Asymptotically stable scheme for the Euler-Poisson system in the quasi-neutral limit

P. Crispel, P. Degond and Marie-Hélène Vignal

Lab. MIP : Mathematics for Industry and Physic

Univ. Paul Sabatier, Toulouse 3

Sponsored by CEA and CNES

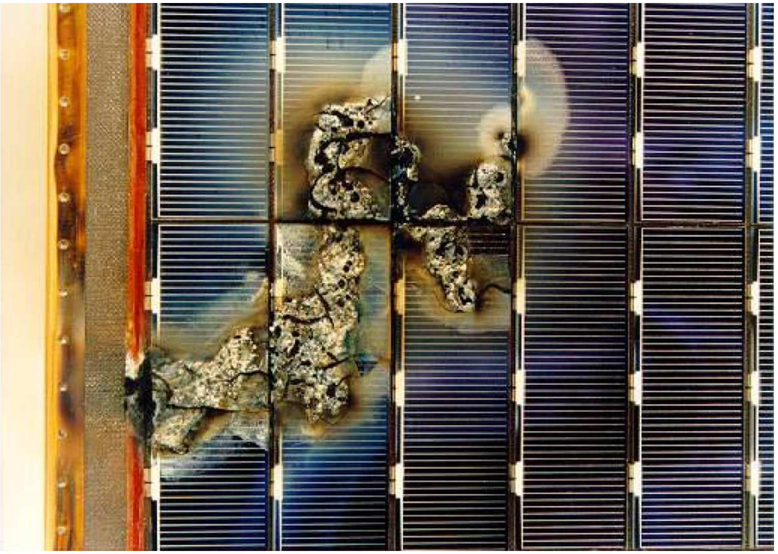
1. Introduction
2. The Euler-Poisson system and its quasi-neutral limit
3. Classical discretization for Euler-Poisson (EP)
4. The new “AP” scheme for (EP)
5. Numerical results
6. Conclusion and works in progress

1. Introduction

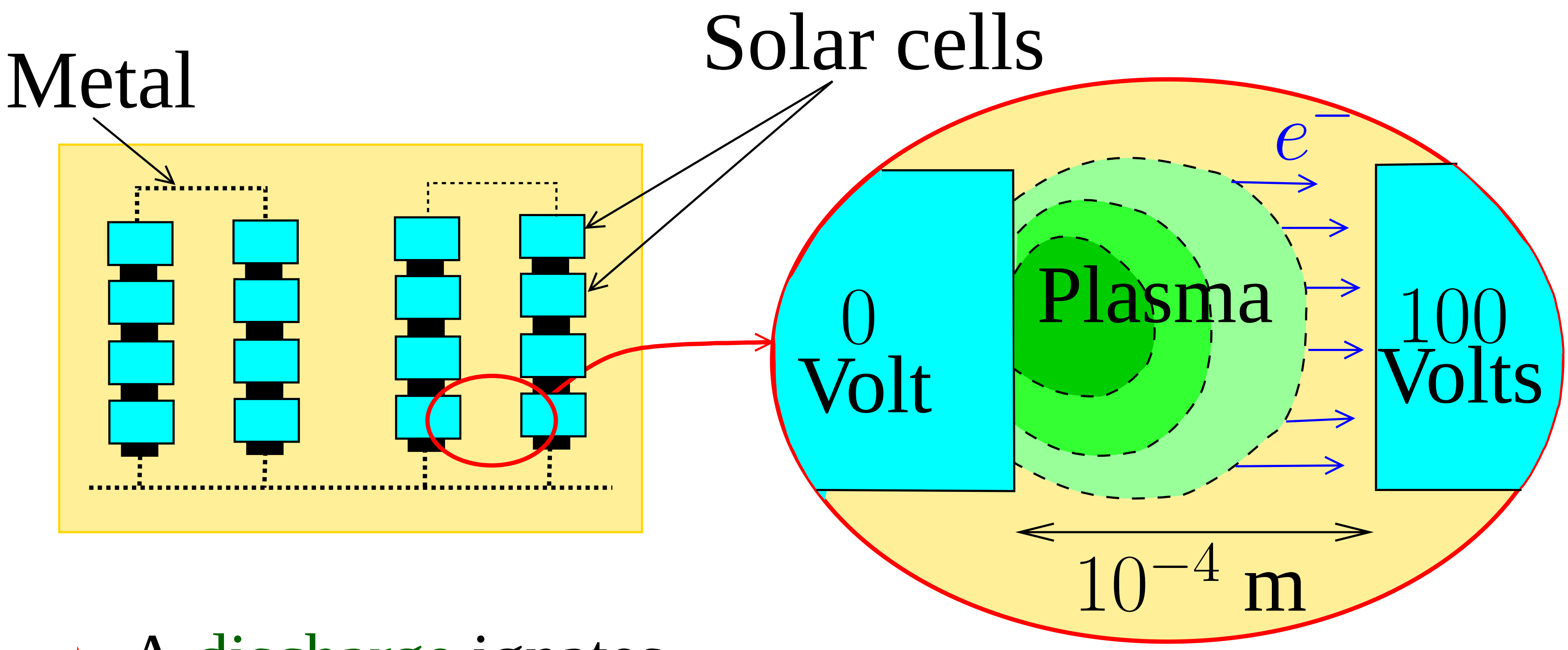
Appl. 1 : Electric arcs on satellites (I)

Picture of a satellite and of its solar panels

Solar panel dammaged by an electric arc



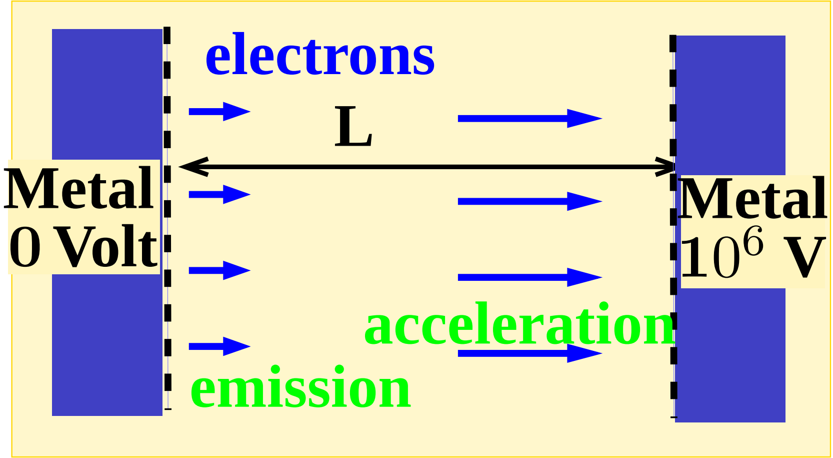
Appl. 1 : Electric arcs on satellites (II)



- A discharge ignates.
- A high density plasma is created and expands.
- The plasma fills the gap, an arc appears.

Appl. 2 : High current diodes (I)

- Conventional plane diode :



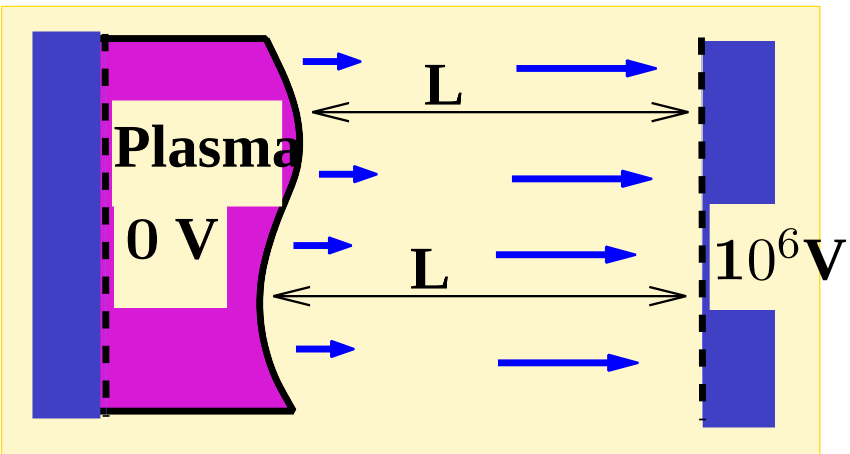
Maximal current
=
Child-Langmuir current

$$J_{CL} = \frac{4}{9}\varepsilon_0 \sqrt{\frac{2\,e}{m}} \frac{\phi^{3/2}}{L^2}$$

for all emission process.

- Problem: How to bypass this limitation?

- High-current plane diode



- – Plasma expands
- – Interface moves
- – Extracted current ↗

- Questions :
  - What is the motion of the interface?
  - What is the emission law for electrons at the plasma-vacuum interface?

What is a plasma ?

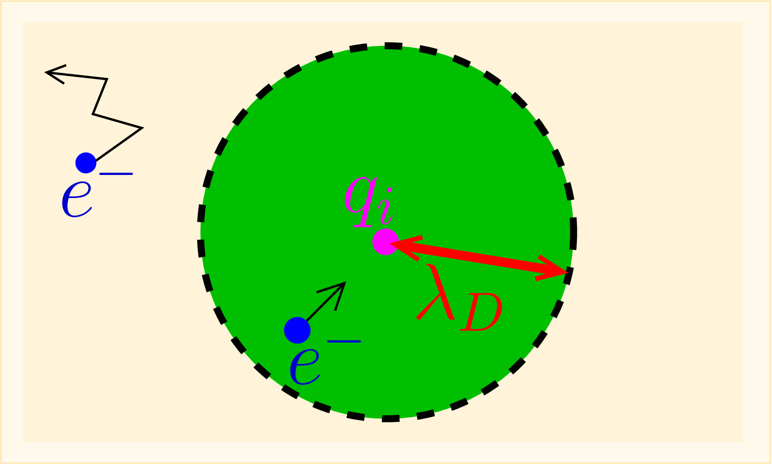
- Gaz containing charged particles
  - Charges < 0 : electrons, negative ions
  - Charges > 0 : positive ions
  - Charges $= 0$: neutral atoms or molecules

- Specificity of plasmas (compared with gazes)
  - Electromagnetic forces between charged particles

Caracteristic scales in plasmas

- Debye length:

$$\lambda_D = \left(\frac{\varepsilon_0 k_B T}{e^2 n} \right)^{1/2}$$



- Electrons are attracted by $q_i > 0$
- A cloud of < 0 charges around q_i
- Screening of q_i beyond the distance λ_D

