2D Benchmark Example

[Tartar's Broken Extremal Example in 2D]

Antiplane sheer model of Erickson-James energy

$$W(F) = |F - F_1|^2 \, |F - F_2|^2 \quad \text{for } F \in \mathbb{R}^2$$

(M) Minimize

$$E(u) := \int_\Omega (W(\nabla u) + |u - f|^2)\, dx$$

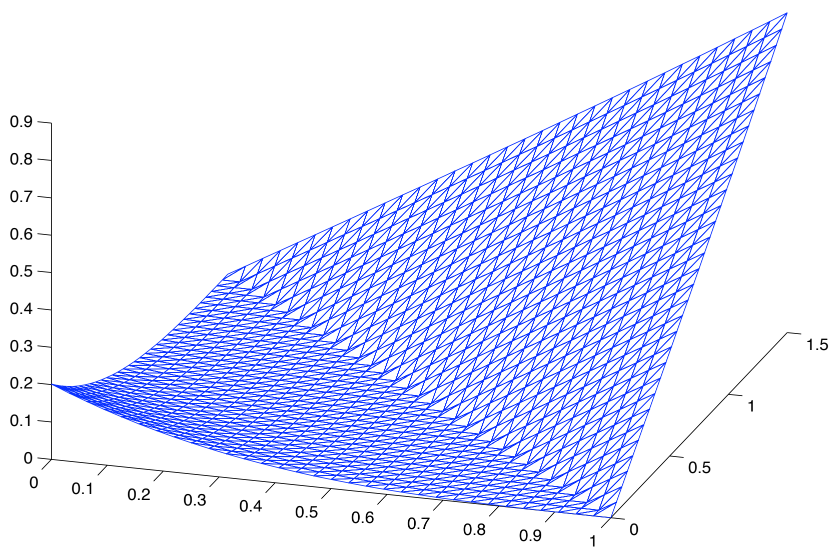
over $u \in \mathcal{A} := u_D + W_0^{1,4}(\Omega)$.

Averaged solution reads $\quad (z = \frac{3(x-1)+2y}{\sqrt{13}} + \frac{1}{2})$

$$u(x,y) = u_D(x,y) = \begin{cases} f_0(z) & \text{für } 0 \le z \le \frac{1}{2}, \\ f_1(z) & \text{für } \frac{1}{2} \le z \le 1, \end{cases}$$

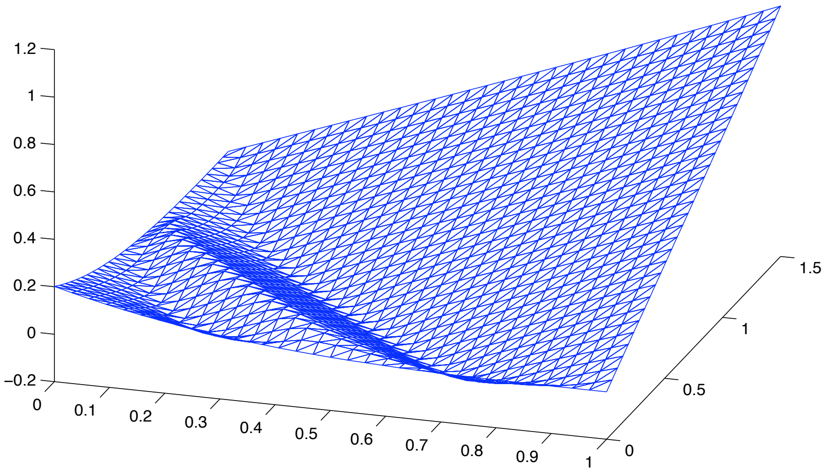
where $f_0(z) = -\frac{3}{128} z^5 - \frac{1}{3} z^3$, $f_1(z) = \frac{1}{24} z^3 + z$.

