Hysteresis in Two-Phase Solids

Numerical Analysis Based on Variational Model due to Mielke, Theil, Levitas

A mixture χ is a N-tuple of characteristic functions

$$\chi = (\chi^{(1)}, \ldots, \chi^{(N)}) \in \Xi \subset L^\infty(\Omega; \{0,1\}^N), i.e.$$

$\chi^{(j)}(x) = 1$ if the phase j is found at material point x and $\chi^{(j)}(x) = 0$ otherwise. Below N=2.

Total free energy of arrangement $\chi \in \Xi$ at time t

$$\mathcal{E}(t,\chi) := \inf_{v \in V} \left\{ \int_\Omega \sum_{j=1}^N \chi^{(j)} W_j(\varepsilon(v))\, dx + \mathcal{L}(t,v) \right\}$$

Re-arrangement of the phase configuration χ to the actual arrangement ζ dissipates energy

$$\mathcal{D}(\chi,\zeta) := \int_\Omega D(\chi,\zeta)\, dx := \int_\Omega \kappa\, |\chi - \zeta|\, dx.$$

Maximum dissipation principle and stability concept result in Incremental Problem.

Cont. Hysteresis in Two-Phase Solids

Incremental Problem

Given initial phase arrangement χ_0 and time-discretization $0 < t_1 < \cdots < t_{j-1} < t_j < \cdots < t_J = T$, seek minimizer $\chi_j \in \Xi$, $j = 1, 2, \ldots, J$, of

$$\text{(IP)} \qquad \left\{\mathcal{E}(t_j, \zeta) + \mathcal{D}(\chi_{j-1}, \zeta)\right\} \quad \text{over } \zeta \in \Xi.$$

Nonconvexity of Ξ results in nonattainment and generalized solutions $\theta_j \in \Xi^{**}$: Seek minimizer $\chi_j \in \Xi$ of

$$\text{(RIP)} \quad \text{minimize } \mathcal{I}^{rlx}(t, \theta_{j-1}, \zeta) \text{ over } \zeta \in \Xi^{**},$$

where

$$\begin{aligned} \mathcal{I}^{rlx}(t, \eta, \theta) \; &= \; \inf_{v \in V} \left\{ \int_\Omega W_\theta^{qc}(Dv)\, dx \right. \\ & \qquad \left. + \mathcal{L}(t, v) \right\} + \int_\Omega \kappa\, |\eta - \theta|\, dx. \end{aligned}$$

Cont. Hysteresis in Two-Phase Solids

Fully Discrete Algorithm

(i) Compute a minimizer $u_j \in V_h$ of

$$\int_\Omega \Big(W_2(\varepsilon(v_h)) + 2\gamma H(\vartheta_{j-1}, \ell(\varepsilon(v_h))) \Big)\, dx + L(t_j, v_h) \qquad \text{over } v_h \in V_h.$$

