and then clearly $\widetilde{q}$ attains its infimum γ on $\widetilde{F}$ at Tx_0.

Similarly, if $\widetilde{q}$ is quasiconvex on $\widetilde{F}$, then q is quasiconvex on F. Therefore the other two statements follow as well. $\square$

Yet another elementary property of FW-sets is the following

Proposition 2. *Suppose F is a FW-set, and let F' be a closed convex set containing F such that $F' \setminus F$ is bounded. Then F' is FW. The analogous statement holds for qFW-sets.*

Proof: Suppose q is a quadratic function with finite infimum γ' on F'. Then q has also a finite infimum γ on F, where obviously $\gamma \geq \gamma'$. There are two cases. If $\gamma = \gamma'$, then we choose $x \in F$ where γ is attained, and then γ' is also attained at x. On the other hand, if $\gamma' < \gamma$, then $\inf_{x \in F'} q(x) = \inf_{x' \in F' \setminus F} q(x')$. Since $F' \setminus F$ is bounded, there exists $x' \in \overline{F' \setminus F} \subset F'$ where the infimum γ' is attained. $\square$

3. F-asymptotes

Following Klee [9], an affine manifold M in $\mathbb{R}^n$ is called an f-asymptote of the closed convex set F if $F \cap M = \emptyset$ and $\operatorname{dist}(F, M) = 0$. The link between f-asymptotes and the Frank-and-Wolfe property is given by the following

Theorem 1. *Let F be a convex set in $\mathbb{R}^n$. Then the following statements are equivalent:*

(i) *Every quadratic function q which is quasiconvex on F and bounded below on F attains its infimum on F. That is, F is qFW.*
(ii) *Every quadratic function q which is convex on F and bounded below on F attains its infimum on F. That is, F is cFW.*
(iii) *F is closed and has no f-asymptotes.*

Proof: The implication $(i) \implies (ii)$ is clear. Consider $(ii) \implies (iii)$. We have to show that F is closed and has no f-asymptotes. Closedness follows readily from Lemma 1. Now let M be an affine manifold with $\operatorname{dist}(F, M) = 0$. We have to show that M is not an f-asymptote of F. Suppose $M = y + U$ for a direction space U and some $y \in U^\perp$. Let P be the orthogonal projection on $U^\perp$, then $P(M) = \{y\}$ and $M = P^{-1}(y)$. Since $\operatorname{dist}(F, M) = 0$, there exist sequences $x_k \in F$, $z_k \in M$, such that $\operatorname{dist}(x_k, z_k) \to 0$. Then $\operatorname{dist}(Px_k, Pz_k) \leq \operatorname{dist}(x_k, z_k) \to 0$, but $Pz_k = y$ for every k, hence $\operatorname{dist}(Px_k, y) \to 0$, so the sequence Px_k converges to y. Now since F has property (ii), its affine image $P(F)$ is closed by Proposition 1, so $y \in P(F)$. Pick $x \in F$ with $y = Px$, then $x \in F \cap P^{-1}(y) = F \cap M$, so that $F \cap M \neq \emptyset$. This shows that F does not have f-asymptotes.

It remains to prove the implication $(iii) \implies (i)$. We will prove this by induction on the dimension n of F. For dimension $n = 1$ the implication is clearly true, because any quadratic function $q : \mathbb{R} \to \mathbb{R}$ which is bounded below on a closed convex set $F \subset \mathbb{R}$ attains its infimum on F. Suppose therefore that the result is true for dimension $n - 1$, and consider a quadratic function $q : \mathbb{R}^n \to \mathbb{R}$ which is quasiconvex on F and bounded below on F. Assume without loss that the dimension of F is n, i.e., F has nonempty interior, as otherwise the result follows directly from the induction hypothesis. Let $\gamma = \inf\{q(x) : x \in F\} > -\infty$, and fix $\alpha > \gamma$. If the sublevel set $S_\alpha := \{x \in F : q(x) \leq \alpha\}$ is bounded, then by the Weierstrass extreme value theorem the infimum of q over S_α is attained. But this infimum is also the infimum of q over F, so in that case we are done. Assume therefore that S_α is unbounded. Since q is quasiconvex on F, the set S_α is closed convex, which means S_α has a direction of recession d, that is, a direction with $x + td \in S_\alpha$ for every $t \geq 0$ and every $x \in S_\alpha$ (see e.g. [12, Theorem 8.4]). Fix $x \in S_\alpha$. Expanding q at $x + td \in S_\alpha$ gives

$$\gamma \leq q(x + td) = \tfrac{1}{2} x^\mathsf{T} A x + b^\mathsf{T} x + c + t d^\mathsf{T}(Ax + b) + \tfrac{1}{2} t^2 d^\mathsf{T} A d \leq \alpha$$

for every $t \geq 0$, and this implies $d^\top Ad = 0$. Substituting this back gives

$$\gamma \leq q(x+td) = \tfrac{1}{2}x^\top Ax + b^\top x + c + td^\top(Ax+b) \leq \alpha,$$

for every $t \geq 0$. That implies $d^\top(Ax+b) = 0$. But the argument is valid for every $x \in S_\alpha$. By assumption F has dimension n, so S_α has nonempty interior, meaning $x + \epsilon B \subset S_\alpha$ for some $\epsilon > 0$, with B the unit ball. That shows $Ad = 0$. Going back with this into $d^\top(Ax+b) = 0$ shows $d^\top b = 0$, too. Altogether we have shown

$$q(x+td) = q(x) \text{ for every } x \in S_\alpha \text{ and every } t \geq 0. \tag{1}$$

Since q is a quadratic function and S_α has nonempty interior, this implies $q(x+td) = q(x)$ for every $x \in \mathbb{R}^n$ and every $t \in \mathbb{R}$.

Now let P be the orthogonal projection onto the hyperplane $H = d^\perp$. Then $\widetilde{q} := q \restriction H$ is quadratic on the $(n-1)$-dimensional space H and takes the same values as q due to (1). In particular, $\widetilde{q} = q \restriction H$ is bounded below on the qFW-set $\widetilde{F} = P(F)$. Since q is quasiconvex on F, $\widetilde{q}$ is quasiconvex on $\widetilde{F}$. Therefore $\widetilde{q}$ attains its infimum on $\widetilde{F}$ by the induction hypothesis, since $\dim(\widetilde{F}) = n-1$, and then q, having the same values, also attains its infimum on F. $\square$

Remark 1. From the implication (iii) $\Longrightarrow$ (i) it is clear that for a quadratic function q bounded below on F to attain its infimum on F, it is sufficient to have just one of its sublevel sets S_α with $\alpha > \gamma = \inf_{x\in F} q(x)$ convex, a condition which is weaker than quasiconvexity on F. An even weaker condition suffices, namely, the existence of a not necessarily convex sublevel set S_α and a direction $d \in \mathbb{R}^n$ with the following property: For every $x \in S_\alpha$ there exists $t_x \in \mathbb{R}$ such that $x + td \in S_\alpha$ for every $t \geq t_x$.

