Well-balanced Finite Volume Evolution Galerkin Scheme for the Shallow Water Equations

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Shallow Water Equations, conservative form

$$W_t + f_1(W)_x + f_2(W)_y = hq(W, b)$$

Conservative variable W, flux function f_1 und f_2, source term q

$$W = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}, f_1 = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix}, f_2 = \begin{bmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{bmatrix}, q = \begin{bmatrix} 0 \\ -g(b_x + S_{f_x}) + fv \\ -g(b_y + S_{f_y}) - fu \end{bmatrix}$$

- h: height of the water level
- hu, hv: momentum in x-, y-direction
- Source q: bottom topography b, friction slopes S_{f_x}, S_{f_y}, Coriolis force
- Coriolis parameter f
- Gravitational constant g
- Friction slopes, e.g. Darcy Weisbach law (M. Lukáčová, U. Teschke, ZAMM 2006)

Shallow water equations, quasi linear form

$$w_t + A_1(w)w_x + A_2(w)w_y = q(w,b)$$

Primitive variable w, Matrices A_1 and A_2, source q

$$w = \begin{bmatrix} h \\ u \\ v \end{bmatrix}, \quad A_1 = \begin{bmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{bmatrix}, \quad A_2 = \begin{bmatrix} v & 0 & h \\ 0 & v & 0 \\ g & 0 & v \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ -gb_x + fv \\ -gb_y - fu \end{bmatrix}$$

Stationary States in Question

- Material derivative $\dot{} := \partial_t + u\partial_x + v\partial_y$

$$(h,u,v)_t = 0$$
$$(\dot{h},\dot{u},\dot{v}) = 0$$

Give

$$uh_x + vh_y = uu_x + vu_y = uv_x + vv_y = 0$$

Equilibrium conditions of aligned streamlines

$$
\begin{aligned}
u &= 0 & & \\
v_y &= 0 & vh_y &= 0 \\
g(h+b)_x &= fv & g(h+b)_y &= 0
\end{aligned}
$$

Potential energies

Define U, V, K and L by

$$
\begin{aligned}
V_x &= \frac{f}{g} v & U_y &= \frac{f}{g} u \\
K &:= g(h+b-V) & L &:= g(h+b+U)
\end{aligned}
$$

Equilibrium conditions

$$
\begin{aligned}
u &= 0 & & \\
v_y &= 0 & vh_y &= 0 \\
K_x &= 0 & L_y &= 0
\end{aligned}
$$

FV update

$$U_{ij}^{n+1} = U_{ij}^{n} - \frac{\Delta t}{\hbar} \sum_{k=1}^{2} \delta_{x_k}^{ij} \bar{f}_k^{n+1/2} + \frac{\Delta t}{\hbar} B_{ij}^{n+1/2}$$

Flux approximation

$$\bar{f}_k^{n+1/2} := \sum_j \omega_j f_k(E_{\Delta t/2} U^n(x^j(\mathcal{E})))$$

- $\hbar$ mesh size
- Δt time step

- $\hat{h}, \ldots$ predicted values at $t = t^{n+1/2}$ at the integration nodes
- δ_{x_k} central difference operator in x_k-direction
- μ_{x_k} average operator in x_k-direction

Discretization of the source term

$$B_{ij}^{n+1/2} = -g \begin{pmatrix} 0 \\ \sum_{j'} \omega_{j'} \mu_{x_1}(\hat{h}) \delta_{x_1}(\hat{b} - \hat{V}) \\ \sum_{i'} \omega_{i'} \mu_{x_2}(\hat{h}) \delta_{x_2}(\hat{b} + \hat{U}) \end{pmatrix}$$

$$\delta_{x_1}(\hat{V}) = \hbar \frac{f}{g} \mu_{x_1}(\hat{v})$$

$$\delta_{x_2}(\hat{U}) = \hbar \frac{f}{g} \mu_{x_2}(\hat{u})$$

Standard argument for the 'Lake at Rest'

Well balancing is here essentialy forced by the discrete product law

$$\delta(ab) = \delta(a)\mu(b) + \mu(a)\delta(b) \quad \Rightarrow \quad \delta(gh^2/2) = g\mu(h)\delta(h)$$

Discrete equilibrium (M. Lukáčová, S. Noelle, M.K.[J. Comp. Phys. 2006])

Suppose that the values $(\hat{h}, \hat{u}, \hat{v})$ given by the predictor step satisfy the equilibrium conditions on the discrete level

$$\hat{u}_{i\pm\frac{1}{2},j+j'} = \hat{u}_{i+i',j\pm\frac{1}{2}} = 0$$

$$\delta_y^{i+i',j}(\hat{v}) = 0 \qquad \hat{v}_{i+i',j\pm\frac{1}{2}}\,\delta_y^{i+i',j}(\hat{h}) = 0$$

$$\delta_x^{i,j+j'}(\hat{K}) = 0 \qquad \delta_y^{i+i',j}(\hat{L}) = 0$$

Then the finite volume step preserves the 'Lake at Rest' and the 'Jet in the Rotational Frame'.

