LEVI DECOMPOSITION FOR SMOOTH POISSON STRUCTURES

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Abstract. We prove the existence of a local smooth Levi decomposition for smooth Poisson structures and Lie algebroids near a singular point. This Levi decomposition is a kind of normal form or partial linearization, which was established in the formal case by Wade [Wad97] and in the analytic case by the second author [Zun03]. In particular, in the case of smooth Poisson structures with a compact semisimple linear part, we recover Conn's smooth linearization theorem [Con85], and in the case of smooth Lie algebroids with a compact semisimple isotropy Lie algebra, our Levi decomposition result gives a positive answer to a conjecture of Weinstein [Wei00] on the smooth linearization of such Lie algebroids. In the appendix of this paper, we show an abstract Nash-Moser normal form theorem, which generalizes our Levi decomposition result, and which may be helpful in the study of other smooth normal form problems.

1. Introduction

In the study of Poisson structures, in particular their local normal forms, one is led naturally to the problem of finding a semisimple subalgebra of the (infinite-dimensional) Lie algebra of functions under the Poisson bracket: such a subalgebra can be viewed as a semisimple Lie algebra of symmetry for the corresponding Poisson structure, and by linearizing it one get a partial linearization of the Poisson structure, which in some case leads to a full linearization. We call it the Levi decomposition problem, because it is an infinite-dimensional analog of the classical Levi decomposition for finite-dimensional Lie algebras.

Recall that, if $\mathfrak{l}$ is a finite-dimensional Lie algebra and $\mathfrak{r}$ is the solvable radical of $\mathfrak{l}$, then there is a semisimple subalgebra $\mathfrak{g}$ of $\mathfrak{l}$ such that $\mathfrak{l}$ is a semi-direct product of $\mathfrak{g}$ with $\mathfrak{r}$: $\mathfrak{l} = \mathfrak{g} \ltimes \mathfrak{r}$. This semidirect product is called the Levi decomposition of $\mathfrak{l}$, and $\mathfrak{g}$ is called the Levi factor of $\mathfrak{l}$. The classical theorem of Levi and Malcev says that $\mathfrak{g}$ exists and is unique up to conjugations in $\mathfrak{l}$, see, e.g., [Bou60].

The Levi-Malcev theorem does not hold for infinite dimensional algebras in general. But a formal version of it holds for filtered pro-finite Lie algebras: if $\mathfrak{L} \supset \mathfrak{L}_1 \supset ... \supset \mathfrak{L}_i \supset ...$ where $\mathfrak{L}_i$ are ideals of a Lie algebra $\mathfrak{L}$ such that $[\mathfrak{L}_i, \mathfrak{L}_j] \subset \mathfrak{L}_{i+j}$ and $\dim \mathfrak{L}/\mathfrak{L}_i$ are finite, then the projective limit $\lim_{i\to\infty} \mathfrak{L}/\mathfrak{L}_i$ admits a Levi factor (which is isomorphic to the Levi factor for $\mathfrak{L}/\mathfrak{L}_1$). The proof

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of this formal infinite dimensional Levi decomposition is absolutely similar to the proof of the classical Levi-Malcev theorem. And the formal Levi decomposition for singular foliations [Cer79] and Poisson structures [Wad97] are instances of this infinite dimensional formal Levi decomposition.

In [Zun03], the second author obtained the local analytic Levi decomposition theorem for analytic Poisson structures which vanish at a point. This theorem generalizes Conn's linearization theorem for analytic Poisson structure with a semisimple linear part [Con84], and is at the base of some new analytic linearization results for Poisson structures and Lie algebroids [Zun03, DZ02].

The aim of this paper is to establish the local smooth Levi decomposition theorem for smooth Poisson structures and Lie algebroids which vanish at a point. Our main theorem (Theorem 1.1) is a generalization of Conn's smooth linearization theorem [Con85] for Poisson structures with a compact semisimple linear part, and provides a local smooth semi-linearization for any smooth Poisson structure whose linear part (when considered as a Lie algebra) contains a compact semisimple subalgebra.

Let Π be a C^p Poisson structure ($p \in \mathbb{N} \cup \{\infty\}$) in a neighborhood of 0 in $\mathbb{R}^n$, which vanishes at the origin. Denote by $\mathfrak{l}$ the n-dimensional Lie algebra of linear functions in $\mathbb{R}^n$ under the Lie-Poisson bracket Π_1 which is the linear part of Π at 0, and by $\mathfrak{g}$ a compact semisimple subalgebra of $\mathfrak{l}$. (Without loss of generality one can assume that $\mathfrak{g}$ is a maximal compact semisimple subalgebra of $\mathfrak{l}$, and we will call $\mathfrak{g}$ a *compact Levi factor* of $\mathfrak{l}$). Denote by $(x_1, ..., x_m, y_1, ..., y_{n-m})$ a linear basis of $\mathfrak{l}$, such that $x_1, ..., x_m$ span $\mathfrak{g}$ ($\dim \mathfrak{g} = m$), and $y_1, ..., y_{n-m}$ span a linear complement $\mathfrak{r}$ of $\mathfrak{g}$ in $\mathfrak{l}$ which is invariant under the adjoint action of $\mathfrak{g}$. Denote by c_{ij}^k and a_{ij}^k the structural constants of $\mathfrak{g}$ and of the action of $\mathfrak{g}$ on $\mathfrak{r}$ respectively: $[x_i, x_j] = \sum_k c_{ij}^k x_k$ and $[x_i, y_j] = \sum_k a_{ij}^k y_k$. We say that Π admits a local C^q-*smooth Levi decomposition with respect to* $\mathfrak{g}$ if there exists a local C^q-smooth system of coordinates $(x_1^\infty, ..., x_m^\infty, y_1^\infty, ..., y_{n-m}^\infty)$, with $x_i^\infty = x_i +$ higher order terms and $y_i^\infty = y_i +$ higher order terms, such that in this coordinate system the Poisson structure has the form

$$\Pi = \frac{1}{2}\Big[\sum c_{ij}^k x_k^\infty \frac{\partial}{\partial x_i^\infty} \wedge \frac{\partial}{\partial x_j^\infty} + \sum a_{ij}^k y_k^\infty \frac{\partial}{\partial x_i^\infty} \wedge \frac{\partial}{\partial y_j^\infty} + \sum F_{ij} \frac{\partial}{\partial y_i^\infty} \wedge \frac{\partial}{\partial y_j^\infty}\Big] \tag{1.1}$$

where F_{ij} are some functions in a neighborhood of 0 in $\mathbb{R}^n$. In other words, we have

$$\{x_i^\infty, x_j^\infty\} = \sum c_{ij}^k x_k^\infty \text{ and } \{x_i^\infty, y_j^\infty\} = \sum a_{ij}^k y_k^\infty, \tag{1.2}$$

i.e. the functions $x_1^\infty, \dots, x_m^\infty$ span a compact Levi factor (isomorphic to $\mathfrak{g}$) and their Hamiltonian vector fields $X_{x_1^\infty}, \dots, X_{x_m^\infty}$ are linear in the coordinate system $(x_1^\infty, ..., x_m^\infty, y_1^\infty, ..., y_{n-m}^\infty)$.

Theorem 1.1. *There exists a positive integer l (which depends only on the dimension n) such that any C^{2q-1}-smooth Poisson structure Π in a neighborhood of 0 in $\mathbb{R}^n$ which vanishes at 0, where $q \in \mathbb{N} \cup \{\infty\}$, $q \geq l$, admits a local C^q-smooth Levi decomposition (with respect to any compact semisimple Lie subalgebra $\mathfrak{g}$ of the Lie algebra $\mathfrak{l}$ which corresponds to the linear part of Π at 0).*

A particular case of the above theorem is when $\mathfrak{g} = \mathfrak{l}$, i.e. when the linear part of Π is compact semisimple. In this case a local Levi decomposition is nothing but a local linearization of the Poisson structure, and we recover the smooth linearization theorem of Conn [Con85] for a smooth Poisson structure with a compact semisimple linear part. When $\mathfrak{l} = \mathfrak{g} \oplus \mathbb{R}$, a Levi decomposition is still a linearization of Π. In general, one may consider a Levi decomposition (we also call it a *Levi normal form*, see [Zun03]) as a partial linearization of Π.

Similarly to the analytic case [Zun03], an analogue of Theorem 1.1 holds for smooth Lie algebroids:

Theorem 1.2. *Let A be a local N-dimensional C^{2q-1}-smooth Lie algebroid over $(\mathbb{R}^n, 0)$ with the anchor map $\# : A \to T\mathbb{R}^n$, such that $\#a = 0$ for any $a \in A_0$, the fiber of A over point 0, where $q = \infty$ or is a natural number which is large enough ($q \geq l$, where l is a natural number which depends only on N and n). Denote by $\mathfrak{l}$ the N-dimensional Lie algebra in the linear part of A at 0 (i.e. the isotropy algebra of A at 0), and by $\mathfrak{g}$ a compact semisimple Lie subalgebra of $\mathfrak{l}$. Then there exists a local C^q-smooth system of coordinates $(x_1^\infty, ..., x_n^\infty)$ of $(\mathbb{R}^n, 0)$, and a local C^q-smooth basis of sections $(s_1^\infty, s_2^\infty, ..., s_m^\infty, v_1^\infty, ..., v_{N-m}^\infty)$ of A, where $m = \dim \mathfrak{g}$, such that we have:*

$$\begin{array}{l} [s_i^\infty, s_j^\infty] = \sum_k c_{ij}^k s_k^\infty \ , \\ [s_i^\infty, v_j^\infty] = \sum_k a_{ij}^k v_k^\infty \ , \\ \# s_i^\infty = \sum_{j,k} b_{ij}^k x_k^\infty \partial / \partial x_j^\infty \ , \end{array} \tag{1.3}$$

where $c_{ij}^k, a_{ij}^k, b_{ij}^k$ are constants, with c_{ij}^k being the structural constants of the compact semisimple Lie algebra $\mathfrak{g}$.

The meaning of the above theorem is that the algebra of sections of A admits a Levi factor (Lie isomorphic to $\mathfrak{g}$), spanned by $s_1^\infty, s_2^\infty, ..., s_m^\infty$, whose action can be linearized. Theorem 1.2 is called the local smooth Levi decomposition theorem for smooth Lie algebroids. As a particular case of this theorem, we obtain the following result, conjectured by A. Weinstein [Wei00]: any smooth Lie algebroid whose anchor vanishes at a point and whose corresponding isotropy Lie algebra at that point is compact semisimple is locally smoothly linearizable.

Remark that, compared to the analytic case, in the smooth case considered in [Con85] and in the present paper we need the additional condition of compactness on our semisimple Lie (sub)algebra $\mathfrak{g}$. In a sense, this compactness condition is necessary, due to the following result of Weinstein [Wei87]: any real semisimple Lie algebra of real rank at least 2 is smoothly degenerate, i.e. there is a smoothly nonlinearizable Poisson structure with a linear part corresponding to it.

We hope that the results of this paper will be useful for finding new smoothly nondegenerate Lie algebras (and Lie algebroids) in the sense of Weinstein [Wei83]. In particular, our smooth Levi decomposition is one of the main steps in the study of smooth linearizability of Poisson structures whose linear part corresponds to a real semisimple Lie algebra of real rank 1 (this case was left out by Weinstein [Wei87]). This problem will be studied in a separate work.

Our proof of Theorem 1.1 is based on the Nash-Moser fast convergence method (see, e.g., [Ham82]) applied to Fréchet spaces of smooth functions and vector fields. In particular, our algorithm for constructing a convergent sequence of smooth coordinate transformations, which is a combination of smoothing operators with the algorithm in [Zun03] for the analytic case, is inspired by Hamilton's "near projections" in his proof of the so-called Nash-Moser theorem for exact sequences [Ham77].

Besides smoothing operators for tame Fréchet spaces, we will need homotopy operators for certain Chevalley-Eilenberg complexes with vanishing first and second cohomologies. The homotopy operators and the smoothing operators are both already present in Conn's paper [Con85], and in a sense the present paper is a further development of [Con85] and follows more or less the same organization.

Using the fact that Lie algebroids can be viewed as fiber-wise linear Poisson structures, one can immediately deduce Theorem 1.2 from the proof given below of Theorem 1.1, simply by restricting some functional spaces, in a way absolutely similar to the analytic case (see Section 6 of [Zun03]). That's why we will mention only briefly the proof of Theorem 1.2, after the full proof Theorem 1.1.

The rest of this paper, except the appendix, is devoted mainly to the proof of Theorem 1.1, and is organized as follows. In Sections 2 and 3 we write down important inequalities involving homotopy operators and smoothing operators that will be used. Then in Section 4 we present our algorithm for constructing the required new systems of coordinates, and give a proof of Theorem 1.1, modulo some technical lemmas. These lemmas are proved in Section 5. In Section 6 we briefly explain how to modify (in an obvious way) the proof of Theorem 1.1 to get a proof of Theorem 1.2.

In the appendix, we present an abstract Nash-Moser smooth normal form theorem, which generalizes Theorems 1.1 and 1.2. We hope that this abstract normal form theorem can be used or easily adapted for the study of other smooth normal form problems (of functions, dynamical systems, various geometric structures, etc.).

2. Homotopy operators

Similarly to the analytic case [Con84, Zun03], in order to prove Theorem 1.1, we will need a normed version of Whitehead's lemma about the vanishing of cohomology of the semisimple algebra $\mathfrak{g}$, with respect to certain orthogonal modules of $\mathfrak{g}$ constructed below. Our modules will be spaces of real functions or vector fields, equipped with Sobolev norms, and the action of $\mathfrak{g}$ will preserve these norms.