$\Rightarrow$ Charge imbalances subsist only at scales $\leq \lambda_D$

Quasi-neutrality

➠ Quasi-neutral plasmas: (very frequent)

$$\lambda = \frac{\lambda_D}{L} \ll 1 \qquad \Rightarrow \qquad \text{Charges imbalances negligible} \quad n_+(x,t) \approx n_-(x,t)$$

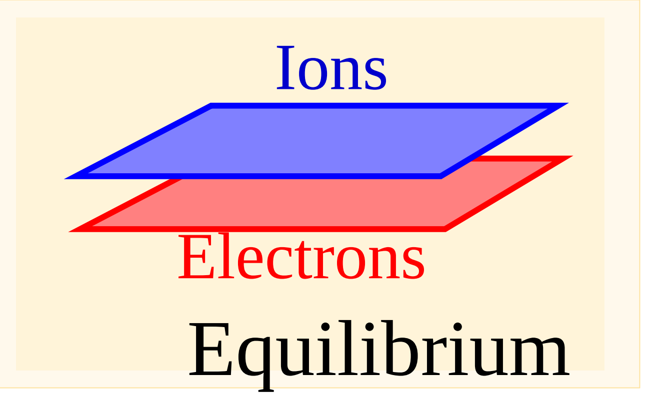
$L =$ caract. length of the problem

➠ Non quasi-neutral plasmas : (sheaths, beams, ...)

$$\lambda \sim 1 \qquad \Rightarrow \qquad \text{Charge imbalances of order 1} \quad n_+(x,t) \neq n_-(x,t)$$

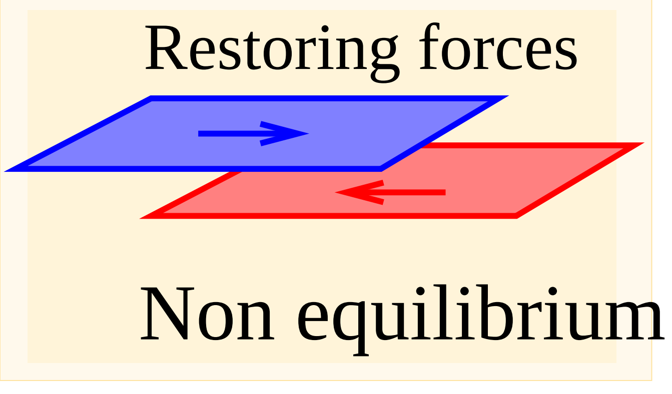
Time scale: plasma oscillations

- Plasma oscillations:
  - Charge imbalances
  - Restoring electric forces
  - Oscillations



- (electronic) Plasma period

$$\tau_e = \left(\frac{\varepsilon_0 m_e}{e^2 n}\right)^{1/2}$$



- In quasi-neutral regime

$$\tau := \frac{\tau_e}{t_0} \ll 1$$

$t_0 =$ caracteristic time of the problem

- Quasi-neutral state = average over a very large number of plasma periods

Numerical constraints

- Non quasi-neutral model, valid in all regime
  - Classical schemes

$$\Delta x \leq \lambda_D \qquad \text{and} \qquad \Delta t \leq \tau_e$$

  - Huge cost in quasi-neutral zones (QN)

$$(\lambda_D/L \ll 1 \text{ and } \tau_e/t_0 \ll 1)$$

$$\Downarrow$$

(Non-QN) models in QN zones unusable in dim. > 1

$$\Downarrow$$

- A (QN) model is necessay for QN regimes

Problems with quasi-neutral models

- Quasi-neutrality constraint uneasy to deals with numerically for non vanishing current.

- If quasi and non quasi- neutral zones
  - (Moving) Interface between (QN) and (Non-QN) models

    Formal derivation of the dynamic:

    [Degond, Parzani, V., SIAM MMS 04]

    [Slemrod, Ha, ARMA 05]

  - Moving interface: difficult numerical pb. in 2-D and 3-D

    Interface tracking [Tryggvason, . . .], Level set [Osher, Sethian, . . .]

    VoF [Youngs, Zaleski, . . .], Ficticious mixture [Karni, Abgrall& Saurel, .

A different strategy

- Use the (Non-QN) model for all zones
  - discretized with a scheme non resolving small scales

$$\Delta x \leq \lambda_D \,, \quad \Delta t \leq \tau_e$$

  - "AP" schemes for "Asymptotic Preserving"

    ([Jin] kinetic $\to$ Hydro)

- Our contribution : an AP scheme for Euler-Poisson
  - with an explicite cost like classical schemes

Additionnal references

- Rigorous quasi-neutral limits
  - Euler-Poisson : [Cordier & Grenier, Wang, Alì & Yungël]

- AP schemes, quasi-neutral limit, fluid models
  - [Fabre]
  - [Choe, Yoon, Kim, Choi]
  - [Colella, Dorr, Wake]
  - [Crispel, Degond, V]

2. The Euler-Poisson model and its quasi-neutral limit

- One species model for clarity

$$(EP)\begin{cases}\partial_t n + \nabla\cdot q = 0,\\ \varepsilon\,\partial_t q + \varepsilon\,\nabla\left(\dfrac{q\otimes q}{n}\right) + \nabla p(n) = n\nabla\phi,\\ \lambda^2\Delta\phi = n - n_0,\end{cases}$$

- $n_0 =$ constant ions density, $n =$ elec. density,

  $q = n\,u =$ elec. momentum, $p(n) =$ elec. pressure,

  $\phi =$ electric potential, $\varepsilon = \dfrac{\text{electron mass}}{\text{ion mass}}$.

- $$\lambda = \frac{\lambda_D}{L} = \frac{\text{Debye length}}{\text{caracteristic length}}$$

$$(QN)\begin{cases} \partial_t n + \nabla \cdot q = 0 \\ \varepsilon\, \partial_t q + \varepsilon\, \nabla \left(\dfrac{q \otimes q}{n} \right) + \nabla(p(n)) = n \nabla \phi, \\ n = n_0. \end{cases}$$

➠ Equivalently:
$$\begin{cases} \nabla \cdot q = 0, \\ \partial_t q + \nabla \left(\dfrac{q \otimes q}{n_0} \right) = \dfrac{n_0 \nabla \phi}{\varepsilon}, \\ n = n_0. \end{cases}$$

$n_0 = 1 \Rightarrow$ Incompressible Euler Eqs. (pressure$= -\phi$)

➠ $\phi =$ Lagrange multiplier of $\nabla \cdot q = 0$

$\lambda \to 0$: Quasi-neutral limit (II)

- Explicite eq. for the potential

$$\nabla \cdot \left(\partial_t q + \nabla \left(\frac{q \otimes q}{n_0} \right) = \frac{n_0 \, \nabla \phi}{\varepsilon} \right)$$

$$\Downarrow \nabla \cdot q = 0$$

- Elliptique eq.:

$$-\nabla \cdot \left(\frac{n_0}{\varepsilon} \nabla \phi \right) = -\nabla^2 : \left(\frac{q \otimes q}{n_0} \right)$$

Unify (EP) and (QN) (I)

- Different eqs. for ϕ
  - (EP) : Poisson $\lambda^2 \Delta\phi = n - n_0$
  - (QN) : Eq. $-\nabla \cdot \left(\frac{n_0}{\varepsilon} \nabla\phi \right) = -\nabla^2 : \left(\frac{q \otimes q}{n_0} \right)$

Not the same homogeneity

$\Downarrow$

Is it possible to unify them?

Unify (EP) and (QN) (II)

- Starting from (EP):
  - Take the $\nabla\cdot$ of the momentum Eq.

$$\nabla \cdot (\partial_t q) + \nabla^2 : f(n,q) = \nabla \cdot \left(\frac{n \nabla \phi}{\varepsilon} \right) \qquad (1)$$

with $\quad f(n,q) = q \otimes q/n + p(n)\mathbf{Id}/\varepsilon$

  - Take the ∂_t of the mass Eq.

$$\partial_{tt}^2 n + \partial_t (\nabla \cdot q) = 0 \qquad (2)$$

  - Take the difference of (1) and (2)

$$-\partial_{tt}^2 n + \nabla^2 : f(n,q) = \nabla \cdot \left(\frac{n \nabla \phi}{\varepsilon} \right)$$

Unify (EP) and (QN) (III)

- Use the Poisson Eq., $n = n_0 + \lambda^2 \Delta\phi$:

$$-\lambda^2 \Delta(\partial^2_{tt}\phi) + \nabla^2 : f(n, q) = \nabla \cdot \left(\frac{n \nabla \phi}{\varepsilon} \right)$$

- The reformulated Poisson Eq.

$$\lambda^2 \partial^2_{tt}(-\Delta\phi) - \nabla \cdot \left(\frac{n \nabla \phi}{\varepsilon} \right) = -\nabla^2 : f(n, q)$$

$$\lambda^2 \partial_{tt}^2(-\Delta\phi) - \nabla \cdot \left(\frac{n\nabla\phi}{\varepsilon}\right) = -\nabla^2 : f(n,q)$$

- New elliptic eq. replaces Poisson eq.
  - Equivalent to Poisson eq. under initial cond.

$$(\lambda^2\Delta\phi = n - n_0)\big|_{t=0} \quad \text{and} \quad \frac{d}{dt}(\lambda^2\Delta\phi = n - n_0)\big|_{t=0}.$$

- Does not degenerate when $\lambda \to 0$

- Reduces to (QN) elliptic eq. when $\lambda = 0$

- $n = constant$

$$\partial^2_{tt}\rho + \frac{n}{\lambda^2\varepsilon}\rho = -\nabla^2 : f(n,q) \qquad (3)$$

  - Harmonic oscillator Eq. on $\rho = -\Delta\phi$
  - $\lambda^2\varepsilon = \tau^2 =$rescaled elec. plasma period
  - Explicit scheme $\Rightarrow$ conditionnal stability
  - Implicit scheme $\Rightarrow$ unconditionnal stability

3. Classical discretization of the Euler-Poisson (EP) system

- (EP) system:

$$\begin{cases} \partial_t n + \nabla \cdot q = 0 \\ \partial_t q + \nabla f(n,q) = \dfrac{n \nabla \phi}{\varepsilon} \\ \lambda^2 \Delta \phi = n - n_0 \end{cases}$$

$$f(n,q) = \frac{q \otimes q}{n} + \frac{p(n)\mathrm{Id}}{\varepsilon}$$

- Rescaled electronic plasma period $= \tau = \sqrt{\varepsilon}\,\lambda$

(EP) time discretization

- Classical scheme:
  - Explicit hydro fluxes
  - Implicit Poisson eq. and electric force terms