Remark 2. Yet another equivalent condition which we could add to the above list is

(iv) $P(F)$ is closed for every orthogonal projection P.

Indeed (ii) $\Longrightarrow$ (iv) is Proposition 1, and (iv) $\Longrightarrow$ (ii) is implicit in the proof of (ii) $\Longrightarrow$ (iii) above. For the equivalence of (iii) and (iv) see also [9].

Corollary 1. *Frank-and-Wolfe sets have no f-asymptotes.* $\square$

We end this section by indicating that the converse of Corollary 1 is not true. Put differently, the absence of f-asymptotes does *not* characterize Frank-and-Wolfe sets. Or put again differently, there exist quasi-Frank-and-Wolfe sets, which are not Frank-and-Wolfe.

Example 3.1. We construct a closed convex set F without f-asymptotes, which is not Frank-and-Wolfe. We use Example 2 of [10], which we reproduce here for convenience. Consider the optimization program

$$\begin{array}{ll} \text{minimize} & q(x) = x_1^2 - 2x_1x_2 + x_3x_4 \\ \text{subject to} & c_1(x) = x_2^2 - x_3 \leq 0 \\ & c_2(x) = x_2^2 - x_4 \leq 0 \\ & x \in \mathbb{R}^4 \end{array}$$

then as Lou and Zhang [10] show the constraint set $F = \{x \in \mathbb{R}^4 : c_1(x) \leq 0, c_2(x) \leq 0\}$ is closed convex, and the quadratic function q has infimum $\gamma = -1$ on F, but this infimum is not attained.

Let us show that F has no f-asymptotes. Note that $F = F_1 \times F_2$, where $F_1 = \{(x_1, x_3) \in \mathbb{R}^2 : x_1^2 - x_3 \leq 0\}$, $F_2 = \{(x_2, x_4) \in \mathbb{R}^2 : x_2^2 - x_4 \leq 0\}$. Observe that $F_1 \cong F_2$, and that F_1 does not have asymptotes, being a parabola. Therefore, F does not have f-asymptotes either. This can be seen from the following

Proposition 3. *Suppose F_1, F_2 do not have f-asymptotes. Then neither does $F_1 \times F_2$ have f-asymptotes.*

Proof: We write $F_1 \times F_2 = (F_1 \times \mathbb{R}^n) \cap (\mathbb{R}^n \times F_2)$. Suppose M is an f-asymptote of $F_1 \times F_2$, then by Klee [9, Theorem 4] the flat M contains either an f-asymptote N_1 of $F_1 \times \mathbb{R}^n$, or it contains an f-asymptote N_2 of $\mathbb{R}^n \times F_2$. Assume without loss that M contains N_1. Let P be the projection on the first coordinate, then $P(N_1)$ is an affine manifold, and it is easy to see that it is an f-asymptote of F_1. $\square$

Example 3.2. Let F be the epigraph of $f(x) = x^2 + \exp(-x^2)$ in $\mathbb{R}^2$. Then $q(x, y) = y - x^2$ is bounded below on F, but does not attain its infimum, so F is not FW. However, F has no f-asymptotes, so it is qFW. $\square$

4. Motzkin decomposable sets

The proof of the classical Frank-and-Wolfe theorem [5] exploits the fact that a polyhedron P can be decomposed as $P = C + D$, where C is a polytope, and D a convex polyhedral cone. This rises the question whether the Frank and Wolfe theorem may be extended to other classes of convex sets F with this type of decomposition. We recall the following

Definition 1. A nonempty closed convex set F in $\mathbb{R}^n$ is called Motzkin decomposable if there exists a compact convex set C and a closed convex cone D such that $F = C + D$. We call (C, D) a Motzkin decomposition of F. $\square$

We start with a disclaimer. Not all Motzkin decomposable sets are Frank-and-Wolfe.

Example 4.1. We put $D = \{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_1x_2 - x_3^2 \geq 0\}$, then D is a closed convex cone, hence is trivially Motzkin decomposable. But D is not Frank-and-Wolfe. In fact, it is not even quasi-Frank-and-Wolfe, as we now show. Indeed, define $q : \mathbb{R}^3 \to \mathbb{R}$ by $q(x) = x_1^2 + (x_3 - 1)^2$, then q is quadratic convex and bounded below by 0. In fact, $\gamma = 0$ is the infimum of q on D, because $q\left(\frac{1}{k}, \frac{(k+1)^2}{k}, 1 + \frac{1}{k}\right) = \frac{2}{k^2} \to 0$, but 0 is not attained on D. In view of Theorem 1, the cone D must have f-asymptotes. $\square$

Example 4.2. In the same vein consider the quadratic function $q : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined as $q(x, y, z) := (x-1)^2 - y + z$ and the ice-cream cone $F := \left\{(x, y, z) \in \mathbb{R}^3 : z \geq \sqrt{x^2 + y^2}\right\}$. Clearly $q \geq 0$ on F since $z \geq y$ for every $(x, y, z) \in F$. But the infimum of q on F is 0, since $\left(1, k, \sqrt{1+k^2}\right) \in F$ and

$$q\left(1, k, \sqrt{1+k^2}\right) = \sqrt{1+k^2} - k \longrightarrow 0,$$

and this infimum is not attained, as for $(x, y, z) \in F$, one has either $x \neq 1$ or $z \geq \sqrt{1+y^2} > y$, which both imply $q(x, y, z) > 0$.

The orthogonal projection of F onto the hyperplane

$$H := \left\{(x, y, z) \in \mathbb{R}^3 : y + z = 0\right\}$$

is not closed. To see this, notice that the orthogonal projection P on H is given by $P(x, y, z) = \left(x, \frac{y-z}{2}, \frac{z-y}{2}\right)$. Consider again $\left(1, k, \sqrt{1+k^2}\right) \in F$, then $P\left(1, k, \sqrt{1+k^2}\right) = \left(1, \frac{k-\sqrt{1+k^2}}{2}, \frac{\sqrt{1+k^2}-k}{2}\right) \in P(F)$, but its limit $(1, 0, 0)$ does not belong to $P(F)$, because $P^{-1}(1, 0, 0) = \{(x, y, z) \in \mathbb{R}^3 : x = 1,\ y = z\}$ does not intersect F. $\square$

These examples raise the question whether a Motzkin decomposable set F is Frank-and-Wolfe as soon as its recession cone 0^+F is Frank-and-Wolfe. A similar question can be asked for quasi-Frank-and-Wolfe sets. For the latter class things have been simplified due to Theorem 1, and we have the following answer.