Consider a Lie algebra $\mathfrak{l}$ of dimension n together with a compact semisimple Lie subalgebra $\mathfrak{g} \subset \mathfrak{l}$ of dimension m. (Our Poisson structure will live in a neighborhood of 0 in the dual space $\mathbb{R}^n = \mathfrak{l}^*$ of $\mathfrak{l}$). Denote by G the simply-connected compact semisimple Lie group whose Lie algebra is $\mathfrak{g}$. Then G acts on $\mathbb{R}^n = \mathfrak{l}^*$ by the coadjoint action. Since G is compact, we can fix a linear coordinate system $(x_1, \ldots, x_m, y_1, \ldots, y_{n-m})$ such that the Euclidean metric on $\mathbb{R}^n$ with respect to this coordinate system is invariant under the action of G, and the first m coordinates $(x_1, \ldots, x_m)$ come from $\mathfrak{g}$. In other words, there is a basis $(\xi_1, \ldots, \xi_m)$ of $\mathfrak{g}$ such that each ξ_i, considered as an element of $\mathfrak{l}$ and viewed as a linear function on $\mathfrak{l}^*$, gives rise to the coordinate x_i.

For each positive number $r > 0$, denote by B_r the closed ball of radius r in $\mathbb{R}^n$ centered at 0. The group G (and hence the algebra $\mathfrak{g}$) acts linearly on the space of functions on B_r via its action on B_r: for each function F and element $g \in G$ we put

$$g(F)(z) := F(g^{-1}(z)) = F(Ad^*_{g^{-1}} z) \quad \forall\, z \in B_r. \tag{2.1}$$

For each nonnegative integer $k \geq 0$ and each pair of real-valued functions F_1, F_2 on B_r, we will define the Sobolev inner product of F_1 with F_2 with respect to the Sobolev H_k-norm as follows:

$$\langle F_1, F_2 \rangle^H_{k,r} := \sum_{|\alpha| \leq k} \int_{B_r} \left(\frac{|\alpha|!}{\alpha!} \right) \left(\frac{\partial^{|\alpha|} F_1}{\partial z^\alpha}(z) \right) \left(\frac{\partial^{|\alpha|} F_2}{\partial z^\alpha}(z) \right) d\mu(z), \tag{2.2}$$

where $d\mu$ is the standard Lebesgue measure on $\mathbb{R}^n$. The Sobolev H_k-norm of a function F on B_r is

$$\|F\|^H_{k,r} := \sqrt{\langle F, F \rangle_{k,r}} \ . \tag{2.3}$$

We will denote by $\mathcal{C}_r$ the subspace of the space $C^\infty(B_r)$ of C^∞-smooth real-valued functions on B_r, which consists of functions vanishing at 0 whose first derivatives also vanish at 0. Then the action of G on $\mathcal{C}_r$ defined by (2.1) preserves the Sobolev inner products (2.2).

Denote by $\mathcal{Y}_r$ the space of C^∞-smooth vector fields on B_r of the type

$$u = \sum_{i=1}^{n-m} u_i \partial / \partial y_i \ , \tag{2.4}$$

such that u_i vanish at 0 and their first derivatives also vanish at 0.

Recall that $(\xi_1, \ldots, \xi_m)$ is the basis of $\mathfrak{g}$ which correspond to the coordinates $(x_1, \ldots, x_m)$ on $\mathbb{R}^n = \mathfrak{l}^*$. The space $\mathcal{Y}_r$ is a $\mathfrak{g}$-module under the following action:

$$\xi_i \cdot \sum_j u_j \partial/\partial y_j := \Big[\sum_{jk} c^k_{ij} x_k \frac{\partial}{\partial x_j} + \sum_{jk} a^k_{ij} y_k \frac{\partial}{\partial y_j} \ , \ \sum_j u_j \partial/\partial y_j \Big] \ , \tag{2.5}$$

where $X_i = \sum_{jk} c^k_{ij} x_k \partial/\partial x_j + \sum_{jk} a^k_{ij} y_k \partial/\partial y_j$ are the linear vector fields which generate the linear orthogonal coadjoint action of $\mathfrak{g}$ on $\mathbb{R}^n$.

Equip $\mathcal{Y}_r$ with Sobolev inner products:

$$\langle u, v \rangle^H_{k,r} := \sum_{i=1}^{n-m} \langle u_i, v_i \rangle_{k,r} \ , \tag{2.6}$$

and denote by $\mathcal{Y}^H_{k,r}$ the completion of $\mathcal{Y}_r$ with respect to the corresponding $H_{k,r}$-norm. Then $\mathcal{Y}^H_{k,r}$ is a separable real Hilbert space on which $\mathfrak{g}$ and G act orthogonally.

The following infinite dimensional normed version of Whitehead's lemma is taken from Proposition 2.1 of [Con85]:

Lemma 2.1 (Conn)**.** *For any given positive number r, and $W = \mathcal{C}_r$ or $\mathcal{Y}_r$ with the above action of $\mathfrak{g}$, consider the (truncated) Chevalley-Eilenberg complex*

$$W \xrightarrow{\delta_0} W \otimes \wedge^1 \mathfrak{g}^* \xrightarrow{\delta_1} W \otimes \wedge^2 \mathfrak{g}^* \xrightarrow{\delta_2} W \otimes \wedge^3 \mathfrak{g}^*.$$

Then there is a chain of operators

$$W \xleftarrow{h_0} W \otimes \wedge^1 \mathfrak{g}^* \xleftarrow{h_1} W \otimes \wedge^2 \mathfrak{g}^* \xleftarrow{h_2} W \otimes \wedge^3 \mathfrak{g}^*$$

such that

$$\begin{aligned} \delta_0 \circ h_0 + h_1 \circ \delta_1 &= \mathrm{Id}_{W \otimes \wedge^1 \mathfrak{g}^*} \ , \\ \delta_1 \circ h_1 + h_2 \circ \delta_2 &= \mathrm{Id}_{W \otimes \wedge^2 \mathfrak{g}^*} \ . \end{aligned} \tag{2.7}$$

Moreover, there exist a constant $C > 0$, which is independent of the radius r of B_r, such that

$$\|h_j(u)\|^H_{k,r} \leq C\|u\|^H_{k,r}\,, \quad j = 0, 1, 2 \tag{2.8}$$

for all $k \geq 0$ and $u \in W \otimes \wedge^{j+1}\mathfrak{g}^*$. If u vanishes to an order $l \geq 0$ at the origin, then so does $h_j(u)$.

Proof. Strictly speaking, Conn [Con85] only proved the above lemma in the case when $\mathfrak{g} = \mathfrak{l}$ and for the module $\mathcal{C}_r$, but his proof is quite general and works perfectly in our situation without any modification. Here, we will just recall the main idea of this proof. The action of $\mathfrak{g}$ on W can be extended to the completion $\tilde{W}_k$ of W with respect to the $H_{k,r}$-norm (this is the Sobolev space $H_k(B_r)$ when $W = \mathcal{C}_r$ and $\mathcal{Y}^H_{k,r}$ when $W = \mathcal{Y}_r$). We can decompose $\tilde{W}_k$ as an orthogonal direct sum of $\mathfrak{g}$-modules $\tilde{W}^0_k \oplus \tilde{W}^1_k$ where $\tilde{W}^0_k$ is a trivial $\mathfrak{g}$-module and $\tilde{W}^1_k$ can be decomposed as a Hilbert direct sum of finite dimensional irreducible $\mathfrak{g}$-invariant subspaces. This decomposition induces a decomposition $W = W^0 \oplus W^1$. We can construct a homotopy operator $h'_i : W^0 \otimes \wedge^{i+1}\mathfrak{g} \longrightarrow W^0 \otimes \wedge^i\mathfrak{g}$ by tensoring the identity mapping of W^0 with a homotopy operator for the trivial $\mathfrak{g}$-module $\mathbb{R}$. To construct the homotopy operator h''_i on $W^1 \otimes \wedge^{i+1}\mathfrak{g}$, we can restrict to the case when W^1 is irreducible. Then we define the h''_i by

$$\begin{aligned} h''_0(w) &= \Gamma^{-1} \cdot (\sum_i \xi_k \cdot w(\xi_k)) \\ h''_1(w) &= \sum_i \xi^* \otimes (\Gamma^{-1} \cdot (\sum_k \xi_k \cdot w(\xi_i \wedge \xi_k))) \\ h''_2(w) &= \sum_{ij} \xi^*_i \wedge \xi^*_j \otimes (\Gamma^{-1} \cdot (\sum_k \xi_k \cdot w(\xi_i \wedge \xi_j \wedge \xi_k))) \end{aligned}$$

where $\{\xi^*_i\}$ is the dual basis of $\{\xi_i\}$ and Γ is the Casimir element of $\mathfrak{g}$. Then one can show that

$$\|h''_i(w)\|^H_{k,r} \leq C\|w\|^H_{k,r}$$

with $C = m(\min_{\gamma\in\mathcal{J}} \|\gamma\|)^{-1}$, where $\mathcal{J}$ is the weight lattice of $\mathfrak{g}$. □

For simplicity, in the sequel we will denote the homotopy operators h_j in the above lemma simply by h. Relation (2.7) will be rewritten simply as follows:

$$\mathrm{Id} - \delta \circ h = h \circ \delta\ . \tag{2.9}$$

The meaning of the last equality is as follows: if u is an 1-cocycle or 2-cocycle, then it is also a coboundary, and $h(u)$ is an explicit primitive of u: $\delta(h(u)) = u$. If u is a "near cocycle" then $h(u)$ is also a "near primitive" for u.

For convenience, in the sequel, instead of Sobolev norms, we will use the following absolute forms:

$$\|F\|_{k,r} := \sup_{|\alpha|\leq k} \sup_{z\in B_r} |D^\alpha F(z)| \tag{2.10}$$

for $F \in \mathcal{C}_r$, where the sup runs over all partial derivatives of degree $|\alpha|$ at most k. More generaly, if $F = (F_1, \ldots, F_m)$ is a smooth maping from B_r to $\mathbb{R}^m$ we can define

$$\|F\|_{k,r} := \sup_i \sup_{|\alpha|\leq k} \sup_{z\in B_r} |D^\alpha F_i(z)|\,. \tag{2.11}$$

Similarly, for $u = \sum_{i=1}^{n-m} u_i \partial/\partial y_i \in \mathcal{Y}_r$ we put

$$\|u\|_{k,r} := \sup_i \sup_{|\alpha|\leq k} \sup_{z\in B_r} |D^\alpha u_i(z)| \,. \tag{2.12}$$

The absolute norms $\|.\|_{k,r}$ are related to the Sobolev norms $\|.\|^H_{k,r}$ as follows:

$$\|F\|_{k,r} \leq C_1 \|F\|^H_{k+s,r} \text{ and } \|F\|^H_{k,r} \leq C_2 (n+1)^k \|F\|_{k,r} \tag{2.13}$$

for any F in $\mathcal{C}_r$ or $\mathcal{Y}_r$ and any $k \geq 0$, where $s = [\frac{n}{2}] + 1$ and C_1 and C_2 are positive constants which do not depend on k. A priori, the constants C_1 and C_2 depend continuously on r (and on the dimension n), but later on we will always assume that $1 \leq r \leq 2$, and so may assume C_1 and C_2 to be independent of r. The above first inequality is a version of the classical Sobolev's lemma for Sobolev spaces. The second inequality follows directly from the definitions of the norms. Combining it with Inequality (2.8), we obtain the following estimate for the homotopy operators h with respect to absolute norms:

$$\|h(u)\|_{k,r} \leq C(n+1)^{k+s} \|u\|_{k+s,r} \tag{2.14}$$

for all $k \geq 0$ and $u \in W \otimes \wedge^{j+1}\mathfrak{g}^*$ ($j = 0, 1, 2$), where $W = \mathcal{C}_r$ or $\mathcal{Y}_r$. Here $s = [\frac{n}{2}] + 1$, C is a positive constant which does not depend on k (and on r, provided that $1 \leq r \leq 2$).

3. Smoothing operators and some useful inequalities

We will refer to [Ham82] for the theory of tame Fréchet spaces used here. It is well-known that the space $C^\infty(B_r)$ with absolute norms (2.10) is a tame Fréchet space. Since $\mathcal{C}_r$ is a tame direct summand of $C^\infty(B_r)$, it is also a tame Fréchet space. Similarly, $\mathcal{Y}_r$ with absolute norms (2.12) is a tame Fréchet space as well. In particular, $\mathcal{C}_r$ and $\mathcal{Y}_r$ admit smoothing operators and interpolation inequalities:

For each $t > 1$ there is a linear operator $S(t) = S_r(t)$ from $\mathcal{C}_r$ to itself, with the following properties:

$$\|S(t)F\|_{p,r} \leq C_{p,q} t^{(p-q)} \|F\|_{q,r} \tag{3.1}$$

and

$$\|(I - S(t))F\|_{q,r} \leq C_{p,q} t^{(q-p)} \|F\|_{p,r} \tag{3.2}$$

for any $F \in \mathcal{C}_r$, where p, q are any nonnegative integers such that $p \geq q$, I denotes the identity map, and $C_{p,q}$ denotes a constant which depends on p and q.

The second inequality means that $S(t)$ is close to identity and tends to identity when $t \to \infty$. The first inequality means that F becomes "smoother" when we apply $S(t)$ to it. For these reasons, $S(t)$ is called the smoothing operator.

Remark. Some authors write $e^{t(p-q)}$ and $e^{t(q-p)}$ instead of $t^{(p-q)}$ and $t^{(q-p)}$ in the above inequalities. The two conventions are related by a simple rescaling $t = e^\tau$.

There is a similar smoothing operator from $\mathcal{Y}_r$ to itself, which by abuse of language we will also denote by $S(t)$ or $S_r(t)$. We will assume that inequalities (3.1) and (3.2) are still satisfied when F is replaced by an element of $\mathcal{Y}_r$.

For any F in $\mathcal{C}_r$ or $\mathcal{Y}_r$, and nonnegative integers $p_1 \geq p_2 \geq p_3$, we have the following interpolation estimate:

$$(\|F\|_{p_2,r})^{p_3-p_1} \leq C_{p_1,p_2,p_3}(\|F\|_{p_1,r})^{p_3-p_2}(\|F\|_{p_3,r})^{p_2-p_1} \tag{3.3}$$

where C_{p_1,p_2,p_3} is a positive constant which may depend on p_1, p_2, p_3.

