Three dimensional modeling and simulations of ionospheric plasma instabilities

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Outline

The earth environment

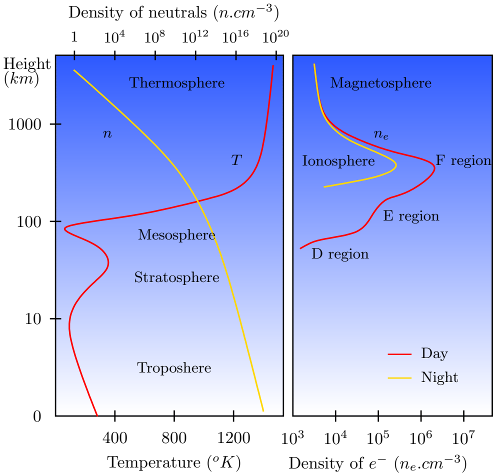


Fig. 1: Stratification of the earth atmosphere.

- the atmosphere is a stratified and ionized medium,
- the ionosphere is characterized by
  - heights ranging from 90 to 1500 kilometers,
  - a neutral density much larger than the plasma density,
  - a maximum in the plasma density 10^6 cm^{-3} in the $\sim 300 km$ heights area (the F-region).

Properties of the ionosphere

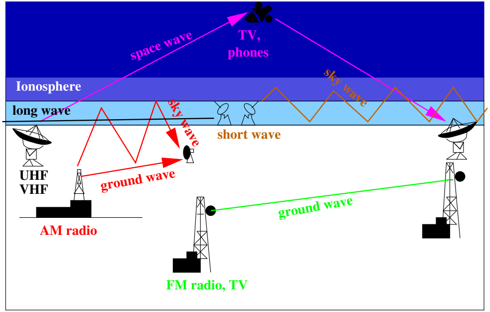


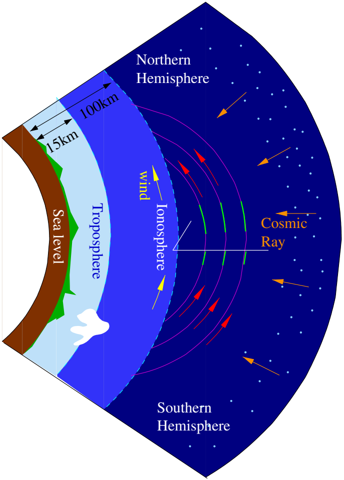
Fig. 2: Radio waves transmissions and interaction with the ionosphere.

The ionospheric plasma :

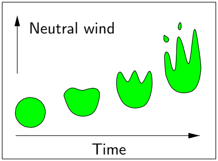
- reflects low frequency waves,
- transmits high frequency waves,
- is subjected to instabilities (solar eruptions, striations, ...).

Striations : ionospheric plasma instability observed along the earth magnetic field lines.

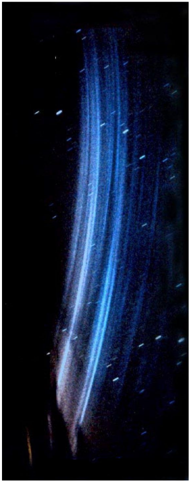
Striations formations



Time evolution of a plasma bubble located at a 300 km altitude.



The plasma bubble is progressively tired up along the magnetic field lines. In a plane perpendicular to the magnetic field the plasma bubble frontier is submitted to an instability referred to as "$E \times B$ instability".



Outline

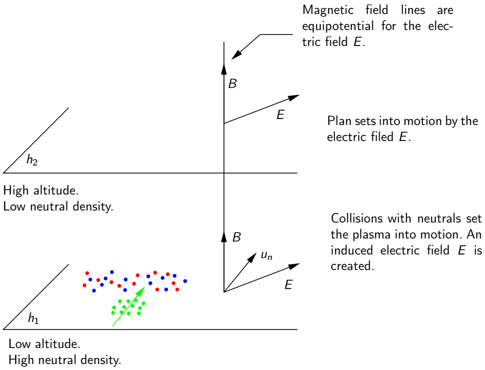


Fig. 4: The Dynamo effect.

The model :

- Maxwell equations,
- Euler equations with collisions against neutrals.

Simplications

- only one ion species : O^+,
- no chemical reaction,
- electron-ions collisions and gravity effect are neglected.

