1D Numerical Analysis

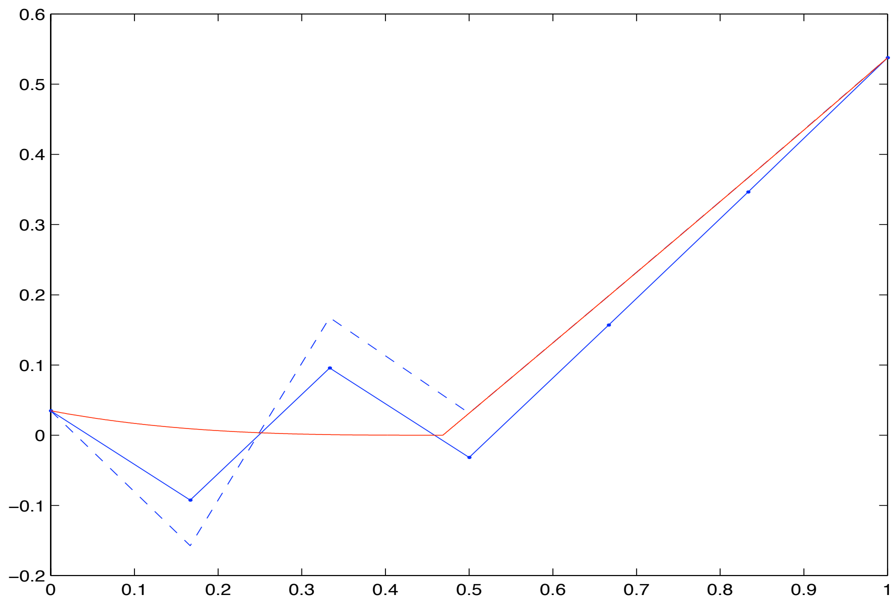
[Shifted Tartar's Broken Extremal Example]

Energy density $W(F) = (F^2 - 1)^2$ for $F \in \mathbb{R}$, f polynomial from Tartar's broken extremal example and modified Dirichlet BC in $\mathcal{A}$.

(M) Minimize

$$E(u) := \int_0^1 W(u'(x))\,dx + \underbrace{\int_0^1 |u(x) - f(x + \frac{\pi}{100})|^2\,dx}_{=:L(u)} \quad \text{over } \mathcal{A}.$$

(M) has no solution, (Q) has solution u, which is is known, picewise polynomial, plotted in red and which shows microstructures in $(0, 1/2 - \frac{\pi}{100})$ classical solutions in $(1/2 - \frac{\pi}{100}, 1)$:



Cont. 1D Numerical Analysis

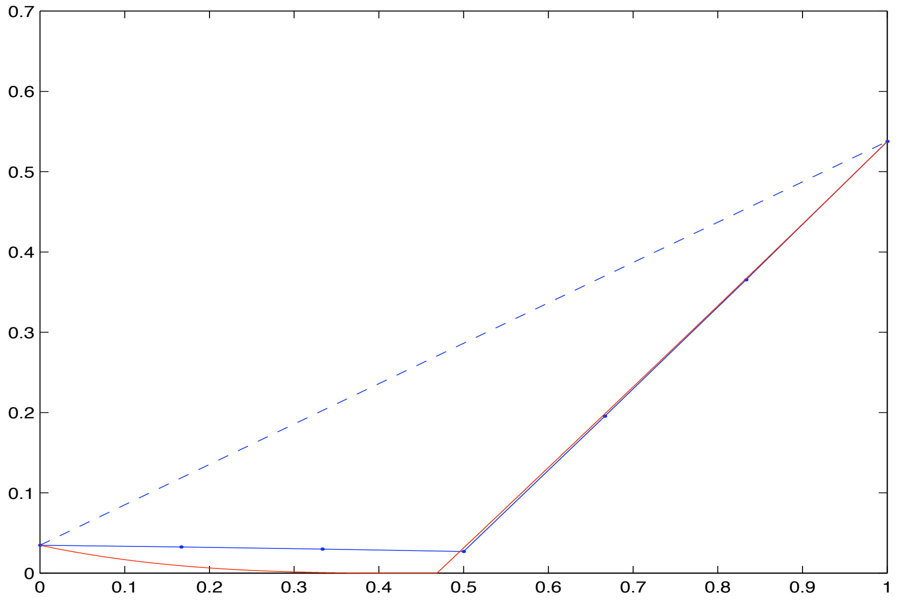
Discretization (M_h)