Proposition 4. *Let F be a Motzkin decomposable set. Then F is quasi-Frank-and-Wolfe if and only if its recession cone 0^+F is quasi-Frank-and-Wolfe.*

Proof: 1) Suppose 0^+F is *qFW*. Assume contrary to what is claimed that F has an f-asymptote M. Write $M = y + U$ for the direction space U of M and $y \in U^\perp$. Let P be the orthogonal projection onto $U^\perp$. Then $M = P^{-1}(y)$. Observe that $P(F)$ is not closed. Indeed, there exist $x_k \in F$, $y_k \in M$, with $\operatorname{dist}(x_k, y_k) \to 0$. Therefore $Px_k \to y$. But $y \notin P(F)$, because if $y = Px$ for some $x \in F$, then $x \in F \cap P^{-1}(y) = F \cap M$, which is impossible due to $F \cap M = \emptyset$.

Since F is Motzkin decomposable, there exist a compact convex C with $F = C + 0^+F$. Then $P(F) = P(C) + P(0^+F)$, while $\overline{P(F)} = P(C) + \overline{P(0^+F)}$. Since $P(F) \neq \overline{P(F)}$, we deduce that $P(0^+F)$ cannot be closed, and that means 0^+F has an f-asymptote parallel to U, contradicting the fact that 0^+F is a *qFW*-set.

2) Conversely, suppose F is *qFW*, but that 0^+F is not *qFW*. Then 0^+F must have an f-asymptote L by Theorem 1. Suppose $L = y + W$ with W the direction space of L and $y \in W^\perp$. Let P be the orthogonal projection on $W^\perp$, then again $P(0^+F)$ is not closed. Now by [8, Proposition 5] F has an f-asymptote parallel to W, and by Theorem 1 this contradicts the fact that F is *qFW*. $\square$

Remark 3. This result is no longer correct if one drops the hypothesis that F is Motzkin decomposable. We take $F = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, xy \geq 1\}$, then F, being a hyperbola, has f-asymptotes, but 0^+F is the positive orthant, which does not have f-asymptotes.

Proposition 4 is a strong incentive to look for similar criteria for the Frank-and-Wolfe property in terms of 0^+F. A first partial answer is the following generalization of the classical Frank and Wolfe theorem.

Theorem 2. *Let F be a Motzkin decomposable convex set, and suppose its recession cone 0^+F is polyhedral. Then F is Frank-and-Wolfe.*

Proof: Write $F = C + 0^+F$ for C compact convex. Now consider a quadratic function $q(x) = \frac{1}{2}x^\top Ax + b^\top x$ bounded below by γ on F. Hence

$$\inf_{x \in F} q(x) = \inf_{y \in C} \inf_{z \in 0^+F} q(y + z) = \inf_{y \in C} \left(q(y) + \inf_{z \in 0^+F} y^\top Az + q(z) \right) \geq \gamma. \tag{2}$$

Now observe that for fixed $y \in C$ the function $q_y : z \mapsto y^\top Az + q(z)$ is bounded below on 0^+F by $\eta = \gamma - \max_{y' \in C} q(y')$. Indeed, for $z \in 0^+F$ we have

$$\begin{aligned} y^\top Az + q(z) &\geq \left(q(y) + \inf_{z' \in 0^+F} y^\top Az' + q(z') \right) - q(y) \\ &\geq \inf_{y \in C} \left(q(y) + \inf_{z' \in 0^+F} y^\top Az' + q(z') \right) - \max_{y' \in C} q(y') \\ &\geq \gamma - \max_{y' \in C} q(y') = \eta. \end{aligned}$$

Since q_y is a quadratic function bounded below on the polyhedral cone 0^+F, the inner infimum is attained at some $z = z(y)$. This is in fact the classical Frank and Wolfe theorem on a polyhedral cone. In consequence the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ defined as

$$f(y) = \inf_{z \in 0^+ F} y^\mathsf{T} Az + q(z),$$

satisfies $f(y) = y^\mathsf{T} Az(y) + q(z(y)) > -\infty$ for $y \in C$, so the compact set C is contained in the domain of f. But now a stronger result holds, which one could call a parametric Frank and Wolfe theorem, and which we shall prove in Lemma 2 below. We show that f is continuous relative to its domain. Once this is proved, the infimum (2) can then be written as

$$\inf_{x \in F} q(x) = \inf_{y \in C} q(y) + f(y),$$

and this is now attained by the Weierstrass extreme value theorem due to the continuity of $q + f$ on the compact C. Continuity on C is now a consequence of the following

Lemma 2. *Let D be a polyhedral convex cone and define*

$$f(c) = \inf_{x \in D} c^\mathsf{T} x + \tfrac{1}{2} x^\mathsf{T} Gx,$$

where $G = G^\mathsf{T}$. Then $\mathrm{dom}(f)$ *is a polyhedral convex cone, and hence f is continuous relative to* $\mathrm{dom}(f)$.

Proof: If $x^\mathsf{T} Gx < 0$ for some $x \in D$, then $\mathrm{dom}(f) = \emptyset$, so we may assume for the remainder of the proof that $x^\mathsf{T} Gx \geq 0$ for every $x \in D$. Clearly then,

$$\mathrm{dom}(f) = \left\{c : c^\mathsf{T} x \geq 0 \text{ for every } x \in D \text{ such that } x^\mathsf{T} Gx = 0\right\}.$$

Now by the Farkas-Minkowski-Weyl theorem (cf. [12, Thm. 19.1] or [13, Cor. 7.1a]) the polyhedral cone D is the linear image of the positive orthant of a space $\mathbb{R}^p$ of appropriate dimension, i.e. $D = \{Zu : u \in \mathbb{R}^p, u \geq 0\}$. This implies

$$\mathrm{dom}(f) = \{c : c^\mathsf{T} Zu \geq 0 \text{ for every } u \geq 0 \text{ such that } u^\mathsf{T} Z^\mathsf{T} GZu = 0\}.$$

Now observe that if $u \geq 0$ satisfies $u^\mathsf{T} Z^\mathsf{T} GZu = 0$, then it is a minimizer of the quadratic function $u^\mathsf{T} Z^\mathsf{T} GZu$ on the cone $u \geq 0$, hence $Z^\mathsf{T} GZu \geq 0$ by the Kuhn-Tucker conditions. Therefore we can write the set $P = \{u \in \mathbb{R}^p : u \geq 0, u^\mathsf{T} Z^\mathsf{T} GZu = 0\}$ as

$$P = \bigcup_{I \subset \{1,\ldots,p\}} P_I,$$

where the P_I are the polyhedral convex cones

$$P_I = \{u \geq 0 : Z^\mathsf{T} GZu \geq 0, u_i = 0 \text{ for all } i \in I, (Z^\mathsf{T} GZu)_j = 0 \text{ for all } j \notin I\}.$$