- n^m, q^m, ϕ^m: known approximations at time t^m

$$\left\{\begin{aligned} &\frac{n^{m+1}-n^m}{\Delta t} + \nabla \cdot q^m = 0, \\ &\frac{q^{m+1}-q^m}{\Delta t} + \nabla f(n^m, q^m) = \frac{n^{m+1}\nabla\phi^{m+1}}{\varepsilon}, \\ &\lambda^2 \Delta\phi^{m+1} = n^{m+1} - n_0. \end{aligned}\right.$$

- Explicit resolution

- Stab. condition related to the quasi-neutrality (S. Fabre):

$$\Delta t \leq \tau = \sqrt{\varepsilon}\,\lambda$$

  - In QN regime: $\lambda \ll 1$,

Huge computationnal cost

- Take the $\nabla\cdot$ of the momentum Eq.

$$\frac{\nabla\cdot q^{m+1}-\nabla\cdot q^{m}}{\Delta t}+\nabla^2 : f(n^m,q^m)=\nabla\cdot\left(\frac{n^{m+1}\nabla\phi^{m+1}}{\varepsilon}\right),$$

- Take the discret ∂_t of the mass Eq.

$$\frac{n^{m+2}-2\,n^{m+1}+n^{m}}{\Delta t^2}+\frac{\nabla\cdot q^{m+1}-\nabla\cdot q^{m}}{\Delta t}=0,$$

- Take the difference and use the discret Poisson Eq.

$$\Downarrow$$

- Discretization of the reformulated Poisson Eq.

$$-\frac{\Delta\phi^{m+2} - 2\,\Delta\phi^{m+1} + \Delta\phi^{m}}{\Delta t^2} - \nabla \cdot \left(\frac{n^{m+1}\nabla\phi^{m+1}}{\lambda^2\,\varepsilon} \right) = -\nabla^2 : f(n^m, q^m)$$

Explicit discretization $\Rightarrow$ Conditionnal stability

4. New approach: “AP” scheme

AP scheme for Euler-Poisson

- "AP" scheme:
  - Implicit Poisson eq. and mass flux
  - Semi-implicit electric force terms
  - Explicit momentum flux

- $$\begin{cases} \dfrac{n^{m+1} - n^m}{\Delta t} + \nabla \cdot q^{m+1} = 0, \\[2ex] \dfrac{q^{m+1} - q^m}{\Delta t} + \nabla f(n^m, q^m) = \dfrac{n^m \nabla \phi^{m+1}}{\varepsilon}, \\[2ex] \lambda^2 \Delta \phi^{m+1} = n^{m+1} - n_0. \end{cases}$$

- Cost and behavior in the quasi-neutral limit?

- Take
  - the $\nabla \cdot$ of the moment. Eq.
  - the discret ∂_t of the mass Eq.
  - the difference
- Use the discret Poisson Eq. $\Rightarrow$

$$-\frac{\Delta\phi^{m+1} - 2\,\Delta\phi^{m} + \Delta\phi^{m-1}}{\Delta t^2} - \nabla \cdot \left(\frac{n^m \nabla \phi^{m+1}}{\varepsilon\,\lambda^2} \right) = -\nabla^2 : f(n^m, q^m)$$

- Implicit discretization for ϕ.
- Explicit resolution

$$\left\{ \begin{array}{l} \dfrac{n^{m+1} - n^m}{\Delta t} + \nabla \cdot q^{m+1} = 0, \\ \dfrac{q^{m+1} - q^m}{\Delta t} + \nabla f(n^m, q^m) = \dfrac{n^m \nabla \phi^{m+1}}{\varepsilon}, \\ -\lambda^2 \, \dfrac{\Delta \phi^{m+1} - 2\,\Delta \phi^m + \Delta \phi^{m-1}}{\Delta t^2} - \nabla \cdot \left(\dfrac{n^m \nabla \phi^{m+1}}{\varepsilon} \right) \\ \qquad = -\nabla^2 : f(n^m, q^m) \end{array} \right.$$

- Explicit resolution
- Stab. of the linearized syst. $\Delta t = O(1)$ even if $\lambda \to 0$

work in collaboration with J-G. Liu (Maryland)

Space discretization (I)

Explicit “AP” scheme iff

$$\text{Mass num. flux} \; = \; \text{q} \; + \; \text{num. viscosity}(n,q) \times n$$

➠ Modified Lax-Friedrichs scheme (LF)

➠ Roe type scheme: degree 0 polynomial scheme ($P0$) (Degond, Peyrard, Villedieu)

5. Numerical results

- Two species: ions and electrons

$$\begin{cases} \partial_t n_i + \nabla \cdot q_i = 0, \\ \partial_t q_i + \nabla \left(\dfrac{q_i \otimes q_i}{n_i} \right) + \nabla p_i(n_i) = -n_i \nabla \phi, \end{cases}$$
$$\begin{cases} \partial_t n_e + \nabla \cdot q_e = 0, \\ \varepsilon\, \partial_t q_e + \varepsilon\, \nabla \left(\dfrac{q_e \otimes q_e}{n_e} \right) + \nabla p_e(n_e) = n_e \nabla \phi \end{cases}$$
$$\lambda^2 \Delta \phi = n_e - n_i,$$

- Perturbation around a quasi-neutral equilibrium

$$n_i = n_e = 1, \quad q_i = 0, \quad q_e = 1.$$

Perturbation of a QN equilibrium (II)

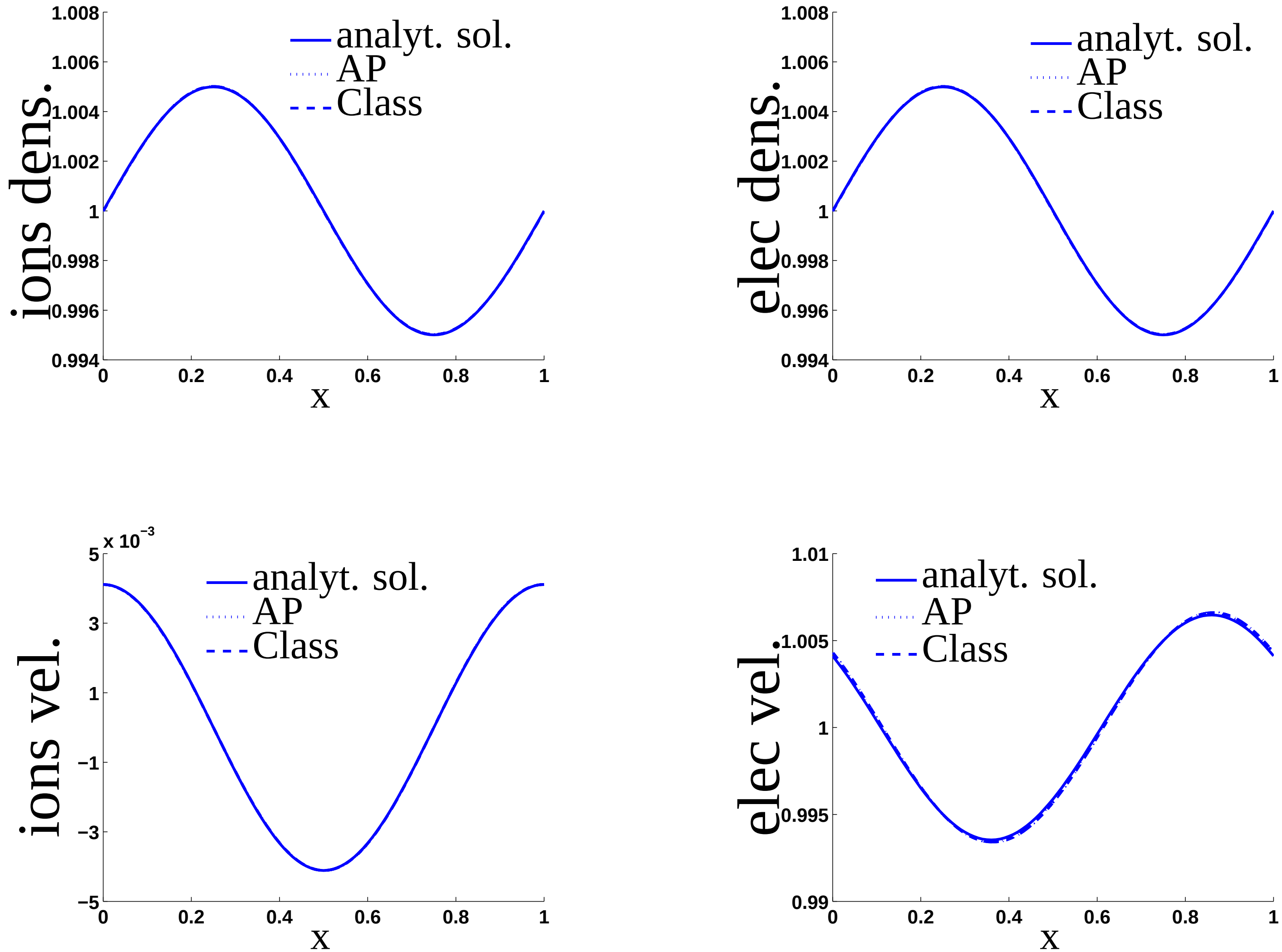
- Initial perturbation:

$$n_i = n_e = 1, \quad q_i = 10^{-2}\cos 2\pi x, \quad q_e = 1+10^{-2}\cos 2\pi x.$$