Remark. A priori, the constants $C_{p,q}$ and C_{p_1,p_2,p_3} also depend on the radius r. But later on, we will always have $1 \leq r \leq 2$ and so we may choose them to be independent of r.

In the proof of Theorem 1.1, we will use local diffeomorphisms of $\mathbb{R}^n$ of type $Id + \chi$ where $\chi(0) = 0$, and Id denotes the identity map from $\mathbb{R}^n$ to itself. The following lemmas allow to control operations on this kind of diffeomorphisms as the composition with a map or the inverse.

Lemma 3.1. *Let r and $\eta < 1$ be two strictly positive real numbers. Consider a smooth map $\Phi : B_r \to \mathbb{R}^n$ of the type $Id+\chi$ with $\chi(0) = 0$. Suppose that $\|\chi\|_{1,r} < \eta$. Then we have*

$$B_{r(1-\eta)} \subset \Phi(B_r) \subset B_{r(1+\eta)}\,. \tag{3.4}$$

Proof : According to the hypotheses we have $\|\chi(x)\| < \eta\|x\|$ for every x in B_r. Therefore, we can write $\|\Phi(x)\| < (1 + \eta)r$ and so, $\Phi(B_r) \subset B_{r(1+\eta)}$.

Now, we consider the map $\hat{\Phi} : B_{r(1+\eta)} \to B_{r(1+\eta)}$ which is Φ on B_r and is defined on $B_{r(1+\eta)} \setminus B_r$ as follows.
Let x be such that $\|x\| = r$. We consider $x_1 = \frac{2+\eta}{2}x$ and $x_2 = (1 + \eta)x$. If $z = \lambda x + (1-\lambda)x_1$ with $0 \leq \lambda \leq 1$, then $\hat{\Phi}(z) = \lambda\Phi(x) + (1-\lambda)x$. If $z = \lambda x_1 + (1-\lambda)x_2$ then $\hat{\Phi}(z) = \lambda x + (1 - \lambda)x_2$.

This map is continous and is the identity on the boundary of $B_{r(1+\eta)}$. According to Brouwer's theorem, the image of $\hat{\Phi}$ is $B_{r(1+\eta)}$.

Now, note that if $z = \lambda x + (1 - \lambda)x_1$ with $0 \leq \lambda \leq 1$ then we have

$$\|\hat{\Phi}(z)\| = \|x + \lambda\chi(x)\| \geq \|x\| - \lambda\|\chi(x)\|\,.$$

Therefore, $\|\hat{\Phi}(z)\| > r(1 - \eta)$.

Moreover, if $z = \lambda x_1 + (1 - \lambda)x_2$ with $0 \leq \lambda \leq 1$ then we have

$$\|\hat{\Phi}(z)\| = \|\lambda x + (1 - \lambda)(1 + \eta)x\| = r(1 + \eta(1 - \lambda)) \geq r\,.$$

We deduce that if y is in $B_{r(1-\eta)}$ then, we have $y = \hat{\Phi}(z)$ where z is, *a priori*, in $B_{r(1+\eta)}$, but according to the previous inequalities, z must belong to B_r. Consequently, y is in $\Phi(B_r)$. □

Lemma 3.2 ([Con85])**.** *Let $r > 0$ and $1 > \eta > 0$ be two positive numbers. Consider two smooth maps*

$$f : B_{r(1+\eta)} \to \mathbb{R}^q \quad \textit{and} \quad \chi : B_r \to \mathbb{R}^n$$

(where the closed balls B_r and $B_{r(1+\eta)}$ are in $\mathbb{R}^n$, and q is a natural number) such that $\chi(0) = 0$ and $\|\chi\|_{1,r} < \eta$. Then the composition $f \circ (id + \chi)$ is a smooth map

from B_r to $\mathbb{R}^n$ which satisfies the following inequalities:

$$\|f\circ(id+\chi)\|_{k,r} \leq \|f\|_{k,r(1+\eta)}(1+P_k(\|\chi\|_{k,r})) \tag{3.5}$$

$$\|f\circ(id+\chi)-f\|_{k,r} \leq Q_k(\|\chi\|_{k,r})\|f\|_{k,r(1+\eta)}+M\|\chi\|_{0,r}\|f\|_{k+1,r(1+\eta)} \tag{3.6}$$

where M is a positive constant and $P_k(t)$,$Q_k(t)$ are polynomials of degree k with vanishing constant term (and which are independent of f and χ).

The proof of the above lemma, which can be found in [Con85], is straightforward and is based solely on the Leibniz rule of derivation. We will call inequalities such as in the above lemma *Leibniz-type inequalities*. Similarly, we have another Leibniz-type inequality, given in the following lemma.

Lemma 3.3. *With the same hypotheses as in the previous lemma, we have*

$$\begin{aligned}\|f\circ(id+\chi)\|_{2k-1,r} &\leq \|f\|_{2k-1,r(1+\eta)}P_k(\|\chi\|_{k,r}) \\ &+ \|\chi\|_{2k-1,r}\|f\|_{k,r(1+\eta)}Q_k(\|\chi\|_{k,r})\,,\end{aligned} \tag{3.7}$$

where $P_k(t)$ and $Q_k(t)$ are polynomials (which are independent of f and χ).

Proof. Denote by θ the map $Id+\chi$. If I is a multiindex such that $|I|\leq 2k-1$ ($|I|$ denotes the sum of the components of I), it is easy to show, by induction on $|I|$, that

$$\frac{\partial^{|I|}(f\circ\theta)}{\partial x^{|I|}} = \sum_{1\leq|\alpha|\leq|I|}\big(\frac{\partial^{|\alpha|}f}{\partial x^{\alpha}}\circ\theta\big)A_\alpha(\theta)\,,$$

where $A_\alpha(\theta)$ is of the type

$$A_\alpha(\theta) = \sum_{\substack{1\leq u_i\leq n\,,\,|\beta_i|\geq 1\\ |\beta_1|+\ldots+|\beta_{|\alpha|}|=|I|}} a_{\beta u}\frac{\partial^{|\beta_1|}\theta_{u_1}}{\partial x^{\beta_1}}\cdots\frac{\partial^{|\beta_{|\alpha|}|}\theta_{u_\alpha}}{\partial x^{\beta_{|\alpha|}}} \tag{3.8}$$

where θ_{u_1} is the u_1-component of θ and the $a_{\beta u}$ are nonnegative integers.

We may write

$$\frac{\partial^{|I|}(f\circ\theta)}{\partial x^{|I|}} = \sum_{k<|\alpha|\leq|I|}\big(\frac{\partial^{|\alpha|}f}{\partial x^{\alpha}}\circ\theta\big)A_\alpha(\theta)+\sum_{1\leq|\alpha|\leq k}\big(\frac{\partial^{|\alpha|}f}{\partial x^{\alpha}}\circ\theta\big)A_\alpha(\theta)\,.$$

When $k<|\alpha|\leq|I|\leq 2k-1$, all the $|\beta_i|$ in the sum (3.8) defining $A_\alpha(\theta)$ are smaller than k. This gives the first term of the right hand side of Inequality (3.7). On the other hand, when $1\leq|\alpha|\leq k$, then in each product in the expression (3.8) of $A_\alpha(\theta)$ there is at most one factor $\frac{\partial^{|\beta|}\theta_u}{\partial x^\beta}$ with $|\beta|>k$ (the others have $|\beta|\leq k$. This gives the second term of the right hand side term of inequality (3.7), and the lemma follows. □

Lemma 3.4. *Let $r>0$ be a real number and $k\geq 1$ a positive integer. There exist a positive real number $\eta<1$ and a polynomial $P_k(t)$ such that if $\Phi: B_r\to\mathbb{R}^n$ is a smooth map of the type $Id+\chi$ with $\chi(0)=0$ and $\|\chi\|_{0,r}<\eta$ then Φ is a smooth local diffeomorphism which possesses an inverse $\Psi=\Phi^{-1}$ of the type $Id+\xi$ with $\xi(0)=0$, which is defined on (a set containing) $B_{r(1-\eta)}$ and satisfies the following inequality:*

$$\|\xi\|_{2k-1,r(1-\eta)} \leq \|\chi\|_{2k-1,r}P_k(\|\chi\|_{k,r})\,. \tag{3.9}$$

Proof. We choose the constant η such that for every smooth map $Id+\chi : B_r \to \mathbb{R}^n$ such that $\|\chi\|_{1,r} < \eta$, the Jacobian matrix of $Id+\chi$ is invertible at each point of B_r.

If Φ is a smooth map as in the theorem, according to the inverse function theorem, it is a local diffeomorphism and has an inverse $\Psi = Id + \xi$ which is smooth on $B_{r(1+\eta)}$ (see Lemma 3.1).

Since $\Phi \circ \Psi = Id$, denoting $\Psi = (\Psi_1, \dots, \Psi_n)$ (and the same thing for Φ), we can write

$$\frac{\partial \Psi_i}{\partial x_j} = \frac{Pol(\{\frac{\partial \Phi_u}{\partial x_v}\})}{Jac\,\Phi} \circ \Psi$$

where $Jac\,\Phi$ is the Jacobian determinant of Φ and $Pol(\{\frac{\partial \Phi_u}{\partial x_v}\})$ is a homogeneous polynomial in the $\{\frac{\partial \Phi_u}{\partial x_v}\}_{uv}$ of degree $n-1$.

By induction, we can see that for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| = \sum \alpha_i > 0$ we can write (trying to simplify the writing)

$$\frac{\partial^{|\alpha|}\Psi_i}{\partial x^\alpha} = \sum_{\substack{1\le|\beta_l|\le|\alpha|\,,\,p\le|\alpha|+1 \\ \sum_l(|\beta_l|-1)=|\alpha|-1}} \Big[\frac{a_{\beta,p}}{(Jac\,\Phi)^p} \times \frac{\partial^{|\beta_1|}\Phi_{u_1}}{\partial x^{\beta_1}} \dots \frac{\partial^{|\beta_k|}\Phi_{u_k}}{\partial x^{\beta_k}}\Big] \circ \Psi$$

where the $a_{\beta,p}$ are non negative integers. In this formula, the term $Jac\,\Phi$ is bounded on B_r, for instance, $0 < b \le |Jac\,\Phi(z)| \le c < 1$ for all z in B_r. This formula is not very explicit but it is sufficient to estimate $\sup_{z\in B_r} |\frac{\partial^{|\alpha|}(\xi)}{\partial x^\alpha}(z)|$ like in (3.9) for $|\alpha| > 1$ (note that in this case, we have $\frac{\partial^{|\alpha|}(\Psi_i)}{\partial x^\alpha} = \frac{\partial^{|\alpha|}(\xi_i)}{\partial x^\alpha}$). Now we have to study the case $|\alpha| = 1$. In this case, writing the Jacobian matrix, we have

$$1 + \frac{\partial \xi}{\partial x} = (1 + \frac{\partial \chi}{\partial x})^{-1} \circ \Phi\,.$$

Denoting by $\| \ \|$ the standard norm of linear operators on a finite dimensional vector space we can assume that $\|\frac{\partial \chi}{\partial x}\| < 1$. Then, since $(1+\frac{\partial \chi}{\partial x})^{-1} = 1 + \sum_{q\ge1}\big(\frac{\partial \chi}{\partial x}\big)^q$, we obtain

$$\frac{\partial \xi}{\partial x} = \big(\sum_{q\ge1}\big(\frac{\partial \chi}{\partial x}\big)^q\big) \circ \Phi\,.$$

We then get

$$\|\frac{\partial \xi}{\partial x}\| \le M \|\frac{\partial \chi}{\partial x}\|$$

where M is a positive constant and we conclude using the equivalence of the norms. □

4. Proof of Theorem 1.1

In order to prove Theorem 1.1, we will construct by recurrence a sequence of local smooth coordinate systems $(x^d, y^d) := (x_1^d, \dots, x_m^d, y_1^d, \dots, y_{n-m}^d)$, where $(x^0, y^0) = (x_1, \dots, x_m, y_1, \dots, y_{n-m})$ is the original linear coordinate system as chosen in Section 2, which converges to a local coordinate system $(x^\infty, y^\infty) =$

$(x_1^\infty, \ldots, x_m^\infty, y_1^\infty, \ldots, y_{n-m}^\infty)$, in which the Poisson structure Π has the required form.

For simplicity of exposition, we will assume that Π is C^∞-smooth. However, in every step of the proof of Theorem 1.1, we will only use differentiability of Π up to some finite order, and that's why our proof will also work for finitely (sufficiently highly) differentiable Poisson structures.

We will denote by Θ_d the local diffeomorphisms of $(\mathbb{R}^n, 0)$ such that

$$(x^d, y^d)(z) = (x^0, y^0) \circ \Theta_d(z)\,, \tag{4.1}$$

where z denotes a point of $(\mathbb{R}^n, 0)$.

Denote by Π^d the Poisson structure obtained from Π by the action of Θ_d:

$$\Pi^d = (\Theta_d)_* \Pi\,. \tag{4.2}$$

Of course, $\Pi^0 = \Pi$. Denote by $\{.,.\}_d$ the Poisson bracket with respect to the Poisson structure Π^d. Then we have

$$\{F_1, F_2\}_d(z) = \{F_1 \circ \Theta_d, F_2 \circ \Theta_d\}({\Theta_d}^{-1}(z))\,. \tag{4.3}$$

Assume that we have constructed $(x^d, y^d) = (x, y) \circ \Theta_d$. Let us now construct $(x^{d+1}, y^{d+1}) = (x, y) \circ \Theta_{d+1}$. This construction consists of two steps : 1) find an "almost Levi factor", i.e. coordinates x_i^{d+1} such that the error terms $\{x_i^{d+1}, x_j^{d+1}\} - \sum_k c_{ij}^k x_k^{d+1}$ are small, and 2) "almost linearize" it, i.e. find the remaining coordinates y^{d+1} such that in the coordinate system (x^{d+1}, y^{d+1}) the Hamiltonian vector fields of the functions x_i^{d+1} are very close to linear ones. In fact, we will define a local diffeomorphism θ_{d+1} of $(\mathbb{R}^n, 0)$ and then put $\Theta_{d+1} = \theta_{d+1} \circ \Theta_d$. In particular, we will have $\Pi^{d+1} = (\theta_{d+1})_* \Pi^d$ and $(x^{d+1}, y^{d+1}) = (x^d, y^d) \circ (\Theta_d)^{-1} \circ \theta_{d+1} \circ \Theta_d$.