Nodal interpolant of $u(x,y)$ $(N = 961)$



Cont. 2D Benchmark Example

Finite Element Solution $u_h(x,y)$ for (M_h) $(N = 961)$

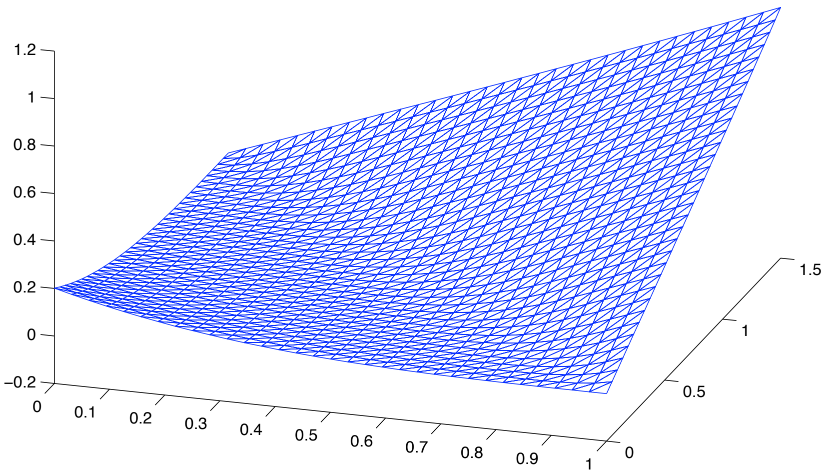


• Oscillations on length scale of discretization?

Lower convex envelope (C. & Plechac, Math Comp 1997)

$$\begin{aligned} W^{qc}(F) &= W^{**}(F) \\ &= \max\{0, |F|^2 - 1\}^2 + 4(|F|^2 - [F_2 \cdot F]^2) \end{aligned}$$

Finite Element Solution $u_h(x,y)$ for (Q_h) $(N = 961)$



• u_h accurate approximation of u, but interface not sharp.

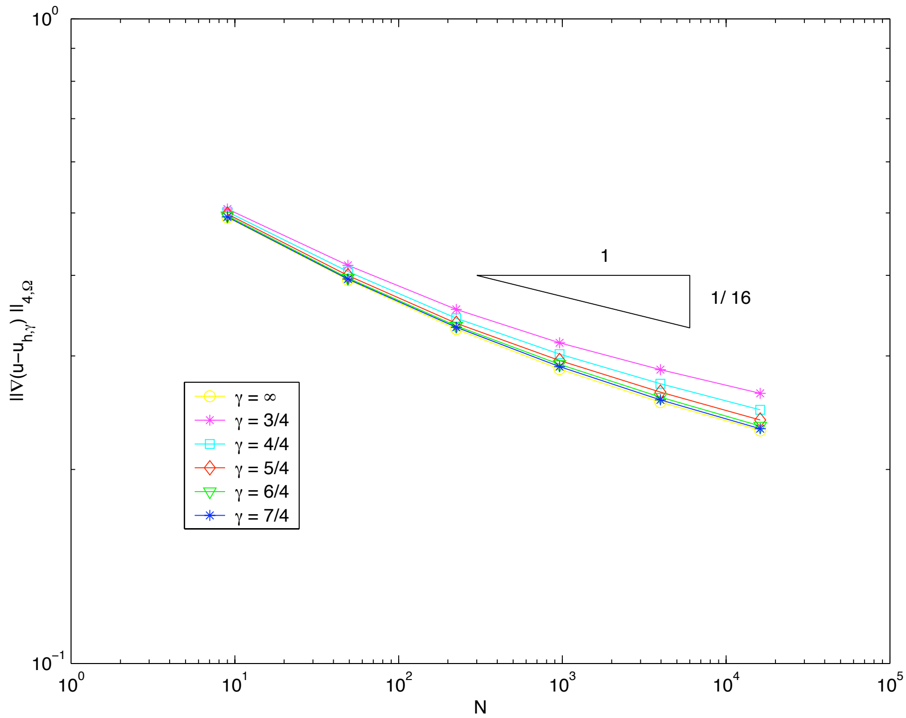
• Proof of weak convergence $\nabla u_h \to \nabla u$ only.

Strong Convergence by Stabilization!

Additional term in (Q_h) reads

$$\sum_{E\in\mathcal{E}} 1/2\, h_E^{\gamma} \int_E [\nabla u_h]^2\, ds$$

[Bartels, C., Prohl, Plechac]