For every $I \subset \{1, \ldots, p\}$ choose m_I generators $u_{I1}, \ldots, u_{Im_I}$ of P_I. Then,

$$\begin{aligned}
\text{(3)} \qquad \mathrm{dom}(f) &= \left\{c : c^\mathsf{T} Zu \geq 0 \text{ for every } u \in P\right\} \\
&= \left\{c : c^\mathsf{T} Zu \geq 0 \text{ for every } u \in \textstyle\bigcup_{I \subset \{1,\ldots,p\}} P_I\right\} \\
&= \textstyle\bigcap_{I \subset \{1,\ldots,p\}} \left\{c : c^\mathsf{T} Zu \geq 0 \text{ for every } u \in P_I\right\} \\
&= \textstyle\bigcap_{I \subset \{1,\ldots,p\}} \left\{c : c^\mathsf{T} Zu_{Ij} \geq 0 \text{ for all } j = 1, \ldots, m_I\right\}.
\end{aligned}$$

Since a finite intersection of polyhedral cones is polyhedral, this proves that $\mathrm{dom}(f)$ is a polyhedral convex cone. To conclude, continuity of f relative to its domain now follows from [12, Thm. 10.2], since f is clearly concave and upper semicontinuous. □

Remark 4. The proof includes the case when $x^\top G x > 0$ for every $x \in D \setminus \{0\}$. In that case one has $P_I = \{0\}$ for every $I \subset \{1, ..., p\}$, and therefore $\{c : c^\top Z u \geq 0$ for every $u \in P_I\} = \mathbb{R}^n$, so that the equality (3) still holds and reduces to $\mathrm{dom}(f) = \mathbb{R}^n$.

Remark 5. We refer to Banks *et al.* [1, Thm. 5.5.1 (4)] or Best and Ding [3] for a related result in the case where $G \succeq 0$. For the indefinite case see also Tam [14].

Remark 6. The example in Remark 3 shows that Theorem 2 is no longer true if F is not Motzkin decomposable.

A second partial answer to the question whether the Frank-and-Wolfe property of 0^+F implies that of F is given in the following

Proposition 5. *Let F have a Motzkin decomposition of the form $F = P + 0^+F$ with P a polytope. If 0^+F is Frank-and-Wolfe, then so is F.*

Proof: Consider a quadratic q which is bounded below on F. Splitting the infimum according to (2), we see as in the proof of Theorem 2 that every $q_y : z \mapsto y^\top A z + q(z)$ is quadratic and bounded below on 0^+F, and since 0^+F is Frank-and-Wolfe by hypothesis, the inner infimum in (2) is attained at $z(y) \in 0^+F$. As in the proof of Theorem 2 define $f(y) = \inf_{z \in 0^+F} q_y(z) = q_y(z(y))$, then f is the infimum of the family of affine functions $y \mapsto y^\top A z + q(z)$ on the polytope P, hence is lower semi-continuous on P by [12, Theorem 10.2]. But then $y \mapsto q(y) + f(y)$ is lower semi-continuous on P, and by compactness of P the outer infimum $y \in P$ in (2) is therefore attained. □

We conclude the section about Motzkin decomposable sets with the following observation.

Proposition 6. *Let F be a Motzkin decomposable qFW-set. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator. Then $T(0^+F)$ is closed in $\mathbb{R}^m$.*

Proof: Let $F = C + 0^+F$ with C convex compact, then $T(F) = T(C) + T(0^+F)$. By Lemma 1 the set $T(F)$ is closed, and this implies closedness of the recession cone $0^+T(F)$ by [12, Theorem 8.2]. But now $(T(C), T(0^+F))$ is a Motzkin decomposition of $T(F)$, hence $T(0^+F) = 0^+T(F)$ by [8, Prop. 6]. That proves the claim. □

5. q-Asymptotes

The discussion in Section 3 shows that the absence of f-asymptotes is only a necessary condition for the Frank-and-Wolfe property. In this section we shall develop a related concept of asymptotes, where we replace affine (flat) surfaces by quadratic surfaces. We start with the following

Definition 2. A quadric in $\mathbb{R}^n$, also called a quadratic surface or a conic, is a set of the form $Q = \{x \in \mathbb{R}^n : x^\top A x + 2b^\top x + c = 0\}$ with $A = A^\top \neq 0$. □

Definition 3. A nonempty closed set A is said to be asymptotic to the nonempty closed convex set F if $A \cap F = \emptyset$ and $\mathrm{dist}(F, A) = 0$. □

If A is an affine subspace of $\mathbb{R}^n$, then A is asymptotic to F iff it is an f-asymptote in the sense of Klee [9] and in the sense of Section 3. Now we can give the central definition of this section.

Definition 4. The quadric $Q = \{x \in \mathbb{R}^n : q(x) = x^\top A x + 2b^\top x + c = 0\}$ is a q-asymptote of the closed convex subset F of $\mathbb{R}^n$ if $F \cap Q = \emptyset$ and $\mathrm{dist}(Q \times \{0\}, \{(x, q(x)) : x \in F\}) = 0$.
□

The condition means $F \cap Q = \emptyset$, and that there exist $x_k \in F$ and y_k with $q(y_k) = 0$ such that $x_k - y_k \to 0$ and $q(x_k) \to 0$. This shows that the notion of a q-asymptote is invariant under an affine change of coordinates in $\mathbb{R}^n$, hence is a concept of affine geometry. The condition could also be expressed as follows: The quadric $Q \times \{0\}$ in $\mathbb{R}^{n+1}$ is asymptotic to $\text{graph}_F(q) := \text{graph}(q) \cap (F \times \mathbb{R})$ in the sense of Definition 3.