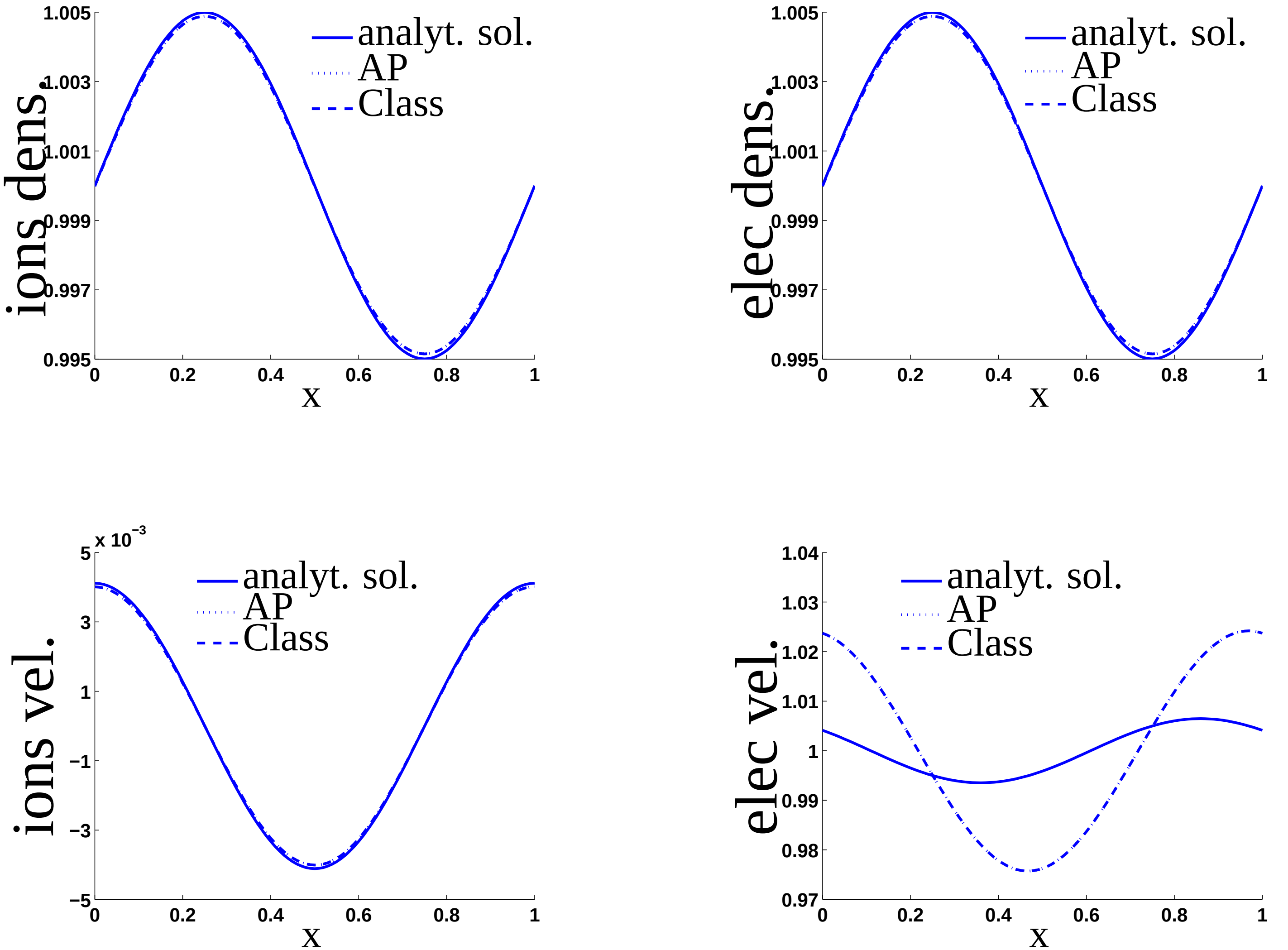
- Explicit solutions of the linearized system

- Parameters of the pb.
  - Mass ratio $= \varepsilon = 10^{-4}$,
  - Rescaled Debye length $= \lambda = 10^{-4}$,
  - Rescaled plasma period $= \tau = \sqrt{\varepsilon}\,\lambda = 10^{-6}$

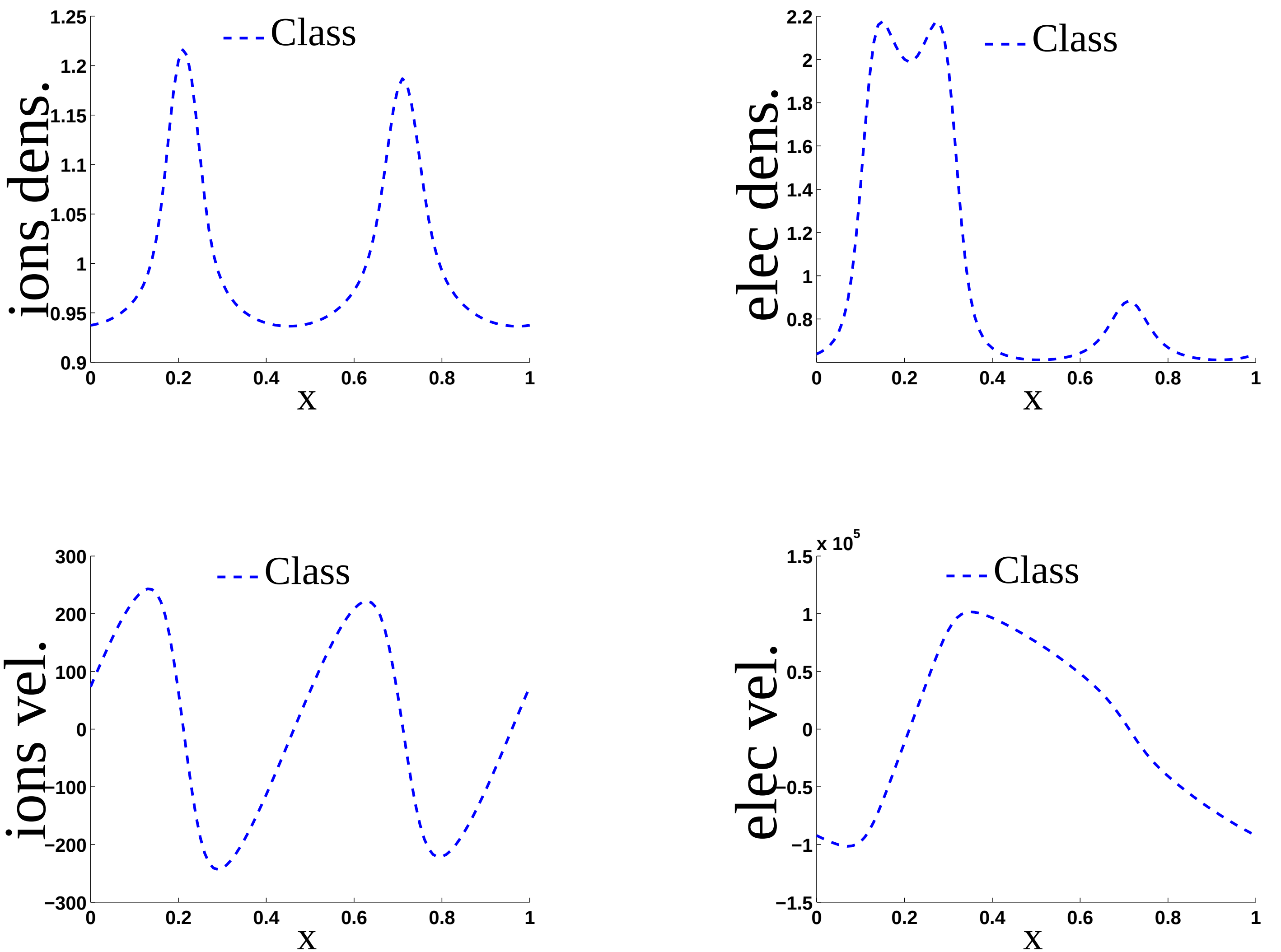
Class. and AP schemes: $\Delta x < \lambda \quad \Delta t < \tau$



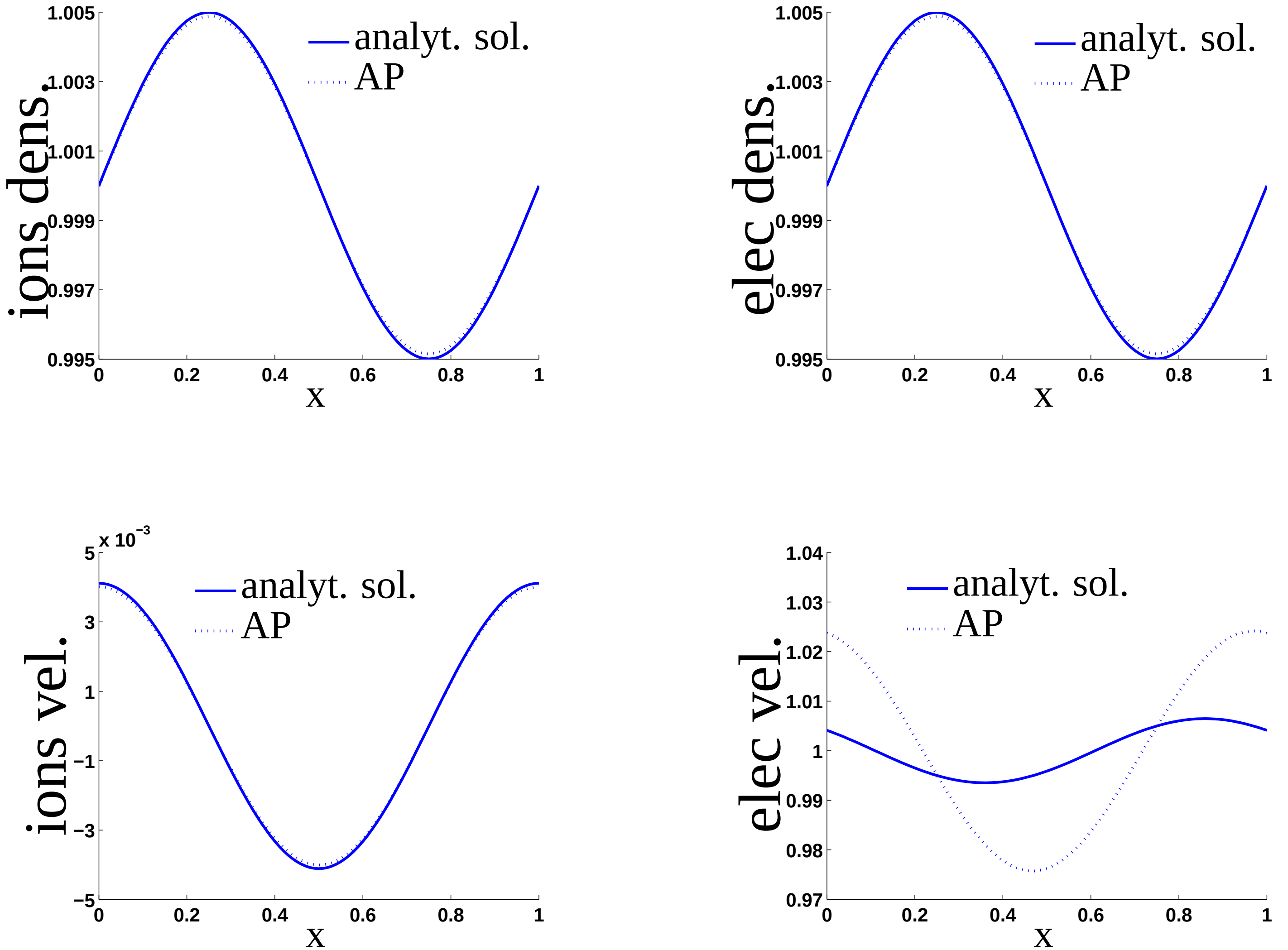
Class. and AP schemes: $\Delta x > \lambda \quad \Delta t < \tau$