- Substitute $\mathcal{A}$ by finite element space $\mathcal{A}_h \subset \mathcal{A}$ on (uniform) triangulation in 6 elements

$$(M_h) \quad \text{Minimize} \quad E(u_h) = \int_0^1 W(u_h'(x))\,dx + L(u_h) \quad \text{over } \mathcal{A}_h.$$

- (M_h) has solution(s) u_h, the discrete global minimizer(s), plotted in blue (cf. generalized solution u in red)

- Newton-Raphson solver computed u_h (solid blue line) is sensitve w.r.t. initial values (dashed blue line)



- Cluster of local minimizers near u_h Newton-Raphson schemes do not work, descent methods stuck at local minimizer:

  numerical analysts' nightmares [C 2001]

  u_h difficult & expensive to compute

- Convergence analysis for particular cases [cf. Luskin Acta Numerica]

- u_h cannot converge strongly in $W^{1,p}(\Omega)$ (Proof: Strong limits solve (M)!)

- No further knowledge required, feasability in question (cf. Tartar's 4-well example analysed in [Chipot 2001])

Cont. 1D Numerical Analysis

Discretization (G_h)

(G_h) Minimize

$$\int_\Omega \langle \nu_h, W(x,\cdot)\rangle \, dx + L(u_h)$$

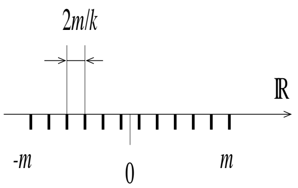
over $u_h \in \mathcal{A}_h$ & GYM ν_h

subj. to $\langle \nu_h, \mathrm{Id}\rangle = Du_h$ on each T.

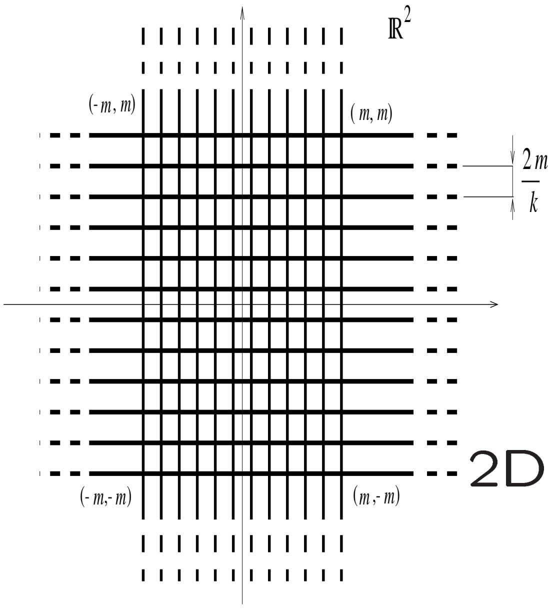
- $\mathcal{A}_h$ as before and discretization ν_h of GYM ν_x on each FE T

