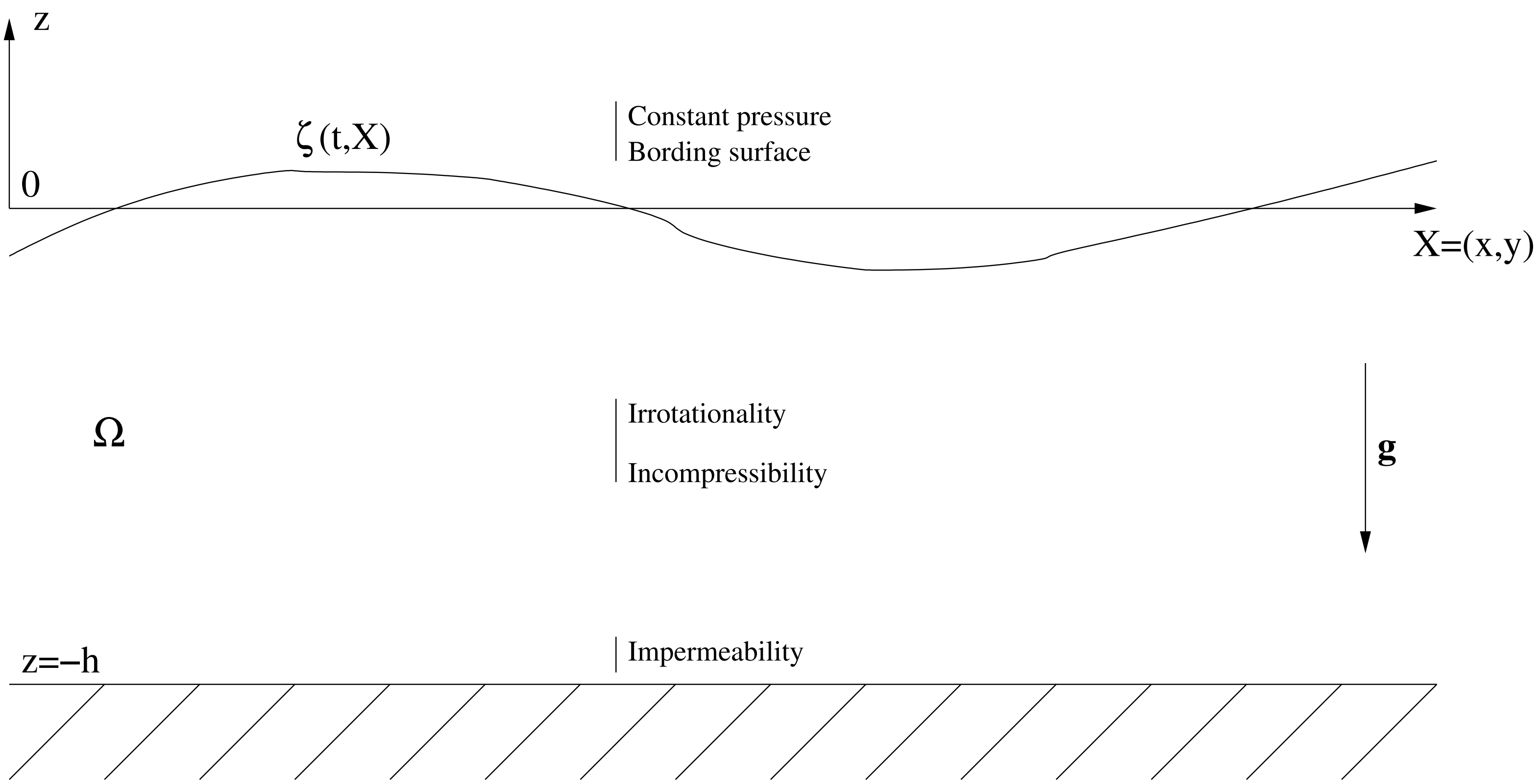
Water-Waves Problem

Goal: Describe the motion of the free surface



Water-Waves Equations

i. *In the fluid*

- Incompressibility:

$$\text{div } \underline{\mathbf{v}} = 0;$$

- Conservation of momentum (Euler equation):

$$\partial_t \underline{\mathbf{v}} + \underline{\mathbf{v}} \cdot \nabla_{X,z} \underline{\mathbf{v}} = -\frac{1}{\rho} \nabla_{X,z} P + \mathbf{g};$$

- Irrotationality:

$$\text{curl } \underline{\mathbf{v}} = 0.$$

Water-Waves Equations

ii. *Boundary conditions*

The surface and the bottom are bounding surfaces: no fluid particule crosses them.

Remark. An implicit surface $\Sigma(t, X, z) = 0$ is bounding iff

$$\frac{d\Sigma}{dt} = (\partial_t + \underline{\mathbf{v}} \cdot \nabla_{X,z})\Sigma = 0.$$

$\rightsquigarrow$ Bottom: $\Sigma(t, X, z) = z + h$ and the condition is

$$v_z = 0 \quad \text{at} \quad z = -h;$$

$\rightsquigarrow$ Surface: $\Sigma(t, X, z) = z - \zeta(t, X)$ and one gets

$$\partial_t \zeta = (-\nabla\zeta, 1)^T \cdot \underline{\mathbf{v}}_{|_{z=\zeta(t,X)}}.$$

Water-Waves equations

ii. *Boundary conditions* Neglecting the surface tension, the pressure is constant at the surface and we can assume

$$P = 0 \quad \text{at} \quad z = \zeta(t, X).$$

Conclusion: *Free surface Euler Equations:*

$$\begin{cases} \text{div } \underline{\mathbf{v}} = 0, & \text{in } \Omega, \\ \text{rot } \underline{\mathbf{v}} = 0, & \text{in } \Omega, \\ \partial_t \underline{\mathbf{v}} + \underline{\mathbf{v}} \cdot \nabla_{X,z} \underline{\mathbf{v}} = -\frac{1}{\rho} \nabla_{X,z} P + \mathbf{g} & \text{in } \Omega, \\ \underline{\mathbf{v}}_z = 0, & z = -h, \\ \partial_t \zeta = (-\nabla \zeta, 1)^T \cdot \underline{\mathbf{v}} & z = \zeta(t, x, y), \\ P = 0 & z = \zeta(t, x, y). \end{cases}$$

Bernouilli's formulation

Remark that

$$\begin{cases} \text{div } \underline{\mathbf{v}} = 0, & \text{in } \Omega, \\ \text{rot } \underline{\mathbf{v}} = 0, & \text{in } \Omega, \\ \underline{\mathbf{v}}_z = 0, & z = -h, \end{cases} \iff \begin{cases} \Delta\Phi = 0, & \text{in } \Omega \\ \partial_z \Phi = 0, & z = -h, \end{cases}$$

where Φ is a *velocity potential*: $\underline{\mathbf{v}} = \nabla_{X,z}\Phi$.
Similarly,

$$\partial_t \underline{\mathbf{v}} + \underline{\mathbf{v}} \cdot \nabla_{X,z} \underline{\mathbf{v}} = -\frac{1}{\rho} \nabla_{X,z} P + \mathbf{g} \quad \text{in } \Omega,$$

$$\iff \partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + gz = -\frac{1}{\rho} P \quad \text{in } \Omega.$$

$\rightsquigarrow$ P disappears at the surface.

Bernouilli's formulation

Water-Waves Equations under Bernouilli's formulation:

$$
\begin{cases}
\Delta\Phi = 0, & -h \leq z \leq \zeta(t,x,y), \\
\partial_z \Phi = 0, & z = -h, \\
\partial_t \zeta + \partial_y \Phi \partial_x \zeta + \partial_y \Phi \partial_y \zeta = \partial_z \Phi, & z = \zeta(t,x,y), \\
\partial_t \Phi + \frac{1}{2} |\nabla\Phi|^2 + gz = -\frac{1}{\rho} P & z = \zeta(t,x,y).
\end{cases}
$$

Nondimensionalization

- $a \rightsquigarrow$ Typical amplitude of the wave;
- $h \rightsquigarrow$ Mean depth;
- $\lambda \rightsquigarrow$ typical wavelength in the x direction;
- $\frac{\lambda}{\gamma} \rightsquigarrow$ typical wavelength in the y direction.

Dimensionless quantities:

$$x' = \frac{x}{\lambda}, \quad y' = \gamma\frac{y}{\lambda}, \quad z' = \frac{z}{h},$$

$$\zeta' = \frac{\zeta}{a}, \quad \Phi' = \frac{\Phi}{\Phi_0}, \quad t' = \frac{\sqrt{gh}}{\lambda}t,$$

where $\Phi_0 = \frac{a}{h}\sqrt{gh}\lambda$.

Nondimensionalized equations

$$\begin{cases} \mu\partial_x^2\Phi + \gamma^2\mu\partial_y^2\Phi + \partial_z^2\Phi = 0, & (-1 \leq z \leq \varepsilon\zeta), \\ \partial_z\Phi = 0, & (z = -1), \\ \partial_t\zeta + \varepsilon\partial_x\Phi\partial_x\zeta + \varepsilon\gamma^2\partial_y\Phi\partial_y\zeta = \dfrac{1}{\mu}\partial_z\Phi, & (z = \varepsilon\zeta), \\ \partial_t\Phi + \dfrac{1}{2}(\varepsilon(\partial_x\Phi)^2 + \varepsilon\gamma^2(\partial_y\Phi)^2 + \dfrac{\varepsilon}{\mu}(\partial_z\Phi)^2) + \zeta = 0. \end{cases}$$

with

- $\varepsilon = \frac{a}{h} \leadsto$ nonlinearities;
- $\mu = \frac{h^2}{\lambda^2} \leadsto$ shallowness parameter;
- $\gamma \leadsto$ lack of isotropy.