- Let $P = (x, y, t_{n+1/2})$ be a point where predictions are desired
- Let $\tilde{w} = (\tilde{h}, \tilde{u}, \tilde{v})$ be a suitable linearization around point P

Shallow water equations, locally linearized form, primitive variables

$$w_t + A_1(\tilde{w})w_x + A_2(\tilde{w})w_y = q(w, b)$$

Definition

Let

$$n = n(\theta) = (n_x, n_y)^t := (\cos(\theta), \sin(\theta)), \quad A(n) := A_1 n_x + A_2 n_y$$

Lemma

$A(n)$ is for all n diagonalizable. Eigenvalues

$$\lambda_1 = \tilde{u}\cos(\theta) + \tilde{v}\sin(\theta) - \tilde{c},$$
$$\lambda_2 = \tilde{u}\cos(\theta) + \tilde{v}\sin(\theta),$$
$$\lambda_3 = \tilde{u}\cos(\theta) + \tilde{v}\sin(\theta) + \tilde{c},$$
$$\tilde{c} := \sqrt{g\tilde{h}},$$

and Eigenvectors

$$r_1 = \begin{bmatrix} -1 \\ g/c\cos(\theta) \\ g/c\sin(\theta) \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0 \\ \sin(\theta) \\ -\cos(\theta) \end{bmatrix}, \quad r_3 = \begin{bmatrix} 1 \\ g/c\cos(\theta) \\ g/c\sin(\theta) \end{bmatrix}.$$

- Transformation to characteristic variables $\bar{w} := R^{-1}w$

Linearized equation in characteristic variables

$$\bar{w}_t + B_1(\tilde{w})\bar{w}_x + B_2(\tilde{w})\bar{w}_y = R^{-1}q$$

B_1 und B_2

$$B_1(\tilde{w}) := R^{-1}A_1(\tilde{w})R = \begin{bmatrix} \tilde{u} - \tilde{c}\cos(\theta) & -\frac{1}{2}\tilde{h}\sin(\theta) & 0 \\ -g\sin(\theta) & \tilde{u} & g\sin(\theta) \\ 0 & \frac{1}{2}\tilde{h}\sin(\theta) & \tilde{u} + \tilde{c}\cos(\theta) \end{bmatrix}$$

$$B_2(\tilde{w}) := R^{-1}A_2(\tilde{w})R = \begin{bmatrix} \tilde{v} - \tilde{c}\sin(\theta) & \frac{1}{2}\tilde{h}\cos(\theta) & 0 \\ g\cos(\theta) & \tilde{v} & -g\cos(\theta) \\ 0 & -\frac{1}{2}\tilde{h}\cos(\theta) & \tilde{v} + \tilde{c}\sin(\theta) \end{bmatrix}$$

- $R := [r_1, r_2, r_3]$
- Not diagonal form

- Non diagonal elements are hidden in quasi source term

Quasi diagonalized system
(M. Lukáčová, K.W. Morton, G. Warnecke [Math. Comp. 2000])

$$\bar{w}_t + \begin{bmatrix} \tilde{u} - \tilde{c}\cos(\theta) & 0 & 0 \\ 0 & \tilde{u} & 0 \\ 0 & 0 & \tilde{u} + \tilde{c}\cos(\theta) \end{bmatrix} \bar{w}_x + \begin{bmatrix} \tilde{v} - \tilde{c}\sin(\theta) & 0 & 0 \\ 0 & \tilde{v} & 0 \\ 0 & 0 & \tilde{v} + \tilde{c}\sin(\theta) \end{bmatrix} \bar{w}_y = S \tag{1}$$

Quasi source term (M. Lukáčová, J. Saibertová[ENUMATH 2002])

$$S = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} \frac{\tilde{h}}{2}\left(\sin(\theta)\frac{\partial \bar{w}_2}{\partial x} - \cos(\theta)\frac{\partial \bar{w}_2}{\partial y}\right) \\ g\sin(\theta)\left(\frac{\partial \bar{w}_1}{\partial x} - \frac{\partial \bar{w}_3}{\partial x}\right) + g\cos(\theta)\left(\frac{\partial \bar{w}_3}{\partial y} - \frac{\partial \bar{w}_1}{\partial y}\right) \\ \frac{\tilde{h}}{2}\left(\cos(\theta)\frac{\partial \bar{w}_2}{\partial y} - \sin(\theta)\frac{\partial \bar{w}_2}{\partial x}\right) \end{bmatrix} + R^{-1}q$$

- Like in the characteristics method a path is sought on which the equations can be better understood.
- This is done for all three equations (1).