Notations

$n_e,\ n_i$	electronic, ionic density
$u_e,\ u_i$	electronic, ionic velocity
u_n	neutral velocity
$P_e,\ P_i$	electronic, ionic pressure
$\nu_e,\ \nu_i$	electronic-neutral, ionic-neutral collision frequency
$m_e,\ m_i$	electronic, ionic mass
$j = e(n_i u_i - n_e u_e)$	current density
$\rho_c = e(n_i - n_e)$	charge density

The Euler-Maxwell system

Euler equations

$$\mathcal{L}_{u_e}(n_e) = 0\,,$$
$$m_e \mathcal{L}_{u_e}(n_e u_{ek}) = -\partial_{x_k} P_e - \overbrace{e n_e (E_k + (u_e \times B)_k)}^{\text{Lorentz}} + n_e F_{ek}\,,$$
$$\mathcal{L}_{u_i}(n_i) = 0\,,$$
$$m_i \mathcal{L}_{u_i}(n_i u_{ik}) = -\partial_{x_k} P_i + q n_i (E_k + (u_i \times B)_k) + n_i F_{ik}\,,$$

The Maxwell system

$$\frac{1}{c^2}\partial_t E - \nabla \times B = -\mu_0 j,$$
$$\partial_t B + \nabla \times E = 0,$$
$$\nabla \cdot E = \rho_c/\varepsilon_0,$$
$$\nabla \cdot B = 0,$$
$$\rho_c = q(n_i - n_e)\,, \quad j = q(n_i u_i - n_e u_e).$$

With the following definition for the transport operator :

$$\mathcal{L}_u(\rho) = \partial_t \rho + \nabla \cdot (\rho\, u)\,,$$

and the friction forces :

$$F_e = -\nu_e m_e (u_e - u_n)\,,$$
$$F_i = -\nu_i m_i (u_i - u_n)\,.$$

Scaling relations

Physical parameter	Typical scale	Dimensionless parameter	Typical value
Time	$\bar{t}$	$t' = t/\bar{t}$	10^3 s
Length	$\bar{x}$	$x' = x/\bar{x}$	10^5 m
Speed	$\bar{u} = \bar{x}/\bar{t}$	$u'_{e,i,n} = u_{e,i,n}/\bar{u}$	10^2 ms^{-1}
Density	$\bar{n}$	$n'_{e,i} = n_{e,i}/\bar{n}$	$10^{12} \mid 10^{15}$ m^{-3}
Magnetic field	$\bar{B}$	$B' = B/\bar{B}$	10^{-5} T
Electric field	$\bar{E} = \bar{u}\bar{B}$	$E' = E/\bar{E}$	10^{-3} Vm^{-1}
e-n collision freq.	$\bar{\nu}_e$	$\nu'_e = \nu_e/\bar{\nu}_e$	10^2 s^{-1}
i-n collision freq.	$\bar{\nu}_i = \frac{m_e}{m_i}\bar{\nu}_e$	$\nu'_i = \nu_i/\bar{\nu}_i$	10^{-2} s^{-1}

The rescaled Euler-Maxwell system

Some dimensionless parameters

- β intensity of the induced magnetic field compared to the ambient earth magnetic field,
- κ number of e-n or i-n collisions during a cyclotron period,
- ε ratio of the electronic and ionic masses,
- α ratio of the typical velocity and the speed of light,
- τ ratio of th typical time and the time between two ions-neutral collisions,
- η measure of the thermic energy.

The rescaled Euler-Maxwell system

Euler equations

$$\partial_t n_e + \nabla \cdot (n_e u_e) = 0,$$

$$\tau\varepsilon \left(\mathcal{L}_{u_e}(n_e u_{ek})\right) = -\eta \partial_{x_k} P_e - \kappa^{-1} \overbrace{n_e(E_k + (u_e \times B)_k)}^{\text{Lorentz}} - \overbrace{\nu_e n_e(u_{ek} - u_{nk})}^{\text{friction}},$$

$$\partial_t n_i + \nabla \cdot (n_i u_i) = 0,$$

$$\tau \left(\mathcal{L}_{u_i}(n_i u_{ik})\right) = -\eta \partial_{x_k} P_i + \kappa^{-1} n_i(E_k + (u_i \times B)_k) - \nu_i n_i(u_{ik} - u_{nk}),$$

Maxwell system

$$\alpha \partial_t E - \nabla \times B = -\beta j,$$

$$\partial_t B + \nabla \times E = 0,$$

$$\frac{\kappa\alpha}{\beta} \nabla \cdot E = \rho_c,$$

$$\nabla \cdot B = 0,$$

$$\rho_c = n_i - n_e, \quad \kappa j = n_i u_i - n_e u_e.$$

Typical values of the plasma density $n_{i,e} = 10^{12} | 10^{15} m^{-3}$

$$\varepsilon = 10^{-4}, \qquad \tau = 10^{-1},$$
$$\eta = 10^{1}, \qquad \kappa = 10^{-4}$$
$$\alpha = 10^{-12}, \qquad \beta = 10^{-5} | 10^{-2}.$$

The rescaled Euler-Maxwell system

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$$\begin{array}{ll} \varepsilon = 10^{-4}, & \tau = 10^{-1}, \\ \eta = 10^{1}, & \kappa = 10^{-4} \\ \alpha = 10^{-12}, & \beta = 10^{-5} | 10^{-2}. \end{array}$$