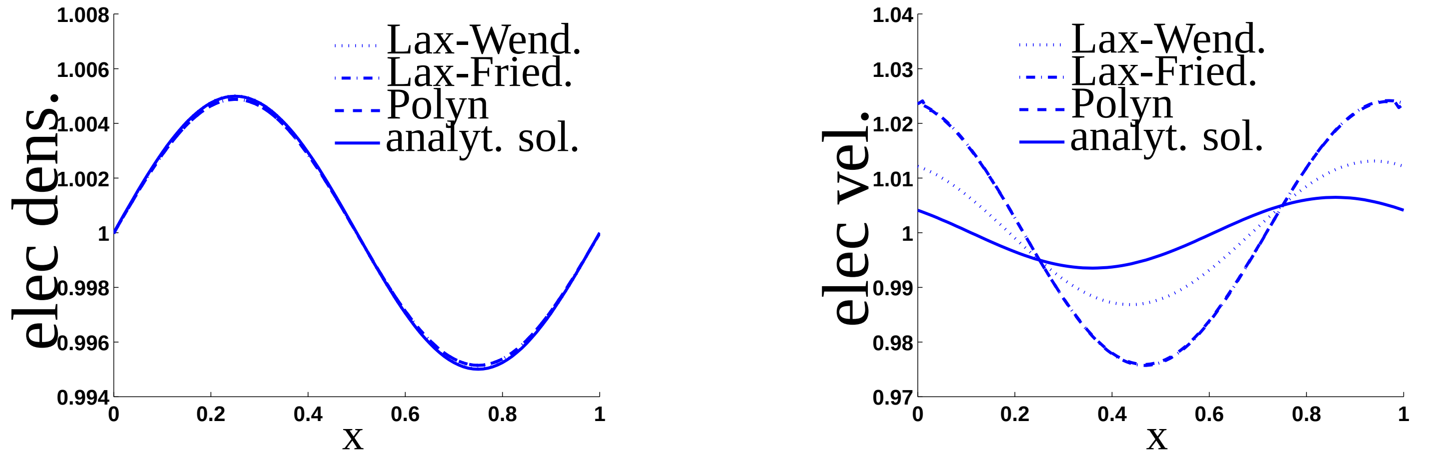
Class. scheme: $\Delta x > \lambda \quad \Delta t > \tau$



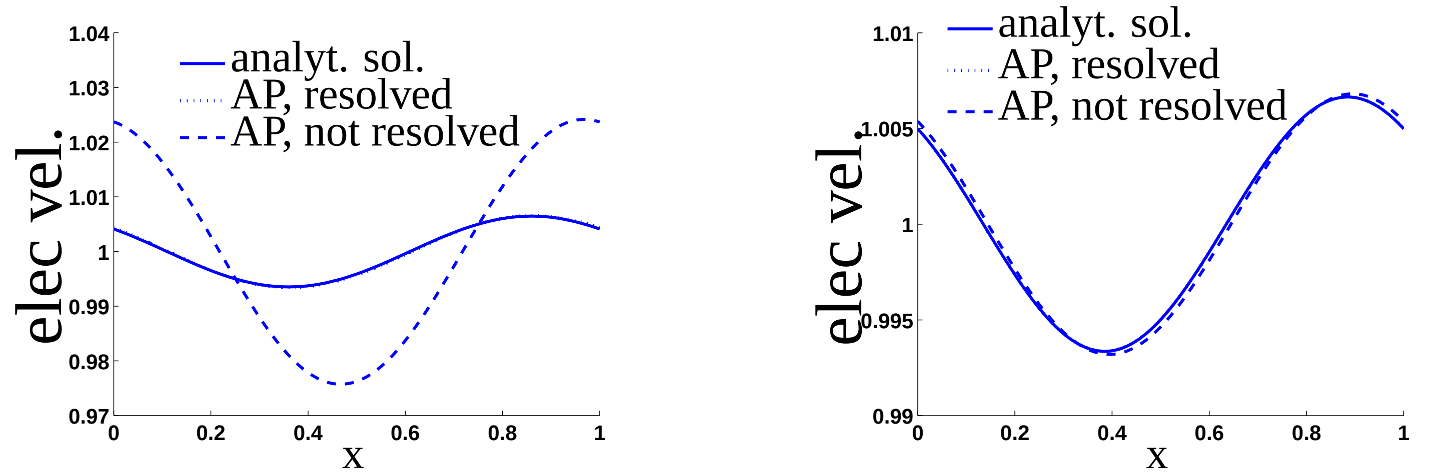
AP scheme: $\Delta x > \lambda \quad \Delta t > \tau$



AP scheme: $\Delta x > \lambda \quad \Delta t > \tau$



Other solvers



$\varepsilon = 10^{-4}$ $\qquad$ $\varepsilon = 10^{-1}$

Variation of ε

Plasma expansion between two electrodes

- Initially, domain devoid of plasma
- Injection at $x = 0$, the cathode:

$$n_i = n_e = 1 \quad u_i = u_e = 1 \quad \phi = 0$$

- Applied D.D.P.

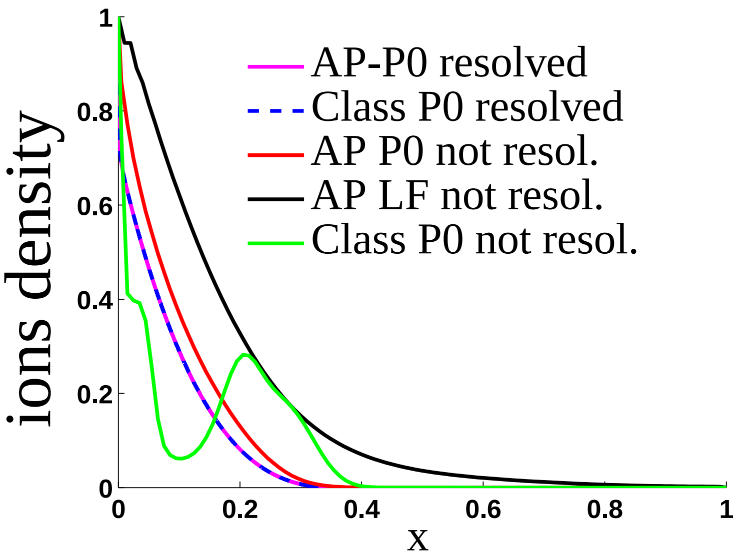
$$\phi(x = 1) = \phi_1$$

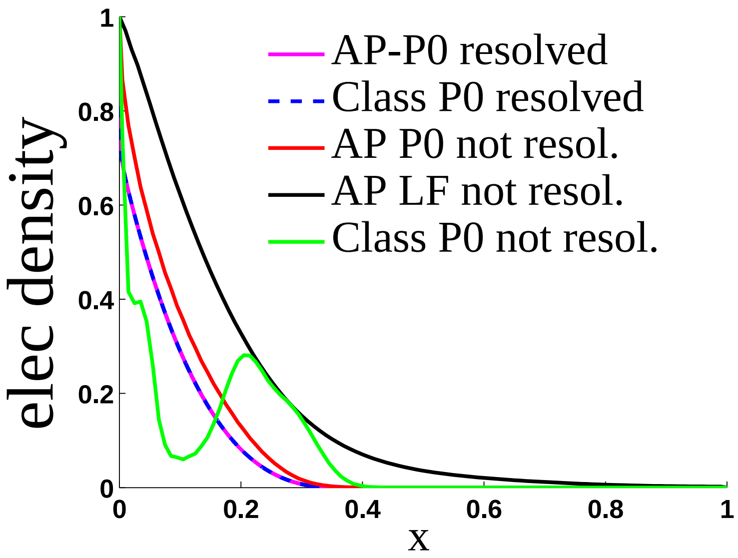
- Parameters

$$\varepsilon = 10^{-4}, \quad \lambda = 10^{-4}, \quad \tau = 10^{-6}, \quad \phi_1 = 100.$$

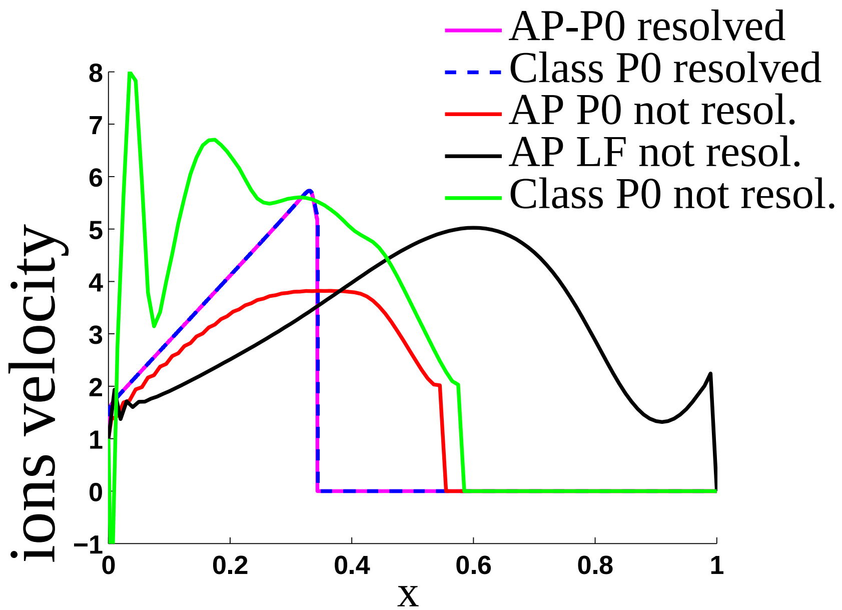
Comparison classical and AP schemes (I)