Remark 7. If Q is a q-asymptote of F, then Q is clearly asymptotic to F, but the converse is not true in general. To see this consider the following example. Let $F = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ be the positive orthant, and let $q(x, y) = xy + 1$, then $Q = \{(x, y) \in \mathbb{R}^2 : q(x, y) = 0\} = \{(x, y) : xy = -1\}$ is a hyperbola with $F \cap Q = \emptyset$. We have $\text{dist}(F, Q) = 0$, because $(-\frac{1}{n}, n) \in Q$ and $(0, n) \in F$, so Q is asymptotic to F in the sense of definition 3. But Q is not a q-asymptote of F, because the sets

$$Q \times \{0\} = \{(x, -\tfrac{1}{x}, 0) : x \neq 0\}$$

and

$$\text{graph}(q) \cap (F \times \mathbb{R}) = \{(x, y, xy + 1) : x \geq 0, y \geq 0\}$$

cannot be close. □

Remark 8. Consider the quadric

$$Q : \quad q(x) = x_1^2 + \cdots + x_{n-1}^2 = 0,$$

then $Q = \{x \in \mathbb{R}^n : q(x) = 0\}$ is the x_n-axis. Suppose the x_n-axis is an f-asymptote of a closed convex set F. This is equivalent to Q being asymptotic to F. However, we argue that Q is then even a q-asymptote of F in the sense of definition 4. Namely, we have

$$Q \times \{0\} = \{(0_{n-1}, \xi, 0) : \xi \in \mathbb{R}\} \subset \mathbb{R}^{n+1}$$

and

$$\text{graph}(q) \cap (F \times \mathbb{R}) = \{(x, q(x)) : x \in F\} = \left\{ \left(x, \sum_{i=1}^{n-1} x_i^2 \right) : x \in F \right\}.$$

Now given $\epsilon > 0$ choose $x \in F$ and $t \in \mathbb{R}$ such that $\|x - (0_{n-1}, t)\|^2 < \epsilon^2$, which is possible because $\text{dist}(Q, F) = 0$. (Naturally, we could take $t = x_n$). Then $q(x) = x_1^2 + \cdots + x_{n-1}^2 < \epsilon^2$ and $(x_n - t)^2 < \epsilon^2$. Therefore

$$\|(0, t, 0) - (x, q(x))\|^2 \leq \|x - (0_{n-1}, t)\|^2 + q(x)^2 \leq \epsilon^2 + q(x)^2 < \epsilon^2 + \epsilon^4.$$

This shows the claim. We can generalize this to a proof that any flat M which is an f-asymptote is also a q-asymptote when interpreted as a quadric:

Proposition 7. *Let F be closed convex in $\mathbb{R}^n$, and let $Q = \{x \in \mathbb{R}^n : q(x) = 0\}$ be a quadric. Suppose Q is flat, that is, degenerates to an affine subspace. Then Q is a q-asymptote of F in the sense of Definition* 4 *if and only if it is an f-asymptote of F in the sense of* [9]*. Moreover, for any f-asymptote M of F there exists a quadric representation of M as $M = \{x \in \mathbb{R}^n : q(x) = 0\}$, and then M is also a q-asymptote of F.*

Proof: The fact that Q is an affine subspace of dimension $k \leq n-1$ means that one can find affine coordinates in $\mathbb{R}^n$ such that Q has the form $Q = \{x \in \mathbb{R}^n : x_{k+1}^2 + \cdots + x_n^2 = 0\} = \mathbb{R}^k \times \{0_{n-k}\}$.

Since being a q-asymptote implies being asymptotic, and since for an affine subspace this coincides with being an f-asymptote, we have but to prove the opposite implication.

Assume therefore that Q is an f-asymptote of F, i.e., $F \cap Q = \emptyset$ and $\text{dist}(F, Q) = 0$. We have to show that $Q \times \{0\} = \{(x, 0) : x \in Q\}$ is asymptotic to $\text{graph}(q) \cap (F \times \mathbb{R}) = \{(y, q(y)) : y \in F\}$. Clearly the two sets are disjoint. Splitting $x = (x', x''), y = (y', y'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, we have $q(y) = y_{k+1}''^2 + \cdots + y_n''^2 = \|y''\|^2$. Now pick $x^r \in Q$, $y^r \in F$ with

$\text{dist}(x^r, y^r) \to 0$ as $r \to \infty$. Then $x^r = (x'^r, x''^r) = (x'^r, 0)$ and $y^r = (y'^r, y''^r)$, hence $\|y''^r\|^2 \leq \|x^r - y^r\|^2 \to 0$, and this implies $q(y^r) = \|y''^r\|^2 \to 0$. Hence Q is a q-asymptote of F, because it now follows that $\|(x^r, 0) - (y^r, q(y^r))\| \to 0$. $\square$

This result shows that the notion of a q-asymptote is a natural extension of Klee's concept of f-asymptotes. We move from flat asymptotes to quadratic asymptotes. We are now ready to state the principal result of this section.

Theorem 3. *A convex set F is Frank-and-Wolfe if and only if it is closed and has no q-asymptotes.*

Proof: 1) Assume that there exists a quadratic function $q : \mathbb{R}^n \to \mathbb{R}$ which is bounded below on F, but does not attain its infimum on F. We have to show that F has a q-asymptote. Assume without loss that the infimum of q on F is 0. Since there exists $x \in F$ with $q(x) > 0$ and $y \notin F$ with $q(y) = 0$, the set $Q = \{x \in \mathbb{R}^n : q(x) = 0\}$ is a quadric in $\mathbb{R}^n$.

Note that if F is not qFW, then by Theorem 1 the set F has an f-asymptote, and then has also a q-asymptote by Proposition 7. So we can assume that F is qFW, and by Proposition 1 we therefore know that orthogonal projections of F are closed.

We clearly have $F \cap Q = \emptyset$, so we have to show that $\text{dist}(\{(x, q(x)) : x \in F\}, Q \times \{0\}) = 0$. Since the statement we have to prove is invariant under an affine change of coordinates in $\mathbb{R}^n$, we may assume that the quadric Q is given by one of the following equations:

$$Q : \quad q(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{r} x_i^2 + \gamma = 0, \quad (p < r \leq n) \tag{4}$$

where $\gamma \in \{0, 1\}$ if Q is a center quadric with 0 as its center, or

$$Q : \quad q(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{r} x_i^2 + x_{r+1} = 0 \quad (p \leq r < n) \tag{5}$$

if Q is a paraboloid.

a) Let us first discuss the easier case of a paraboloid (5). Since q is a quadratic function, it satisfies a Łojasiewicz inequality at infinity. In other words, following [15, Theorem 2.1] there exist constants $\delta > 0$, $c > 0$ and a Łojasiewicz exponent $\alpha > 0$ at infinity such that for every $x \in \mathbb{R}^n$ with $|q(x)| < \delta$ we have

$$|q(x)| \geq c \, \text{dist}\left(x, \widehat{Q}\right)^{\alpha},$$

where $\widehat{Q} = Q \cup Q_1$ with

$$Q = \{x \in \mathbb{R}^n : q(x) = 0\}, \quad Q_1 = \left\{x \in \mathbb{R}^n : \tfrac{\partial}{\partial x_{r+1}} q(x) = 0\right\}.$$

This result uses the fact that q is a monic polynomial of degree $m = 1$ in the variable x_{r+1}. Since $\partial / \partial x_{r+1} q(x) = 1$, the set Q_1 is empty, hence we obtain

$$|q(x)| \geq c \, \text{dist}\,(x, Q)^{\alpha}$$

for $|q(x)| < \delta$. Now choose a sequence $x_k \in F$ with $q(x_k) \to 0$. Then from some k onward, $q(x_k) \geq c \, \text{dist}(x_k, Q)^{\alpha} \to 0$, which proves $\text{dist}(\{(x, q(x)) : x \in F\}, Q \times \{0\}) = 0$. This settles the case where Q is a paraboloid.

b) Let us next consider the more complicated case where Q is a center quadric. Choose a sequence $x_k \in F$ such that $q(x_k) \to 0$. We want to show $\text{dist}(x_k, Q) \to 0$, at least for a subsequence. Assume on the contrary that $\text{dist}(x_k, Q) > d > 0$ for every k. Write $x_k = (\xi_1^k, \ldots, \xi_n^k)$, and note that $\|x_k\| \to \infty$. We now have two principal cases.