The rescaled Euler-Maxwell system

Euler equations

$$\partial_t n_e + \nabla \cdot (n_e u_e) = 0,$$

$$0 = -\eta \partial_{x_k} P_e - \kappa^{-1} n_e(E_k + (u_e \times B)_k) - \nu_e n_e(u_{ek} - u_{nk}),$$

$$\partial_t n_i + \nabla \cdot (n_i u_i) = 0,$$

$$\tau (\mathcal{L}_{u_i}(n_i u_{ik})) = -\eta \partial_{x_k} P_i + \kappa^{-1} n_i(E_k + (u_i \times B)_k) - \nu_i n_i(u_{ik} - u_{nk}),$$

Maxwell system

$$\alpha \partial_t E - \nabla \times B = -\beta j,$$

$$\partial_t B + \nabla \times E = 0,$$

$$\frac{\kappa\alpha}{\beta} \nabla \cdot E = \rho_c,$$

$$\nabla \cdot B = 0,$$

$$\rho_c = n_i - n_e, \quad \kappa j = n_i u_i - n_e u_e.$$

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$$\partial_t n_e + \nabla \cdot (n_e u_e) = 0,$$

$$0 = -\eta \partial_{x_k} P_e - \kappa^{-1} n_e (E_k + (u_e \times B)_k) - \nu_e n_e (u_{ek} - u_{nk}),$$

$$\partial_t n_i + \nabla \cdot (n_i u_i) = 0,$$

$$\tau (\mathcal{L}_{u_i}(n_i u_{ik})) = -\eta \partial_{x_k} P_i + \kappa^{-1} n_i (E_k + (u_i \times B)_k) - \nu_i n_i (u_{ik} - u_{nk}),$$

Maxwell system

$$-\nabla \times B = -\beta j,$$

$$\partial_t B + \nabla \times E = 0,$$

$$0 = \rho_c, \Rightarrow \text{quasineutrality}$$

$$\nabla \cdot B = 0,$$

$$\rho_c = n_i - n_e, \quad \kappa j = n_i u_i - n_e u_e.$$

Typical values of the plasma density $n_{i,e} = 10^{12} | 10^{15} m^{-3}$

$$\varepsilon = 10^{-4}, \qquad \tau = 10^{-1},$$
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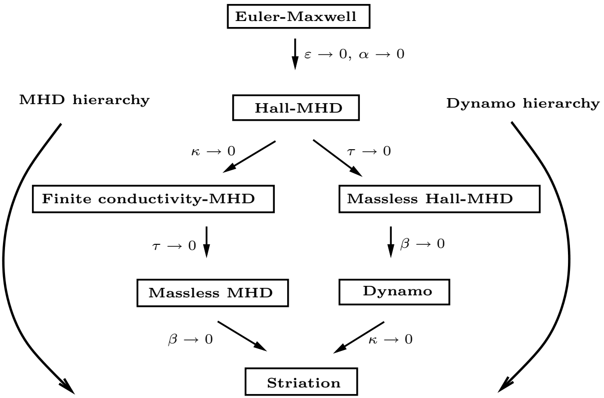


Fig. 5: Model hierarchy.

- Dynamo hierarchy : standard plasma density,
- MHD hierarchy : high plasma density.

Outline

Main assumptions and properties

Assumptions :

- quasi-neutral plasma : $n_e = n_i = n$,
- the magnetic field is reduced to the earth magnetic field and $\partial B / \partial t = 0$. We choose B to be aligned with x_3 (or $x_{\|}$) coordinate,
- the aligned conductivity is infinite : $\kappa \longrightarrow 0$,
- no incoming or outgoing current flowing through the boundary with normal parallel to x_3 (reconnection to the neutral atmosphere).

The "Striation model" main properties :

- the electric potential depends only on the two first coordinate (plane perpendicular to the magnetic field $x_{\perp}$),
- by integrating the elliptic equation along the magnetic field line the problem is reduced to a bi-dimensional one.

The multi-layer "Striation model" (rescaled variables)

$$\frac{\partial n}{\partial t} + \nabla \cdot (nu) = 0$$

$$u = \frac{E \times B}{|B|^2} + \left(\left(u_n - \eta \frac{\nabla p}{n} \right) \cdot \frac{B}{|B|} \right) \frac{B}{|B|} ; \quad E = -\nabla_\perp \phi(x_\perp)$$

$$\nabla_\perp \cdot J_\perp = 0 \,,$$

$$J_\perp = \frac{1}{|B|^2} \left(-\sigma(x_\perp) \nabla_\perp \phi + U_n \times B - \eta \nabla_\perp P_\perp \times B \right) \,,$$

$$\sigma(x_\perp) = \int n\nu \, dx_3 \,, \quad U_n = \int n\nu u_n \, dx_3 \,, \quad P_\perp = \int P(n) dx_3 \,.$$

- $\sigma/|B|^2$: Pedersen conductivity integrated along a magnetic field line,
- 3D transport equation, 2D elliptic equation.

