Stabilization for convergence of iterative solvers

- In general, $E^c(u)$ with multipla minima and D^2E^c positive semidefinite
- Need for stabilization $\Rightarrow E^c_\gamma(v) = E^c(v) + \gamma\|\nabla v\|^2_{L^2(\Omega)}$
- Global convergence for the damped Quasi-Newton scheme

For any $u,\ u_0 \in V$,

(a) Input u_0, $j := 0$

(b) Solve $a(u_j - u_{j+1}, v) = (DE^c_\gamma(u_j), v)$

(c) Set $j := j + 1$ and goto (a)

with a elliptic and continuous, E^c_γ uniformly convex, and DE^c_γ Lipschitz.

Theorem [Bartels *et al* 04] There holds

$$\boxed{E^c_\gamma(u_{j+1}) - E^c_\gamma(u) \leq \alpha(E^c_\gamma(u_j) - E^c_\gamma(u)) \text{ and } \|u - u_j\|^2 \leq \beta(E^c_\gamma(u_j) - E^c_\gamma(u_{j+1}))}$$

with $\alpha < 1$ provided some side restrictions are met.

- For u minimizer of E^c_γ in $V \Rightarrow$

$$\lim E^c_\gamma(u_j) = E^c_\gamma(u) \quad \lim\|u - u_j\| = 0$$

Stabilization for strong convergence of gradients

- (u_h) infimizing sequence for $E^c := \int_\Omega W^{**}(Du)\,dx + \mathcal{L}(u)$ such that

$$u_h \rightharpoonup u \text{ in } W^{1,p} \text{ with } u_h \to u \text{ in } L^p \text{ but } Du_h \not\to Du \text{ in } L^p$$

- For each $h > 0$, let u_h minimize $E^c + J_h$ over $\mathcal{A}_h \Rightarrow Du_h \to Du$ in L^p (for some J_h)
- Sketch of the proof for $p = 2,\ u \in H^2(\Omega; \mathbb{R}^m)$ with
  $J_h(v) = h^2/2\|\Delta v\|^2_{L^2}$ and $D\mathcal{L}$ uniformly monotone.

  Monotony of $D\mathcal{L}$, Galerkin orthogonality, and standard arguments give
  $$h^2\|\Delta e\|^2_{L^2} + \|e\|^2_{L^2} \lesssim h^2 \tag{1}$$
  Since $e = 0$ on $\partial\Omega$, Green formula gives
  $$\|De\|^2_{L^2} = -\int_\Omega e \cdot \Delta e\,dx$$
  Use Cauchy, Young inequalities and (1)
  $$\|De\|^2_{L^2} \le \|e\|_{L^2}\|\Delta e\|_{L^2} \le h/2\|\Delta e\|^2_{L^2} + 1/(2h)\|e\|^2_{L^2} \lesssim h$$
- Proof in [Bartels *et al.* 04] for the following stabilization terms for standard low-order FEM
  - $J_h(v_h) = \sum_{E\in\mathcal{E}_\Omega} h_E^\gamma \int_E |[Dv_h]|^2\,ds$
  - $J_h(v_h) = \int_\Omega h_{\mathcal{T}}^{\gamma-1}|Dv_h - \mathcal{A}Dv_h|^2\,dx$
  - $J_h(v_h) = h^\gamma \int_\Omega |Dv_h|^2\,dx$