Definition

The l-th Bicharacteristic x_l, belonging to the l-th equation of (1) is defined by

$$\frac{dx_l(s)}{ds} = \begin{bmatrix} b^1_{ll} \\ b^2_{ll} \end{bmatrix}.$$

Here b^1_{ll} and b^2_{ll} are the diagonal elements of B_1 and B_2.

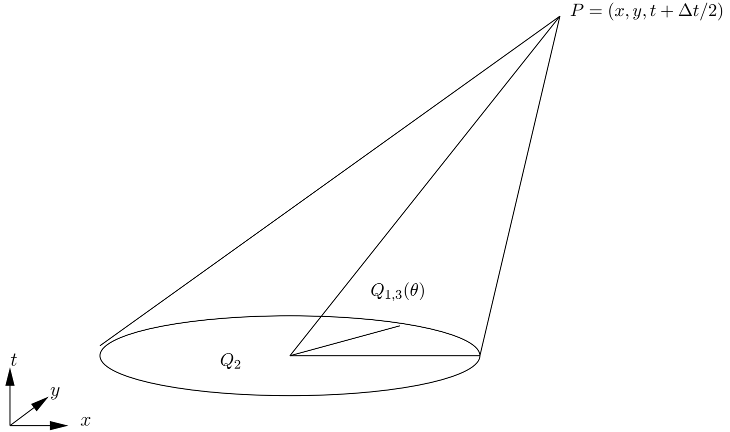
Comparison to characteristic method

This directly gives:

$$\frac{d\bar{w}_l(x^1_l(t), x^2_l(t), t)}{dt} = S_l \tag{2}$$

- Bicharacteristics build the bicharacteristic cone
- $P = (x, y, t + \Delta t/2)$ apex of the cone

$$Q_1(\theta) = (x - (\tilde{u} - \tilde{c}\cos(\theta))\Delta t/2, y - (\tilde{v} - \tilde{c}\sin(\theta))\Delta t/2, t),$$
$$Q_2 = (x - \tilde{u}\Delta t/2, y - \tilde{v}\Delta t/2, t),$$
$$Q_3(\theta) = (x - (\tilde{u} + \tilde{c}\cos(\theta))\Delta t/2, y - (\tilde{v} + \tilde{c}\sin(\theta))\Delta t/2, t) \quad \forall\theta \in [0, 2\pi].$$



Implicit formula of the solution!

Let $\bar{w}$ be characteristic variables of the locally linearized equations

$$\bar{w}_l(P) = \bar{w}_l(Q_l) + \int_t^{t+\Delta t/2} S_l(Q_l(\tilde{t}))\, d\tilde{t} \quad l = 1, 2, 3 \quad \forall \theta,$$

or in averaging

$$\bar{w}_l(P) = \frac{1}{2\pi} \int_0^{2\pi} \left[\bar{w}_l(Q_l) + \int_t^{t+\Delta t/2} S_l(Q_l(\tilde{t}))\, d\tilde{t} \right] d\theta \quad l = 1, 2, 3.$$

Implicit formula of the solution, the example h

$$
\begin{aligned}
h(P) = & \frac{1}{2\pi}\int_0^{2\pi} h(Q) - \frac{\tilde c}{g} u(Q)\cos(\theta) - \frac{\tilde c}{g}\sin(\theta)\, d\theta \\
& - \frac{1}{2\pi}\int_t^{t_{n+1/2}} \frac{1}{t_{n+1/2}-\tilde t}\int_0^{2\pi} \frac{\tilde c}{g}\Big(u(\tilde Q)\cos(\theta) + v(\tilde Q)\sin(\theta)\Big)\, d\theta\, d\tilde t \\
& + \frac{1}{2\pi}\tilde c \int_{t_n}^{t_{n+1/2}} \int_0^{2\pi} \Big(b_x(\tilde Q)\cos(\theta) + b_y(\tilde Q)\sin(\theta)\Big)\, d\theta\, d\tilde t \\
& - \frac{1}{2\pi}\frac{\tilde c f}{g}\int_{t_n}^{t_{n+1/2}} \int_0^{2\pi} \Big(v(\tilde Q)\cos(\theta) - u(\tilde Q)\sin(\theta)\Big)\, d\theta\, d\tilde t
\end{aligned}
$$

- Rectangular rule in time could give explicit approximation.