Dirichlet-Neumann operator

Let $\psi = \phi_{|_{z=\varepsilon\zeta}}$ and define

$$G_{\gamma,\mu}[\varepsilon\zeta] : \psi \mapsto \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_n\phi_{|_{z=\varepsilon\zeta}}$$

with

$$\partial_n\phi_{|_{z=\varepsilon\zeta}} := \mathbf{n}\cdot\begin{pmatrix}\mu\partial_x\phi\\ \gamma^2\mu\partial_y\phi\\ \partial_z\phi\end{pmatrix}_{|_{z=\varepsilon\zeta}}$$

and

$$\begin{cases}\partial_z^2\phi+\mu\partial_x^2\phi+\gamma^2\mu\partial_y^2\phi=0\\ \phi_{|_{z=\varepsilon\zeta}}=\psi, \qquad \partial_z\phi_{|_{z=-1}}=0.\end{cases}$$

Zakharov formulation

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G_{\gamma,\mu}[\varepsilon\zeta]\psi & = & 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2}|\nabla^\gamma \psi|^2 - \frac{\varepsilon}{\mu} \frac{(G_{\gamma,\mu}[\varepsilon\zeta]\psi + \varepsilon\mu\nabla^\gamma\zeta \cdot \nabla^\gamma\psi)^2}{2(1+\varepsilon^2\mu|\nabla^\gamma\zeta|^2)} & = & 0, \end{cases}$$

where

$$\nabla^\gamma := (\partial_x, \gamma\partial_y)^T.$$

Example: Long-Waves Regime:

- Isotropic waves: $\varepsilon = \mu << 1$ and $\gamma = 1$.
- Weakly transverse waves: $\varepsilon = \mu << 1$ and $\gamma^2 = \varepsilon$.

Linear water-waves equations

For small amplitude waves, one obtains at first order, ie when $\varepsilon = 0$:

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G_{\gamma,\mu}[0]\psi & = & 0, \\ \partial_t \psi + \zeta & = & 0. \end{cases}$$

where $G_{\gamma,\mu}[0]\psi = \partial_z \phi_{|_{z=0}}$ and

$$\begin{cases} \partial_z^2 \phi + \mu \partial_x^2 \phi + \gamma^2 \mu \partial_y^2 \phi = 0, & -1 < z < 0 \\ \phi_{|_{z=0}} = \psi, \qquad \partial_z \phi_{|_{z=-1}} = 0. \end{cases}$$

Computation of $G_{\gamma,\mu}[0]\psi$

After horizontal Fourier transform:

$$\begin{cases} \partial_z^2 \widehat{\phi} - \mu(k^2+\gamma^2 l^2)\widehat{\phi} = 0, \\ \widehat{\phi}_{|_{z=0}} = \widehat{\psi}, \qquad \partial_z \widehat{\phi}_{|_{z=-1}} = 0. \end{cases}$$

$$\rightsquigarrow \quad \widehat{\phi}(k,l,z) = \frac{\cosh((z+1)\sqrt{\mu}\sqrt{k^2+\gamma l^2})}{\cosh(\sqrt{\mu}\sqrt{k^2+\gamma l^2})}\widehat{\psi}(k,l),$$

$$\rightsquigarrow \quad \partial_z \widehat{\phi}_{|_{z=0}} = \sqrt{\mu}\sqrt{k^2+\gamma l^2}\tanh(\sqrt{\mu}\sqrt{k^2+\gamma l^2})\widehat{\psi}(k,l),$$

Thus, with $|D^\gamma| = \sqrt{D_x^2+\gamma^2 D_y^2}$,

$$G_{\gamma,\mu}[0]\psi = \sqrt{\mu}|D^\gamma|\tanh(\sqrt{\mu}|D^\gamma|)\psi.$$

Linear water-waves equations

The linearized equations are therefore:

$$\begin{cases} \partial_t \zeta - \frac{1}{\sqrt{\mu}} |D^\gamma| \tanh(\sqrt{\mu}|D^\gamma|)\psi & = & 0, \\ \partial_t \psi + \zeta & = & 0. \end{cases}$$

$$\rightsquigarrow \qquad \partial_t^2 \zeta + \frac{1}{\sqrt{\mu}} |D^\gamma| \tanh(\sqrt{\mu}|D^\gamma|)\zeta = 0.$$

Shallow water approximation

Shallow water condition:

$$\mu << 1 \iff h^2/\lambda^2 << 1.$$

⇝ Costal flows:

$$h = 10m, \qquad \lambda = 100m \Longrightarrow \mu = 0.01.$$

⇝ Indian ocean tsunami:

$$h = 6000m, \qquad \lambda = 100km \Longrightarrow \mu = 0.0036.$$

Linear shallow water eqs (first order)

Isotropic case: $\gamma = 1$

$$\partial_t^2 \zeta - \Delta \zeta = 0 \qquad \text{(dimensionless)},$$

$$\partial_t^2 \zeta - gh\Delta \zeta = 0 \qquad \text{(with dimensions)}.$$

Weakly transverse waves: $\gamma = \sqrt{\mu}$

$$\partial_t^2 \zeta - \partial_x^2 \zeta = 0 \qquad \text{(dimensionless)},$$

$$\partial_t^2 \zeta - gh\partial_x^2 \zeta = 0 \qquad \text{(with dimensions)}.$$

Linear shallow water eqs (second order)

Isotropic case: $\gamma = 1$

$\frac{1}{\sqrt{\mu}}\sqrt{k^2+l^2}\tanh(\sqrt{\mu}\sqrt{k^2+l^2}) \sim (k^2+l^2) - \frac{1}{3}(k^2+l^2)^2$.

$$\rightsquigarrow \partial_t^2\zeta - \Delta\zeta - \frac{\mu}{3}\Delta^2\zeta = 0.$$

Weakly transverse waves: $\gamma = \sqrt{\mu}$

$\frac{1}{\sqrt{\mu}}\sqrt{k^2+\mu l^2}\tanh(\sqrt{\mu}\sqrt{k^2+\mu l^2}) \sim k^2 + \mu(l^2 - \frac{1}{3}k^4)$.

$$\rightsquigarrow \partial_t^2\zeta - \partial_x^2\zeta - \mu(\frac{1}{3}\partial_x^4 + \partial_y^2)\zeta = 0.$$

Nonlinear shallow water models

Different regimes ($\gamma = 1$):

- $\varepsilon = 1$, $\mu << 1$
  $\rightsquigarrow$ Shallow-water (Saint-Venant) equations;
- $\varepsilon = \mu << 1$
  $\rightsquigarrow$ Boussinesq equations;
- $\varepsilon^2 = \mu << 1$
  $\rightsquigarrow$ Serre equations.

The method

Recall the equations:

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G_{\gamma,\mu}[\varepsilon\zeta]\psi & = & 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2}|\nabla^\gamma \psi|^2 - \frac{\varepsilon}{\mu} \frac{(G_{\gamma,\mu}[\varepsilon\zeta]\psi + \varepsilon\mu\nabla^\gamma\zeta \cdot \nabla^\gamma\psi)^2}{2(1+\varepsilon^2\mu|\nabla^\gamma\zeta|^2)} & = & 0, \end{cases}$$

$\rightsquigarrow$ Asymptotic expansion of $G[\varepsilon\zeta]\psi$ when

- $\varepsilon = 1$, $\mu << 1$ (Shallow-water)
- $\varepsilon = \mu << 1$ (Boussinesq)
- $\varepsilon^2 = \mu << 1$ (Serre)

Shallow water equations

Proposition 1 *One has*

$$G_\mu[\zeta]\psi = -\mu\nabla\cdot((1+\zeta)\nabla\psi) + O(\mu^2).$$

Plugging this into the ww equations yields:

$$\begin{cases} \partial_t\psi + \zeta + \frac{1}{2}|\nabla\psi|^2 & = & 0 \\ \partial_t\zeta + \nabla\cdot((1+\zeta)\nabla\psi) & = & 0, \end{cases}$$

or, with $V = \nabla\psi$,

$$\begin{cases} \partial_t V + \nabla\zeta + \frac{1}{2}|V|^2 & = & 0 \\ \partial_t\zeta + \nabla\cdot((1+\zeta)V) & = & 0. \end{cases}$$

A Boussinesq system

Proposition 2 *One has*

$$G_\varepsilon[\varepsilon\zeta]\psi = -\varepsilon\Delta\Psi - \varepsilon^2(\frac{1}{3}\Delta^2\psi + \nabla\cdot(\zeta\nabla\psi)) + O(\varepsilon^3).$$