The 2D "Striations model" (rescaled variables)

Assumptions :

- All quantities only depend on $x_\perp$,
- $u_{n3} = 0$ and $\nabla P = 0$,

$$\frac{\partial n}{\partial t} + \nabla \cdot (nu) = 0$$

$$-\nabla\left(\frac{n}{|B|^2}\nabla\phi\right) = -\nabla \cdot \left(n\frac{u_n \times B}{|B|^2}\right),$$

$$u = -\frac{E \times B}{|B|^2},$$

$$E = -\nabla\phi.$$

The "$E \times B$" instability

$$\boxed{\begin{aligned} &\partial_t n + \nabla \cdot (nu) = 0, \\ &\nabla \cdot (nh) = 0, \\ &h = -\nabla\phi - u_n^\perp . \end{aligned}}$$

Recall that $u = (\nabla\phi)^\perp$ and
$h = -\nabla\phi - u_n^\perp = u_i - u_e$.

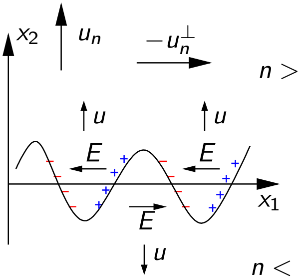


Fig. 6: Stable configuration

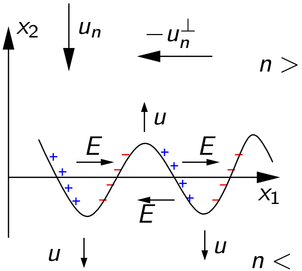


Fig. 7: Unstable configuration

The Striations model is widely used by physicists e.g.

- Ronchi, Similon, Farley (1989),
- Zalesak, Ossakov, Chaturvedi (1982).

☞ Mainly 2D model,

☞ poor description of the variations along the third coordinate,

☞ Cartesian geometry.

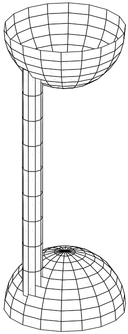


Fig. 8: Magnetic field tube in Cartesian geometry.

A curvilinear coordinates for non uniform magnetic field.

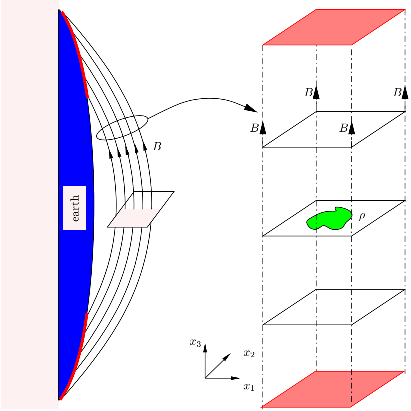


Fig. 9: Magnetic field tube with Cartesian coordinates.

Dipolar model for the earth magnetic field :

- Axisymmetric field $B = (B_r(r,z), 0, B_z(r,z)$, (r,θ,z) being the cylindrical coordinates ;
- use of Euler Potentials : $\nabla \cdot B = \nabla \times B = 0 \Rightarrow \exists$ $\beta(r,z), \gamma(r,z)$ such that

$$B = -\nabla\gamma = (\nabla^{\perp}\beta)\,,$$

- magnetic field lines are described by $(\alpha,\beta) =$ constant.

A curvilinear coordinate set

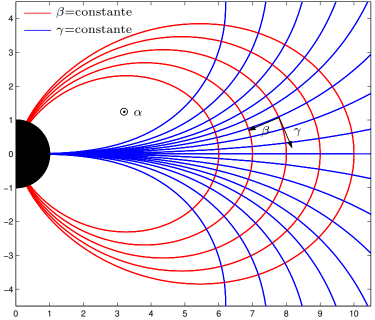


Fig. 10: Curvilinear coordinate system associated to the dipolar magnetic field.

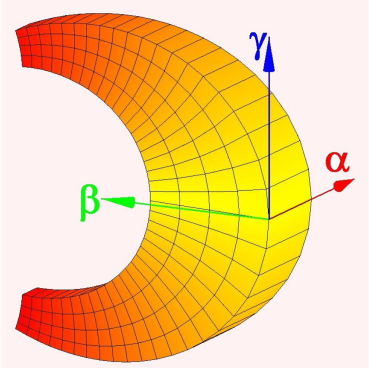


Fig. 11: Magnetic field tube using a dipolar model.

The Striation model in a non-uniform magnetic field (physical variables)

Assumptions : no transport in the direction aligned with the magnetic fields and no pressure term.

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \alpha}\left(\frac{v_{i\alpha}}{r} n\right) + |B|^2 \frac{\partial}{\partial \beta}\left(\frac{r\, v_{i\beta}}{|B|} n\right) = 0\,,$$

$$v_{i\alpha} = r\frac{\partial \phi}{\partial \beta}\,, \qquad v_{i\beta} = -\frac{1}{r|B|}\frac{\partial \phi}{\partial \alpha}\,,$$

$$\frac{\partial}{\partial \alpha}\left(A_\alpha \frac{\partial \phi}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(A_\beta \frac{\partial \phi}{\partial \beta}\right) = \frac{\partial J_{n\alpha}}{\partial \alpha} + \frac{\partial J_{n\beta}}{\partial \beta}\,,$$