• Guaranteed strong convergence (& less accuracy) of gradients

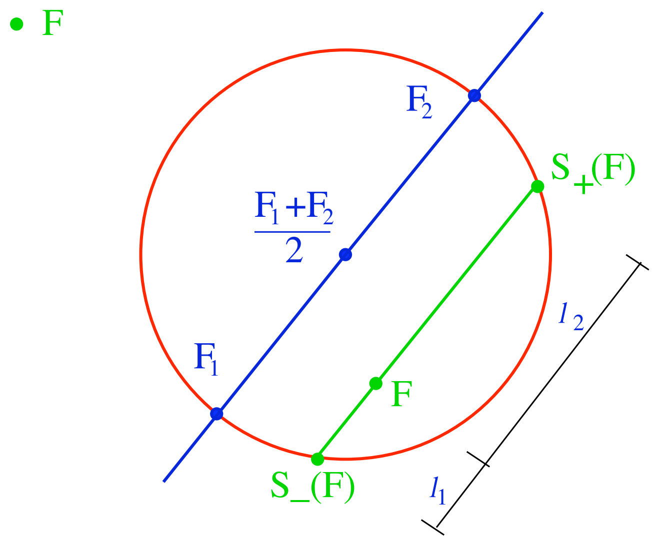
• Stability improves effective solution of (Q_h)!

E.g. for 3-Well Problem

Cont. 2D Benchmark Example

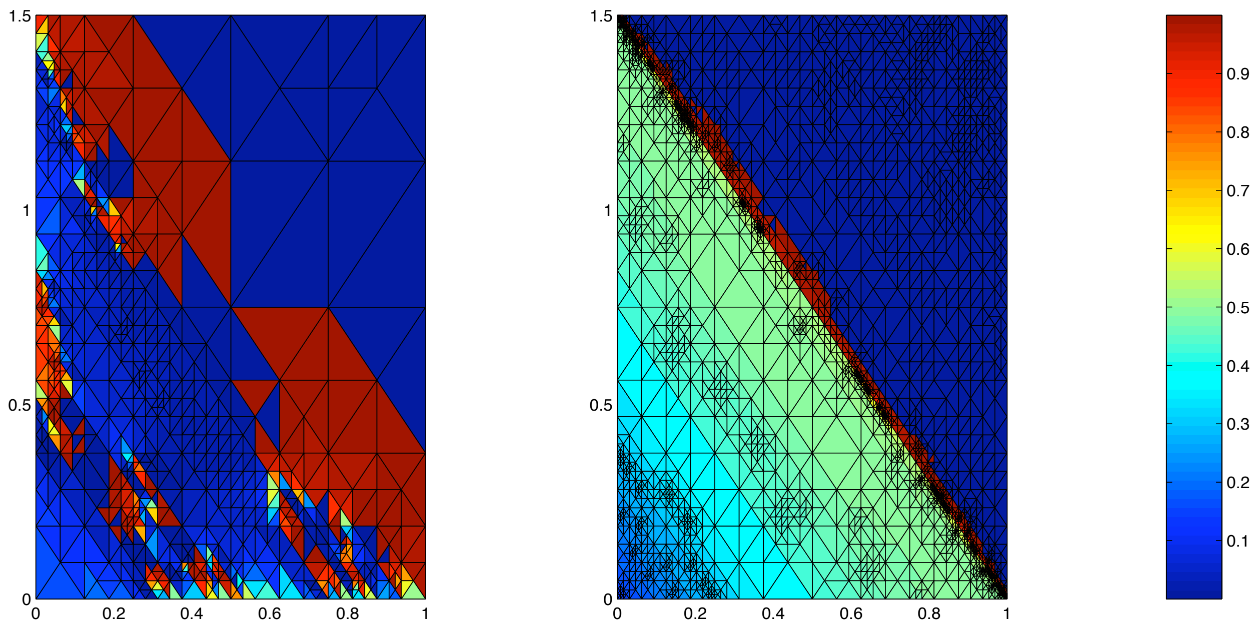
Recovery of Young measure

$$\nu = \Lambda(\nabla u) \cdot \delta_{S_+(\nabla u)} + (1 - \Lambda(\nabla u)) \cdot \delta_{S_-(\nabla u)}$$