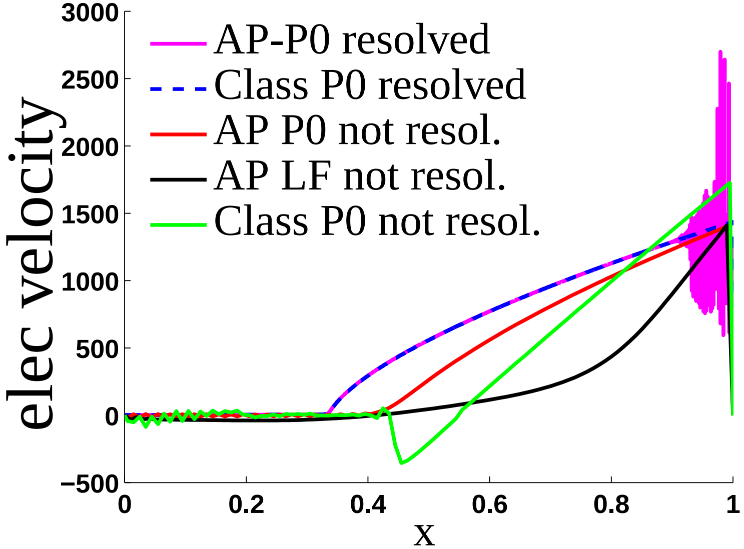
- Resolved $\Leftrightarrow$ $(\Delta x \leq \lambda$ and $\Delta t \leq \tau)$
- Not resolved $\Leftrightarrow$ $(\Delta x > \lambda$ and $\Delta t > \tau)$

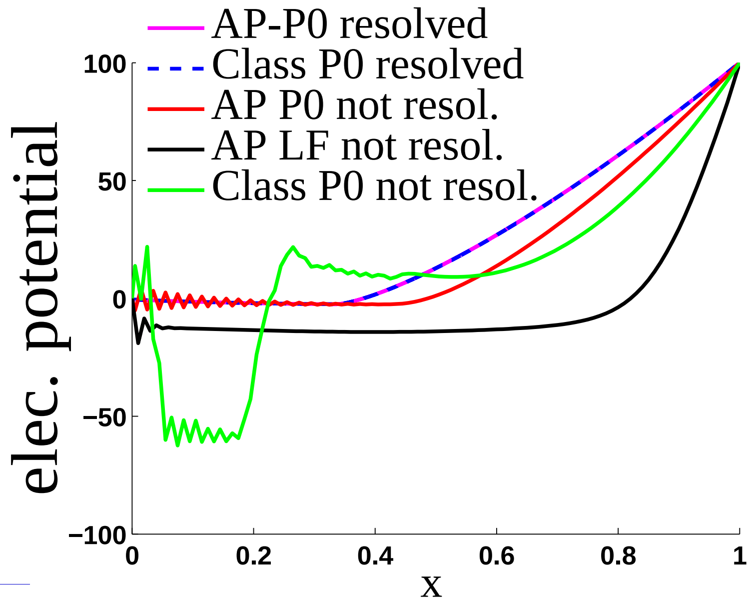




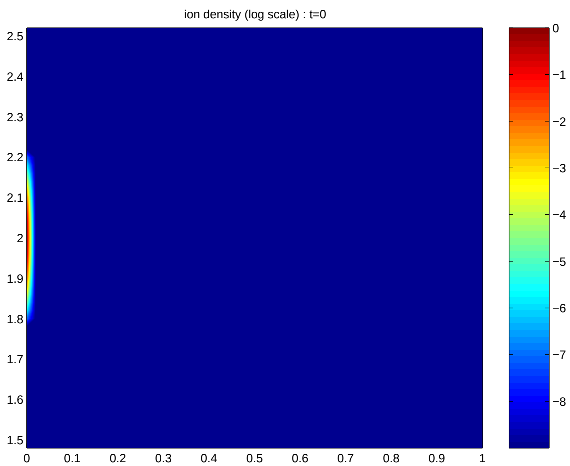
Comparison classical and AP schemes (II)