$$A_\alpha = \int_{\gamma_{min}}^{\gamma_{max}} n\,\zeta\,\frac{d\gamma}{r^2|B|^4}\,, \qquad A_\beta = \int_{\gamma_{min}}^{\gamma_{max}} n\,\zeta\,\frac{r^2\,d\gamma}{|B|^2}\,,$$

$$J_{n\alpha} = -\int_{\gamma_{min}}^{\gamma_{max}} n\zeta\, u_{n\beta}\,\frac{d\gamma}{r|B|^3}\,, \qquad J_{n\beta} = \int_{\gamma_{min}}^{\gamma_{max}} n\zeta\, u_{n\alpha}\,\frac{r\,d\gamma}{|B|^2}\,,$$

$$\zeta = \frac{m_i}{e}\nu_i + \frac{m_e}{e}\nu_e\,.$$

Simulations with non-uniform magnetic

Simulation domain : One magnetic field tube,

- The domain is discretized thanks to several layers,
- each layer being discretized with a Cartesian mesh,

Initial condition :

- earth ionospheric model : IRI (International Reference Ionosphere),
- earth neutral atmosphere model : MSISE-1990 (Extended version of the Mass Spectrometer Incoherent Scatter Model).

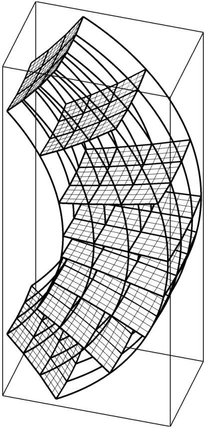


Fig. 12: Discretized magnetic field tube.

Simulations with non-uniform magnetic field

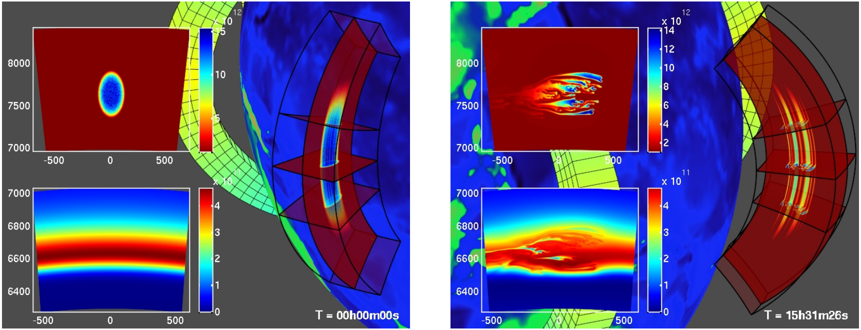


Fig. 13: Plasma bubble evolution.
(The wind blows from the left to the right)

Some assumptions may be relaxed :

1. quantities may vary along the magnetic field line,
2. a constant ions-neutrals collision frequency is not acceptable : (see Fig. 14),

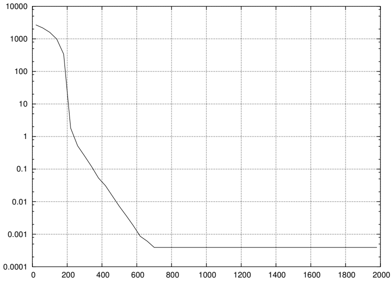


Fig. 14: Ions-neutrals collision frequency (s^{-1}) versus altitude (km).

Transport in the aligned direction and pressure term

Assumptions : $T = T_e = T_i$, hence $P = nk_B\, 2T$.

Striation model with pressure term and transport in the aligned direction

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \alpha}\left(\frac{v_{i\alpha}}{r} n\right) + |B|^2 \left(\frac{\partial}{\partial \beta}\left(\frac{r\, v_{i\beta}}{|B|} n\right) + \frac{\partial}{\partial \gamma}\left(\frac{v_{i\gamma}}{|B|} n\right)\right) = 0\,,$$

$$v_{i\alpha} = r\frac{\partial \phi}{\partial \beta}\,, \qquad v_{i\beta} = -\frac{1}{r|B|}\frac{\partial \phi}{\partial \alpha}\,, \qquad v_{i\gamma} = u_{n\gamma} - \frac{|B|}{e\; n\; \zeta}\frac{\partial P}{\partial \gamma}\,,$$

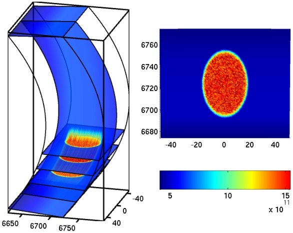
$$\frac{\partial}{\partial \alpha}\left(A_\alpha \frac{\partial \phi}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(A_\beta \frac{\partial \phi}{\partial \beta}\right) = \frac{\partial}{\partial \alpha}\left(J_{n\alpha} + J_{P\alpha}\right) + \frac{\partial}{\partial \beta}\left(J_{n\beta} + J_{P\beta}\right)\,,$$

$$A_\alpha = \int_{\gamma_{min}}^{\gamma_{max}} n\,\zeta\, \frac{d\gamma}{r^2|B|^4}\,, \qquad A_\beta = \int_{\gamma_{min}}^{\gamma_{max}} n\,\zeta\, \frac{r^2\, d\gamma}{|B|^2}\,,$$