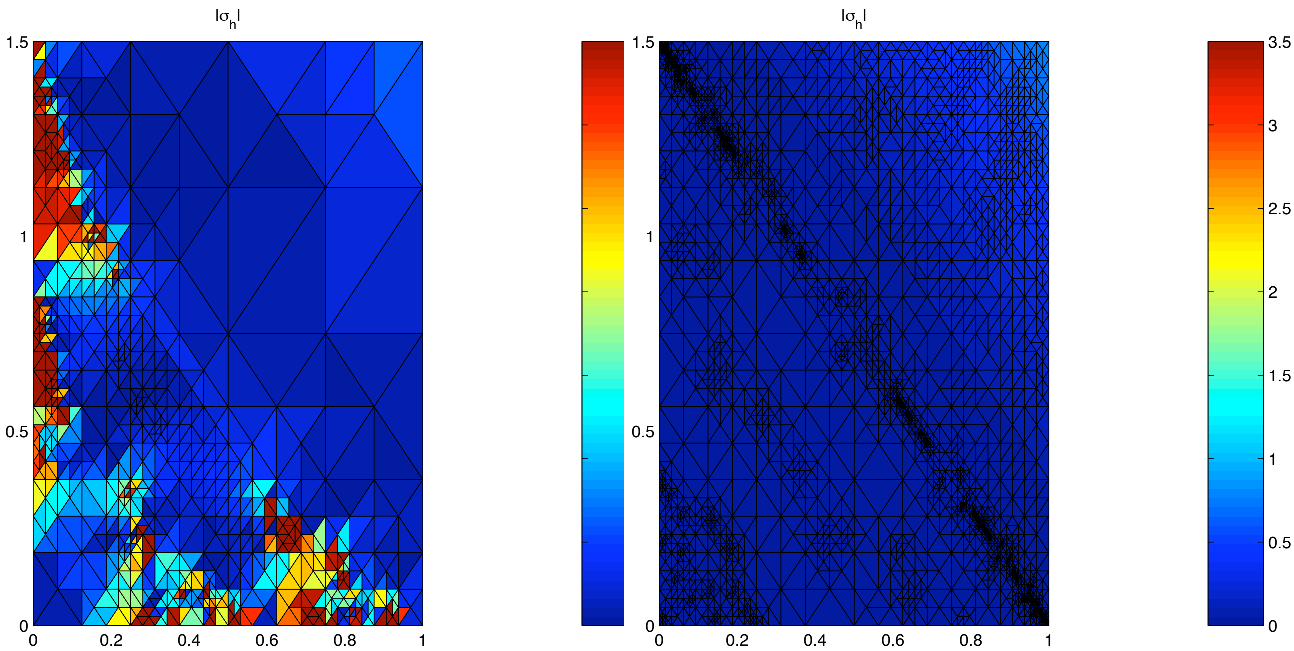
$$\Lambda(F) = \frac{\ell_1}{\ell_1+\ell_2}$$

Volume Fractions from (M_h) and from (Q_h)



$\mathcal{T}_{10}\,(N = 473)$ $\mathcal{T}_{15}\,(N = 2485)$

Stress Field $\sigma_h = DW(\nabla u_h) \to \sigma = DW^{**}(\nabla u)$
[Ball, Kirchheim & Kristensen, Calc Var 2000]



$\mathcal{T}_{10}\,(N = 473)$ $\mathcal{T}_{15}\,(N = 2485)$

Cont. 2D Benchmark Example

Adaptive Mesh Refining

1. Start with a coarse mesh $\mathcal{T}_0$, $k = 0$.
2. Solve (Q_{h_k}) with N unknowns.
3. Compute all η_T w.r.t. $\mathcal{T}_k$, i.e., η_T from a posteriori error estimate or for averaging scheme

$$\begin{aligned} \eta_T &:= \|\sigma_h - A\,\sigma_h\|_{L^{4/3}(T)}, \\ (A\,\sigma_h)(z) &:= \fint_{\omega_z} \sigma_h(x)\,dx, \\ \omega_z &:= \bigcup \Big\{ T \in \mathcal{T} \,\Big|\, z \in T \Big\}. \end{aligned}$$