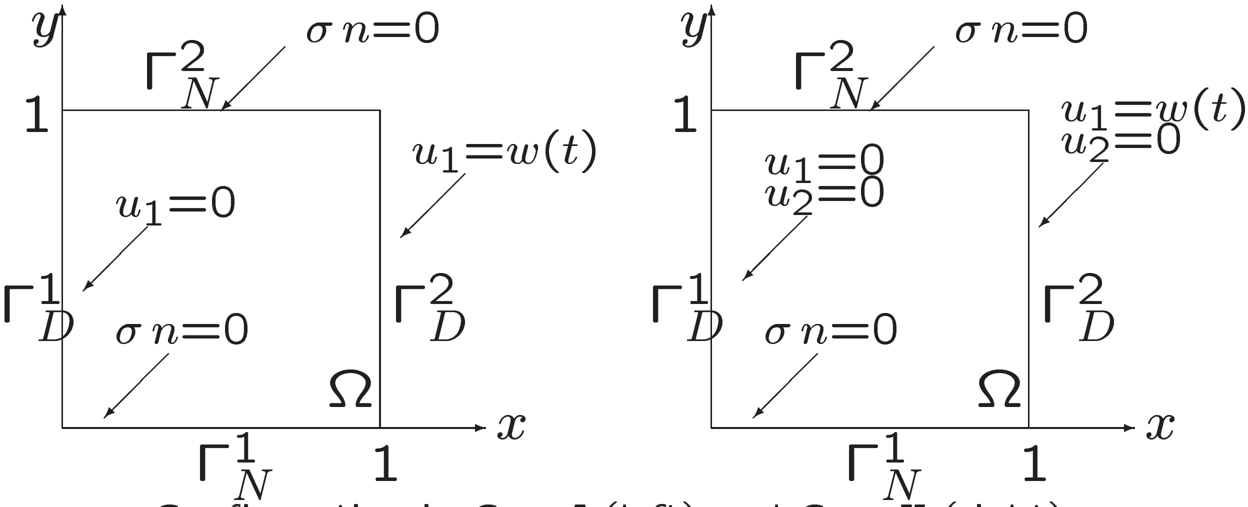
(ii) Compute new arrangements $\theta_j =: (\vartheta_j, 1-\vartheta_j)$

$$\vartheta_j = M(\vartheta_{j-1}, \ell(\varepsilon(u_j))).$$

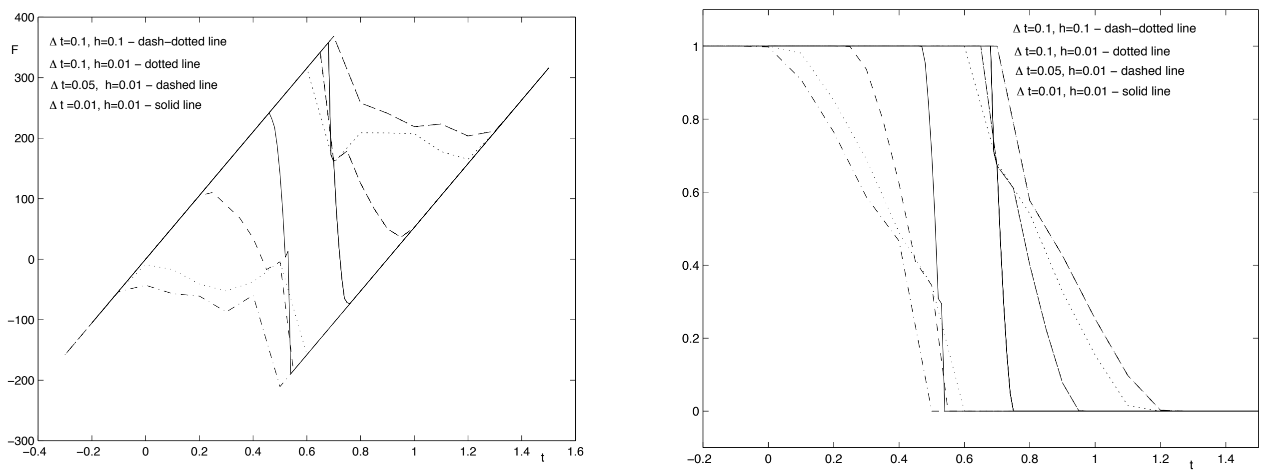
Explicit formulae include, for each $E \in \mathbb{R}^{d\times d}_{sym}$ and real r and s,