Case I is when $(\xi_1^k,\dots,\xi_r^k)\to 0$, while the part $(\xi_{r+1}^k,\dots,\xi_n^k)$ on which q given by (4) does not depend satisfies $\|(\xi_{r+1}^k,\dots,\xi_n^r)\|\to\infty$. In this case we necessarily have $r<n$.

Case II is when there exists $i\in\{1,\dots,r\}$ such that $\xi_i^k\to\xi_i\neq 0$ for a subsequence $k\in\mathcal{K}$, including the possibilities $\xi_i=\pm\infty$.

We start by discussing case II. Suppose ξ_i is finite and the signature of i is negative (i.e. $i\in\{p+1,\dots,r\}$). Then there must also exist another index with positive signature $j\in\{1,\dots,p\}$ say, for which $\xi_j^k\to\xi_j\neq 0$. (This is because in (4) the $-\xi_i^2$ and ξ_j^2 have to sum to $\gamma\geq 0$. Therefore if there is a non-vanishing contribution from an index i with negative signature, there is necessarily also one from an index j with positive signature.) We may without loss assume that this contribution with positive signatures comes from $j=1$. A similar argument applies when $\xi_i=\pm\infty$. We now have two subcases. Case II.1 is when $\xi_1^k\to\xi_1\in(0,+\infty]$, case II. 2 is when $\xi_1^k\to\xi_1\in[-\infty,0)$.

Let us discuss case II.1. Shrinking d if need be, we assume $\xi_1-d>0$, and then also $\xi_1^k>d$ for all k large enough. (This works also for $\xi_1=+\infty$). Now define $f_k(t)=q(t,\xi_2^k,\dots,\xi_n^k)$, then $f_k(t)=t^2+r(\xi_2^k,\dots,\xi_n^k)$. We have $f_k(t)>0$ for every $t\in I_k:=[\xi_1^k-d,\xi_1^k+d]$, because $d<\operatorname{dist}(x_k,Q)$. Moreover, $f_k'(t)=2t\geq 2(\xi_1^k-d)>0$ for $t\in I_k$. So f_k is positive and increasing on I_k. Therefore

$$\max_{t\in[\xi_1^k-d,\xi_1^k]} f_k(t)=f_k(\xi_1^k)=q(x_k)\to 0.$$

Now define $g_k(t)=f_k(t+\xi_1^k)$, then

$$\max_{s\in[-d,0]} g_k(s)=q(x_k)\to 0.$$

Therefore the sequence g_k converges to 0 in the space of quadratic polynomials in the variable t. But that implies its coefficients tend to 0, a contradiction with $g_k(t)=(t+\xi_1^k)^2+r(\xi_2^k,\dots,\xi_n^k)$, because the coefficient of t^2 is 1 and does not tend to 0. That is a contradiction in case II. 1, and therefore settles that case.

Now consider case II. 2. Here we arrange $\xi_1+d<0$, and then also $\xi_1^k+d<0$ for k sufficiently large, and that works also for $\xi_1=-\infty$. So here f_k is positive and decreasing on I_k. We use an analogous argument, and get a similar contradiction. That settles case II.

c) It remains to discuss case I. Note that here we must have $\gamma=0$, so Q is a cone (in the sense of quadric theory). Suppose $r>0$, then the sublevel set $\{x\in F: q(x)\leq r\}$ is nonempty and unbounded. Fix x in this set, then $q(x)=q(x+td)$ for every d of the form $d=(0,\dots,0,\xi_{r+1},\dots,\xi_n)$, because q does not depend on the coordinates $\xi_{r+1},\dots,\xi_n$. Now let P be the orthogonal projection on $d^\perp$, then $P(F)$ is convex and, in addition, closed by what was observed at the beginning of the proof. But the infimum of q on $P(F)$ is still 0, and it is not attained. With regard to the form (4) we have therefore reduced the dimension n by 1, but the quadric is still of the form (4) with the same r. Continuing in this way, we end up with the case where $r=n$ in (4). But then we are in case II, because remember that case I can only occur when $r<n$. That settles case I and therefore completes the first part of the proof.

2) Let us now prove that if F has a q-asymptote Q, then it is not Frank-and-Wolfe. From the definition of a q-asymptote we have $F\cap Q=\emptyset$. We may therefore assume without loss that $F\subset\{x\in\mathbb{R}^n: q(x)>0\}$, because F is connected and q is continuous. Now there exists a sequence $x_k\in F$ and a sequence $y_k\in Q$ such that $\operatorname{dist}((x_k,q(x_k)),(y_k,0))\to 0$. That means 0 is the infimum of q, and it is not attained. $\square$

Remark 9. One might be tempted to guess that F is Frank-and-Wolfe iff there is no quadratic Q which is asymptotic to F. The example of the positive orthant in remark 7 shows that this guess is incorrect. The corresponding condition is too strong.

Remark 10. Note that if we remove the condition $\operatorname{dist}(Q,F)=0$ from the definition of q-asymptotes, then the statement becomes trivial. The sole difficulty in the first part of the proof of Theorem 3 is indeed to establish $\operatorname{dist}(Q,F)=0$.

This observation also indicates why there is little hope for an extension of Theorem 3 to higher degree polynomials. Consider for instance $f(x,y,z)=(y^2+(xy-1)^2)z$, then $Q=\{(x,y,z):f(x,y,z)=0\}=\{z=0\}$ is an affine manifold. We have $f(k,\frac{1}{k},1)\to 0$, but $\operatorname{dist}((k,\frac{1}{k},1),Q)\to\frac{1}{\sqrt{2}}$ as $k\to\infty$. Putting $F=\{(x,y,1)\in\mathbb{R}^3: xy\geq 1\}$, we see that f does not attain its infimum 0 on F, yet the affine manifold $Q=\{f=0\}$ is not asymptotic to F. What is missing is an argument to infer from $f(x_k)\to 0$ for $x_k\in F$ that also $\operatorname{dist}(x_k,Q)\to 0$, and for higher order polynomials such an argument may not exist.