$\rightsquigarrow$

$$\begin{cases} \partial_t\psi + \zeta + \varepsilon(\frac{1}{2}|\nabla\psi|^2) & = & 0 \\ \partial_t\zeta + \Delta\psi + \varepsilon(\frac{1}{3}\Delta^2\psi + \nabla\cdot(\zeta\nabla\psi)) & = & 0, \end{cases}$$

or, with $V = \nabla\psi$,

$$\begin{cases} \partial_t V + \nabla\zeta + \varepsilon\frac{1}{2}|V|^2 & = & 0 \\ \partial_t\zeta + \nabla\cdot V + \varepsilon(\frac{1}{3}\nabla\cdot\Delta V + \nabla\cdot(\zeta V)) & = & 0. \end{cases}$$

Serre equations

Proposition 3 *One has*

$$G_\mu[\sqrt{\mu}\zeta]\psi = -\mu\Delta\Psi - \mu^{3/2}\nabla\cdot(\zeta\nabla\psi) + O(\mu^2).$$

$\rightsquigarrow$

$$\begin{cases} \partial_t\psi + \zeta + \sqrt{\mu}(\frac{1}{2}|\nabla\psi|^2) &= 0 \\ \partial_t\zeta + \Delta\psi + \sqrt{\mu}\nabla\cdot(\zeta\nabla\psi) &= 0, \end{cases}$$

or, with $V = \nabla\psi$,

$$\begin{cases} \partial_t V + \nabla\zeta + \sqrt{\mu}\frac{1}{2}|V|^2 &= 0 \\ \partial_t\zeta + \nabla\cdot((1+\sqrt{\mu}\zeta))V &= 0. \end{cases}$$

Expanding the DN operator

Recall that

$$G_{\gamma,\mu}[\varepsilon\zeta] : \psi \mapsto \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\,\partial_n\phi_{|_{z=\varepsilon\zeta}}$$

with

$$\partial_n\phi_{|_{z=\varepsilon\zeta}} := \mathbf{n}\cdot\begin{pmatrix}\mu\partial_x\phi\\ \gamma^2\mu\partial_y\phi\\ \partial_z\phi\end{pmatrix}_{|_{z=\varepsilon\zeta}}$$

and

$$\begin{cases}\partial_z^2\phi+\mu\partial_x^2\phi+\gamma^2\mu\partial_y^2\phi=0 & \text{in }\Omega\\ \phi_{|_{z=\varepsilon\zeta}}=\psi, \qquad \partial_z\phi_{|_{z=-1}}=0. & \end{cases}$$

Expanding the DN operator

Proposition 4 *One has* $G_{\gamma,\mu}[\varepsilon\zeta]\psi = \partial_n^P \widetilde{\Phi}_{|z=0}$, *with*

$$\begin{cases} \nabla^\gamma \cdot P_{\varepsilon,\gamma,\mu} \nabla^\gamma \widetilde{\Phi} = 0 & \textit{in } \mathbb{R}^2 \times (-1,0), \\ \widetilde{\Phi}_{|z=0} = \psi, & \partial_n^P \widetilde{\Phi}_{|z=-1} = 0, \end{cases}$$

and

$$P_{\varepsilon,\gamma,\mu} = \begin{pmatrix} \mu(1+\varepsilon\zeta) & 0 & -\mu\varepsilon(z+1)\partial_x\zeta \\ 0 & \mu(1+\varepsilon\zeta) & -\gamma\mu\varepsilon(z+1)\partial_y\zeta \\ -\mu\varepsilon(z+1)\partial_x\zeta & -\gamma\mu\varepsilon(z+1)\partial_y\zeta & \frac{1+\mu\varepsilon^2(z+1)^2|\nabla^\gamma\zeta|^2}{1+\varepsilon\zeta} \end{pmatrix}$$

Expanding the DN operator

Proposition 5 *If Φ_{app} satisfies*

$$\begin{cases} \nabla^\gamma\cdot P_{\varepsilon,\gamma,\mu}\nabla^\gamma\Phi_{app}\sim 0 & \textit{in } \mathbb{R}^2\times(-1,0), \\ \Phi_{app\,|_{z=0}}=\psi, & \partial_n^P\Phi_{app|_{z=-1}}=0, \end{cases}$$

then

$$G_{\gamma,\mu}[\varepsilon\zeta]\psi\sim\partial_n^P\Phi_{app\,|_{z=0}}.$$

$\rightsquigarrow$ We want to construct an explicit Φ_{app}.

Expanding the DN operator

Example: In the Boussinesq, $\varepsilon = \mu << 1$ and $\gamma = 1$:

$$\Phi_{app} = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + ...$$

One choses the Φ_j to cancel the leading terms of the expansion of Φ_{app} into powers of ε:

- Order $O(1)$: $\partial_z^2\Phi_0 = 0$
  $\rightsquigarrow$ $\Phi_0(X, z) = \psi(X)$.
- Order $O(\varepsilon)$: $\partial_z^2\Phi_1 = -\Delta\Phi_0$
  $\rightsquigarrow$ $\Phi_1 = -(\frac{z^2}{2} + z)\Delta\psi$.
- Etc.

Provisional conclusion

We have so far proved *consistency results*:
If for some $T > 0$, there exists a unique family of solutions $(\psi, \zeta)_{\varepsilon,\gamma,\mu}$ such that $(V := \nabla^\gamma \psi, \zeta)_{\varepsilon,\gamma,\mu}$ is bounded over times $[0, \frac{T}{\varepsilon}]$ then:

- (V, ζ) solves the Boussinesq system up to a $O(\varepsilon^2)$ residual;
- (V, ζ) solves the Shallow-Water system up to a $O(\mu)$ residual;
- (V, ζ) solves the Serre eqs up to a $O(\mu)$ residual.

Remarks

- In the weakly transverse case, we assume that $\sqrt{\varepsilon}\partial_y\psi$ is bounded $\rightsquigarrow$ $\partial_y\psi$ may grow as $1/\sqrt{\varepsilon}$.
- Consistency is not convergence !
- Keypoint: large time existence result for the ww equations.
- Boussinesq is linearly ill-posed!
- One-way asymptotic models (KdV,KP) are a step further.

Formally equivalent Boussinesq systems

From the Boussinesq system

$$\begin{cases} \partial_t V + \nabla\zeta + \varepsilon\frac{1}{2}|V|^2 & = & 0 \\ \partial_t\zeta + \nabla\cdot V + \varepsilon(\frac{1}{3}\nabla\cdot\Delta V + \nabla\cdot(\zeta V)) & = & 0, \end{cases}$$

one can derive an infinity of formally equivalent Boussinesq systems:

- $\partial_t V = -\nabla\zeta + O(\varepsilon) \rightsquigarrow \partial_t V = (1-\mu)\partial_t V - \mu\nabla\zeta + O(\varepsilon)$.
- $\partial_t\zeta = -\nabla\cdot V + O(\varepsilon)$
  $\rightsquigarrow \nabla\cdot V = \lambda\nabla\cdot V - (1-\lambda)\partial_t\zeta + O(\varepsilon)$.
- $V_\theta := (1 - \frac{\varepsilon}{2}(1-\theta^2)\Delta)^{-1}V$.

Formally equivalent Boussinesq systems

A first class of systems:

$$\begin{cases} (1-\varepsilon b\Delta)\partial_t V + \nabla\zeta + \varepsilon(\frac{1}{2}\nabla|V|^2 + a\Delta\nabla\zeta) = 0 \\ (1-\varepsilon d\Delta)\partial_t \zeta + \nabla\cdot V + \varepsilon(\nabla\cdot(\zeta V) + c\nabla\cdot\Delta V) = 0. \end{cases}$$

Remark: 4 coefficients depending on the 3 parameters θ, λ, μ.
Remark: Linearly well-posed or ill-posed depending on a, b, c and d.
Remark: One can take $a = c$, $b \geq 0$, $d \geq 0$.
$\rightsquigarrow$ Symmetric linear part.

Formally equivalent Boussinesq systems

A second class of systems:

Make the *nonlinear* change of variables

$$\widetilde{V} = (1 + \frac{\varepsilon}{2}\zeta)V.$$

$$\begin{cases} (1 - \varepsilon b\Delta)\partial_t V + \nabla\zeta \\ \quad +\varepsilon(\frac{1}{4}\nabla|V|^2 + \frac{1}{2}(V\cdot\nabla)V + \frac{1}{2}V\nabla\cdot V + \frac{1}{2}\zeta\nabla\zeta + a\Delta\nabla\zeta) = 0 \\ (1 - \varepsilon d\Delta)\partial_t\zeta + \nabla\cdot V + \varepsilon(\frac{1}{2}\nabla\cdot(\zeta V) + c\nabla\cdot\Delta V) = 0. \end{cases}$$

Remark: The nonlinear part is always symmetric.
Remark: When the linear part is also symmetric, the system is quasilinear symmetric hyperbolic
$\rightsquigarrow$ Good energy estimates.

Rigorous justification

Let $(\psi^\varepsilon, \zeta^\varepsilon)_\varepsilon$ be a family of solution such that $(V^\varepsilon, \zeta^\varepsilon)_\varepsilon$ is bounded over $[0, \frac{T}{\varepsilon}]$.

Step 1. We saw that $(V^\varepsilon, \zeta^\varepsilon)$ is consistent with ONE Boussinesq system.