$$\nu_h|_T = \sum_{j=1}^{j_T} \mu_{T,j}\, \delta_{F_{T,j}}$$

for given atoms $F_{T,j}$ and unknown convex coefficients $\mu_{T,j}$



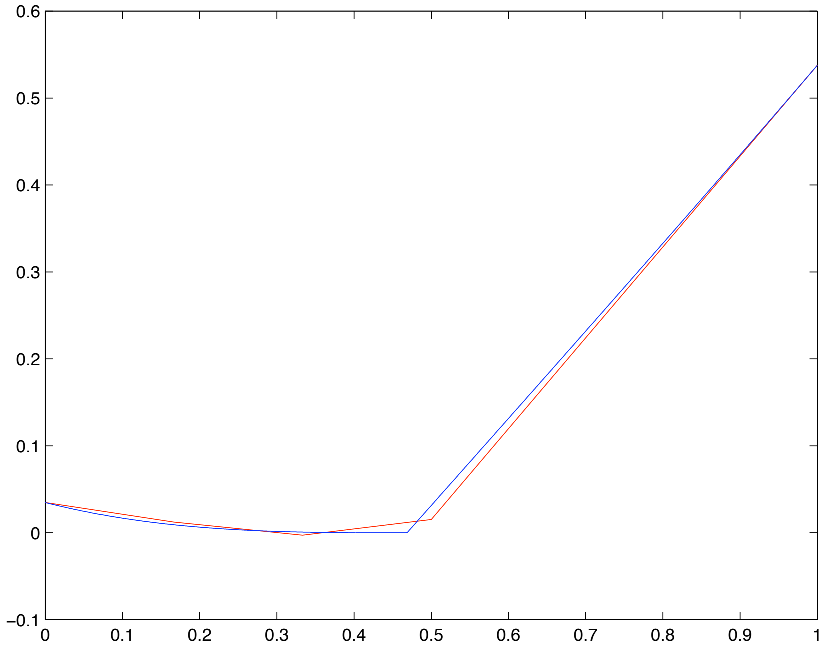
1D

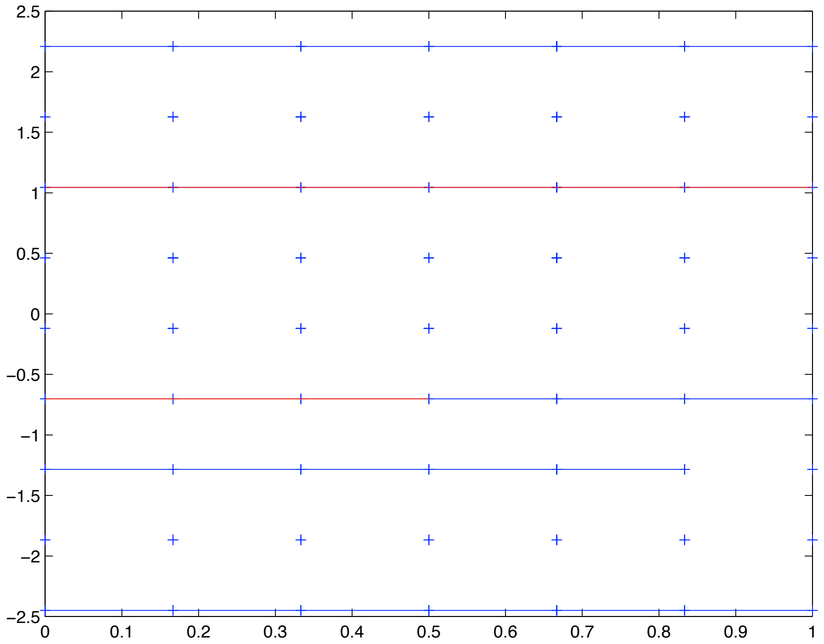


2D

- Requires sqp with huge number of unknowns

Shifted Tartar's Broken Extremal Example
$[(u_h, \nu_h)$ from $(G_h)]$





u_h and u (top) and support of Young measure ν_h in $\mathbb{R} \times (0, 1)$ (bottom)

Cont. 1D Numerical Analysis

Discretization (Q_h)