We write the current error terms (that we want to make smaller by going from (x^d, y^d) to (x^{d+1}, y^{d+1})) as follows:

$$f_{ij}^d(x, y) = \{x_i, x_j\}_d - \sum_{k=1}^m c_{ij}^k x_k, \tag{4.4}$$

and

$$g_{i\alpha}^d(x, y) = \{x_i, y_\alpha\}_d - \sum_{\beta=1}^{n-m} a_{i\alpha}^\beta y_\beta. \tag{4.5}$$

Consider the 2-cochain

$$f^d = \sum_{ij} f_{ij}^d \otimes \xi_i^* \wedge \xi_j^* \tag{4.6}$$

of the Chevalley-Eilenberg complex associated to the $\mathfrak{g}$-module $\mathcal{C}_r$, where $r = r_d$ depends on d and is chosen as follows:

$$r_d = 1 + \frac{1}{d+1}. \tag{4.7}$$

In particular, $r_0 = 2$, $r_d/r_{d+1} \sim 1 + \frac{1}{d^2}$, and $\lim_{d\to\infty} r_d = 1$ is positive. This choice of radii r_d means in particular that we will be able to arrange so that the Poisson structure $\Pi^d = (\Theta_d)_* \Pi$ is defined in the closed ball of radius r_d. (For this to hold, we will have to assume that Π is defined in the closed ball of radius 2, and show by recurrence that $B_{r_d} \subset \theta_d(B_{r_{d-1}})$ for all $d \in \mathbb{N}$).

Put

$$\varphi^{d+1} := \sum_i \varphi_i^{d+1} \otimes \xi_i^* = -S(t_d)\big(h(f^d)\big)\,, \tag{4.8}$$

where h is the homotopy operator as given in Lemma 2.1, S is the smoothing operator and the parameter t_d is chosen as follows: take a real constant $t_0 > 1$ (which later on will be assumed to be large enough) and define the sequence $(t_d)_{d\geq 0}$ by $t_{d+1} = t_d^{3/2}$. In other words, we have

$$t_d = \exp\left(\left(\frac{3}{2}\right)^d \ln t_0\right)\,,\ \ln t_0 > 0\,. \tag{4.9}$$

The above choice of smoothing parameter t_d is a standard one in problems involving the Nash-Moser method, see, e.g., [Ham77, Ham82]. The number $\frac{3}{2}$ in the above formula is just a convenient choice. The main point is that this number is greater than 1 (so we have a very fast increasing sequence) and smaller than 2 (where 2 corresponds to the fact that we have a fast convergence algorithm which "quadratizes" the error term at each step, i.e. go from an "ε-small" error term to an "ε^2-small" error term).

According to Inequality (2.14), in order to control the C^k-norm of $h(f^d)$ we need to control the C^{k+s}-norm of f^d, i.e. we face a "loss of differentiability". That's why in the above definition of φ^{d+1} we have to use the smoothing operator S, which will allow us to compensate for this loss of differentiability. This is a standard trick in the Nash-Moser method.

Next, consider the 1-cochains

$$g^d = \sum_i \big(\sum_\alpha g^d_{i\alpha}\frac{\partial}{\partial y_\alpha}\big) \otimes \xi_i^*, \tag{4.10}$$

$$\hat{g}^d = g^d - \sum_i \big(\sum_\alpha \{h(f^d)_i, y_\alpha\}_d \frac{\partial}{\partial y_\alpha}\big) \otimes \xi_i^* \tag{4.11}$$

of the differential of the Chevalley-Eilenberg complex associated to the $\mathfrak{g}$-module $\mathcal{Y}_r$, where $r = r_d = 1 + \frac{1}{d+1}$, and put

$$\psi^{d+1} := \sum_\alpha \psi_\alpha^{d+1} \frac{\partial}{\partial y_\alpha} = -S(t_d)\big(h(\hat{g}^d)\big)\,, \tag{4.12}$$

where h is the homotopy operator as given in Lemma 2.1, and $S(t_d)$ is the smoothing operator (with the same t_d as in the definition of φ^{d+1}).

Now define θ_{d+1} to be a local diffeomorphism of $\mathbb{R}^n$ given by

$$\theta_{d+1} := Id + \chi^{d+1} := Id + (\varphi^{d+1}, \psi^{d+1})\,, \tag{4.13}$$

where $(\varphi^{d+1}, \psi^{d+1})$ now means $(\varphi_1^{d+1}, \dots, \varphi_m^{d+1}, \psi_1^{d+1}, \dots, \psi_{n-m}^{d+1})$. This finishes our construction of $\Theta_{d+1} = \theta_{d+1} \circ \Theta_d$ and $(x^{d+1}, y^{d+1}) = (x, y) \circ \Theta_{d+1}$. This construction is very similar to the analytic case [Zun03], except mainly for the use of the smoothing operator. Another difference is that, for technical reasons, in the smooth case considered in this paper we use the original coordinate system and the transformed Poisson structures Π^d for determining the error terms, while in the analytic case the original Poisson structure and the transformed coordinate systems are used. (In particular, the closed balls used in this paper are always balls with respect to the original coordinate system – this allows us to easily compare the Sobolev norms of functions on them, i.e. bigger balls correspond to bigger norms).

In order to show that the sequence of diffeomorphisms defined above converges to a smooth local diffeomorphism Θ_∞ and that the limit Poisson structure $(\Theta_\infty)_*\Pi$ is in Levi normal form, we will have to control the norms of δf and $\delta \hat{g}^d$, where δ denotes the differential of the corresponding Chevalley-Eilenberg complexes. This will be done with the help of the following two simple lemmas:

Lemma 4.1. *For every i, j and k, we have*

$$\delta f^d(\xi_i \wedge \xi_j \wedge \xi_k) = \oint_{ijk} \Big(\sum_u f^d_{iu} \frac{\partial f^d_{jk}}{\partial x_u} + \sum_\alpha g^d_{i\alpha} \frac{\partial f^d_{jk}}{\partial y_\alpha} \Big) , \tag{4.14}$$

where $\oint$ denotes the cyclic sum.

Lemma 4.2. *For every i, j and α, the coefficient of $\frac{\partial}{\partial y_\alpha}$ in $\delta \hat{g}^d(\xi_i \wedge \xi_j)$ is*

$$\begin{aligned}
- \quad & \sum_u f^d_{iu} \frac{\partial g^d_{j\alpha}}{\partial x_u} - \sum_\beta g^d_{i\beta} \frac{\partial g^d_{j\alpha}}{\partial y_\beta} + \sum_u f^d_{ju} \frac{\partial g^d_{i\alpha}}{\partial x_u} + \sum_\beta g^d_{j\beta} \frac{\partial g^d_{i\alpha}}{\partial y_\beta} \\
+ \quad & \sum_u f^d_{iu} \frac{\partial \{h(f^d)_j, y_\alpha\}_d}{\partial x_u} + \sum_\beta g^d_{i\beta} \frac{\partial \{h(f^d)_j, y_\alpha\}_d}{\partial y_\beta} \\
- \quad & \sum_u f^d_{ju} \frac{\partial \{h(f^d)_i, y_\alpha\}_d}{\partial x_u} - \sum_\beta g^d_{j\beta} \frac{\partial \{h(f^d)_i, y_\alpha\}_d}{\partial y_\beta} \\
+ \quad & \{h(f^d)_i, g^d_{j\alpha}\}_d - \{h(f^d)_j, g^d_{i\alpha}\}_d \\
+ \quad & \{y_\alpha, \sum_u f_{iu} \frac{\partial h(f^d)_j}{\partial x_u} - \sum_u f_{ju} \frac{\partial h(f^d)_i}{\partial x_u} + \sum_\beta g_{i\beta} \frac{\partial h(f^d)_j}{\partial y_\beta} - \sum_\beta g_{j\beta} \frac{\partial h(f^d)_i}{\partial y_\beta}\} \\
- \quad & \{y_\alpha, h(\delta f^d)_{ij}\} \ .
\end{aligned}$$

The first lemma is a direct consequence of the Jacobi identity $\{x_i, \{x_j, x_k\}_d\}_d + \{x_j, \{x_k, x_i\}_d\}_d + \{x_k, \{x_i, x_j\}_d\}_d = 0$. The second one follows from the Jacobi identity $\{x_i - h(f^d)_i, \{x_j - h(f^d)_j, y_\alpha\}_d\}_d + \{x_j - h(f^d)_j, \{y_\alpha, x_i - h(f^d)_i\}_d\}_d + \{y_\alpha, \{x_i - h(f^d)_i, x_j - h(f^d)_j\}_d\}_d = 0$ and the homotopy relation (2.9). □

Roughly speaking, the above lemmas say that δf^d and $\delta \hat{g}^d$ are "quadratic functions" in f^d, g^d and their first derivatives, so if f^d and g^d are "ε-small" then δf^d and $\delta \hat{g}^d$ are "ε^2-small".

Let us now give some expressions for the new error terms, which will allow us to estimate their norms. Recall that the new error terms after Step d are

$$f^{d+1}_{ij}(x, y) \quad = \quad \{x_i, x_j\}_{d+1} - \sum_k c^k_{ij} x_k \ , \tag{4.15}$$

$$g^{d+1}_{i\alpha}(x, y) \quad = \quad \{x_i, y_\alpha\}_{d+1} - \sum_\beta a^\beta_{i\alpha} y_\beta \ . \tag{4.16}$$

We can also write, for instance,

$$f_{ij}^{d+1}(x,y) = [\{x_i + \varphi_i^{d+1}, x_j + \varphi_j^{d+1}\}_d - \sum_k c_{ij}^k (x_k + \varphi_k^{d+1})](\theta_{d+1}^{-1}(x,y))\,. \tag{4.17}$$

A simple direct computation shows that

$$f_{ij}^{d+1} = \left[(\delta\varphi^{d+1})_{ij} + f_{ij}^d + Q_{ij}^d\right] \circ (\theta_{d+1})^{-1}\ , \tag{4.18}$$

$$g_{i\alpha}^{d+1} = \left[(\delta\psi^{d+1})_{i\alpha} + \hat{g}_{i\alpha}^d + T_{i\alpha}^d + U_{i\alpha}^d\right] \circ (\theta_{d+1})^{-1}\ , \tag{4.19}$$

where Q_{ij}^d and $T_{i\alpha}^d$ are "quadratic functions", namely

$$Q_{ij}^d = \sum_u (f_{iu}^d \frac{\partial \varphi_j^{d+1}}{\partial x_u} - f_{ju}^d \frac{\partial \varphi_i^{d+1}}{\partial x_u}) + \sum_\beta (g_{i\beta}^d \frac{\partial \varphi_j^{d+1}}{\partial y_\beta} - g_{j\beta}^d \frac{\partial \varphi_i^{d+1}}{\partial y_\beta}) + \{\varphi_i^{d+1}, \varphi_j^{d+1}\}_d\,, \tag{4.20}$$

and

$$T_{i\alpha}^d = \sum_u f_{iu}^d \frac{\partial \psi_\alpha^{d+1}}{\partial x_u} + \sum_\beta g_{i\beta}^d \frac{\partial \psi_\alpha^{d+1}}{\partial y_\beta} + \{\varphi_i^{d+1}, \psi_\alpha^{d+1}\}_d\,, \tag{4.21}$$

and $U_{i\alpha}^d$ is defined by

$$U_{i\alpha}^d = \{h(f^d)_i - S(t_d)h(f^d)_i, y_\alpha\}_d\,. \tag{4.22}$$

Putting

$$Q^d = \sum_{ij} Q_{ij}^d \otimes \xi_i^* \wedge \xi_j^*\ ,$$

$$T^d = \sum_i (\sum_\alpha T_{i\alpha}^d \frac{\partial}{\partial y_\alpha}) \otimes \xi_i^*\ ,$$

$$U^d = \sum_i (\sum_\alpha U_{i\alpha}^d \frac{\partial}{\partial y_\alpha}) \otimes \xi_i^*\ ,$$

we can write

$$f^{d+1} = (\delta\varphi^{d+1} + f^d + Q^d) \circ (\theta_{d+1})^{-1}\ , \tag{4.23}$$

$$g^{d+1} = (\delta\psi^{d+1} + \hat{g}^d + T^d + U^d) \circ (\theta_{d+1})^{-1}\,. \tag{4.24}$$

Equality (2.9) allows us to give another expression for f^{d+1} and g^{d+1}, which will be more convenient:

$$f^{d+1} = \left[\delta\big(\varphi^{d+1} + h(f^d)\big) + h(\delta f^d) + Q^d\right] \circ (\theta_{d+1})^{-1}\ , \tag{4.25}$$

$$g^{d+1} = \left[\delta\big(\psi^{d+1} + h(\hat{g}^d)\big) + h(\delta \hat{g}^d) + T^d + U^d\right] \circ (\theta_{d+1})^{-1}\,. \tag{4.26}$$

The following two technical lemmas about the norms will be the key points of the proof of Theorem 1.1. In order to formulate them, we need to introduce some positive constants A, l and L. Recall that we denote $s = [\frac{n}{2}] + 1$ (this number s appears in the Sobolev inequality and measures the "loss of differentiability" in our algorithm). Put $A = 6s + 9$. We will use the fact that

$$A > 6s + 8\ . \tag{4.27}$$

Choose an auxilliary positive constant $\varepsilon < 1$ such that

$$-(1-\varepsilon) + A\varepsilon < -\frac{3}{4}\,. \tag{4.28}$$

Choose an integer $l > s$ such that

$$\frac{3s+5}{l-1} \quad < \quad \varepsilon \tag{4.29}$$

(this is the number l which appears in the formulation of Theorem 1.1), and put

$$L = 2l - 1\ . \tag{4.30}$$

Recall also that $t_0 > 1$, $t_d = \exp((3/2)^d \ln t_0)$ and $r_d = 1 + \frac{1}{d+1}$ (note that we have $r_{d+1} = r_d(1 - \frac{1}{(d+2)^2})$). By choosing t_0 large enough, we can assume that $t_d^{-1/2} < \frac{1}{(d+2)^2}$ for every d.