Step 2. Linear manipulations:

$$(V^\varepsilon, \zeta^\varepsilon) \rightsquigarrow (V_1^\varepsilon, \zeta^\varepsilon),$$

consistent with ANY system of the first class.

Step 3. Nonlinear change of variables:

$$(V_1^\varepsilon, \zeta^\varepsilon) \rightsquigarrow (V_2^\varepsilon, \zeta^\varepsilon),$$

consistent with the corresponding system with symmetric nonlinearity.

Rigorous justification

In particular: *One can transform* $(V^\varepsilon, \zeta^\varepsilon)$ *into a family* $(V_2^\varepsilon, \zeta^\varepsilon)$ *consistent with a completely symmetric system.*

Step 4. There exist an exact solution $(V^\sharp, \zeta^\sharp)$ to this system, with same ICs.

Step 5. Energy estimates on the completely symmetric system:

$\rightsquigarrow |(V_2^\varepsilon, \zeta^\varepsilon) - (V^\sharp, \zeta^\sharp)|_{L^\infty([0,t]\times\mathbb{R}^2)} \leq$ Cst $\varepsilon^2 t.$

Step 6. Inverting the changes of variables, one gets $(V^\sharp, \zeta^\sharp) \rightsquigarrow (V_{app}, \zeta_{app})$.

Step 7. Conclusion:

$$|(V^\varepsilon, \zeta^\varepsilon) - (V_{app}, \zeta_{app})|_{L^\infty([0,t]\times\mathbb{R}^2)} \leq \text{Cst } \varepsilon^2 t.$$

Rigorous Justification

Theorem 1 *Any family* $(V^\varepsilon = \nabla\psi^\varepsilon, \zeta^\varepsilon)_\varepsilon$ *bounded on* $[0, \frac{T}{\varepsilon}]$ *solution to the ww equations can be approximated with a precision* $O(\varepsilon^2 t)$ *using ANY of the Boussinesq systems seen above.*

Remark. Same approach for weakly transverse waves.
Remark. Same kind of result for shallow water, Serre ...
Remark. What about the large time existence of solutions for the ww equations?

Uncoupled models

KdV approximation: (for 1D surface waves)
$\zeta(t,x) \sim \zeta_+(\varepsilon t, x-t) + \zeta_-(\varepsilon t, x+t)$, with

$$\partial_\tau \zeta_+ + \frac{1}{6}\partial_x^3 \zeta_+ + \frac{3}{2}\zeta_+ \partial_x \zeta_+ = 0.$$

Kadomtsev-Petviashvili (KP) approximation: (for weakly transverse, 2D surface waves)

$$\zeta(t,x) \sim \zeta_+(\varepsilon t, \sqrt{\varepsilon} y, x-t) + \zeta_-(\varepsilon t, \sqrt{\varepsilon} y, x+t),$$

with

$$\partial_\tau \zeta_+ + \partial_x^{-1}\partial_y^2 \zeta_+ + \frac{1}{6}\partial_x^3 \zeta_+ \frac{3}{2}\zeta_+ \partial_x \zeta_+ = 0.$$

Justification of KdV

$\rightsquigarrow$ Enough to justify the KdV approximation for one of the Boussinesq systems:

$$\begin{cases} \partial_t V + \partial_x \zeta + \varepsilon\big(\frac{3}{2} V \partial_x V + \frac{1}{2}\zeta\partial_x\zeta + \frac{1}{3}\partial_x^3 \zeta\big) = 0 \\ \partial_t \zeta + \partial_x V + \varepsilon\big(\frac{1}{2}\partial_x(\zeta V) + \frac{1}{3}\partial_x^3 V\big) = 0. \end{cases}$$

Ansatz: $(V,\zeta)(t,x) = (\widetilde{V},\widetilde{\zeta})(\varepsilon t, t, x)$, with

$$(\widetilde{V},\widetilde{\zeta})(T,t,x) = (V_0,\zeta_0) + \varepsilon(V_1,\zeta_1) + \dots$$

Plug the ansatz into the equations and BKW...

Justification of KdV

Order $O(1)$:

$$\partial_t V_0 + \partial_x \zeta_0 = 0, \qquad \partial_t \zeta_0 + \partial_x V_0 = 0.$$

$\rightsquigarrow \zeta_0(T,t,x) = \zeta^+(T, x-t) + \zeta^-(T, x+t)$
$\rightsquigarrow V_0(T,t,x) = \zeta^+(T, x-t) - \zeta^-(T, x+t)$.

Order $O(\varepsilon)$:

$$\partial_t \zeta_1 + \partial_x V_1 = -\partial_T V_0 - \frac{1}{2}\partial_x(\zeta_0 V_0) - \frac{1}{3}\partial_x^3 V_0$$
$$= \quad -\partial_T(\zeta^+ + \zeta^-) - \frac{1}{2}\partial_x((\zeta^+ + \zeta^-)(\zeta^+ - \zeta^-)) - \frac{1}{3}\partial_x^3(\zeta^+ - \zeta^-)$$
$$\partial_t V_1 + \partial_x \zeta_1 = \dots$$

Justification of KdV

Lemma 1 *One has*

$$\begin{cases} (\partial_t + \partial_x)u = f(x-t) + g(x+t) + h_1(x-t)h_2(x+t) \\ \lim_{t\to\infty} \frac{1}{t}u(t) = 0 \end{cases}$$

is equivalent to

$$\begin{cases} f = 0 \\ (\partial_t + \partial_x)u = g(x+t) + h_1(x-t)h_2(x+t). \end{cases}$$

Justification of KdV

We deduce:

$$\left\{ \begin{array}{l} (\partial_t + \partial_x)(\zeta_1 + V_1) = L^+(\zeta^-, \partial_x)\zeta^- + \alpha^+\zeta^+\partial_x\zeta^- + \beta^+\zeta^-\partial_x\zeta^+ \\ \partial_T\zeta^+ + \frac{1}{3}\partial_x^3\zeta^+ + \frac{3}{2}\zeta^+\partial_x\zeta^+ = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} (\partial_t - \partial_x)(\zeta_1 - V_1) = L^-(\zeta^+, \partial_x)\zeta^+ - \alpha^-\zeta^+\partial_x\zeta^- + \beta^-\zeta^-\partial_x\zeta^+ \\ \partial_T\zeta^- + \frac{1}{3}\partial_x^3\zeta^- - \frac{3}{2}\zeta^-\partial_x\zeta^- = 0 \end{array} \right.$$

$\rightsquigarrow$ Solving KdV gives $\zeta^\pm$ and thus (V_0, ζ_0)

$\rightsquigarrow$ Solving the above eqs gives (V_1, ζ_1).

Justification of KdV

Conclusion:

Step 1. We have constructed an approximate solution with $O(\varepsilon^2)$ residual.

Step 2. Energy estimates over times $O(1/\varepsilon)$

$\rightsquigarrow$ Our approximate solution (V, ζ) gives a $O(\varepsilon)$ error term with respect to the exact solution of the Boussinesq system.

Step 3. The error with respect to the exact solution of the water waves equation is $O(\varepsilon^2 t) + O(\varepsilon) = O(\varepsilon)$.

Step 4. One has $(V, \zeta) = (V_0, \zeta_0) + \varepsilon(V_1, \zeta_1)$ and the KdV approximation is (V_0, ζ_0).

$\rightsquigarrow$ Under appropriate assumptions, $\varepsilon(V_1, \zeta_1) = O(\varepsilon)$ and the precision of KdV is $O(\varepsilon)$ (difficulty= *secular growth*).

Justification of KdV

Theorem 2 *Any family* $(V^\varepsilon = \nabla\psi^\varepsilon, \zeta^\varepsilon)_\varepsilon$ *bounded on* $[0, \frac{T}{\varepsilon}]$ *solution to the ww equations can be approximated with a precision* $O(\varepsilon)$ *using the uncoupled KdV approximation.*

Remark. For Boussinesq, we had $O(\varepsilon^2 t)$
$\rightsquigarrow$ The coupling effects are of order $O(\varepsilon)$.

Remark. False on bounded domains.

Remark. Decay assumptions not necessary for Boussinesq.

Justification of KP

Same method:

Step 1. Expansion of the DN operator ($\varepsilon = \mu << 1$, $\gamma = \sqrt{\varepsilon}$).

Step 2. Derivation of a *Weakly Transverse Boussinesq system*.

Step 3. Consistency of any family of solution to the ww equations such that $(\partial_x\psi, \sqrt{\varepsilon}\partial_y\psi, \zeta)$ is bouded over $[0, \frac{T}{\varepsilon}]$.

Step 4. Formally equivalent systems and symetrization.

Step 5. Justification of KP from a weakly transverse Boussinesq system under additional (restrictive) zero mass assumptions.

Bibliography

Boussinesq:

- M. W. DINGEMANS, *Water Wave Propagation over uneven bottoms. Part II: Non-linear Wave Propagation*, Adanced Series on Ocean Engineering **13**, World Scientific.
- J. L. BONA, M. CHEN, J.-C. SAUT, *Boussinesq Equations and Other Systems for Small-Amplitude Long Waves in Nonlinear Dispersive Media. I: Derivation and Linear Theory*, J. Nonlinear Sci. **12** (2002), 283-318.
- J. L. BONA, T. COLIN, D. LANNES, *Long Waves Approximations for Water Waves*, Arch. Rational Mech. Anal. **178** (2005) 373-410.