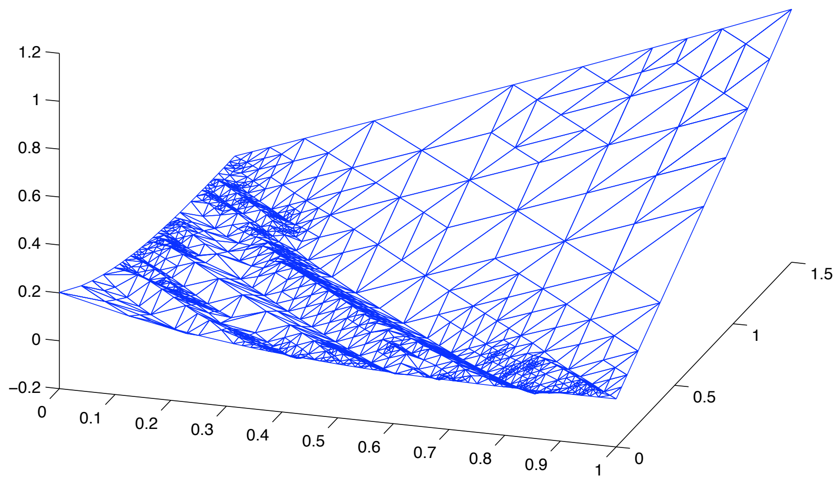
4. Mark T if

$$\eta_T \geq \frac{1}{2} \max_{K \in \mathcal{T}_k} \eta_K .$$

5. Refine other triangles for regular triangulation $\mathcal{T}_{k+1}$, update k, go to 2.

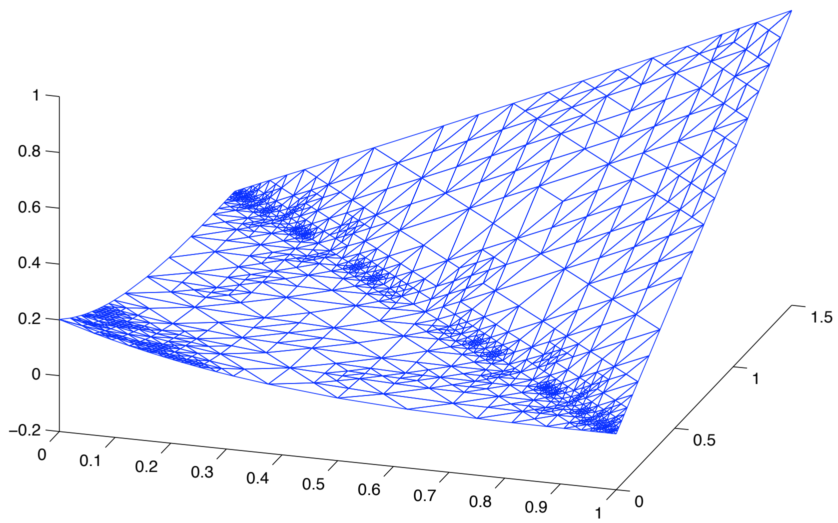
Cont. 2D Benchmark Example
η_Z-Adaptive Mesh Refining

Deformation on η_Z Adapted Mesh for (M_h)



$\mathcal{T}_{15}\,(N = 998)$

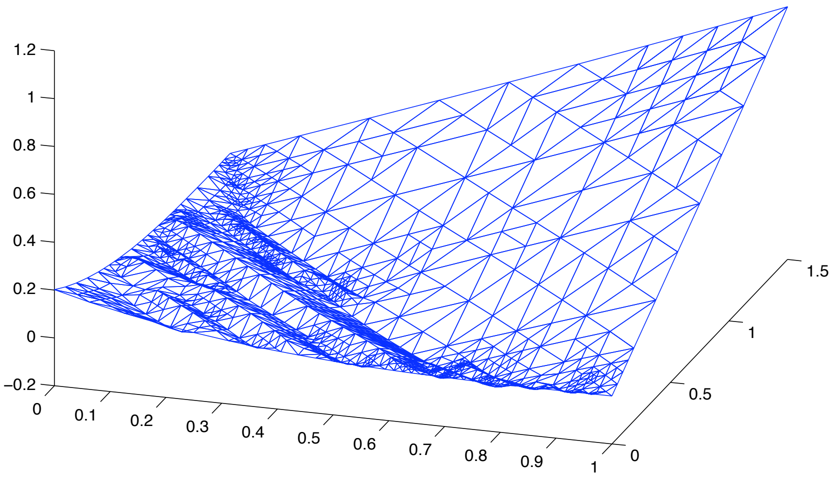
Deformation on η_Z Adapted Meshes for (Q_h)



$\mathcal{T}_{15}\,(N = 918)$

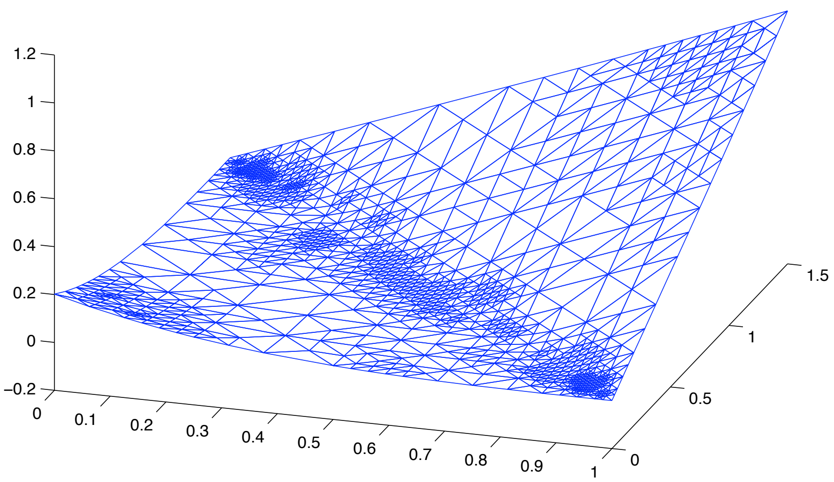
Cont. 2D Benchmark Example
η_R-Adaptive Mesh Refining