Lemma 4.3. *Suppose that Π is defined on B_{r_0} and satisfies the following inequalities:*

(4.31)
$$\|f^0\|_{l,r_0} < t_0^{-1},\quad \|g^0\|_{l,r_0} < t_0^{-1},\quad \|\Pi\|_{L,r_0} < t_0^A,\quad \|f^0\|_{L,r_0} < t_0^A,\quad \|g^0\|_{L,r_0} < t_0^A,$$

where $t_0 > 1$ is a sufficiently large number. Then for every nonnegative integer d, Π^d is well-defined on B_{r_d} and we have the following estimates:

- *(1_d) $\|\chi^{d+1}\|_{l,r_d} < t_d^{-1/2}$ (recall that $\chi^{d+1} = -(\varphi_1^{d+1}, \dots, \psi_{n-m}^{d+1})$)*
- *(2_d) $\|\Pi^d\|_{L,r_d} < t_d^A$*
- *(3_d) $\|\Pi^d\|_{l,r_d} < C\frac{d+1}{d+2}$, where C is a positive constant independent of d.*
- *(4_d) $\|f^d\|_{L,r_d} < t_d^A$ and $\|g^d\|_{L,r_d} < t_d^A$*
- *(5_d) $\|f^d\|_{l,r_d} < t_d^{-1}$ and $\|g^d\|_{l,r_d} < t_d^{-1}$*

Roughly speaking, Inequality (1_d) is the one which ensures the convergence of Θ_d when $d \to \infty$ in C^l-topology. Inequality (3_d) says that $\|\Pi^d\|_l$ stays bounded. Inequality (5_d) means that the error terms converge to 0 very fast in C^l-topology, while Inequalities (2_d) and (4_d) mean that things don't "get bad" too fast in C^L-topology.

Lemma 4.4. *Suppose that for an integer $k \geq l$, there exists a constant $C_k > 0$ and an integer $d_k \geq 0$ such that for any $d \geq d_k$, the following inequalities are satisfied:*

$$\begin{aligned} &\|f^d\|_{k,r_d} < C_k t_d^{-1},\quad \|g^d\|_{k,r_d} < C_k t_d^{-1},\quad \|f^d\|_{2k-1,r_d} < C_k t_d^A,\\ &\|g^d\|_{2k-1,r_d} < C_k t_d^A,\quad \|\Pi^d\|_{2k-1,r_d} < C_k t_d^A,\quad \|\Pi^d\|_{k,r_d} < C_k(1 - \frac{1}{d+2}). \end{aligned} \tag{4.32}$$

Then there exists a constant $C_{k+1} > 0$ and an integer $d_{k+1} > d_k$ such that, for any $d \geq d_{k+1}$, we have

- i) $\|\chi^{d+1}\|_{k+1,r_d} < C_{k+1} t_d^{-1/2}$
- ii) $\|\Pi^d\|_{k+1,r_d} < C_{k+1}(1 - \frac{1}{d+2})$
- iii) $\|f^d\|_{k+1,r_d} < C_{k+1} t_d^{-1}$ *and* $\|g^d\|_{k+1,r_d} < C_{k+1} t_d^{-1}$
- iv) $\|f^d\|_{2k+1,r_d} < C_{k+1} t_d^A$, $\|g^d\|_{2k+1,r_d} < C_{k+1} t_d^A$ *and* $\|\Pi^d\|_{2k+1,r_d} < C_{k+1} t_d^A$

The above two technical lemmas will be proved in Section 5. Let us now finish the proof of Theorem 1.1 modulo them.

Proof of Theorem 1.1. Assume for the moment that Π is sufficiently close to its linear part, more precisely, that the conditions of Lemma 4.3 are satisfied. Let p be a natural number greater or equal to l such that Π is at least C^{2p-1}-smooth. Applying Lemma 4.3 to Π, and then applying Lemma 4.4 repetitively, we get the following inequality: there exist an integer d_p and a positive constant C_p such that for every $d \geq d_p$ we have

$$\|\chi^{d+1}\|_{p,r_d} \leq C_p t_d^{-1/2} = C_p \exp\Big(-\frac{1}{2}\Big(\frac{3}{2}\Big)^d \ln t_0\Big). \tag{4.33}$$

The right hand side of the above inequality tends to 0 exponentially fast when $d \to \infty$. This, together with Lemmas 3.2 and 3.4, implies that

$$(\Theta_d)^{-1} = (\theta_1)^{-1} \circ \ldots \circ (\theta_d)^{-1}, \tag{4.34}$$

where $\theta_d = \mathrm{Id} + \chi^d$, converges in C^p-topology on the ball B_1 of radius 1 (we show in Lemma 4.3 that $(\Theta_d)^{-1}$ is well-defined on the ball of radius $r_d > 1$). The fact that $\Theta_\infty = \lim_{d\to\infty} \Theta_d$ is a local C^p-diffeomorphism should now be obvious. It is also clear that $\Pi^\infty = (\Theta_\infty)_* \Pi$ is in Levi normal form. (Inequalities in (5_d) of Lemma 4.3 measure how far is Π^d from a Levi normal form; these estimates tend to 0 when $d \to \infty$).

If Π does not satisfy the conditions of Lemmas 4.3 and 4.4, then we may use the following homothety trick: replace Π by $\Pi^t = \frac{1}{t} G(t)_* \Pi$ where $G(t) : z \mapsto tz$ is a homothety, $t > 0$. The limit $\lim_{t\to\infty} \Pi^t$ is equal to the linear part of Π. So by choosing t high enough, we may assume that Π^t satisfies the conditions of Lemmas 4.3 and 4.4. If Φ is a required local diffeomorphism (coordinate transformation) for Π^t, then $G(1/t) \circ \Phi \circ G(t)$ will be a required local smooth coordinate transformation for Π. □

5. Proof of the technical lemmas

Proof of Lemma 4.3: We prove this lemma by induction on d. The main tools used are Leibniz-type inequalities, and interpolation inequalities (Inequality (3.3)) involving C^l-norms, C^L-norms and the norms in between. Roughly speaking, (2_d) and (4_d) will follow from Leibniz-type inequalities. The proof of (1_d) and (5_d) will make substantial use of interpolation inequalities. Point (3_d) will from from an analog of (1_d) and Leibniz-type inequalities.

In order to simplify the notations, we will use the letter M to denote a constant, *which does not depend on* d but which varies from inequality to inequality (i.e. it depends on the line where it appears).

We begin the reduction at $d = 0$. For $d = 0$, the only point to be checked is (1_0). We will use a property of the smoothing operator (equation (3.1)), the estimate of the homotopy operator (2.14) and the interpolation relation (3.3).

Recall that $\varphi^1 = -S(t_0)\big(h(f^0)\big)$. We then have

$$\begin{aligned}\|\varphi^1\|_{l,r_0} &\leq M\|h(f^0)\|_{l,r_0} \quad \text{by (3.1)}\\ &\leq M\|f^0\|_{l+s,r_0} \quad \text{by (2.14)}\\ &\leq M\|f^0\|_{l,r_0}^{\frac{l-s-1}{l-1}}\|f^0\|_{L,r_0}^{\frac{s}{l-1}} \quad \text{by (3.3)}\\ &\leq Mt_0^{-\frac{l-s-1}{l-1}}t_0^{A\frac{s}{l-1}}\end{aligned}$$

On the other hand, we have $\psi^1 = -S(t_0)\big(h(\hat{g}^0)\big)$, then

$$\begin{aligned}\|\psi^1\|_{l,r_0} &\leq M\|\hat{g}^0\|_{l+s,r_0} \quad \text{by (3.1) and (2.14)}\\ &\leq M\|g^0 + \{h(f^0), y\}_0\|_{l+s,r_0}\\ &\leq M(\|g^0\|_{l+s,r_0} + \|\Pi^0\|_{l+s,r_0}\|h(f^0)\|_{l+s+1,r_0})\,;\end{aligned}$$

note that since the definition of $\hat{g}^0$ involves the first derivatives of $h(f^0)$ we have to estimate by $\|h(f^0)\|_{l+s+1}$. Now, using (2.14) and the interpolation relation (3.3), we get

$$\begin{aligned}\|\psi^1\|_{l,r_0} &\leq M(\|g^0\|_{l+s,r_0} + \|\Pi^0\|_{l+s,r_0}\|f^0\|_{l+2s+1,r_0})\\ &\leq M(\|g^0\|_{l,r_0}^{\frac{l-s-1}{l-1}}\|g^0\|_{L,r_0}^{\frac{s}{l-1}} + \|\Pi^0\|_{l,r_0}^{\frac{l-s-1}{l-1}}\|\Pi^0\|_{L,r_0}^{\frac{s}{l-1}}\|f^0\|_{l,r_0}^{\frac{l-2s-2}{l-1}}\|f^0\|_{L,r_0}^{\frac{2s+1}{l-1}})\\ &\leq M(t_0^{-\frac{l-s-1}{l-1}+A\frac{s}{l-1}} + t_0^{-\frac{l-2s-2}{l-1}+A\frac{2s+1}{l-1}+A\frac{s}{l-1}})\,.\end{aligned}$$

Since $\frac{s}{l-1} < \frac{2s+1}{l-1}$ we have $-\frac{l-s-1}{l-1} + A\frac{s}{l-1} < -\frac{l-2s-2}{l-1} + A\frac{2s+1}{l-1} + A\frac{s}{l-1}$. Therefore, we obtain

$$\|\chi^1\|_{l,r_0} < Mt_0^{-\frac{l-2s-2}{l-1}+A\frac{3s+1}{l-1}}\,.$$

By assumptions (4.29), we see that $\frac{2s+1}{l-1}$ and $\frac{3s+1}{l-1}$ are strictly smaller than ε. Therefore, $-\frac{l-2s-2}{l-1} + A\frac{3s+1}{l-1}$ is strictly smaller than $-(1-\varepsilon)+A\varepsilon$. Then, according to inequality (4.28), we have $\|\chi^1\|_{l,r_0} \leq Mt_0^{-\mu}$ with $-\mu < -3/4 < -1/2$. We may choose t_0 sufficiently large such that $Mt_0^{-\mu} < t_0^{-1/2}$, which gives

$$\|\chi^1\|_{l,r_0} < t_0^{-1/2}\,. \tag{5.1}$$

Now, by induction, we suppose that for some $d \geq 0$, Π^d is well defined on B_{r_d} and that the inequalities $(1_d), \ldots, (5_d)$ are true. We will show that they still hold when we replace d by $d+1$. To simplify the writing we will omit the index r_d in the norms, unless the radius in question is different from r_d.

Since $r_{d+1} = r_d(1 - \frac{1}{(d+2)^2})$, according to Inequality (1_d) ($\|\chi^{d+1}\|_{l,r_d} < t_d^{-1/2}$ which is, by assumption, strictly smaller than $\frac{1}{(d+2)^2}$ for every d) and Lemma 3.1, we know that $B_{r_{d+1}}$ is included in $\theta_{d+1}(B_{r_d})$ and so Π^{d+1} will be well defined on $B_{r_{d+1}}$.

- Proof of (1_{d+1}): absolutely similar to the proof of (1_0) given above.
- Proof of (2_{d+1}): Recall that, due to the fact that $\Pi^{d+1} = (\Theta_{d+1})_*\Pi = (\theta_{d+1})_*\Pi^d$, we have

$$\{x_i, x_j\}_{d+1} = \{x_i + \varphi_i^{d+1}, x_j + \varphi_j^{d+1}\}_d \circ (\theta_{d+1})^{-1},$$

and similar formulas for $\{x_i, y_\alpha\}_{d+1}$ and $\{y_\alpha, y_\beta\}_{d+1}$.