Bibliography

KdV:

- W. CRAIG, *An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits*, Comm. Partial Diff. Eqs **10** (1985), no. 8, 787–1003.
- T. KANO, T. NISHIDA, *A mathematical justification for Korteweg-de Vries equation and Boussinesq equation of water surface waves*, Osaka J. Math. **23** (1986), no. 2, 389–413.
- G. SCHNEIDER, C. E. WAYNE, *The long-wave limit for the water wave problem. I. The case of zero surface tension.* Comm. Pure Appl. Math., **53** (2000), no. 12, 1475–1535.
- J. L. BONA, T. COLIN, D. LANNES, *Long Waves Approximations for Water Waves*, Arch. Rational Mech. Anal. **178** (2005) 373-410.

Bibliography

KP:

- T. KANO, *L'équation de Kadomtsev-Petviashvili approchant les ondes longues de surface de l'eau en écoulement trois-dimensionnel*, Patterns and waves, 431–444, Stud. Math. Appl., **18**, North-Holland, Amsterdam, 1986.
- W. BEN YOUSSEF, D. LANNES, *The long wave limit for a general class of 2D quasilinear hyperbolic problems*, Comm. Partial Differential Equations. **27** (2002), 979-1020.
- T. GALLAY, G. SCHNEIDER, *KP description of unidirectional long waves. The model case*, Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), no. 4, 885–898.

Bibliography

KP:

- L. PAUMOND, *A rigorous link between KP and a Benney-Luke equation*, Differential Integral Equations **16** (2003), no. 9, 1039–1064.
- D. LANNES, J.-C. SAUT, *Weakly transverse Boussinesq systems and the KP approximation*, submitted.
- B. ALVAREZ-SAMANIEGO, D. LANNES, *Large time existence results for water-waves and asymptotics*, LaTex-ing...

Existence results

Ovsjannikov, Nalimov,Yosihara...

Theorem 3 (S. Wu JAMS'99, D.L. JAMS '05) *For all (ζ_0, ψ_0) such that $(\zeta_0, \nabla\psi_0) \in H^s(\mathbb{R}^d)$ ($s > s_0$), there exists $T > 0$ and a unique solution (ζ, ψ) to the water-waves equations on the time interval $[0, T)$.*

Question: How does the existence time T depend on the parameters ε, μ and γ?

Partial answer (Csq of Craig, Kano-Nishida, Schneider-Wayne):
In $1D$, and when $\varepsilon = \mu << 1$, then $T = O(1/\varepsilon)$.

Conclusion

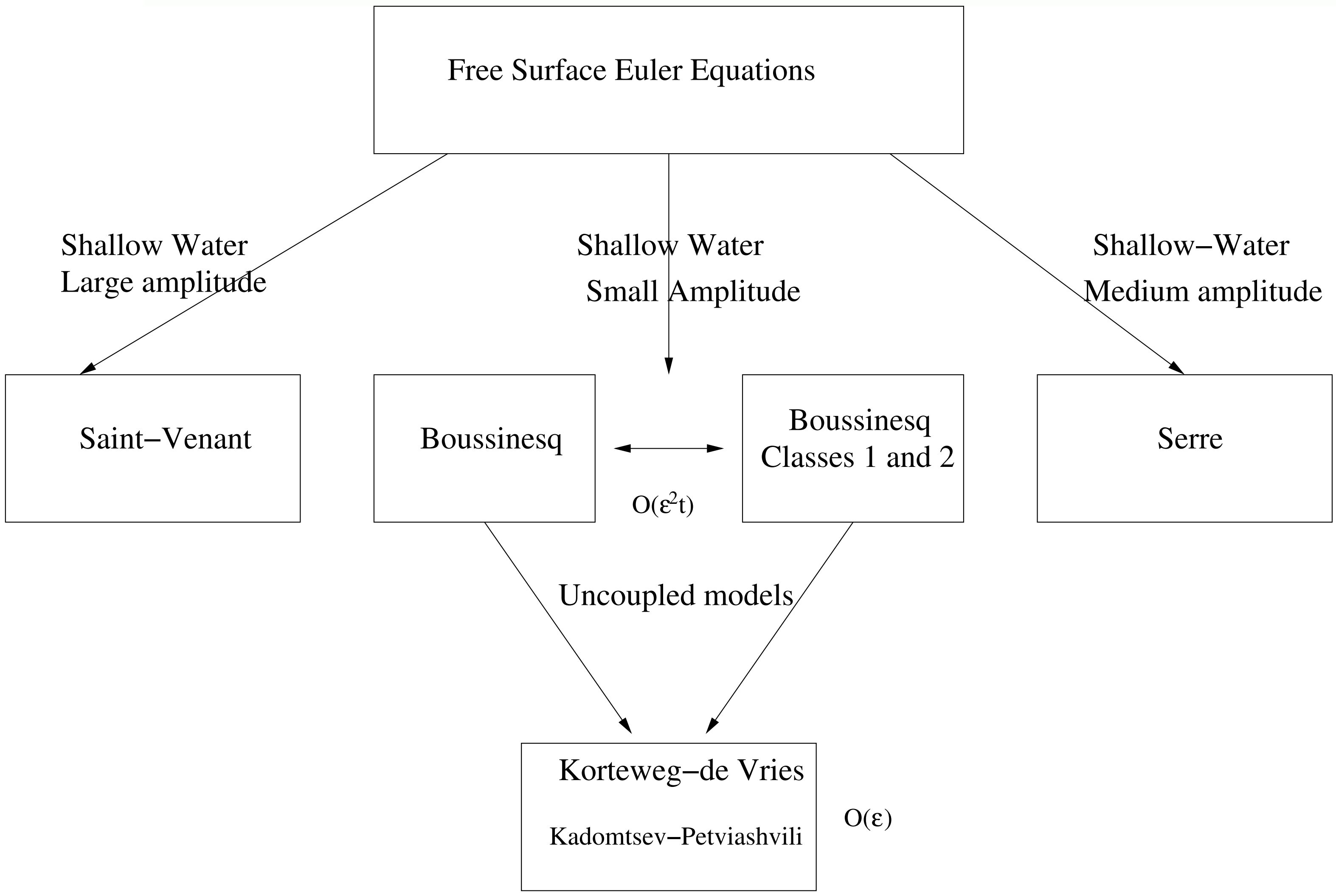
Theorem 4 (B. Alvarez, D.L.) *For all $\zeta^0, \nabla\psi^0 \in H^s$, there exist $T > 0$ and a unique solution (ζ, ψ) to the water-waves eqs defined on $[0, \frac{T}{\varepsilon}]$.*
Moreover, $|\zeta|_{H^s}$ and $|\nabla^\gamma \psi|_{H^s}$ remain bounded on $[0, \frac{T}{\varepsilon}]$.

Corollary 1 $\varepsilon = \mu$, $\gamma = 1$*: 2D Boussinesq approx justified.*

Corollary 2 $\varepsilon = \mu = \gamma^2$*: control of ζ, $\partial_x \psi$ and $\sqrt{\varepsilon}\partial_y \psi$.*
$\rightsquigarrow$ *Weakly transverse Boussinesq systems justified;*
$\rightsquigarrow$ *KP approximation justified.*

Similarly: **i.** $\varepsilon = \sqrt{\mu} << 1$, $\gamma = 1 \rightsquigarrow$ Serre.
ii. $\varepsilon = \gamma = 1$, $\mu << 1$, $\rightsquigarrow$ Saint-Venant (=shallow water eqs).

Summary


Free Surface Euler Equations
Shallow Water
Large amplitude
Shallow Water
Small Amplitude
Shallow−Water
Medium amplitude
Saint−Venant
Boussinesq
Boussinesq
Classes 1 and 2
Serre
O(ε²t)
Uncoupled models
Korteweg−de Vries
Kadomtsev−Petviashvili
O(ε)

So what?

Dispersion relation:

- Water-waves equations
  $\rightsquigarrow \omega^2 = \frac{1}{\sqrt{\mu}}\sqrt{k^2+\gamma^2 l^2}\tanh(\sqrt{\mu}\sqrt{k^2+\gamma^2 l^2})$;
- Weakly-transverse Boussinesq systems ($\mu = \varepsilon = \gamma^2$):

$$\omega^2 \quad = \quad k^2\frac{(1-\varepsilon a k^2)(1-\varepsilon c k^2)}{(1+\varepsilon b k^2)(1+\varepsilon d k^2)} + \varepsilon l^2 \frac{(1-\varepsilon g k^2)(1-\varepsilon f k^2)}{(1+\varepsilon e k^2)(1+\varepsilon d k^2)}$$

- KP:

$$\omega^2 = (k + \varepsilon\frac{l^2}{2k} - \frac{k^3}{6})^2.$$

$\rightsquigarrow$ Completely different behaviour, especially for small and large frequencies!

