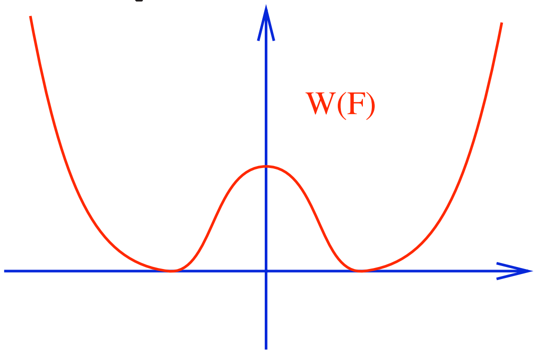
1D Examples

[Bolza, Young, etc.]

Example A:



Energy density W,
$W(F) = (F^2 - 1)^2$
for $F \in \mathbb{R}$.

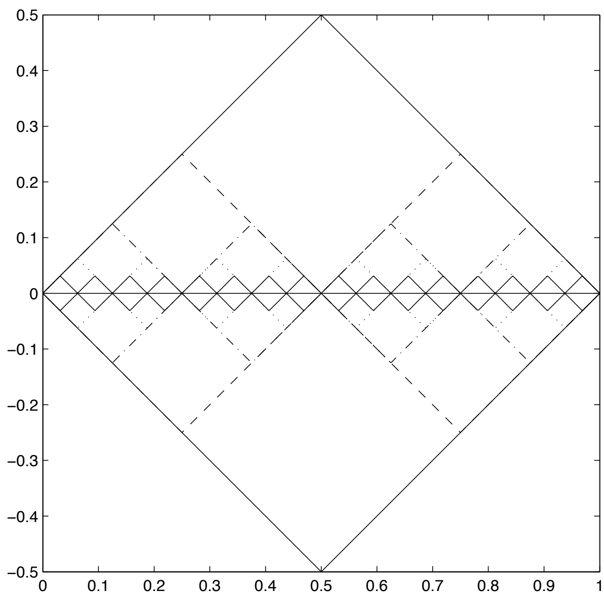
(M) Minimize

$$E(u) := \int_0^1 W(u')\,dx \quad \text{over } u \in \mathcal{A} \subseteq W^{1,4}(0,1).$$

Theorem A.1: There exist infinitely many minimizers

$$u \in \mathcal{A} := W_0^{1,4}(0,1)$$

characterised by $u' = \pm 1$.



Theorem A.2: There exists a unique minimizer u of (M) in
$\mathcal{A} := \left\{ u \in W^{1,4}(0,1) \,\middle|\, u(0) = 0, u(1) = 2 \right\}$,
namely $u(x) = 2x$ for $x \in (0,1)$.

Cont. 1D Examples

Example B: $W(F)$ as before.
(M) Minimize

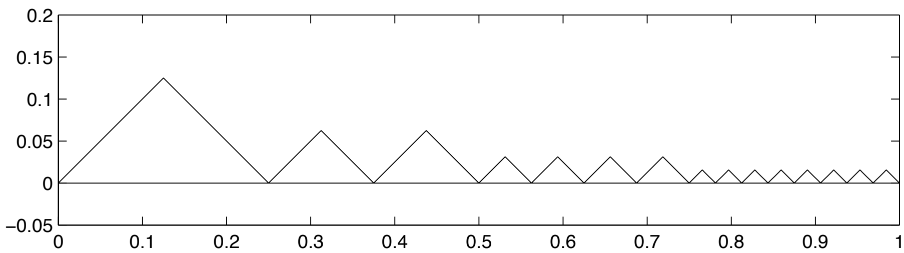
$$E(u) := \int_0^1 W(u'(x))\,dx + \int_0^1 u(x)^2\,dx \quad \text{over } \mathcal{A}.$$

Theorem B: Let $W_0^{1,4}(0,1) \subseteq \mathcal{A} \subseteq W^{1,4}(0,1)$. Then there exists no minimizer of (M).

Proof: (a) $\inf E(\mathcal{A}) \geq 0$.