Applying Lemmas 3.3 and 3.4 we obtain

$$\begin{aligned}\|\{x_i,x_j\}_{d+1}\|_{L,r_{d+1}} &\leq \|\{x_i+\varphi_i^{d+1},x_j+\varphi_j^{d+1}\}_d\|_{L,r_d}P(\|\chi^{d+1}\|_{l,r_d}) \\ &+ \|\{x_i+\varphi_i^{d+1},x_j+\varphi_j^{d+1}\}_d\|_{l,r_d}\|\chi^{d+1}\|_{L,r_d}Q(\|\chi^{d+1}\|_{l,r_d})\,,\end{aligned}$$

where P and Q are polynomials functions which do not depend on d. By the Leibniz rule of derivation, the term $\|\{x_i+\varphi_i^{d+1},x_j+\varphi_j^{d+1}\}_d\|_l$ may be estimated by $M\|\Pi^d\|_l(1+\|\varphi^{d+1}\|_{l+1})^2$ and, using the same technic as in the proof of (1_0), we can write $\|\varphi^{d+1}\|_{l+1}<t_d^{-1/2}$. Therefore, using (3_d), we can write $\|\{x_i+\varphi_i^{d+1},x_j+\varphi_j^{d+1}\}_d\|_l \leq M$. Consequently, we have

$$\|\{x_i,x_j\}_{d+1}\|_{L,r_{d+1}} \leq M\|\{x_i+\varphi_i^{d+1},x_j+\varphi_j^{d+1}\}_d\|_{L,r_d}+M\|\chi^{d+1}\|_{L,r_d}\,.$$

We first study the term χ^{d+1}. Actually, we will estimate $\|\chi^{d+1}\|_{L+1}$ rather than $\|\chi^{d+1}\|_L$ because it will be useful for the estimation of $\|\{x_i+\varphi_i^{d+1},x_j+\varphi_j^{d+1}\}_d\|_L$. We first write $\|\varphi^{d+1}\|_{L+1}\leq Mt_d^{s+1}\|h(f^d)\|_{L-s}$ by the property (3.1) of the smoothing operator. Using the estimate (2.14) for the homotopy operator, we obtain $\|\varphi^{d+1}\|_{L+1}\leq Mt_d^{s+1}\|f^d\|_L\leq Mt_d^{A+s+1}$. Now, we have

$$\begin{aligned}\|\psi^{d+1}\|_{L+1} &\leq Mt_d^{3s+2}\|\hat{h}(\hat{g}^d)\|_{L-3s-1} \text{ by (3.1)} \\ &\leq Mt_d^{3s+2}\|\hat{g}^d\|_{L-2s-1} \text{ by (2.14)}\end{aligned}$$

Then the definition of $\hat{g}^d$, the Leibniz rule of derivation (recall that $L=2l-1$) and Inequality (2.14) give

$$\begin{aligned}\|\psi^{d+1}\|_{L+1} &\leq Mt_d^{3s+2}(\|g^d\|_{L-2s-1}+\|\Pi^d\|_{L-2s-1}\|h(f^d)\|_{l-s-1+1} \\ &\quad +\|\Pi^d\|_{l-s-1}\|h(f^d)\|_{L-2s-1+1}) \\ &\leq Mt_d^{3s+2}(\|g^d\|_L+\|\Pi^d\|_L\|f^d\|_l+\|\Pi^d\|_l\|f^d\|_L) \\ &\leq Mt_d^{A+3s+2}\,.\end{aligned}$$

Therefore, we can write

$$\|\chi^{d+1}\|_{L+1,r_d}\leq Mt_d^{A+3s+2}\,.$$

Note that in the same way as in the proof of (1_0), one can show that $\|\chi^{d+1}\|_{l+1,r_d}<t_d^{-1/2}$ and then, using once more the Leibniz formula of the derivation of a product, we get

$$\begin{aligned}\|\{x_i+\varphi_i^{d+1},x_j+\varphi_j^{d+1}\}_d\|_{L,r_d} &\leq M\big(\|\Pi^d\|_L(1+\|\varphi^{d+1}\|_{l+1})^2 \\ &\quad +\|\Pi^d\|_l(1+\|\varphi^{d+1}\|_{l+1})(1+\|\varphi^{d+1}\|_{L+1})\big) \\ &\leq M(\|\Pi^d\|_L+\|\varphi^{d+1}\|_{L+1}+1) \\ &\leq M(\|\Pi^d\|_L+t_d^{A+3s+2}+1) \\ &\leq Mt_d^{A+3s+2}\,.\end{aligned}$$

Exactly in the same way, we can estimate the terms $\|\{x_i+\varphi_i^{d+1},y_\alpha+\psi_\alpha^{d+1}\}_d\|_{L,r_d}$ and $\|\{y_\alpha+\psi_\alpha^{d+1},y_\beta+\psi_\beta^{d+1}\}_d\|_{L,r_d}$ by Mt_d^{A+3s+2}. To conclude, since by our choice $A=6s+9$ we have $A+3s+2<3A/2$, these estimates lead to $\|\Pi^{d+1}\|_{L,r_{d+1}}\leq Mt_d^D$ where D is a positive constant such that $D<3A/2$. Therefore, we may choose t_0 large enough (in a way which does not depend on d) in order to obtain

$\|\Pi^{d+1}\|_{L,r_{d+1}} < t_d^{3A/2} = t_{d+1}^A$.

• Proof of (3_{d+1}): Recall again that we have

$$\{x_i, x_j\}_{d+1} = \{x_i + \varphi_i^{d+1}, x_j + \varphi_j^{d+1}\}_d \circ (\theta_{d+1})^{-1},$$

and similar formulas involving also y_i-components.

The estimates in Lemmas 3.2 and 3.4 give

$$\|\Pi^{d+1}\|_{l,r_{d+1}} \leq \|\Lambda^{d+1}\|_{l,r_d}(1 + P(\|\chi^{d+1}\|_{l,r_d})), \tag{5.2}$$

where p is a polynomial (which does not depend on d) with vanishing constant term, and

$$\Lambda^{d+1} = \sum_{ij}\{x_i + \varphi_i^{d+1}, x_j + \varphi_j^{d+1}\}_d \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \tag{5.3}$$

$$+\sum_{i\alpha}\{x_i + \varphi_i^{d+1}, y_\alpha + \psi_\alpha^{d+1}\}_d \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_\alpha} + \sum_{\alpha\beta}\{y_\alpha + \psi_\alpha^{d+1}, y_\beta + \psi_\beta^{d+1}\}_d \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta}.$$

Notice that Λ^{d+1} is equal to Π^d plus terms which involve χ^{d+1} and the Π^d-bracket. Hence, by the Leibniz formula, we can write

$$\|\Lambda^{d+1}\|_{l,r_d} \leq \|\Pi^d\|_{l,r_d}(1 + M\|\chi^{d+1}\|_{l+1,r_d})^2, \tag{5.4}$$

which implies that

$$\|\Pi^{d+1}\|_{l,r_{d+1}} \leq \|\Pi^d\|_{l,r_d}(1 + M\|\chi^{d+1}\|_{l+1,r_d})^2(1 + p(\|\chi^{d+1}\|_{l,r_d})). \tag{5.5}$$

Similarly to the proof of (1_{d+1}), it is easy to see that $\|\chi^{d+1}\|_{l+1,r_d} < t_d^{-1/2}$, which is exponentially small when $d \to \infty$. By choosing the constant t_0 large enough, we may assume that

$$(1 + M\|\chi^{d+1}\|_{l+1,r_d})^2(1 + p(\|\chi^{d+1}\|_{l,r_d})) < 1 + \frac{1}{(d+1)(d+3)} \tag{5.6}$$

Together with the induction hypothesis $\|\Pi^d\|_{l,r_d} < \frac{C(d+1)}{(d+2)}$, we get

$$\|\Pi^{d+1}\|_{l,r_{d+1}} < \frac{C(d+1)}{(d+2)}(1 + \frac{1}{(d+1)(d+3)}) = \frac{C(d+2)}{(d+3)}. \tag{5.7}$$

• Proof of (4_{d+1}): Recall that

$$f_{ij}^{d+1} = \{x_i, x_j\}_{d+1} - \sum_k c_{ij}^k x_k \ .,$$

It is easy to check that for every i and j,

$$\|\sum_k c_{ij}^k x_k\|_{L,r_{d+1}} \leq B\|\Pi\|_{1,r_{d+1}} \leq B\|\Pi\|_{L,r_{d+1}} \ ,$$

where B is a positive constant which only depends on the dimension n. This implies immediately that

$$\|f^{d+1}\|_{L,r_{d+1}} \leq (B+1)\|\Pi^{d+1}\|_{L,r_{d+1}} \ .$$

In Point (2_{d+1}), we showed that $\|\Pi^{d+1}\|_{L,r_{d+1}} < Mt_d^D$ where D is a positive constant such that $D < 3A/2$ therefore, replacing t_0 by a larger real number (which of course does not depend on d) if necessary, we have $\|f^{d+1}\|_{L,r_{d+1}} < t_d^{3A/2} = t_{d+1}^A$.

The estimate of $\|g^{d+1}\|_{L,r_{d+1}}$ can be done in the same way.

- Proof of (5_{d+1}) :

Recall the formula (4.25)

$$f^{d+1} = \big[\delta\big(\varphi^{d+1} + h(f^d)\big) + h(\delta f^d) + Q^d\big] \circ (\theta_{d+1})^{-1} .$$

We then have, using lemmas 3.2 and 3.4

$$\|f^{d+1}\|_{l,r_{d+1}} \le M\|\delta\big(\varphi^{d+1} + h(f^d)\big) + h(\delta f^d) + Q^d\|_{l,r_d}(1 + P(t_d^{-1/2})) , \tag{5.8}$$

where P is a polynomial function.

Thus, we only have to estimate $\|\delta\big(\varphi^{d+1}+h(f^d)\big)+h(\delta f^d)+Q^d\|_{l,r_d}$. To do that, we use the second property of the smoothing operator (3.2), the estimate of the homotopy operator (2.14) and the interpolation inequality.

We first write

$$\begin{aligned} \|\delta\big(\varphi^{d+1} + h(f^d)\big)\|_l &\le M\|h(f^d) - S(t_d)h(f^d)\|_{l+1} \\ &\le Mt_d^{-1}\|h(f^d)\|_{l+2} \quad \text{by (3.2)} \\ &\le Mt_d^{-1}\|f^d\|_{l+2+s} \quad \text{by (2.14)} \\ &\le Mt_d^{-1}\|f^d\|_l^{\frac{l-s-3}{l-1}}\|f^d\|_L^{\frac{s+2}{l-1}} \quad \text{by (3.3)} \end{aligned}$$

Then, we have

$$\|\delta\big(\varphi^{d+1} + h(f^d)\big)\|_l \le Mt_d^{-1-\frac{l-s-3}{l-1}+A\frac{s+2}{l-1}} . \tag{5.9}$$

Next, we write

$$\begin{aligned} \|h(\delta f^d)\|_l &\le M\|\delta f^d\|_{l+s} \quad \text{by (3.1)} \\ &\le M(\|f^d\|_{l+s}\|f^d\|_{l+s+1} + \|g^d\|_{l+s}\|f^d\|_{l+s+1}) \quad \text{by Lemma 4.1} \\ &\le M(\|f^d\|_{l+s+1}^2 + \|g^d\|_{l+s+1}\|f^d\|_{l+s+1}) \\ &\le M(\|f^d\|_l^{2\frac{l-s-2}{l-1}}\|f^d\|_L^{2\frac{s+1}{l-1}} + \|g^d\|_l^{\frac{l-s-2}{l-1}}\|g^d\|_L^{\frac{s+1}{l-1}}\|f^d\|_l^{\frac{l-s-2}{l-1}}\|f^d\|_L^{\frac{s+1}{l-1}}) . \end{aligned}$$

Thus,

$$\|h(\delta f^d)\|_l \le Mt_d^{-2\frac{l-s-2}{l-1}+2A\frac{s+1}{l-1}} . \tag{5.10}$$

Finally, by the definition (4.20) of Q^d , we have

$$\|Q^d\|_l \le M(\|f^d\|_l\|\varphi^{d+1}\|_{l+1} + \|g^d\|_l\|\varphi^{d+1}\|_{l+1} + \|\Pi^d\|_l\|\varphi^{d+1}\|_{l+1}^2) .$$

In the same way as in the proof of the point (1_0), we can easily show that $\|\chi^{d+1}\|_{l+1} < Mt_d^{-\frac{l-2s-3}{l-1}+A\frac{3s+3}{l-1}}$. Therefore, we can write

$$\|Q^d\|_l \le M(t_d^{-1-\frac{l-2s-3}{l-1}+A\frac{3s+3}{l-1}} + t_d^{-2\frac{l-2s-3}{l-1}+2A\frac{3s+3}{l-1}}) . \tag{5.11}$$

Combining (5.9), (5.10) and (5.11) we obtain

$$\|f^{d+1}\|_{l,r_{d+1}} < Mt_d^{-2\frac{l-2s-3}{l-1}+2A\frac{3s+3}{l-1}} .$$

Now, by (4.29), $\frac{2s+2}{l-1}$ and $\frac{3s+3}{l-1}$ are strictly smaller than ε, and then $-2\frac{l-2s-3}{l-1} + 2A\frac{3s+3}{l-1}$ is strictly smaller than $-2(1-\varepsilon) + 2A\varepsilon$. To finish, the inequality (4.28) gives $\|f^{d+1}\|_{l,r_{d+1}} < Mt_d^{-\alpha}$ where $-\alpha < -\frac{3}{2}$. We may choose t_0 large enough (in a way which depends on α but not on d) in order to obtain $\|f^{d+1}\|_{l,r_{d+1}} < t_d^{-\frac{3}{2}} = t_{d+1}^{-1}$.

Now, we apply the same technic to estimate $\|g^{d+1}\|_{l,r_{d+1}}$. Recall the formula (4.26)

$$g^{d+1} = \left[\delta\left(\psi^{d+1} + h(\hat{g}^d)\right) + h(\delta\hat{g}^d) + T^d + U^d\right] \circ (\theta_{d+1})^{-1}.$$

In the same way as above, according to Lemmas 3.2 and 3.4, we just have to estimate $\|\delta\left(\psi^{d+1} + h(\hat{g}^d)\right) + h(\delta\hat{g}^d) + T^d + U^d\|_l$. We first write

$$\begin{aligned}
\|\delta\left(\psi^{d+1} + h(\hat{g}^d)\right)\|_l &\leq M\| - S(t_d)h(\hat{g}^d) + h(\hat{g}^d)\|_{l+1} \\
&\leq Mt_d^{-1}\|h(\hat{g}^d)\|_{l+2} \quad \text{by (3.2)} \\
&\leq Mt_d^{-1}\|\hat{g}^d\|_{l+s+2} \quad \text{by (2.14)} \\
&\leq Mt_d^{-1}\|g^d + \{h(f^d), y\}_d\|_{l+s+2} \\
&\leq Mt_d^{-1}(\|g^d\|_{l+s+2} + \|\Pi^d\|_{l+s+2}\|h(f^d)\|_{l+s+3}) \\
&\leq Mt_d^{-1}(\|g^d\|_{l+s+2} + \|\Pi^d\|_{l+s+2}\|f^d\|_{l+2s+3})
\end{aligned}$$

Using the interpolation inequality (3.3), we obtain

$$\begin{aligned}
\|\delta\left(\psi^{d+1} + h(\hat{g}^d)\right)\|_l &\leq Mt_d^{-1}(\|g^d\|_l^{\frac{l-s-3}{l-1}}\|g^d\|_L^{\frac{s+2}{l-1}} \\
&\qquad + \|\Pi^d\|_l^{\frac{l-s-3}{l-1}}\|\Pi^d\|_L^{\frac{s+2}{l-1}}\|f^d\|_l^{\frac{l-2s-4}{l-1}}\|f^d\|_L^{\frac{2s+3}{l-1}}) \\
&\leq M(t_d^{-1-\frac{l-s-3}{l-1}+A\frac{s+2}{l-1}} + t_d^{-1-\frac{l-2s-4}{l-1}+A\frac{3s+5}{l-1}})
\end{aligned}$$

and then, since $\frac{s+2}{l-1} < \frac{2s+3}{l-1}$,

$$\|\delta\left(\psi^{d+1} + h(\hat{g}^d)\right)\|_l \leq Mt_d^{-1-\frac{l-2s-4}{l-1}+A\frac{3s+5}{l-1}}. \tag{5.12}$$