Dispersive properties

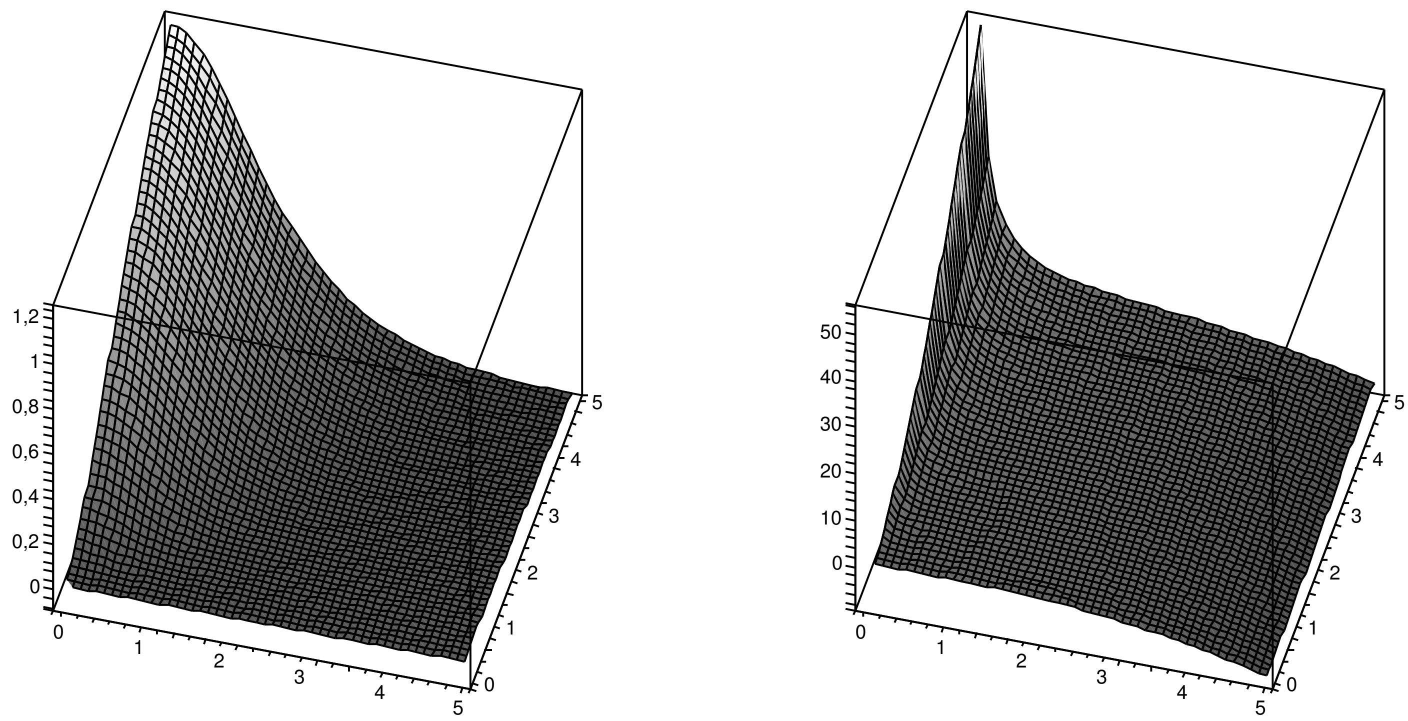


Figure 1: Left=δ_{bouss}, Right=δ_{kp}

Range of validity of the model

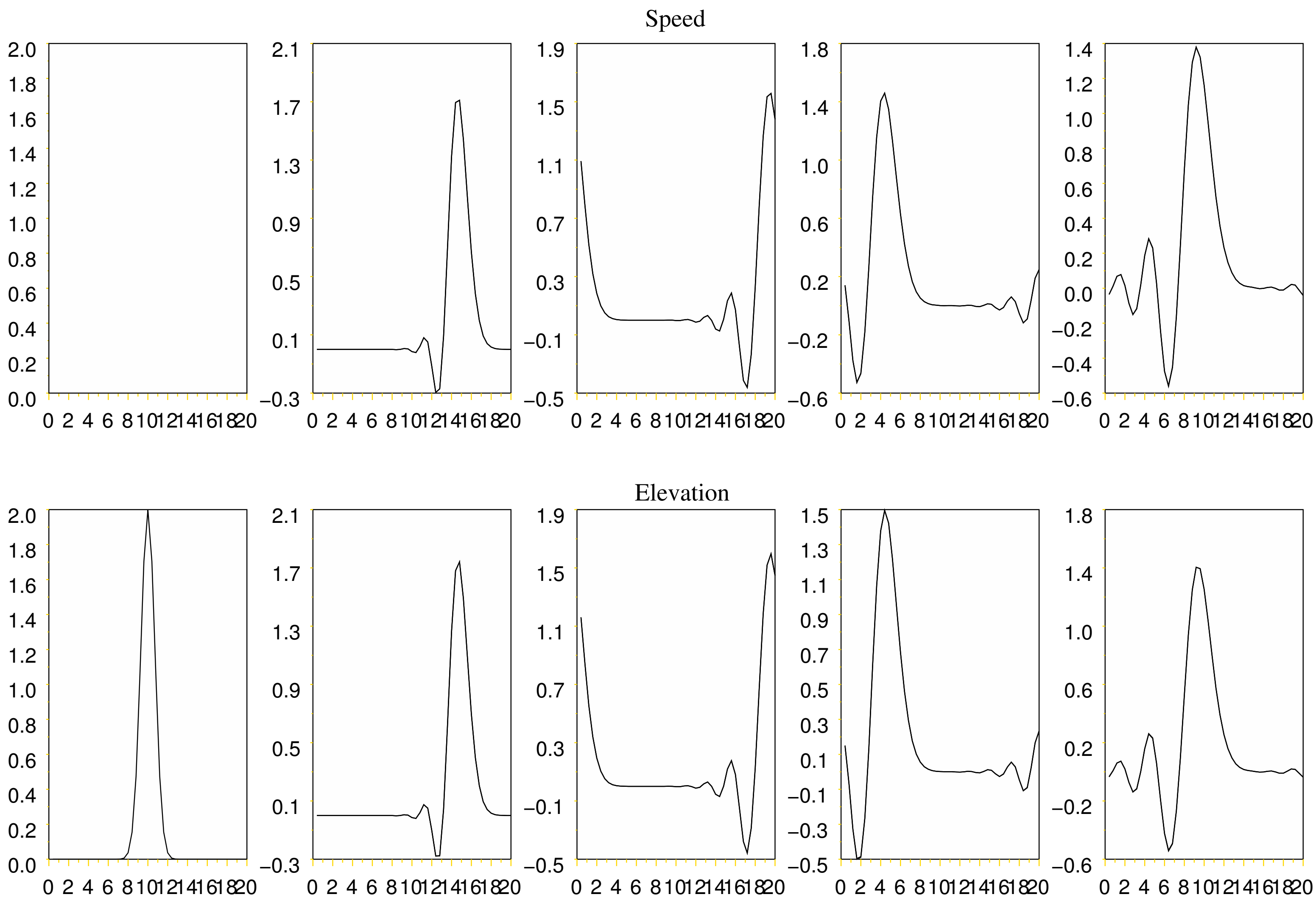
Asymptotically, all the Boussinesq models are equivalent, and are also equivalent to the KdV approximation (unbounded domain).

Example: Coastal flows:
i. $\lambda = 100m$, $h = 10m \rightsquigarrow \mu = 0.01$.
ii. $\lambda = 80m$, $h = 30m \rightsquigarrow \mu = 0.14$.

$\rightsquigarrow$ Is the asymptotic regime reached with the physical values of the small parameters?