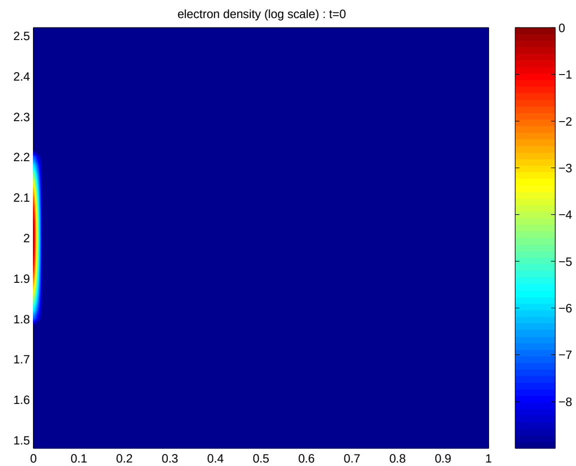




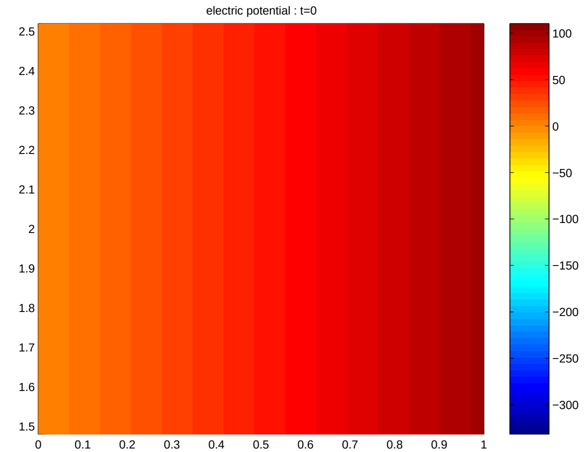
AP scheme: 2D results $t = 0.00$



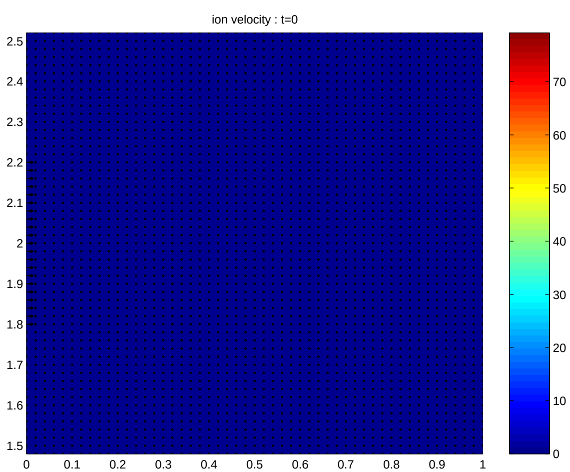
Ions density



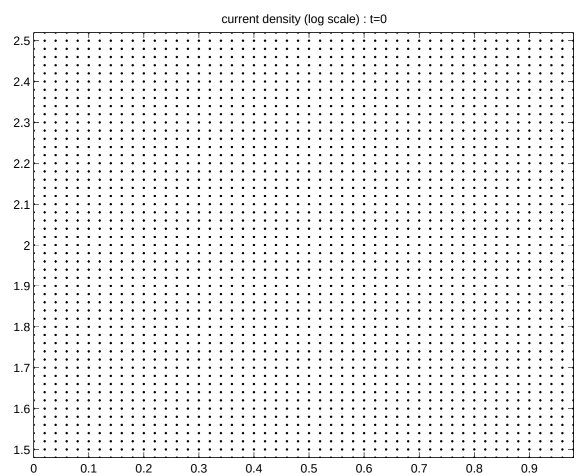
Electrons density



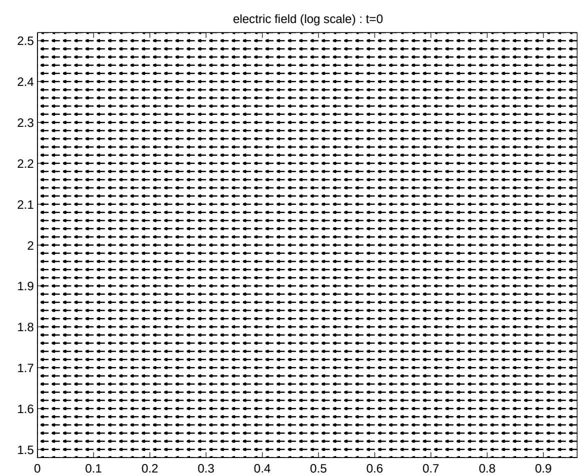
Potential



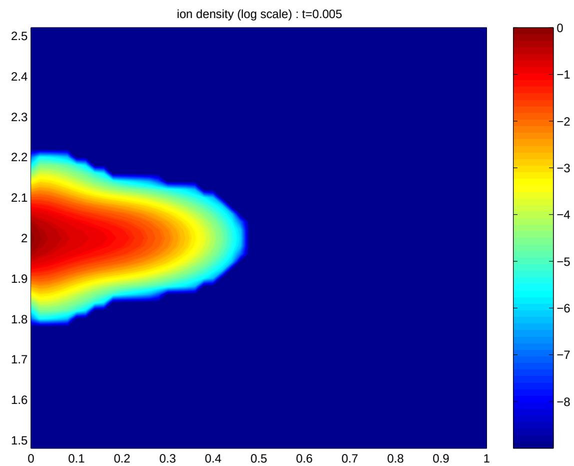
Ions velocity



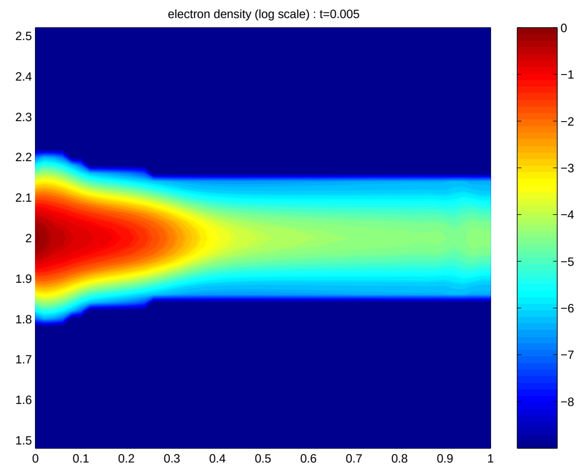
Current



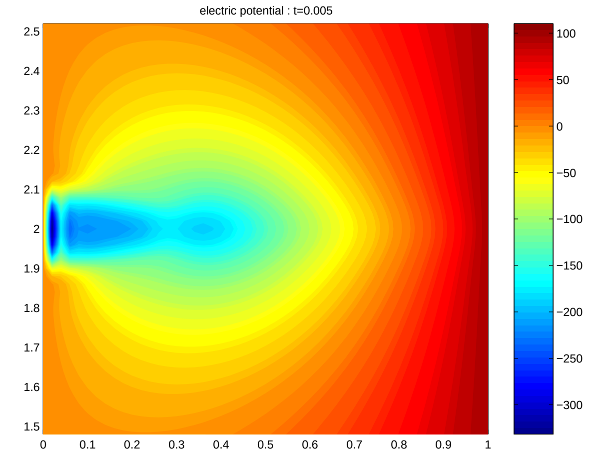
Electric field



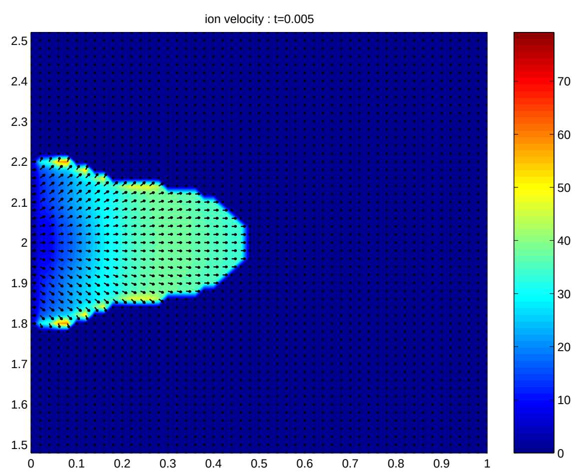
Ions density



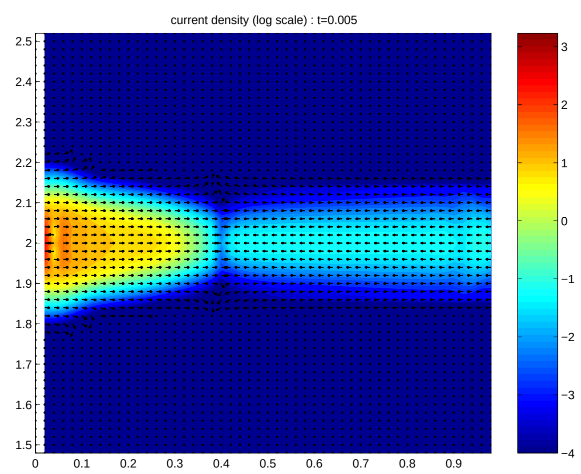
Electrons density



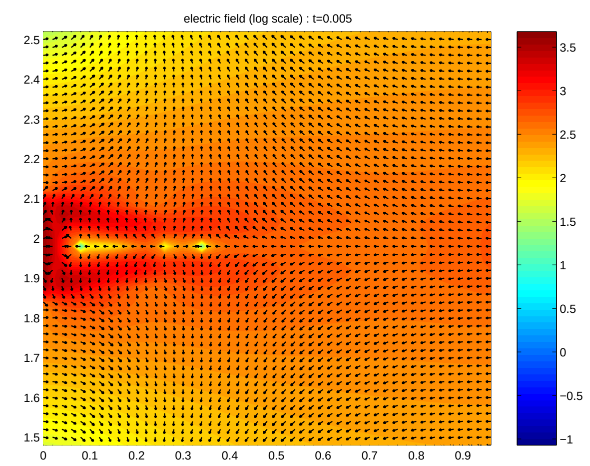
Potential



Ions velocity

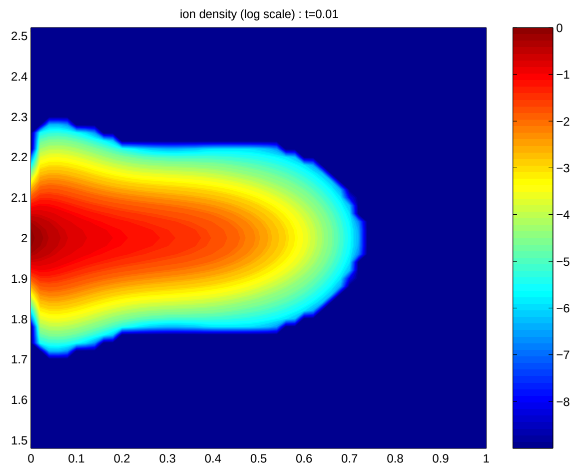


Current

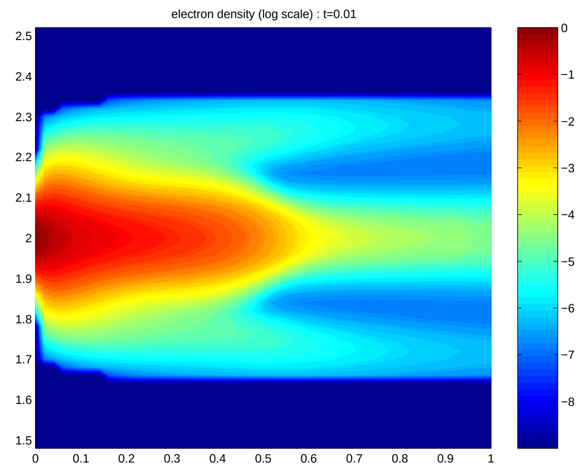


Electric field

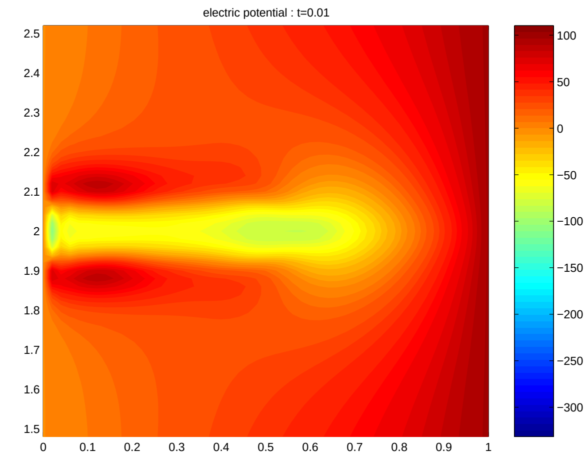
AP scheme: 2D results $t = 0.01$



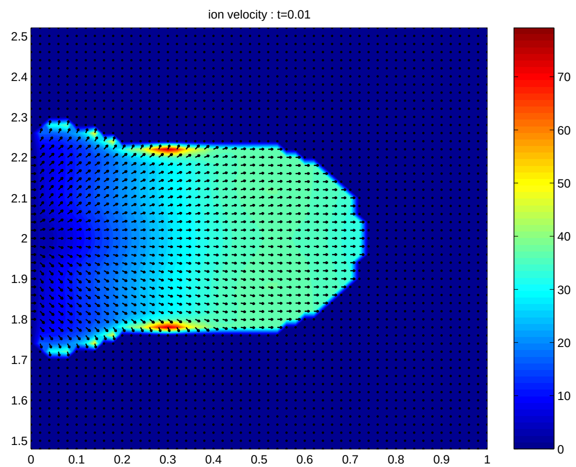
Ions density



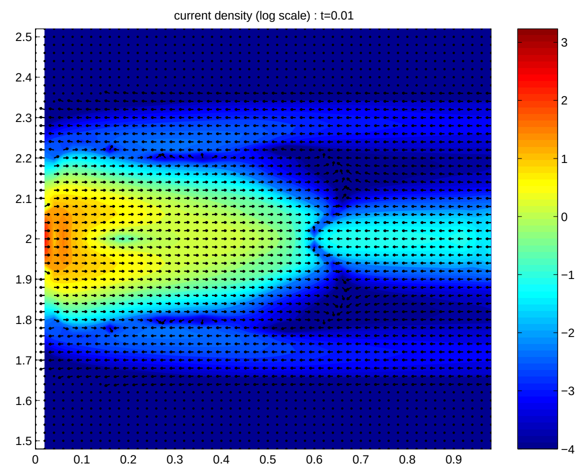
Electrons density



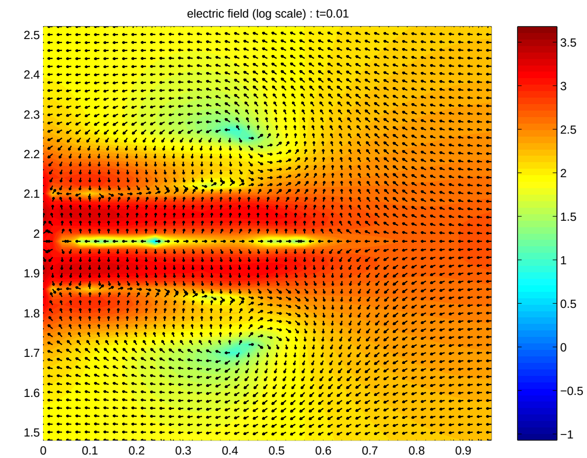
Potential



Ions velocity

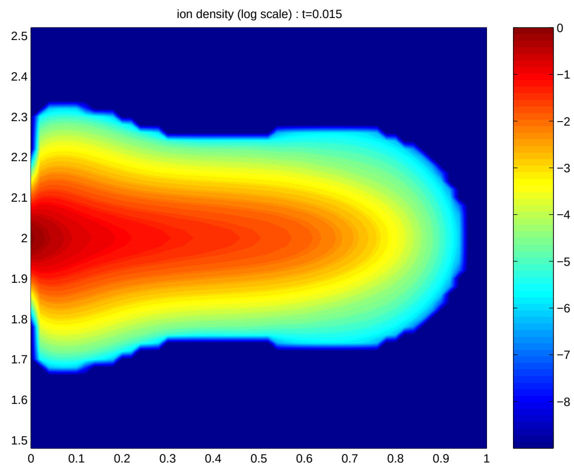


Current

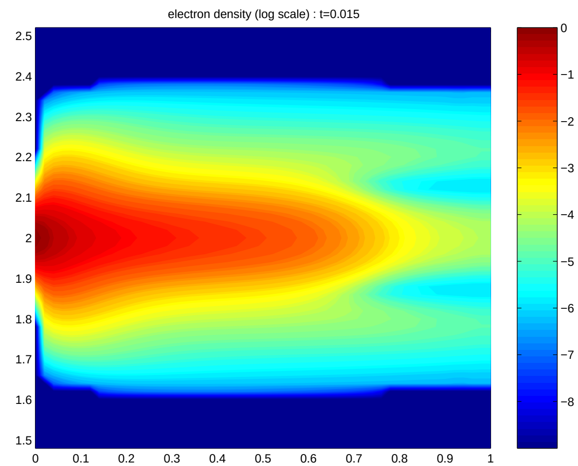


Electric field

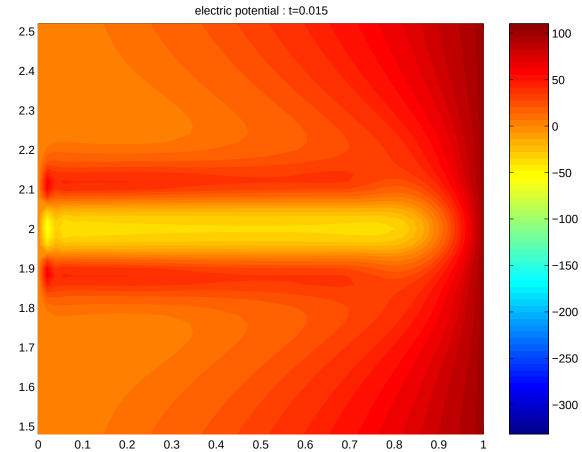
AP scheme: 2D results $t = 0.015$



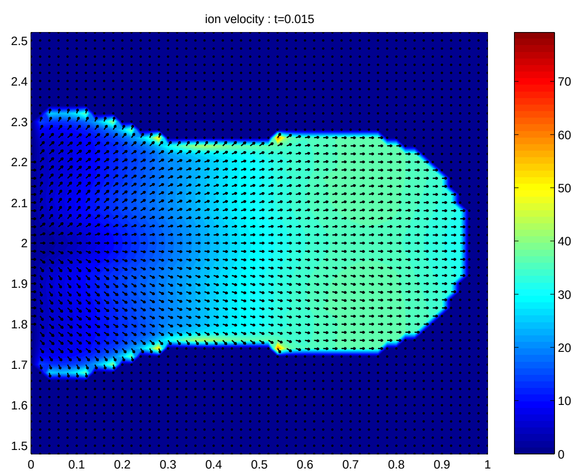
Ions density



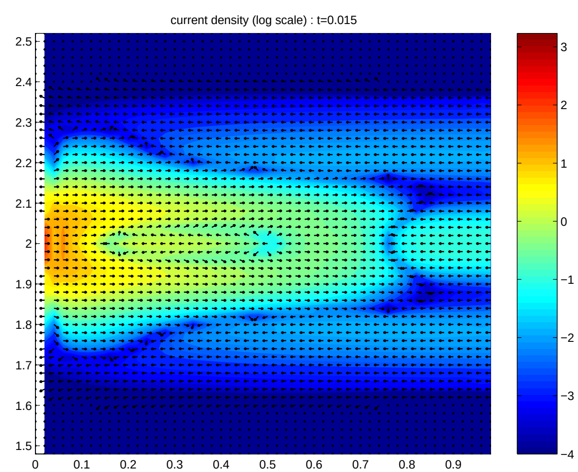
Electrons density



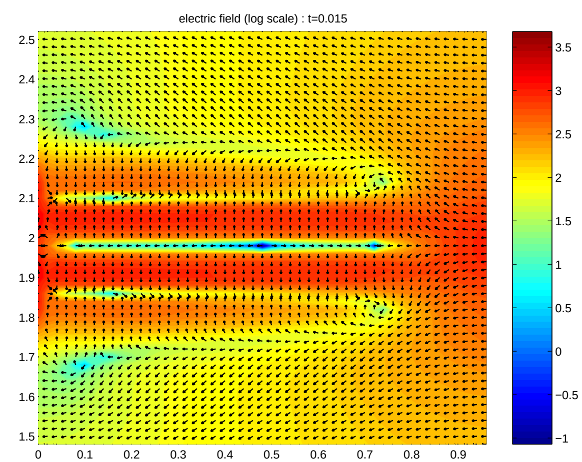