We also have, by the estimate of the homotopy operator (2.14),

$$\|h(\delta\hat{g}^d)\|_l \leq M\|\delta\hat{g}^d\|_{l+s},$$

and using Lemma 4.2 and the interpolation inequality (3.3), we obtain

$$\begin{aligned}
\|h(\delta\hat{g}^d)\|_l &\leq M\Big(\|f^d\|_{l+s}\|g^d\|_{l+s+1} + \|g^d\|_{l+s}\|g^d\|_{l+s+1} + \|\Pi^d\|_{l+s+1}\|f^d\|_{l+s}\|h(f^d)\|_{l+s+2} \\
&\quad + \|\Pi^d\|_{l+s+1}\|g^d\|_{l+s}\|h(f^d)\|_{l+s+2} + \|\Pi^d\|_{l+s}\|h(f^d)\|_{l+s+1}\|g^d\|_{l+s+1} \\
&\quad + \|\Pi^d\|_{l+s}\|f^d\|_{l+s+1}\|h(f^d)\|_{l+s+2} + \|\Pi^d\|_{l+s}\|g^d\|_{l+s+1}\|h(f^d)\|_{l+s+2} \\
&\quad + \|\Pi^d\|_{l+s}\|h(\delta f^d)\|_{l+s+1}\Big) \\
&\leq M\Big(\|f^d\|_{l+2s+2}\|g^d\|_{l+2s+2} + \|g^d\|_{l+2s+2}^2 + \|\Pi^d\|_{l+s+1}\|f^d\|_{l+2s+2}^2 \\
&\qquad + \|\Pi^d\|_{l+s+1}\|g^d\|_{l+2s+2}\|f^d\|_{l+2s+2} + \|\Pi^d\|_{l+s}\|h(\delta f^d)\|_{l+s+1}\Big) \\
&\leq M\Big(t_d^{-2\frac{l-2s-3}{l-1}+2A\frac{2s+2}{l-1}+A\frac{s+1}{l-1}} + \|\Pi^d\|_{l+s}\|h(\delta f^d)\|_{l+s+1}\Big)
\end{aligned}$$

In the same way as in the proof of (5.10) one can show that

$$\|h(\delta f^d)\|_{l+s+1} \leq Mt_d^{-2\frac{l-2s-3}{l-1}+2A\frac{2s+2}{l-1}}$$

This gives, applying the interpolation inequality to $\|\Pi^d\|_{l+s}$,

$$\|h(\delta\hat{g}^d)\|_l \le M t_d^{-2\frac{l-2s-3}{l-1}+2A\frac{2s+2}{l-1}+A\frac{s+1}{l-1}} . \tag{5.13}$$

Now, recalling the definition of T (4.21), we have

$$\|T^d\|_l \le M(\|f^d\|_l + \|g^d\|_l + \|\Pi^d\|_l\|\varphi^{d+1}\|_{l+1})\|\psi^{d+1}\|_{l+1} ,$$

and using the estimate of $\|\chi^{d+1}\|_{l+1,r_d}$ given above, we can show that

$$\|T^d\|_l \le M(t_d^{-1-\frac{l-2s-3}{l-1}+A\frac{3s+3}{l-1}} + t_d^{-2\frac{l-2s-3}{l-1}+2A\frac{3s+3}{l-1}}) . \tag{5.14}$$

Finally, by the definition of U^d (4.22), we can write

$$\begin{aligned} \|U^d\|_l &\le M\|\Pi^d\|_l\|h(f^d) - S(t_d)h(f^d)\|_{l+1} \\ &\le M\|\Pi^d\|_l t_d^{-1}\|h(f^d)\|_{l+2} \quad \text{by (3.2)} \\ &\le M t_d^{-1}\|f^d\|_{l+s+2} \quad \text{by } (3_d) \text{ and (2.14)} \\ &\le M t_d^{-1}\|f^d\|_l^{\frac{l-s-3}{l-1}}\|f^d\|_L^{\frac{s+2}{l-1}} \quad \text{by (3.3)} . \end{aligned}$$

We then obtain

$$\|U^d\|_l \le M t_d^{-1-\frac{l-s-3}{l-1}+A\frac{s+2}{l-1}} . \tag{5.15}$$

Combining (5.12), (5.13), (5.14) and (5.15), we obtain

$$\|g^{d+1}\|_{l,r_{d+1}} < M t_d^{-2\frac{l-2s-4}{l-1}+2A\frac{3s+5}{l-1}} ,$$

and we can conclude in the same way as for the estimate of $\|f^{d+1}\|_{l,r_{d+1}}$.

Lemma 4.3 is proved. □

Proof of Lemma 4.4: The main tools used in the proof of this lemma are the same as in the previous lemma: Leibniz-type inequalities and interpolation inequalities. To simplify the notations, we will denote by M_k a positive constant which depends on k *but not on* d and which varies from inequality to inequality.

• Proof of *(i)*: If $d \ge d_k$, we have

$$\begin{aligned} \|\varphi^{d+1}\|_{k+1,r_d} = \|S(t_d)h(f^d)\|_{k+1} &\le M_k\|f^d\|_{k+s+1} \quad \text{by (3.1) and (2.14)} \\ &\le M_k\|f^d\|_k^{\frac{k-s-2}{k-1}}\|f^d\|_{2k-1}^{\frac{s+1}{k-1}} \quad \text{by (3.3)} \\ &\le M_k t_d^{-\frac{k-s-2}{k-1}+A\frac{s+1}{k-1}} . \end{aligned}$$

In the same way, we get

$$\begin{aligned} \|\psi^{d+1}\|_{k+1,r_d} = \|S(t_d)\hat{h}(\hat{g}^d)\|_{k+1} &\le M_k\|\hat{g}^d\|_{k+s+1} \\ &\le M_k\|g^d + \{h(f^d), y\}_d\|_{k+s+1} \\ &\le M_k(\|g^d\|_{k+s+1} + \|\Pi^d\|_{k+s+1}\|h(f^d)\|_{k+s+2}) \\ &\le M_k(\|g^d\|_{k+s+1} + \|\Pi^d\|_{k+s+1}\|f^d\|_{k+2s+2}) \\ &\le M_k(t_d^{-\frac{k-s-2}{k-1}+A\frac{s+1}{k-1}} + t_d^{A\frac{s+1}{k-1}-\frac{k-2s-3}{k-1}+A\frac{2s+2}{k-1}}) \end{aligned}$$

Therefore, we have

$$\|\chi^{d+1}\|_{k+1,r_d} \le M_k t_d^{-\frac{k-2s-3}{k-1}+A\frac{3s+3}{k-1}} .$$

According to the inequality (4.29), the terms $\frac{2s+2}{k-1}$ and $\frac{3s+3}{k-1}$ are strictly smaller than ε. Then, $-\frac{k-2s-3}{k-1} + A\frac{3s+3}{k-1}$ is strictly smaller than $-(1-\varepsilon) + A\varepsilon$. Therefore, by (4.28), we can write $\|\chi^{d+1}\|_{k+1,r_d} < Mt_d^{-\mu}$ with $-\mu < -3/4 < -1/2$. We conclude that there exists a positive integer $d_{k+1} > d_k$ such that $\forall d \geq d_{k+1}$, $\|\chi^{d+1}\|_{k+1,r_d} < t_d^{-1/2}$.

Moreover, in the same way, we can prove that

$$\|\chi^{d+1}\|_{k+2,r_d} \leq M_k t_d^{-\frac{k-2s-4}{k-1}+A\frac{3s+5}{k-1}},$$

and we can assume (replacing d_{k+1} by a higher integer if necessary), that $\|\chi^{d+1}\|_{k+2,r_d} < t_d^{-1/2}$ for every $d \geq d_{k+1}$.

• Proof of *(ii)* : Let $d \geq d_{k+1}$. Proceeding in the same way as in the proof of Point (3_d) of the previous lemma, we get

$$\|\Pi^{d+1}\|_{k+1} \leq \|\Pi^d\|_{k+1}(1 + M_k\|\chi^{d+1}\|_{k+2})^2(1 + p(\|\chi^{d+1}\|_{k+1})), \tag{5.16}$$

where p is a polynomial with vanishing constant term. Now, since $\|\chi^{d+1}\|_{k+1}$ and $\|\chi^{d+1}\|_{k+2}$ are strictly smaller than $t_d^{-1/2}$, replacing d_{k+1} by a higher integer if necessary, we can assume that $\forall d \geq d_{k+1}$, we have

$$(1 + M_k\|\chi^{d+1}\|_{k+2})^2(1 + p(\|\chi^{d+1}\|_{k+1})) < 1 + \frac{1}{(d+1)(d+2)}. \tag{5.17}$$

We choose a positive constant $\tilde{C}_{k+1}$ such that $\|\Pi^{d_{k+1}}\|_{k+1} < \tilde{C}_{k+1}\left(\frac{d_{k+1}+1}{d_{k+1}+2}\right)$ and we can conclude by induction, as in the previous lemma, that for all $d \geq d_{k+1}$,

$$\|\Pi^d\|_{k+1,r_d} < \tilde{C}_{k+1}(1 - \frac{1}{d+2}). \tag{5.18}$$

Note that the constant $\tilde{C}_{k+1}$ is not the C_{k+1} of the lemma. Later, we will choose C_{k+1} to be greater than $\tilde{C}_{k+1}$ and satisfying other conditions.

• Proof of *(iii)* : The idea is exactly the same as in the previous lemma, using the interpolation inequality (3.3) with k and $2k-1$. Let $d \geq d_{k+1} - 1 \geq d_k$. By Lemmas 3.2 and 3.4, in order to estimate $\|f^{d+1}\|_{k+1,r_{d+1}}$ we just have to estimate $\|\delta\big(\varphi^{d+1} + h(f^d)\big) + h(\delta f^d) + Q^d\|_{k+1,r_d}$. As above, we write

$$\begin{aligned}
\|\delta\big(\varphi^{d+1} + h(f^d)\big)\|_{k+1} &\leq M_k\|h(f^d) - S(t_d)h(f^d)\|_{k+2} \\
&\leq M_k t_d^{-1}\|h(f^d)\|_{k+3} \quad \text{by (3.2)} \\
&\leq M_k t_d^{-1}\|f^d\|_{k+s+3} \quad \text{by (2.14)} \\
&\leq M_k t_d^{-1}\|f^d\|_k^{\frac{k-s-4}{k-1}}\|f^d\|_{2k-1}^{\frac{s+3}{k-1}} \quad \text{by (3.3)}
\end{aligned}$$

Then, since $\|f^d\|_k < C_k t_d^{-1}$ and $\|f^d\|_{2k-1} < C_k t_d^A$ we have

$$\|\delta\big(\varphi^{d+1} + h(f^d)\big)\|_{k+1} \leq M_k t_d^{-1-\frac{k-s-4}{k-1}+A\frac{s+3}{k-1}}.$$

In the same way as in Point (5_d) of the previous lemma, we can estimate $\|h(\delta f^d)\|_{k+1}$ by $M_k t_d^{-2\frac{k-s-3}{k-1}+2A\frac{s+2}{k-1}}$. Now, we just have to estimate $\|Q^d\|_{k+1}$. By the definition of Q^d, we have

$$\begin{aligned}\|Q^d\|_{k+1} &\leq M_k(\|f^d\|_{k+1}+\|g^d\|_{k+1}+\|\Pi^d\|_{k+1}\|\varphi^{d+1}\|_{k+1})\|\varphi^{d+1}\|_{k+2}\\ &\leq M_k(\|f^d\|_k^{\frac{k-2}{k-1}}\|f^d\|_{2k-1}^{\frac{1}{k-1}}+\|g^d\|_k^{\frac{k-2}{k-1}}\|g^d\|_{2k-1}^{\frac{1}{k-1}}\\ &\quad+\|\Pi^d\|_{k+1}\|\varphi^{d+1}\|_{k+1})\|\varphi^{d+1}\|_{k+2}\end{aligned}$$

We saw in *(ii)* that $\|\Pi^d\|_{k+1} \leq \tilde{C}_{k+1}$. Moreover we saw in *(i)* that $\|\chi^{d+1}\|_{k+2,r_d} \leq M_k t_d^{-\frac{k-2s-4}{k-1}+A\frac{3s+5}{k-1}}$. Since $\frac{1}{k-1} < \frac{2s+3}{k-1}$, we obtain

$$\|Q^d\|_{k+1} \leq M_k t_d^{-2\frac{k-2s-4}{k-1}+2A\frac{3s+5}{k-1}}\,.$$

We then obtain

$$\|f^{d+1}\|_{k+1,r_{d+1}} \leq M_k t_d^{-2\frac{k-2s-4}{k-1}+2A\frac{3s+5}{k-1}}\,,$$

and we deduce, in the same way as in the proof of Point (5_d) of the previous lemma that for all $d \geq d_{k+1}-1$,

$$\|f^{d+1}\|_{k+1,r_{d+1}} < M_k t_d^{-\mu}\,,$$

where $-\mu < -3/2$. Therefore, replacing d_{k+1} by a greater integer if necessary, we have for every $d \geq d_{k+1}-1$

$$\|f^{d+1}\|_{k+1,r_{d+1}} < t_d^{-3/2} = t_{d+1}^{-1}\,. \tag{5.19}$$

In the same way, we can show that

$$\|g^{d+1}\|_{k+1,r_{d+1}} < t_{d+1}^{-1}\,. \tag{5.20}$$

• Proof of *(iv)* : First recall that we have

$$\begin{aligned}f_{ij}^d &= \{x_i,x_j\}_d - \sum_u c_{ij}^u x_u\\ g_{i\alpha}^d &= \{x_i,y_\alpha\}_d - \sum_\beta a_{i\alpha}^\beta y_\beta\end{aligned}$$

and, as in the proof of Point (4_{d+1}) of the previous lemma, we can write

$$\begin{aligned}\|f^d\|_{2k+1,r_d} &< V\|\Pi^d\|_{2k+1,r_d}\,,\\ \|g^d\|_{2k+1,r_d} &< V\|\Pi^d\|_{2k+1,r_d}\,,\end{aligned} \tag{5.21}$$

where $V > 1$ is a positive constant independent of d and k.