- Due to (M. Lukáčová, K.W. Morton, G. Warnecke [Math. Comp. 2000, J. Sci. Comp. 2004]) and (M. Lukáčová, G. Warnecke, Y. Zahaykah[J. Numer. Anal. 2006]) application of classical quadrature rules can decrease the stability range of the scheme

Numerical quadrature for constant data (M. Lukáčová, K.W. Morton, G. Warnecke [J. Sci. Comp. 2004])

$$\frac{1}{2\pi}\int_0^{2\pi} f(Q)\cos(\theta)d\theta + \frac{1}{2\pi}\int_{t_n}^{t_n+\Delta/2} \frac{1}{t_n+\Delta t/2-\tilde{t}}\int_0^{2\pi} f(\tilde{Q})\cos(\theta)d\theta d\tilde{t}$$
$$\approx \frac{1}{2\pi}\int_0^{2\pi} f(Q)\mathrm{sgn}(\cos(\theta))d\theta$$

Well balanced explicit approximation for constant data, example h

$$h(P) = -b(P) + \frac{1}{2\pi}\int_0^{2\pi} (h(Q) + b(Q)) - \frac{\tilde{c}}{g}(u(Q)\mathrm{sgn}(\cos(\theta)) + v(Q)\mathrm{sgn}(\sin(\theta)))d\theta$$

- A counterpart for bilinear data exists accordingly.

Well balanced Operator (M. Lukáčová, S. Noelle, M.K.[J. Comp. Phys. 2006])

Suppose thast the reconstructions at time t^n satisfy for all (x, y) the conditions of the 'Lake at Rest' or of the 'Jet in the Rotational Frame', namely

$$u^n \equiv v^n \equiv 0 \quad \wedge \quad h^n + b^n \equiv \text{const}$$

or

$$u^n \equiv 0 \quad \wedge \quad K^n \equiv \text{const} \quad \wedge \quad \partial_y h^n \equiv \partial_y v^n \equiv \partial_y b \equiv 0$$

then the EG-Operator preserves the conditions neccessary for the FV-Update to preserve the 'Lake at Rest' and the 'Jet in the Rotational Frame'.

Two dimensional example (Y. Xing, C.W. Shu[To appear in J. Comp. Phys.])

$$
\begin{aligned}
b(x,y) &:= \sin(2\pi x) + \cos(2\pi y),\\
h(x,y,0) &:= 10 + e^{\sin(2\pi x)} \cdot \cos(2\pi y),\\
hu(x,y,0) &:= \sin(\cos(2\pi x)) \cdot \sin(2\pi y),\\
hv(x,y,0) &:= \cos(2\pi x)\cos(\sin(2\pi y)).
\end{aligned}
$$

- Gravitational constant $g = 9.812$.
- Coriolis parameter $f = 10$.
- Computational domain: $(x,y) \in [0,1] \times [0,1]$
- End of virtual time $t = 0.05$.
- Periodic boundary condition.

Reference Solution

- Reference Solution calculated by Well balanced FV-Scheme (4th Order)
- Special Thanks to Normann Pankratz of the RWTH Aachen

EOC measurement, FVEG scheme: Convergence in the L^1 norm, CFL=0.8

N	L^1 error in h	EOC	L^1 error in hu	EOC	L^1 error in hv	EOC
25	1.04e-02		3.56e-02		8.52e-02	
50	2.42e-03	2.10	8.71e-03	2.03	2.15e-02	1.99
100	6.01e-04	2.01	2.23e-03	1.96	5.50e-03	1.96
200	1.54e-04	1.96	5.76e-04	1.95	1.44e-03	1.93
400	3.97e-05	1.96	1.47e-04	1.97	3.69e-04	1.96
800	1.02e-05	1.97	3.71e-05	1.98	9.40e-05	1.97