Potential



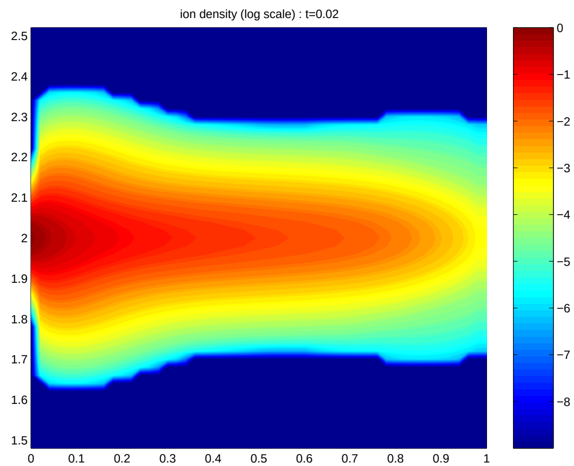
Ions velocity



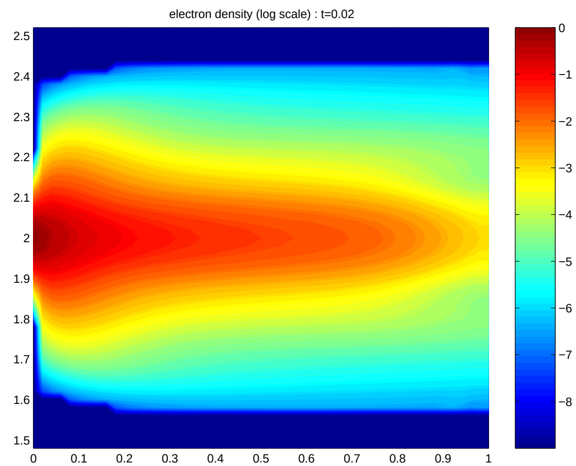
Current



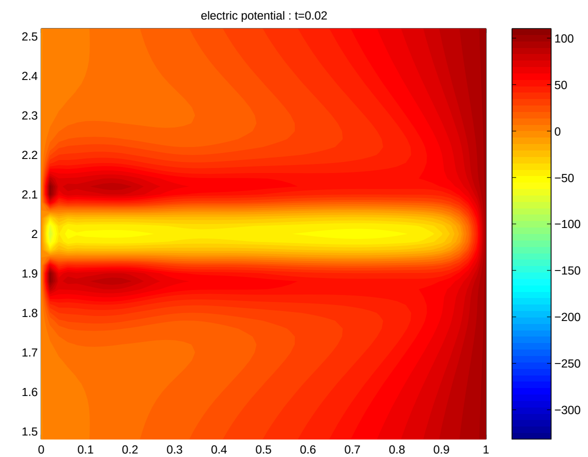
Electric field



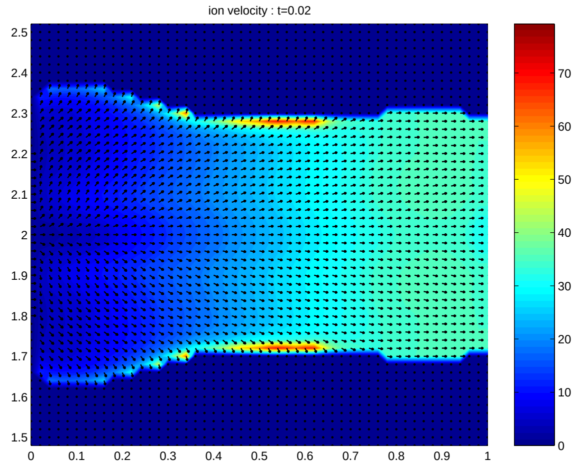
Ions density



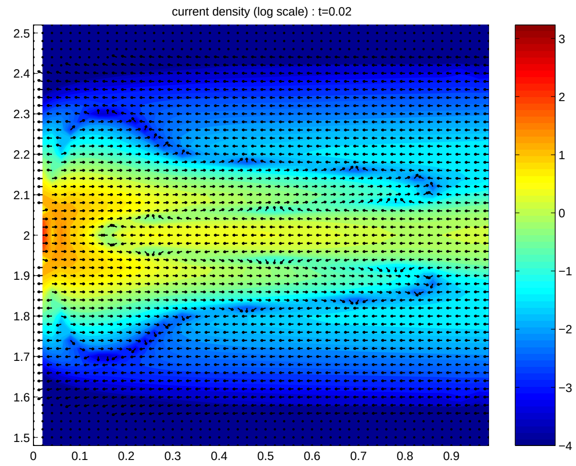
Electrons density



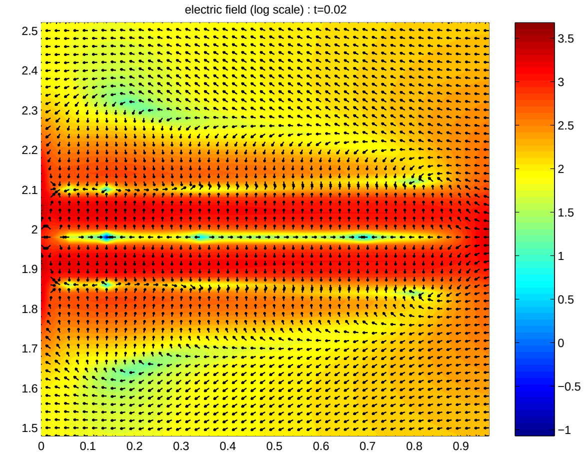
Potential



Ions velocity



Current



Electric field

6. Work in progress

Our scheme:

- is asymptotically stable in the QN limit
  - Does not need to resolve small QN scales

$$\Delta x \not\lesssim \lambda \qquad \text{and} \qquad \Delta t \not\lesssim \tau$$

- has an explicit cost like class. schemes

AP scheme : comments (II)

- Still constrained by the hydro. C.F.L. condition
  - Can be penalizing for electrons

$$\Delta t \leq u \pm \sqrt{\frac{p'(n)}{\varepsilon}}$$

with a small ε

  - Can be dealt with the same methodology

Work in progress with J-G. Liu (Maryland)

- Law Mach number limit of compressible Euler:
  - Same idea in progr. with N. Lemarchant (MIP)

Appli.: ITER (CEA Cadarache and Saclay)

Works in progress

- High order space discretizations
  - Order two schemes : Lax-Wendroff solver
  - Discontinue Galerkin method

  with S. Wang Shu (Brown)

- Other models
  - Full Euler (including energy eqs.)
  - Vlasov-Poisson (P. Degond, F. Deluzet, L. Novaret))

- Euler-Maxwell (P. Degond, F. Deluzet, F. Loret)