Deformation on η_R Adapted Meshes for (M_h)



$\mathcal{T}_{14}\,(N = 936)$

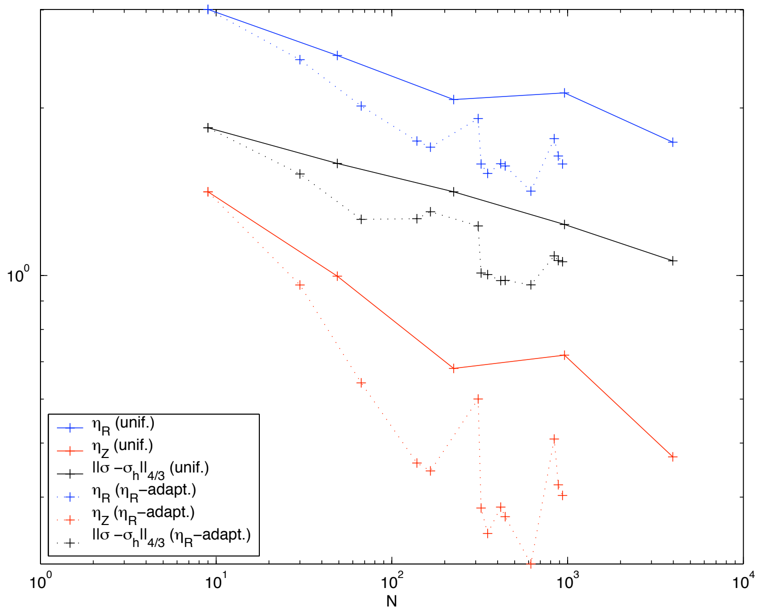
Deformation on η_R Adapted Meshes for (Q_h)



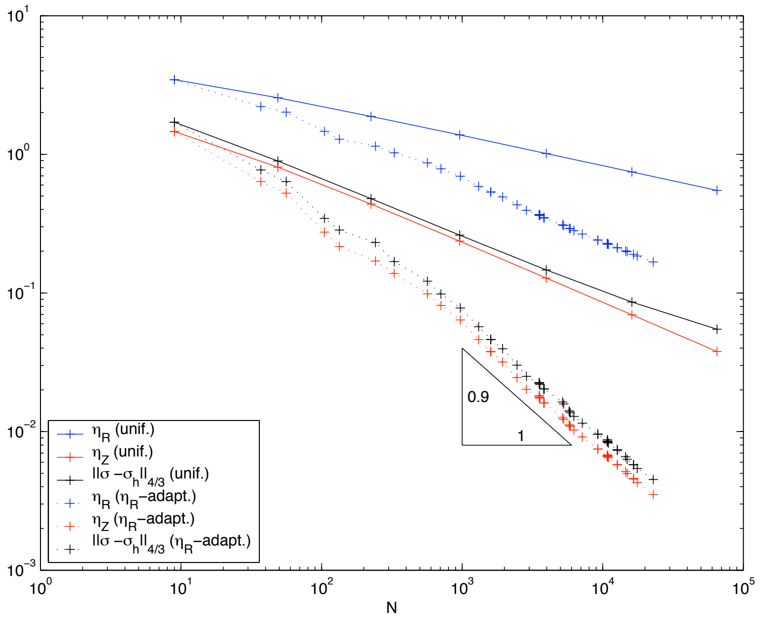
$\mathcal{T}_{10}\,(N = 971)$

Cont. 2D Benchmark Example
A Posteriori Error Estimators

Error estimators for (M_h) on uniform and η_R adapted meshes



Error estimators for (Q_h) on uniform and η_R adapted meshes



Conclusions

- Reliability–Efficiency–Gap
- Averaging error estimator extremely accurate
- Adaptive mesh refining more effective
- Justification of adaptive (M_h) unclear
- Effective solvers for (M_h) needed!