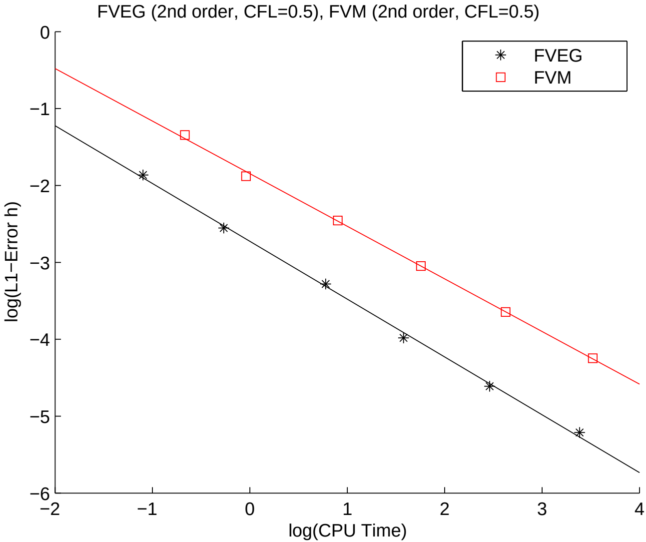
EOC measurement, FVEG scheme: Convergence in the L^1 norm, CFL=0.5

N	L^1 error in h	EOC	L^1 error in hu	EOC	L^1 error in hv	EOC
25	1.37e-02		6.19e-02		1.18e-01	
50	2.80e-03	2.29	1.05e-02	2.56	2.33e-02	2.34
100	5.23e-04	2.42	1.80e-03	2.54	4.25e-03	2.45
200	1.04e-04	2.33	3.63e-04	2.31	8.12e-04	2.39
400	2.45e-05	2.09	8.79e-05	2.05	1.80e-04	2.17
800	6.14e-06	1.99	2.20e-05	2.00	4.36e-05	2.04

EOC measurement, FV scheme: Convergence in the L^1 norm, CFL=0.5

N	L^1 error in h	EOC	L^1 error in hu	EOC	L^1 error in hv	EOC
25	4.53e-02		2.13e-01		3.40e-01	
50	1.32e-02	1.77	5.57e-02	1.94	9.51e-02	1.84
100	3.50e-03	1.92	1.42e-02	1.97	2.52e-02	1.92
200	8.95e-04	1.97	3.58e-03	1.99	6.46e-03	1.96
400	2.26e-04	1.99	8.96e-04	2.00	1.63e-03	1.99
800	5.67e-05	1.99	2.24e-04	2.00	4.10e-04	1.99

- Comparison, computational time to accuracy



Rossby adjustment problem
(F. Bouchut, J. Le Sommer, V. Zeitlin[J. Fluid Mech. 2004])

$$h(x,y,0) = 1, \quad u(x,y,0) = 0, \quad v(x,y,0) = 2N_L(x).$$

Where

$$N_L(x) = \frac{(1 + \tanh(4x/L + 2))(1 - \tanh(4x/L - 2))}{(1 + \tanh(2))^2}.$$

- Typical length scale $L = 2$
- Coriolis parameter $f = 1$
- $g = 1$
- Extrapolation boundary condition

Dambreak over a rectangular obstacle (S. Vukovic, L. Sopta[J. Comp. Phys. 2002]), (Y. Xing, C.W. Shu[To appear in J. Comp. Phys.])

$$b(x) = \begin{cases} 8 & \text{if } |x - 1500/2| \leq 1500/8 \\ 0 & \textit{otherwise} \end{cases}$$

$$h(x,0) + b(x) = \begin{cases} 20 & \text{if } 0 \leq x \leq 750 \\ 15 & \textit{otherwise} \end{cases}$$

$$u(x,0) = 0.$$

- Gravitational constant: $g = 9.81$
- Coriolis parameter: $f = 0$
- Computational domain: $x \in [0, 1500]$
- Extrapolation boundary condition.

Two dimensional example quasi stationary problem (R. J. Leveque[J. Comp. Phys. 1998])

$$
\begin{aligned}
b(x,y) &:= 0.8\exp(-5(x-0.9)^2 - 50(y-0.5)^2 \\
h(x,y,0) &:= \begin{cases} 1 - b(x,y) + \varepsilon & \text{if } 0.05 < x < 0.15 \\ 1 - b(x,y) & \text{otherwise} \end{cases} \\
hu(x,y,0) &:= hv(x,y,0) := 0.
\end{aligned}
$$

- Parameter $\varepsilon = 0.01$
- Gravitational constant $g = 1.$
- Coriolis parameter $f = 0.$
- Computational domain: $(x,y) \in [0,2] \times [0,1]$
- End of virtual time $t = 0.05.$
- Extrapolation boundary condition.