(b) $\inf E(\mathcal{A}) = 0$.
In fact, for each $\varepsilon > 0$, design $u_\varepsilon \in \mathcal{A}$ with $u'_\varepsilon = \pm 1$ for a.e. $x \in \Omega$ and $|u_\varepsilon(x)| < \varepsilon$.

$0 < E(u_\varepsilon) < \varepsilon^2$



Then, for $\varepsilon \to 0$, $\inf E(\mathcal{A}) \leq 0$. $\square$

(c) Infimal energy $\inf E(\mathcal{A}) \leq 0$ is not attained.
In fact, given $u \in W^{1,4}(0,1)$ with $E(u) = 0$ yields

$$\int_0^1 u(x)^2\,dx = 0 \Rightarrow u = 0 \text{ a.e.} \Rightarrow W(u') = W(0) = 1.$$

Hence $E(u) = 1$. This is a contradiction. $\square$

Cont. 1D Examples

General Observations

• Existence and uniqueness of minimizers depend on boundary conditions in $\mathcal{A}$ and on low-order terms in E.

• Infimizing sequences (u_j) do exist in $\mathcal{A}$, are bounded and weakly convergent,

$$u_j \rightharpoonup u \quad \text{in } W^{1,p}(\Omega)$$

converge in weaker norms,

$$u_j \to u \quad \text{in } L^q(\Omega),$$

but are <u>not</u> strongly convergent in $W^{1,p}(\Omega)$,

$$u_j \not\to u \quad \text{in } W^{1,p}(\Omega).$$

• Weak limit u describes macroscopic displacement, $D\,u$ models averaged strain.

• Microscopic oszillations of $D\,u_j$ are <u>not</u> passed to average $D\,u$; their limit is a measure rather than a Sobolev function.

Cont. 1D Examples

Fundamental Theorem of Young Measures (Ball): If (u_k) bdd in $W^{1,p}(\Omega)^m$ then a subsequence $(D\,u_j)$ generates gradient Young Measure (GYM) $(\nu_x : x \in \Omega)$, i.e. for a.e. $x \in \Omega$, ν_x is probability measure, and, given sample $\omega \subset \Omega$ and a function $f \in C_0(\mathbb{R}^{m\times n})$,

$$\lim_{j\to\infty} \int_\omega f(Du_j)\,dx = \int_\omega \langle \nu_x, f\rangle\,dx.$$

Example: In 1D Example B $(m = n = 1)$

$$\begin{aligned} \nu_x &= 1/2(\delta_{-1} + \delta_{+1}), \text{ i.e.} \\ \langle \nu_x, f\rangle &= 1/2\,f(-1) + 1/2\,f(+1). \end{aligned}$$

Remark:

$$\begin{aligned} \nu_x &:= \lim_{\delta\to 0}\lim_{k\to\infty} \nu^k_{x,\delta} \text{ (weak-}*\text{ in measure)}, \\ \nu^k_{x,\delta} &:= \text{ distribution of } u_k'(y), \text{ when y from} \\ & \qquad B(x,\delta) \text{ uniformly at random.} \end{aligned}$$

Cont. 1D Examples

[Tartar's Broken Extremal Example]

Example C (Tartar): $W(F)$ as before and
$f(x) = -3/128\,(x-1/2)^5 - (x-1/2)^3/3$,

$$\mathcal{A} = \{v \in W^{1,4}(0,1) : v(0) = \frac{521}{12288}, v(1) = \frac{97}{192}\}$$

Minimize

$$E(u) := \int_0^1 \left(W(u'(x)) + |u-f|^2 \right) dx \text{ over } \mathcal{A}$$

(M) has no solution, (Q) has solution

$$u(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq 1/2, \\ x - 1/2 + (x-1/2)^3/24 & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

with oszilations (microstructures) in $(0,1/2)$ and classical solution (i.e. $\nu_x = \delta_{u'(x)}$) in $(1/2,1)$.

Numerical examples in 1D involve a shifted version where $u(x)$ is replaced by $u(x+\pi/100)$ etc.