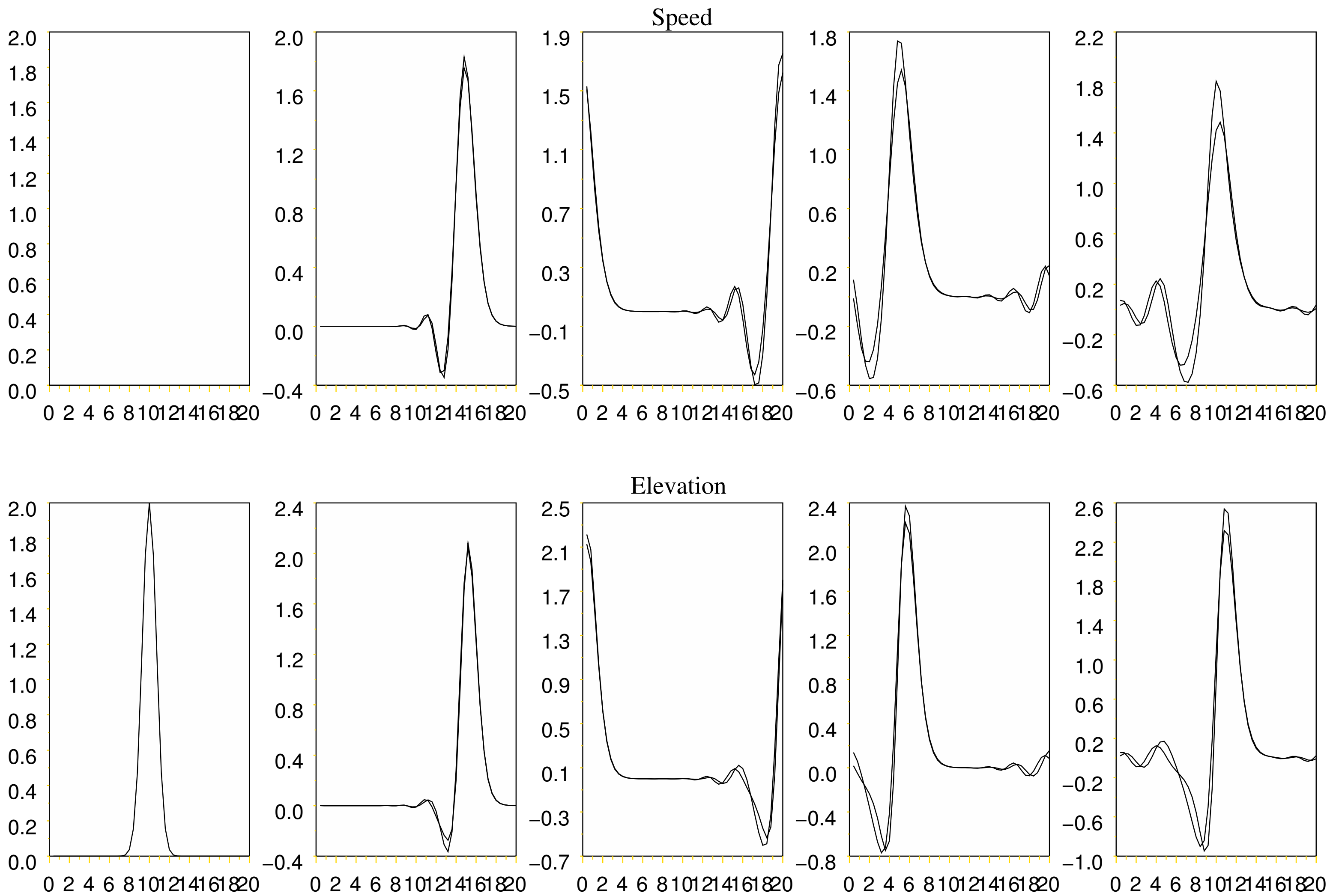
$\rightsquigarrow$ Do some Boussinesq models have a wider range of validity than others?

Equivalent models?



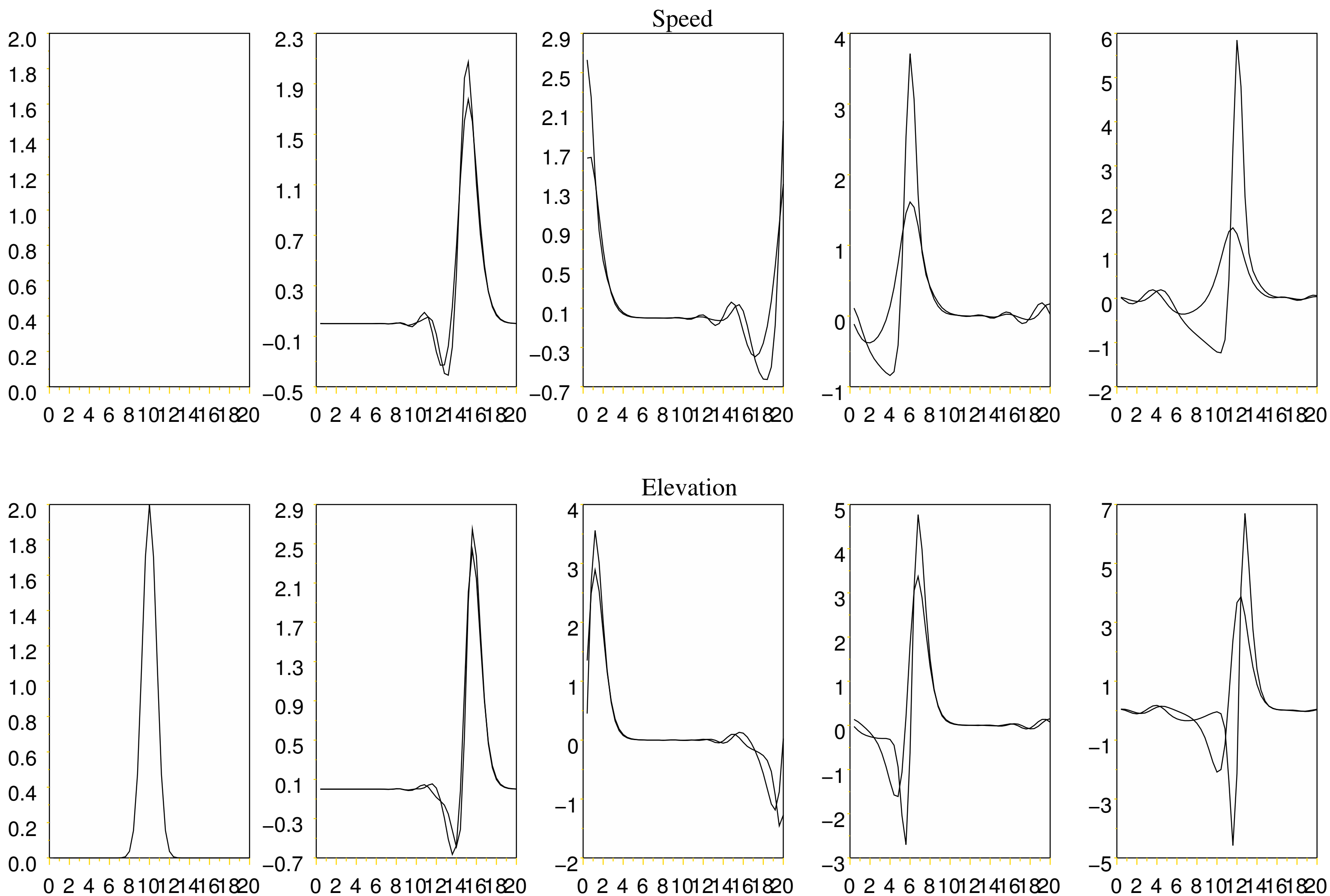
ε=u=0.01

Equivalent models?



ε=u=◊.1

Equivalent models?



Numerical simulation of water-waves

Goal: Develop a general code for the Free Surface Euler equations and use it to check the range of validity of the different Boussinesq models.

$\rightsquigarrow$ No assumption on the physical regime should be made (eg small amplitude).

$\rightsquigarrow$ General method for computing the DN operator.

Numerical computation of the DN

$\rightsquigarrow$ Method inspired by
K. DOMELEVO, P. OMNES, *A finite volume method for the Laplace equation on almost arbitrarytwo-dimensional grids*, M2AN, **39** (2006), no. 6, pp. 1203-1249.

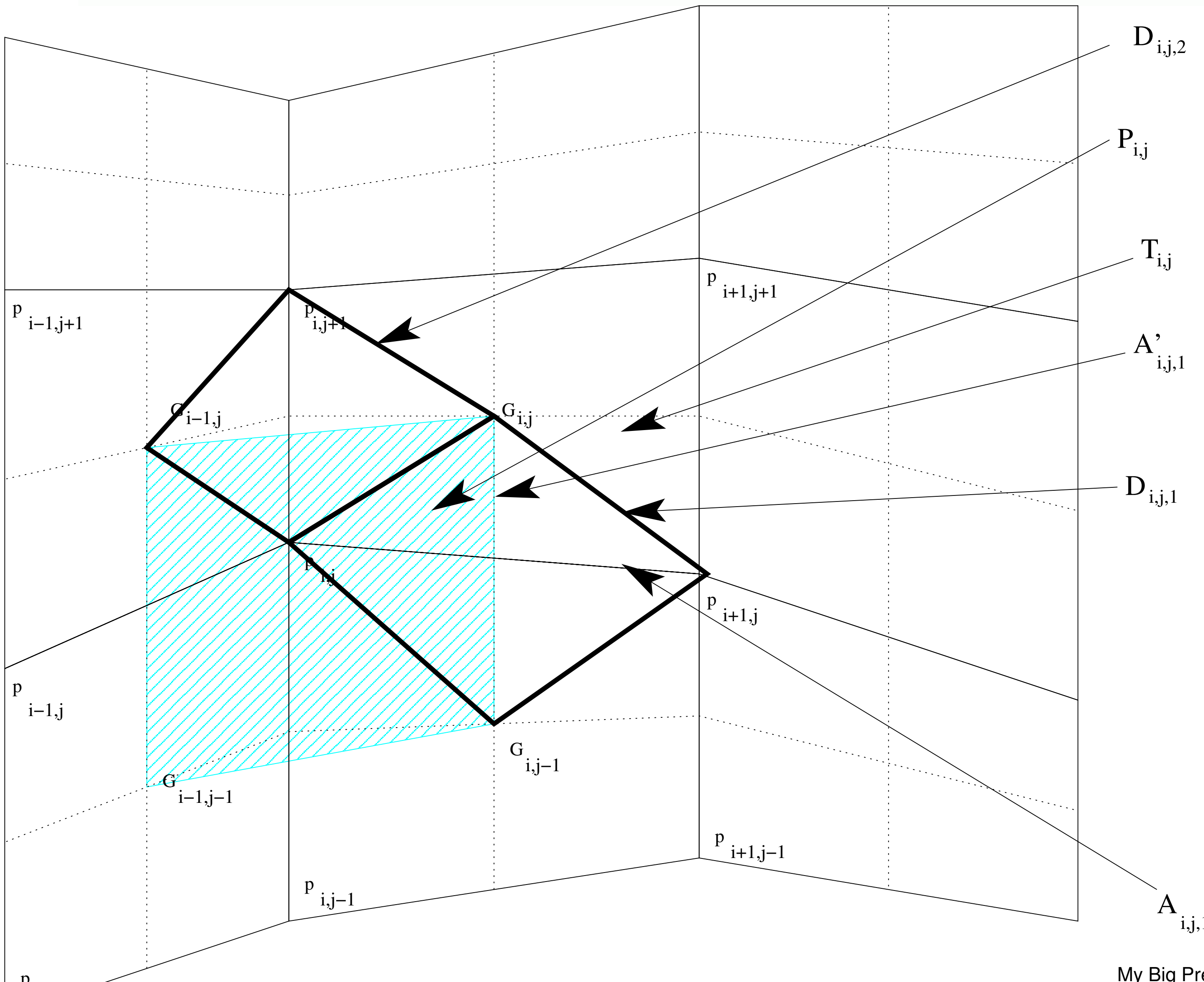
Recall

$$G[\zeta]\psi = \sqrt{1+|\nabla\zeta|^2}\partial_{\mathbf{n}_+}\Phi|_{y=\zeta(t,x)},$$