$$\begin{aligned}
W_j(E) &= \langle E - E_j, \mathbb{C}(E - E_j)\rangle + W_j^0,\\
\ell(E) &= \frac{1}{2\gamma}(W_2(E) - W_1(E)) + \frac{1}{2},\\
H(r,s) &= \begin{cases} \frac{\kappa}{2\gamma} r & \text{if } s \le -\frac{\kappa}{2\gamma},\\ \frac{\kappa}{2\gamma} r - \frac{1}{2}(s + \frac{\kappa}{2\gamma})^2 & \text{if } -\frac{\kappa}{2\gamma} \le s \le r - \frac{\kappa}{2\gamma},\\ \frac{1}{2} r^2 - rs & \text{if } r - \frac{\kappa}{2\gamma} \le s \le r + \frac{\kappa}{2\gamma},\\ -\frac{\kappa}{2\gamma} r - \frac{1}{2}(s - \frac{\kappa}{2\gamma})^2 & \text{if } r + \frac{\kappa}{2\gamma} \le s \le 1 + \frac{\kappa}{2\gamma},\\ \frac{\kappa}{2\gamma}(1-r) + \frac{1}{2} - s & \text{if } s \ge 1 + \frac{\kappa}{2\gamma}, \end{cases}\\
M(r,s) &= \begin{cases} 0 & \text{if } s \le -\frac{\kappa}{2\gamma},\\ s + \frac{\kappa}{2\gamma} & \text{if } -\frac{\kappa}{2\gamma} \le s \le r - \frac{\kappa}{2\gamma},\\ r & \text{if } r - \frac{\kappa}{2\gamma} \le s \le r + \frac{\kappa}{2\gamma},\\ s - \frac{\kappa}{2\gamma} & \text{if } r + \frac{\kappa}{2\gamma} \le s \le 1 + \frac{\kappa}{2\gamma},\\ 1 & \text{if } s \ge 1 + \frac{\kappa}{2\gamma}. \end{cases}
\end{aligned}$$

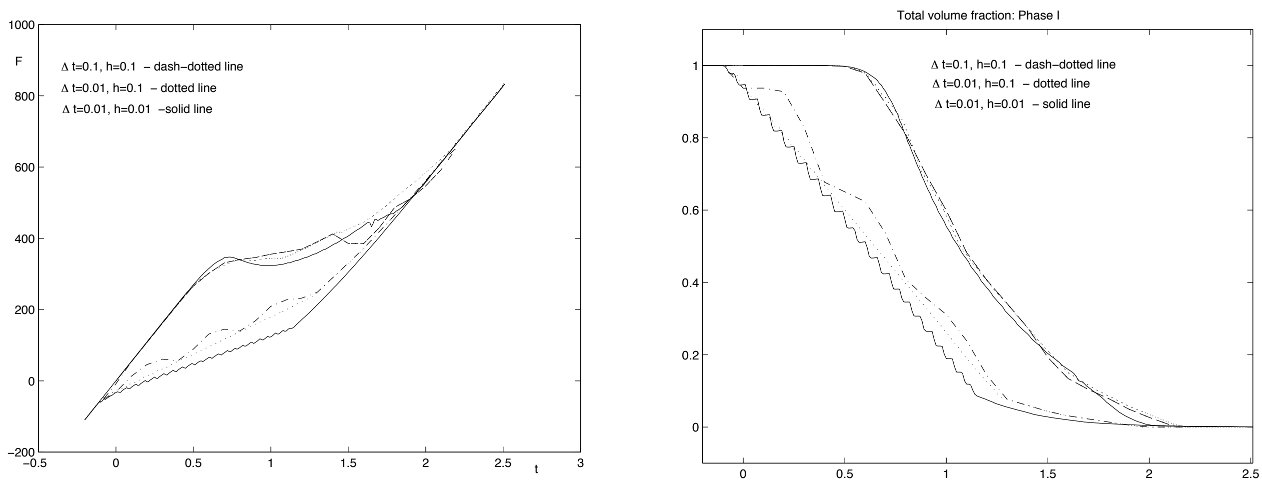
Cont. Hysteresis in Two-Phase Solids
Numerical Results



Configuration in Case I (left) and Case II (right)



Load-displacement diagrams (left) and global volume fractions along the loop (right) for Material B in Case I.



Load-displacement diagrams (left) and global volume fractions along the loop (right) for Material B in Case II.