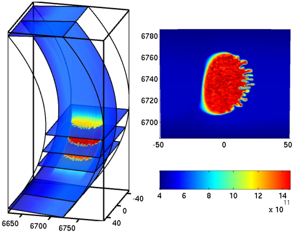
$$J_{n\alpha} = -\int_{\gamma_{min}}^{\gamma_{max}} n\zeta\, u_{n\beta}\, \frac{d\gamma}{r|B|^3}\,, \qquad J_{n\beta} = \int_{\gamma_{min}}^{\gamma_{max}} n\zeta\, u_{n\alpha}\, \frac{r\, d\gamma}{|B|^2}\,,$$

$$J_{P\alpha} = \int_{\gamma_{min}}^{\gamma_{max}} \frac{1}{e}\frac{\partial P}{\partial \beta}\frac{d\gamma}{|B|^2}\,, \qquad J_{P\beta} = -\int_{\gamma_{min}}^{\gamma_{max}} \frac{1}{e}\frac{\partial P}{\partial \alpha}\frac{d\gamma}{|B|^2}\,.$$

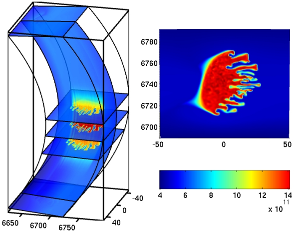
Plasma density evolution (“Aligned transport”)



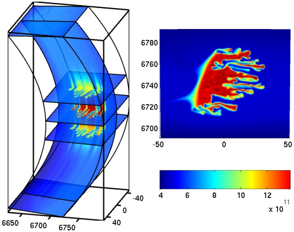
T=0



T = 30 min



T = 1 h 00 min



T = 1 h 30 min

Non constant i-n collision frequency

The Striation model

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \alpha}\left(\frac{v_{i\alpha}}{r} n\right) + |B|^2 \frac{\partial}{\partial \beta}\left(\frac{r\, v_{i\beta}}{|B|} n\right) = 0\,,$$

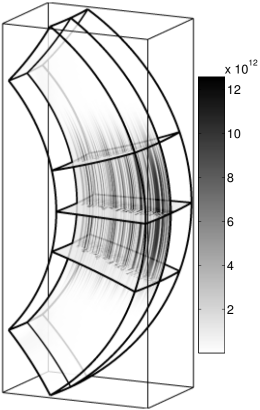
$$v_{i\alpha} = r \frac{\partial \phi}{\partial \beta}\,, \qquad v_{i\beta} = -\frac{1}{r|B|}\frac{\partial \phi}{\partial \alpha}\,,$$

$$\frac{\partial}{\partial \alpha}\left(A_\alpha \frac{\partial \phi}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(A_\beta \frac{\partial \phi}{\partial \beta}\right) = \frac{\partial J_{n\alpha}}{\partial \alpha} + \frac{\partial J_{n\beta}}{\partial \beta}\,,$$

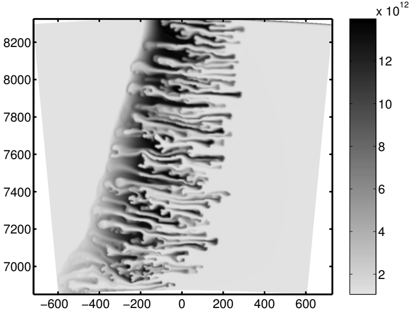
$$A_\alpha = \int_{\gamma_{min}}^{\gamma_{max}} n\,\zeta\, \frac{d\gamma}{r^2|B|^4}\,, \qquad A_\beta = \int_{\gamma_{min}}^{\gamma_{max}} n\,\zeta\, \frac{r^2\, d\gamma}{|B|^2}\,,$$

$$J_{n\alpha} = -\int_{\gamma_{min}}^{\gamma_{max}} n\zeta\, u_{n\beta}\, \frac{d\gamma}{r|B|^3}\,, \qquad J_{n\beta} = \int_{\gamma_{min}}^{\gamma_{max}} n\zeta\, u_{n\alpha}\, \frac{r\, d\gamma}{|B|^2}\,,$$

$$\zeta = \frac{m_i}{e}\nu_i\,.$$

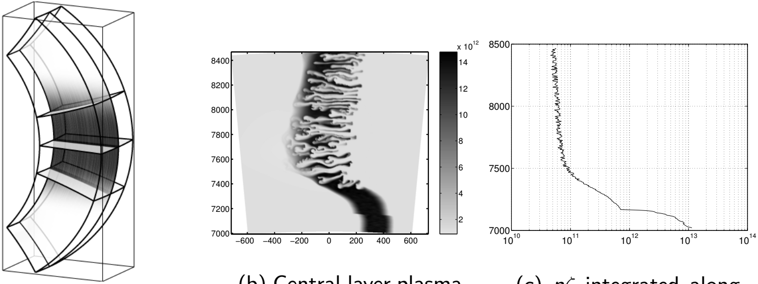


(a) Field tube



(b) Central layer

Fig. 15: Simulation carried out with a constant i-n collision frequency.