where Φ is the periodic solution of period L in x to

$$\begin{cases} \triangle_{x,y}\Phi = 0, \ \forall(x,y)\in\Omega, \\ \partial_{\mathbf{n}_-}\Phi|_{y=b(x)} = 0, \ \forall 0\le x\le L \\ \Phi(y=\zeta(x)) = \psi(x), \ \forall 0\le x\le L. \end{cases}$$

The mesh



D i,j,2
P i,j
T i,j
A' i,j,1
D i,j,1
A i,j,1
p i−1,j+1
p i,j+1
p i+1,j+1
G i−1,j
G i,j
p i,j
p i+1,j
p i−1,j
G i,j−1
G i−1,j−1
p i+1,j−1
p i,j−1

Numerical computation of the DN

$(\Phi^T, \Phi^P) \rightsquigarrow$ approximations of Φ on grids T and P.

Approximate gradient:
$\nabla_h \Phi = ((\nabla_h \Phi)_1, (\nabla_h \Phi)_2) \rightsquigarrow$ approximations of $\nabla \Phi$ on diamond grids D_1 and D_2:

$$(\nabla_h \Phi)_{i,j,1} = \frac{1}{2\,|D_{i,j,1}|} \quad \Big(\quad (\Phi^T_{i,j} - \Phi^T_{i,j-1})|A_{i,j,1}|n_{i,j,1}$$

$$+(\Phi^P_{i+1,j} - \Phi^P_{i,j})|A'_{i,j,1}|n'_{i,j,1});$$

$$(\nabla_h \Phi)_{i,j,2} = \frac{1}{2\,|D_{i,j,2}|} \quad \Big(\quad (\Phi^T_{i,j} - \Phi^T_{i-1,j})|A_{i,j,2}|n_{i,j,2}$$

$$+(\Phi^P_{i,j+1} - \Phi^P_{i,j})|A'_{i,j,2}|n'_{i,j,2}).$$

Numerical computation of the DN

We have constructed:

$$\nabla_h : \quad (P,T) \to (D_1, D_2)$$

Similarly:

$$\mathrm{div}_h : \quad (D_1, D_2) \to (P,T)$$

Thus:

$$\Delta_h : \quad (P,T) \to (P,T).$$

$\rightsquigarrow$ One can compute $G[\zeta]\psi$.

Numerical scheme

With $U = (\zeta, \psi)^T$, the ww eqs can be written

$$\partial_t U + \mathcal{F}(U) = 0,$$

with

$$\mathcal{F}(U) = \Big(- G[\zeta]\psi, g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(G[\zeta]\psi + \nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)}\Big)^T.$$

Equivalently:

$$\begin{cases} \partial_t V + d_U\mathcal{F}\cdot V = 0 \\ U(t) = U_{|t=0} + \int_0^t V(s)ds \\ V_{|t=0} = -\mathcal{F}(U_{|t=0}). \end{cases}$$

Trigonalized equations

There exists explicit $Z(U)$, $\mathbf{v}(U)$ and $\mathfrak{a}(U)$ such that $W = (V_1, V_2 - Z(U)V_1)^T$ solves

$$\begin{cases} \partial_t W + M(U)W = 0 \\ W|_{t=0} = (V_1|_{t=0}, V_2|_{t=0} - Z(U)|_{t=0}V_1|_{t=0})^T, \end{cases}$$

with

$$M(U) = \begin{pmatrix} \nabla \cdot (\cdot \mathbf{v}(U)) & -G(\zeta)\cdot \\ \mathfrak{a}(U) & \mathbf{v}(U) \cdot \nabla \end{pmatrix}.$$

Splitting the trigonalized system

One has $M(U) = M_1(U) + M_2(U)$, with

$$M_1(U) := \begin{pmatrix} \nabla \cdot (\cdot \mathbf{v}(U)) & 0 \\ 0 & \mathbf{v}(U) \cdot \nabla \end{pmatrix}$$

and

$$M_2(U) := \begin{pmatrix} 0 & -G[U_1]\cdot \\ \mathfrak{a}(U) & 0 \end{pmatrix}$$

$\rightsquigarrow$ We want to solve $\partial_t W + M_j(U)W = 0$, $j = 1, 2$.

Solving $\partial_t W + M_1(U)W = 0$

Let $S_1(t)$ be the solution operator.
$\rightsquigarrow$ We seek an an approximation $W^{n,1}$of $S(k/2)W^n$:

$$W^{n,1} = W^n + \frac{k}{4}\,M_1^{n+1/2}\,(W^n + W^*)\,.$$

with

$$W^* = W^n + \frac{k}{2}\,M_1^{n+1/2}\,W^n.$$

Solving $\partial_t W + M_2(U)W = 0$

Let $S_2(t)$ be the solution operator.

$\rightsquigarrow W^{n,2} := S_2(k)W^{n,1} \sim W^{\sharp}_{|t=k}$, with

$$\begin{cases} \partial_t W^\sharp + M_2^{n+1/2} W^\sharp = 0, \\ W^\sharp|_{t=0} = W^{n,1} \end{cases}$$

$\rightsquigarrow \frac{W^{n,2}-W^{n,1}}{k} + M_2^{n+1/2}(\theta W^{n,2} + (1-\theta)W^{n,1}) = 0$.

Iterative method: $\widetilde{W}^0 = W^{n,1}$ and

$$\frac{\widetilde{W}^{k+1} - W^{n,1}}{k} + M_2^{n+1/2}(\theta \widetilde{W}^k + (1-\theta)W^{n,1}) = 0.$$

$$\rightsquigarrow W^{n,2} = \lim_{k\to\infty} \widetilde{W}^k.$$

Numerical scheme

$\rightsquigarrow$ Compute by induction $V^n \sim V(nk)$ and $U^n \sim U(nk)$.

- As in [Besse-Bruneau], compute $U^{n+1/2}$ by

$$\frac{U^{n+1/2} + U^{n-1/2}}{2} = U^n.$$

- Approximations of $Z(U)$ at time n and $n + 1/2$:

$$Z(U^{n+1/2}) \quad \text{and} \quad Z^n = \frac{Z(U^{n+1/2}) + Z(U^{n-1/2})}{2}.$$

- Approximation of W at time n:

$$W^n := (V_1^n, V_2^n - Z^n V_1^n)^T.$$

Numerical scheme

- $U^{n+1/2}$, $Z(U^{n+1/2})$, Z^n, W^n.
- Approximation of the operator $M(U)$ at time $n+1/2$:

$$M^{n+1/2} := \begin{pmatrix} \nabla\cdot(\cdot\mathbf{v}(U^{n+1/2})) & -G[\zeta^{n+1/2}]\cdot \\ \mathfrak{a}^{n+1/2} & \mathbf{v}(U^{n+1/2})\cdot\nabla \end{pmatrix}$$

- Compute $W^{n+1} = W_{|_{(n+1)k}}$ with

$$\begin{cases} \partial_t W + M^{n+1/2}W = 0 \\ W|_{t=nk} = W^n, \end{cases}$$

Numerical scheme

- $U^{n+1/2}$, $Z(U^{n+1/2})$, Z^n, W^n, W^{n+1}.
- Recall that $W = P(U)V$

$$\rightsquigarrow \frac{V^{n+1} + V^n}{2} = P(U^{n+1/2})^{-1} \frac{W^{n+1} + W^n}{2}.$$

- Finally

$$U^{n+1} = U^n + k \frac{V^{n+1} + V^n}{2}.$$

To do

- Boundary value problems;
- Models coupling;
- Good $2DH$ numerics;
- Layers;
- ...

Pub

Challenges actuels en mécanique des fluides :
modélisation et analyse

du 23 au 27 Octobre 2006, CIRM, (Marseille)
http://www.cmi.univ-mrs.fr/ fboyer/MECAFLU2006/

- Water-Waves
- Microfluidics
- Fluid mechanics in biology