Example 5.1. To illustrate Theorem 3 we consider the set $F=\{(x,y)\in\mathbb{R}^2: y\geq x^2\}$ and claim that it is Frank-and-Wolfe. We check this by showing that F has no q-asymptotes. Suppose $Q=\{q=0\}$ is a q-asymptote of F. If Q is a hyperbola or consists of two lines, then F itself has lines as asymptotes, which is impossible, because F is a parabola. It is equally impossible that Q is an ellipse, so Q must be a parabola, too. But it is intuitively clear that no other parabola can be a q-asymptote of $y=x^2$.

To prove this rigorously, suppose $q(x,y)=ax^2+bxy+cy^2+dx+ey+f$. By the definition of a q-asymptote there exist $(x_k,y_k)\in Q$ and $(x_k,x_k^2)\in F$, such that $\|(x_k,x_k^2)-(x_k,y_k)\|\to 0$ and $q(x_k,x_k^2)\to 0$. Picking a subsequence, we may without loss assume $x_k\to+\infty$. Then $q(x_k,x_k^2)=ax_k^2+bx_k^3+cx_k^4+dx_k+ex_k^2+f\to 0$ implies successively $c=0$, then $b=0$, then $a=-e$, then $d=0$ and $f=0$, and finally $a\neq 0$. Hence $Q=\{(x,y): a(x^2-y)=0\}$, but this is the boundary curve of F, which contradicts $F\cap Q=\emptyset$. $\square$

Remark 7 suggests an equivalent geometric characterization of q-asymptotes, which we now develop. Let $Q=\{x\in\mathbb{R}^n: q(x)=0\}$ be a quadric and consider the associated one-parameter family $\mathscr{Q}=\{Q_\alpha\}_{\alpha\in\mathbb{R}}$ of quadrics $Q_\alpha=\{x\in\mathbb{R}^n: q(x)-\alpha=0\}$. Note that $\mathscr{Q}$ is a geometric object, as an affine change of coordinates leads to the same family of sets. Informally, we intend to show that $Q\in\mathscr{Q}$ is a q-asymptote of the closed convex set F if and only if Q,F are asymptotic, and no other element Q' of the bundle $\mathscr{Q}$ can be *squeezed in between* F and Q.

Definition 5. Let F,Q be closed sets with $F\cap Q=\emptyset$ and $\operatorname{dist}(F,Q)=0$. We say that the closed set Q' is squeezed in between F and Q if $F\cap Q'=\emptyset=Q\cap Q'$ and if every segment $[x,y]$ with $x\in F$ and $y\in Q$ contains a point $z\in Q'$, i.e., $[x,y]\cap Q'\neq\emptyset$. $\square$

We now have the following

Proposition 8. *Let F be closed convex and let $Q=\{x\in\mathbb{R}^n: q(x)=0\}$ be a quadric. Then Q is a q-asymptote of F if and only if Q is asymptotic to F and no other member Q' of the bundle $\mathscr{Q}$ can be squeezed in between F and Q. In other words, Q is a tight quadric asymptote to F.*

Proof: 1) Suppose Q is a q-asymptote of F. Then there exist $x_k\in F$, $y_k\in Q$ such that $x_k-y_k\to 0$ and $q(x_k)\to 0$. Clearly Q is asymptotic to F. Since $F\cap Q=\emptyset$ and F is connected, we either have $F\subset\{x:q(x)>0\}$ or $F\subset\{x:q(x)<0\}$. Assume without loss that $F\subset\{x:q(x)>0\}$. Suppose $Q'=\{x:q(x)=\alpha\}$ can be squeezed in between Q and F. Since $Q\cap Q'=\emptyset$, we have $\alpha\neq 0$. There are two cases to be discussed.

Suppose first that $\alpha < 0$. Then we find a point z_k in the open segment (x_k, y_k) such that $q(z_k) = \alpha < 0$. But $q(x_k) > 0$, hence by the mean value theorem there exists another point v_k in the open segment (x_k, z_k) with $q(v_k) = 0$. Now we repeat the argument on $[x_k, v_k]$, which must also contain a point with value $q = \alpha$. That leads to a contradiction, because we thereby find a third root of q on the segment $[x_k, y_k]$, which is impossible as q is quadratic. In consequence the squeezing value must be $\alpha > 0$.

Suppose therefore that the quadric Q' which may be squeezed in between F and Q has $\alpha > 0$. Then we have the following situation on the segment $[x_k, y_k]$. There exists $z_k \in (x_k, y_k)$ with $q(z_k) = \alpha > 0$, while $q(y_k) = 0$ and $q(x_k) \to 0$, $0 < q(x_k) \ll \alpha$. Let L_k be the line generated by $[x_k, y_k]$. Since q is a quadratic function on L_k, there exists a point $v_k \in L_k$ preceding x_k where $q(v_k) = 0$. Here preceding means that $x_k \in [v_k, y_k]$. Since $F \subset \{q > 0\}$, we have $v_k \notin F$. In particular, $F \cap L_k$ is contained in the segment $[v_k, x_k]$. But $v_k \in Q$, $x_k \in F$, hence the segment $[v_k, x_k]$ must also contain an element w_k of Q', i.e., with $q(w_k) = \alpha$, and that is impossible because q is quadratic. Namely, the arrangement on the line L_k is now $v_k < w_k < x_k < z_k < y_k$ with $q(y_k) = 0$, $q(z_k) = \alpha > 0$, $q(x_k) \ll \alpha$, $q(w_k) = \alpha$, $q(v_k) = 0$. But $q \upharpoonright L_k$ is concave, so this is impossible. This proves that $Q' \in \mathscr{Q}$ could not possibly be squeezed in between F and Q.

2) Conversely, suppose Q is asymptotic to F and is tight in the sense that no other member Q' of the bundle $\mathscr{Q}$ can be squeezed in between F and Q. Since $F \cap Q = \emptyset$, we may assume $F \subset \{x : q(x) > 0\}$. Let $\gamma := \inf_{x \in F} q(x)$. We claim that $\gamma = 0$. For suppose we had $\gamma > 0$ then on choosing $0 < \alpha < \gamma$ we find that $Q' = \{x : q(x) - \alpha = 0\}$ is squeezed in between F and Q, which is impossible. Hence $\gamma = 0$. Now pick $x_k \in F$ with $q(x_k) \to 0$ and $y_k \in Q$. Using the argument of part 1) of the proof of Theorem 4, it follows that $y_k - x_k \to 0$. Hence $(x_k, q(x_k)) - (y_k, 0) \to 0$. That shows $\operatorname{dist}(Q \times \{0\}, \operatorname{graph}_F(q)) = 0$, hence Q is a q-asymptote of F. $\square$

Remark 11. In view of the new characterization of q-asymptotes we have the following description of Frank-and-Wolfe sets. Whenever Q is a quadric asymptote of a Frank and Wolfe set F, then there exists another quadric Q' in the bundle $\mathscr{Q}$ associated with Q that can be squeezed in between F and Q. We could say that Q' is a tighter asymptote than Q. As this argument can be repeated, the FW-set F has no tightest asymptote among the quadrics in $\mathscr{Q}$.