(a) Simulation domain.

(b) Central layer plasma density (m^{-3}) at t=2 h.

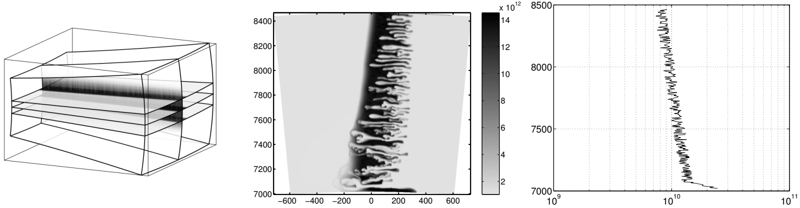
(c) $n\zeta$ integrated along magnetic field lines.

Fig. 16: Plasma density evolution with a non constant i-n collision frequency.

Collisions at low altitudes are very frequent : the plasma is set into motion with the neutral wind without any instability development.

Solution :

Use the Striation model in a small domain where it is valid.



(a) Simulation Domain.

(b) Central layer plasma density (m^{-3}) at t=2 h.

(c) $n\,\kappa$ integrated along magnetic field lines.

Fig. 17: Plasma density with a non-constant i-n collision frequency.

Conclusion :

Relax the infinite aligned mobility assumption : the Dynamo model.

Outline

1. Physical context, motivations
2. Modeling
3. The Striation model
4. The 3D-Dynamo model
   - Presentation
   - Properties of the Dynamo model
   - Perspectives

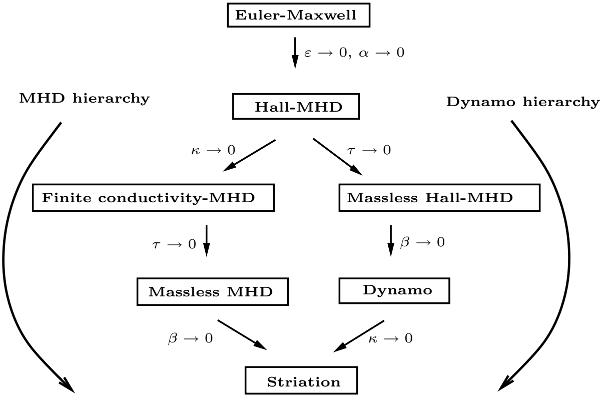


Fig. 18: Model hierarchy.

The 3D Dynamo model

- quasi-neutral plasma : $n = n_e = n_i$,
- the magnetic field reduces to te earth magnetic field,
- on the typical time scale, we assume, $\partial B/\partial t = 0$,

$$\begin{aligned}
&\partial_t n + \nabla \cdot (n u_i) = 0,\\
&u_e = \mathbb{M}_e(-E + \kappa \nu_e u_n),\\
&u_i = \mathbb{M}_i(E + \kappa \nu_i u_n),\\
&E = -\nabla \phi,\\
&\nabla \cdot j = 0,\\
&\kappa j = n(u_i - u_e),
\end{aligned}$$

$$\mathbb{M}_e = \begin{pmatrix} \mu_e^P & -\mu_e^H & 0 \\ \mu_e^H & \mu_e^P & 0 \\ 0 & 0 & \mu_e^{\parallel} \end{pmatrix},$$

$$\mathbb{M}_i = \begin{pmatrix} \mu_i^P & \mu_i^H & 0 \\ -\mu_i^H & \mu_i^P & 0 \\ 0 & 0 & \mu_i^{\parallel} \end{pmatrix},$$

with

$$\mu_{e,i}^P = \frac{\kappa \nu_{e,i}}{(\kappa \nu_{e,i})^2 + |B|^2}, \quad \mu_{e,i}^H = \frac{|B|}{(\kappa \nu_{e,i})^2 + |B|^2}, \quad \mu_{e,i}^{\parallel} = \frac{1}{\kappa \nu_{e,i}}.$$

The 3D-Dynamo model

Expression in Cartesian geometry.

$$\frac{\partial n}{\partial t} + \nabla \cdot (n u_i) = 0\,,$$

$$-\nabla \cdot \Big[n \big(\mathbb{M}_i + \mathbb{M}_e \big) \nabla \phi \Big] = -\nabla \cdot \Big[n \mathbb{M}_i \Big(\kappa_i u_n - \frac{\nabla P_i}{e\, n} \Big) - n \mathbb{M}_e \Big(\kappa_e u_n - \frac{\nabla P_e}{e\, n} \Big) \Big]\,,$$

$$u_i = \mathbb{M}_i \Big(E + \kappa_i u_n - \frac{\nabla P_i}{e\, n} \Big)\,, \qquad E = -\nabla \phi\,,$$

$$\mu^P_{e,i} = \frac{\kappa_{e,i}}{\kappa^2_{e,i} + |B|^2}\,, \quad \mu^H_{e,i} = \frac{|B|}{\kappa^2_{e,i} + |B|^2}\,, \quad \mu^{\parallel}_{e,i} = \frac{1}{\kappa_{e,i}}\,, \quad \kappa_{e,i} = \frac{m_{e,i} \nu_{e,i}}{e}\,,$$