(Q_h) Minimize

$$E(u_h) = \int_0^1 W^{**}(u_h'(x))\,dx + L(u_h)$$

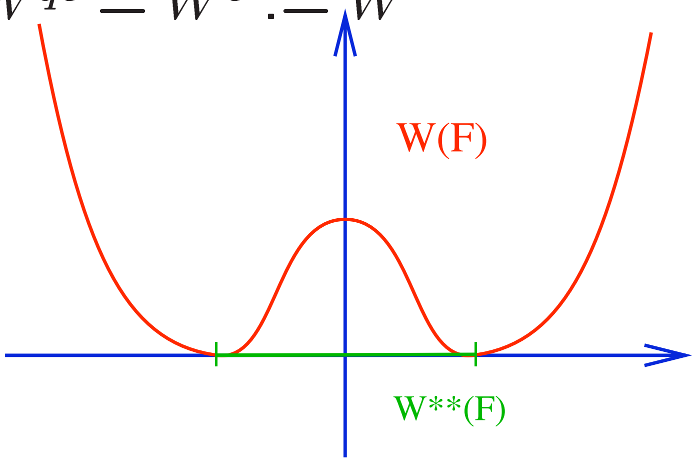
over $\mathcal{A}_h$.

- Computation of

$$W^{qc}(F) := \inf_{\psi \in W_0^{1,\infty}(\omega)} \fint_\omega W(F + D\psi(x))\,dx$$

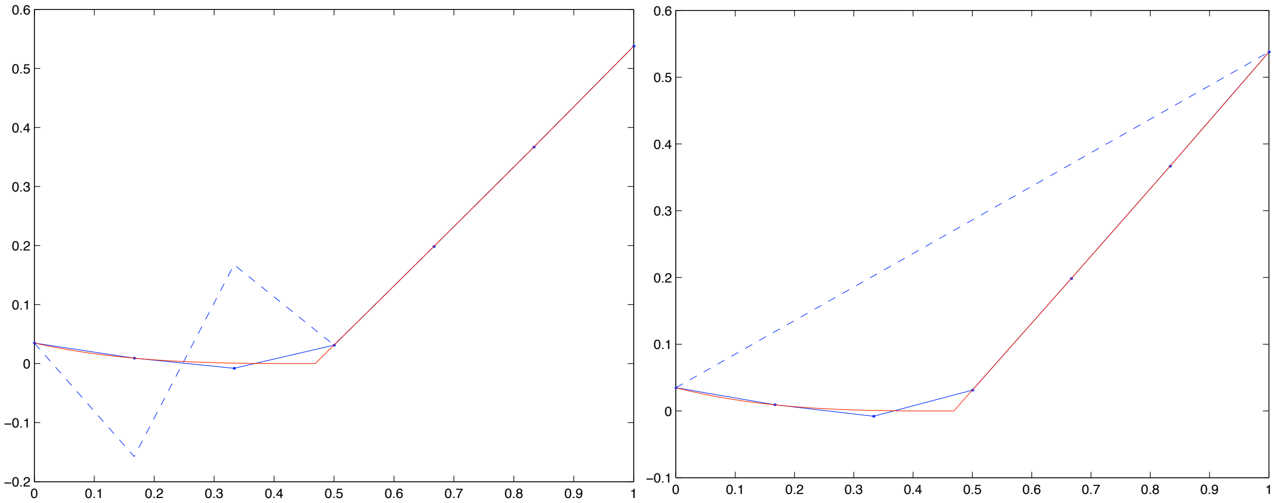
  requires solve of local minimization problem (M) and, hence, numerical approximation of W^{qc} can be extremely complicated

- Scalar Case $W^{qc} = W^c := W^{**}$



- Minimization of Convex Functional E^{**}: Each local extremum of $E^{**} : \mathcal{A}_h \to \mathbb{R}$ is a global discrete minimizer, standard software works

Shifted Tartar's Broken Extremal Example
[u_h from (Q_h)]



- For lumped FE version, strong convergence

$$u_h \to u \text{ strongly in } W^{1,4}(0,1)$$

  (Nicolaides & Walkington, Math Comp 1995)

- Recovery of oscillations possible

- Limitation by request W^{qc}