Remark 12. It is instructive to give a direct argument for the fact that an f-asymptote in the sense of Klee is tight in the sense of the previous remark, hence is a q-asymptote. To see this, suppose $M = \{x \in \mathbb{R}^n : Ax - b = 0\}$ is an f-asymptote of F and represent M as the quadric $M = \{x : q(x) = \|Ax - b\|^2 = 0\} = Q$. Consider the associated bundle $\mathscr{Q} = \{Q_\alpha\}$ and suppose some Q_α with $\alpha \neq 0$ can be squeezed in between $Q = M$ and F. Clearly this means $\alpha > 0$, as the $Q_{\alpha'}$ with $\alpha' < 0$ are empty. But q is convex, hence $F \subset \{x : q(x) = \|Ax - b\|^2 > \alpha\}$, because $F \subset \{x : \|Ax - b\|^2 < \alpha\}$ implies that q is concave on a segment $[x, y]$ with $x \in F$ and $y \in M$. But now we have a contradiction with the fact that $\operatorname{dist}(M, F) = 0$, as this implies $\inf_{x \in F} \|Ax - b\| = 0$.

6. Generalized linear complementarity problem

Let F be a closed convex cone in $\mathbb{R}^n$, let $A = A^\intercal \in \mathbb{R}^{n \times n}$, and $b \in \mathbb{R}^n$. Then we consider the following generalized linear complementarity problem on F with data (A, b):

(6) Find $x^* \in F$ such that $(Ax^* + b)^\intercal x \geq 0$ for every $x \in F$, and $Ax^* + b \perp x^*$.

Every $x^* \in F$ satisfying (6) is called a solution of the problem. We say that the generalized linear complementarity problem (6) is feasible if $\gamma = \inf_{x \in F}(Ax+b)^\mathsf{T} x > -\infty$ and if there exists $x_0 \in F$ such that $(Ax_0 + b)^\mathsf{T} x \geq 0$ for every $x \in F$.

Theorem 4. *Suppose problem* (6) *is feasible. If F is a Frank-and-Wolfe cone, then* (6) *has a solution x^*.*

Proof: Let x_0 be a feasible solution, then $(Ax_0 + b)^\mathsf{T} x \geq 0$ for every $x \in F$. Since F is a cone we have $2x_0 \in F$, and $(2Ax_0 + 2b)^\mathsf{T} x \geq 0$ for every $x \in F$. Due to feasibility the quadratic function $q(x) = (Ax + 2b)^\mathsf{T} x$ is now bounded below by 2γ, and since F is Frank-and-Wolfe, there exists $x^* \in F$ such that

$$(Ax + 2b)^\mathsf{T} x \geq (Ax^* + 2b)^\mathsf{T} x^* \tag{7}$$

for every $x \in F$. For $x \in F$ and $0 < t \leq 1$ we have $\widetilde{x} = x^* + t(x - x^*) \in F$, hence on substituting $\widetilde{x}$ in (7) and expanding, we get

$$t(Ax^* + 2b)^\mathsf{T}(x - x^*) + t(x - x^*)^\mathsf{T} Ax^* + t^2(x - x^*)^\mathsf{T} A(x - x^*) \geq 0.$$

Dividing by t and letting $t \to 0$ gives $2(Ax^*+b)^\mathsf{T}(x-x^*) \geq 0$, hence $(Ax^*+b)^\mathsf{T}(x-x^*) \geq 0$ for every $x \in F$. Putting $x = 0 \in F$ we get $(Ax^* + b)^\mathsf{T} x^* \leq 0$, while putting $x = 2x^* \in F$ gives $(Ax^* + b)^\mathsf{T} x^* \geq 0$, so together we get complementarity $Ax^* + b \perp x^*$. From that follows $(Ax^*+b)^\mathsf{T}(x-x^*) = (Ax^*+b)^\mathsf{T} x \geq 0$ for all $x \in F$, hence x^* is a solution of (6). $\square$

For sufficient conditions guaranteeing $\inf_{x \in F}(Ax + b)^\mathsf{T} x > -\infty$ we refer to [6] and the references given there. Links with the linear complementarity problem can already be found in the original work [5].

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References

[1] B. Banks, J. Guddat, D. Klatte, B. Kummer, K. Tammer. Non-linear parametric optimization. Springer Verlag 1983.
[2] E.G. Belousov, D. Klatte. A Frank-and-Wolfe theorem for convex polynomial programs. Comp. Optim. Appl. 22, 2002, 37 – 48.
[3] M. J. Best, B. Ding. On the continuity of the minimum in parametric quadratic programs. J. Optim. Theory Appl. 86(1):1995,245-250.
[4] E. Blum, W. Oettli. Direct proof of the existence theorem in quadratic programming, Operations Research 20 (1972) 165-167
[5] M. Frank and P. Wolfe. An algorithm for quadratic programming, Naval Research Logistics Quarterly 3 (1956), 95-110.
[6] M.S. Gowda. Minimising quadratic functionals over closed convex cones. Bull. Austr. Math. Soc. 39:1989,15-20.
[7] W.M. Hirsch, A.J. Hoffman. Fixed charge problems. Communications on Pure and Applied Math. XIV, 1961, 355 – 369.

[8] A.N. Iusem, J.E. Martínez-Legaz, M.I. Todorov. Motzkin predecomposable sets. J. Global Optim. 60 (2014), no. 4, 635–647.
[9] V. Klee. Asymptotes and projections of convex sets. Math. Scand., 8 (1960), pp. 356–362.
[10] Z.-Q. Luo, S. Zhang. On extensions of the Frank-Wolfe theorem. Comput. Optim. Appl. 13 (1999), 87-110.
[11] A.F. Perold. A generalization of the Frank and Wolfe theorem. Math. Programming 18:1980, 215-227.
[12] R.T. Rockafellar. Convex Analysis. Princeton University Press. 1970.
[13] A. Schrijver. Theory of Linear and Integer Programming. John Wiley & Sons, 1986.
[14] N. N. Tam. Continuity of the optimal value function in indefinite quadratic programming. J. Global Optim. 23(1):2002,43-61.
[15] D.S. Tiep, K. Kurdyka, O. Le Gal. Łojasiewicz inequality on non-compact domains and singularities at infinity. Int. J. Math. 24(10):2013,8 pages.