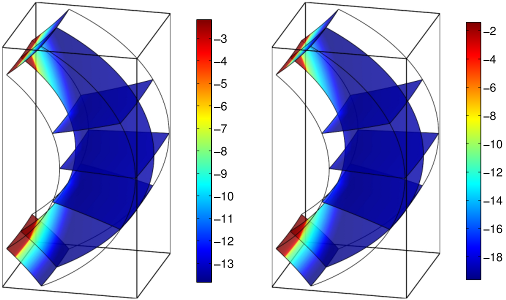
$$\mathbb{M}_e = \begin{pmatrix} \mu^P_e & -\mu^H_e & 0 \\ \mu^H_e & \mu^P_e & 0 \\ 0 & 0 & \mu^{\parallel}_e \end{pmatrix}, \qquad \mathbb{M}_i = \begin{pmatrix} \mu^P_i & \mu^H_i & 0 \\ -\mu^H_i & \mu^P_i & 0 \\ 0 & 0 & \mu^{\parallel}_i \end{pmatrix}$$

The 3D-Dynamo model

- ☞ The Dynamo model does not assume on infinite aligned conductivity.
- ☞ The Dynamo model is valid on the whole range of altitudes.

- The 3D-Dynamo model is a full 3D model : the electric potential is the solution of a three dimensional elliptic equation.
- Furthermore, the discretized equation is ill-conditioned due to the large value of aligned mobilities compared to Perdersen and Hall mobilities.

Ill-conditioness of the discretized 3D-Dynamo elliptic equation



(a) Perdersen and aligned mobilities ratio : $\mu^P/\mu^{\|}$.

(b) Ratio of the Hall and aligned mobilities $\mu^H/\mu^{\|}$.

Fig. 19: Mobilities ratios in decimal log scale.

3D-Dynamo elliptic equation :

$$-\nabla\cdot(\mathbb{M}\nabla\phi) = -\nabla\cdot J\,,$$

with

$$\mathbb{M} = \mathbb{M}_e + \mathbb{M}_i$$

$$= \begin{pmatrix} \mu^P & -\mu^H & 0 \\ \mu^H & \mu^P & 0 \\ 0 & 0 & \mu^{\|} \end{pmatrix},$$

and

$$\mu^P = \mu_e^P + \mu_i^P\,,$$
$$\mu^H = \mu_e^H - \mu_i^H\,,$$
$$\mu^{\|} = \mu_e^{\|} + \mu_i^{\|}\,.$$

Ill-conditioness of the discretized 3D-Dynamo elliptic equation

Solve : $\mathcal{A}_h \, \phi_h = b_h$, a discretization of $-\nabla \cdot (\mathbb{M} \nabla \phi) = b$ (+ Boundary cond.).
Then compute the residual $R = (\mathcal{A}_h \, \phi_h - b_h)/b_h$ for $b_h = 1$ (R_1) and $b_h \approx -\nabla \cdot J$ (R_J).

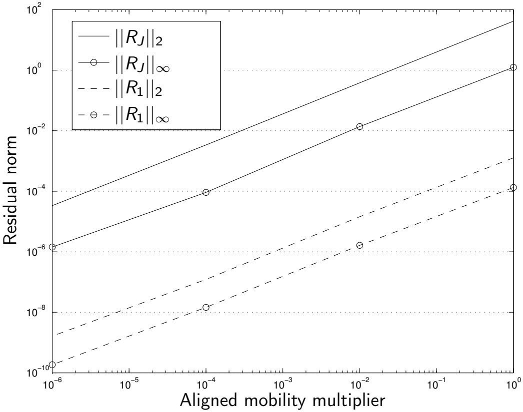


Fig. 20: Residual norms versus aligned mobility multiplier. Linear system solver : ILU preconditioned conjugate gradient method, with $50 \times 50 \times 50$ nodes.

Models coupling

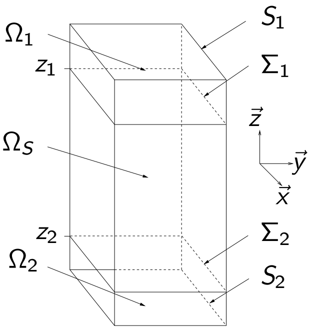


Fig. 21: Domain decomposition.

Strategy :

- Striation model for high altitudes : $\Omega_S \rightarrow \phi^S$,
- 3D-Dynamo model for lower altitudes $\Omega_i \rightarrow \phi^i$, $i = 1, 2$,
- Matching conditions on the boundaries Σ_i, $i = 1, 2$ $\phi^S\big|_{\Sigma_i} = \phi^i\big|_{\Sigma_i}$.