Now, we estimate $\|\Pi^{d+1}\|_{2k+1,r_{d+1}}$ for $d \geq d_{k+1}$. Recall that we have

$$\{x_i,x_j\}_{d+1} = \{x_i+\varphi_i^{d+1}, x_j+\varphi_j^{d+1}\}_d \circ \theta_{d+1}^{-1}$$

and the same type of equality for $\{x_i,y_\alpha\}_{d+1}$ and $\{y_\alpha,y_\beta\}_{d+1}$.

Applying Lemmas 3.3 and 3.4 we obtain

$$\begin{aligned}\|\{x_i,x_j\}_{d+1}\|_{2k+1,r_{d+1}} &\leq \|\{x_i+\varphi_i^{d+1},x_j+\varphi_j^{d+1}\}_d\|_{2k+1,r_d}P_k(\|\chi^{d+1}\|_{k+1,r_d})\\ &+ \|\{x_i+\varphi_i^{d+1},x_j+\varphi_j^{d+1}\}_d\|_{k+1,r_d}\|\chi^{d+1}\|_{2k+1,r_d}Q_k(\|\chi^{d+1}\|_{k+1,r_d})\,,\end{aligned}$$

where P_k and Q_k are polynomials functions which do not depend on d. In the same way as in the proof of (2_d), since $\|\Pi^d\|_{k+1,r_d} < \tilde{C}_{k+1}\frac{d+1}{d+2}$ and $\|\chi^{d+1}\|_{k+2,r_d} < t_d^{-1/2}$, we can show that the term $\|\{x_i+\varphi_i^{d+1}, x_j+\varphi_j^{d+1}\}_d\|_{k+1}$ is bounded. Therefore, we can write

$$\|\{x_i,x_j\}_{d+1}\|_{2k+1,r_{d+1}} \leq M_k\big(\|\{x_i+\varphi_i^{d+1}, x_j+\varphi_j^{d+1}\}_d\|_{2k+1,r_d} + \|\chi^{d+1}\|_{2k+1,r_d}\big)\,.$$

As in the proof of (2_d) of the previous lemma, we first study the term χ^{d+1}. Actually, we will estimate $\|\chi^{d+1}\|_{2k+2}$ rather than $\|\chi^{d+1}\|_{2k+1}$ because it will be used to estimate the terms of type $\|\{x_i+\varphi_i^{d+1}, x_j+\varphi_j^{d+1}\}_d\|_{2k+1}$. We first write $\|\varphi^{d+1}\|_{2k+2} \leq M_k t_d^{s+3}\|h(f^d)\|_{2k-1-s}$ by the property (3.1) of the smoothing operator. Using the estimate of the homotopy operator (2.14), we obtain $\|\varphi^{d+1}\|_{2k+2} \leq M_k t_d^{s+3}\|f^d\|_{2k-1} \leq M_k t_d^{A+s+3}$. Now, we have

$$\begin{aligned}\|\psi^{d+1}\|_{2k+2} &\leq M_k t_d^{3s+4}\|\hat h(\hat g^d)\|_{2k-3s-2} \text{ by } (3.1)\\ &\leq M_k t_d^{3s+4}\|\hat g^d\|_{2k-2-2s} \text{ by } (2.14)\end{aligned}$$

Then the definition of $\hat g^d$, the Leibniz rule of derivation and (2.14) give

$$\begin{aligned}\|\psi^{d+1}\|_{2k+2} &\leq M_k t_d^{3s+4}(\|g^d\|_{2k-2-2s} + \|\Pi^d\|_{2k-2-2s}\|h(f^d)\|_{k-s-1+1}\\ &\quad +\|\Pi^d\|_{k-s-1}\|h(f^d)\|_{2k-2-2s+1})\\ &\leq M_k t_d^{3s+4}(\|g^d\|_{2k-1} + \|\Pi^d\|_{2k-1}\|f^d\|_k + \|\Pi^d\|_k\|f^d\|_{2k-1})\\ &\leq M_k t_d^{A+3s+4}\,.\end{aligned}$$

Therefore, we can write

$$\|\chi^{d+1}\|_{2k+2} \leq M_k t_d^{A+3s+4}\,.$$

Now, in the same way as in the proof of Point (2_d) of the previous lemma, using the Leibniz formula of the derivation of a product and the estimate $\|\chi^{d+1}\|_{k+2,r_d} < t_d^{-1/2}$, we get

$$\begin{aligned}\|\{x_i+\varphi_i^{d+1}, x_j+\varphi_j^{d+1}\}_d\|_{2k+1,r_d} &\leq M_k\big(\|\Pi^d\|_{2k+1}(1+\|\varphi^{d+1}\|_{k+2})^2\\ &\quad +\|\Pi^d\|_{k+1}(1+\|\varphi^{d+1}\|_{k+2})(1+\|\varphi^{d+1}\|_{2k+2})\big)\\ &\leq M_k(\|\Pi^d\|_{2k+1} + \|\varphi^{d+1}\|_{2k+2}+1)\\ &\leq M_k(\|\Pi^d\|_{2k+1} + t_d^{A+3s+4})\,.\end{aligned}$$

Consequently, we have

$$\|\{x_i,x_j\}_{d+1}\|_{2k+1,r_{d+1}} \leq M_k(\|\Pi^d\|_{2k+1} + t_d^{A+3s+4})\,.$$

In the same way, we can estimate $\|\{x_i,y_\alpha\}_{d+1}\|_{2k+1,r_{d+1}}$ and $\|\{y_\alpha,y_\beta\}_{d+1}\|_{2k+1,r_{d+1}}$ by $M_k(\|\Pi^d\|_{2k+1} + t_d^{A+3s+4})$, which implies

$$\|\Pi^{d+1}\|_{2k+1,r_{d+1}} \leq M_k(\|\Pi^d\|_{2k+1,r_d} + t_d^{A+3s+4})\,.$$

Finally, since $A > 6s+8$, we can assume, replacing d_{k+1} by a higher integer if necessary, that $M_k t_d^{A+3s+4} < \frac{1}{2V}t_d^{3A/2}$ for every $d \geq d_{k+1}$ (which also implies that $M_k < \frac{1}{2V}t_d^{A/2}$). We then obtain, $\forall d \geq d_{k+1}$,

$$\|\Pi^{d+1}\|_{2k+1,r_{d+1}} \leq \frac{1}{2V}t_d^{A/2}\|\Pi^d\|_{2k+1,r_d} + \frac{1}{2V}t_d^{3A/2}\,. \tag{5.22}$$

To conclude, if we choose a positive constant C_{k+1} such that

$$C_{k+1} > \mathrm{Max}\Big(1, \tilde{C}_{k+1}, \frac{\|\Pi^{d_{k+1}}\|_{2k+1, r_{d_{k+1}}}}{t^A_{d_{k+1}}}\Big),$$

we then obtain, using (5.22) and an induction,

$$\|\Pi^d\|_{2k+1,r_d} < \frac{C_{k+1}}{V} t^A_d < C_{k+1} t^A_d ,$$

for all $d \geq d_{k+1}$.

Finally, the estimates in (5.21) give, for all $d \geq d_{k+1}$,

$$\begin{array}{rcl} \|f^d\|_{2k+1,r_d} & < & C_{k+1} t^A_d \\ \|g^d\|_{2k+1,r_d} & < & C_{k+1} t^A_d . \end{array}$$

Moreover, the definition of C_{k+1} completes the proof of the points (i), (ii) and (iii).

Lemma 4.4 is proved. □

6. The case of Lie algebroids

In this section we briefly mention the proof of Theorem 1.2. Similarly to the analytic case (see [Zun03]), it is almost the same as the proof of Theorem 1.1.

Let A be a local N-dimensional smooth Lie algebroid (or C^{2q-1}-smooth) over $(\mathbb{R}^n, 0)$. We suppose that the anchor map $\# : A \to T\mathbb{R}^n$, vanishes on A_0, the fiber of A over point 0. It is well-known (see for instance [CW99]) that the Lie algebroid A induces and is, in fact, determined by a fiber-wise linear Poisson structure on the dual bundle A^*. More precisely, if $(x_1, \ldots, x_n)$ is a local coordinate system on $\mathbb{R}^n$ and $(e_1, \ldots, e_N)$ is a local basis of sections, then $(x_1, \ldots, x_n, e_1, \ldots, e_N)$ can be seen as a coordinate system for A^*, which is linear on the fibers. The Poisson structure on A^* is given by

$$\begin{array}{l} \{e_i, e_j\} = [e_i, e_j] \ , \\ \{e_i, x_j\} = \# e_i(x_j) \ , \\ \{x_i, x_j\} = 0 \ . \end{array} \tag{6.1}$$

This Poisson structure is fiber-wise linear in the sense that the bracket of two fiber-wise linear functions is again a fiber-wise linear function, the bracket of a fiber-wise linear function with a base function is a base function and the bracket of two base functions is zero.

As in the statement of Theorem 1.2, we denote by $\mathfrak{l}$ the N-dimensional Lie algebra in the linear part of A at 0 (i.e. the isotropy algebra of A at 0), and by $\mathfrak{g}$ a compact semisimple Lie subalgebra of $\mathfrak{l}$. We can rewrite the basis of sections $(e_1, \ldots, e_N)$ as $(s_1, \ldots, s_m, v_1, \ldots, s_{N-m})$ (m is the dimension of $\mathfrak{g}$) where $(s_1, \ldots, s_m)$ span $\mathfrak{g}$ and $(v_1, \ldots, v_{N-m})$ span a linear complement of $\mathfrak{g}$ in $\mathfrak{l}$ which is invariant under the adjoint action of $\mathfrak{g}$.

To prove Theorem 1.2, it suffices to find a Levi factor for the dual Lie-Poisson structure which consists of fiber-wise linear functions. The existence of a Levi factor is given by Theorem 1.1 and we only have to make sure that this Levi factor can be chosen so that it consists of fiber-wise linear functions. Actually the proof is the same as for Theorem 1.1 with few modifications :

The symbol $\mathcal{C}_r$ denotes now the subspace of the space $C^\infty(B_r)$ of C^∞-smooth real-valued functions on B_r (where $B_r \subset B^n_r \times \mathbb{R}^N$ is the closed ball centered at 0 and of radius r in $\mathbb{R}^{n+N} = \mathbb{R}^n \times \mathbb{R}^N$), which consists of fiber-wise linear functions vanishing at 0 whose first derivatives also vanish at 0.

The symbol $\mathcal{Y}_r$ denotes now the space of C^∞-smooth vector fields on B_r of the type

$$\sum_{i=1}^{N-m} p_i \frac{\partial}{\partial v_i} + \sum_{i=1}^{n} q_i \frac{\partial}{\partial x_i} ,$$

such that p_i and q_i vanish at 0 and their first derivatives also vanish at 0 and, p_i are fiber-wise linear functions and q_i are base functions.

One can check that these spaces are tame Fréchet spaces and $\mathfrak{g}$-modules with the same actions as defined in Section 2. We then still have the homotopy operators and all the properties we saw in Sections 2 and 3. The algorithm of construction of the sequence of diffeomorphisms is the same as for Theorem 1.1 and one can check that if the Poisson structure $\{\ ,\ \}_d$ is fiber-wise linear then $\{\ ,\ \}_{d+1}$ is fiber-wise linear too.

7. Appendix: a Nash-Moser normal form theorem

In this appendix we will generalize Theorem 1.1 into an abstract smooth normal form theorem, which we call a *Nash-Moser normal form theorem*, because its proof is similar to the proof of Theorem 1.1 and is based on the Nash-Moser fast convergence method. Of course, Conn's smooth linearization theorem [Con85], as well as our smooth Levi decomposition theorems, can be viewed as particular cases of this abstract smooth normal form theorem, modulo Lemma 2.1 about the norm of homotopy operators. It is hoped that our abstract Nash-Moser normal form theorem can be used or easily adapted for the study of other smooth normal form problems as well.

7.1. **The setting.** Grosso modo, the situation is as follows: we have a group $\mathcal{G}$ (say of diffeomorphisms) which acts on a set $\mathcal{F}$ (of structures). Inside $\mathcal{F}$ there is a subset $\mathcal{N}$ (of structures in normal form). We want to show that, under some appropriate conditions, each structure can be put into normal form, i.e. for each element $f \in \mathcal{F}$ there is an element $\Phi \in \mathcal{G}$ such that $\Phi.f \in \mathcal{N}$. We will assume that $\mathcal{F}$ is a subset of a linear space $\mathcal{H}$ (a space of tensors) on which $\mathcal{G}$ acts, and $\mathcal{N}$ is the intersection of $\mathcal{F}$ with a linear subspace $\mathcal{V}$ of $\mathcal{H}$. To formalize the situation involving smooth *local* structures (defined in a neighborhood of something), let us introduce the following notions of *SCI-spaces* and *SCI-groups*. Here SCI stands for *scaled C^∞ type*. Our aim here is not to create a very general setting, but just a setting which works and which can hopefully be adjusted to various situations. So our definitions below (especially the inequalities appearing in them) are probably not "optimal", and can be improved, relaxed, etc.

SCI-spaces. An *SCI-space* $\mathcal{H}$ is a collection of Banach spaces $(\mathcal{H}_{k,\rho}, \|\ \|_{k,\rho})$ with $0 < \rho \le 1$ and $k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ (ρ is called the *radius* parameter, k is called the *smoothness parameter*; we say that $f \in \mathcal{H}$ if $f \in \mathcal{H}_{k,\rho}$ for some k and ρ, and in that case we say that f is k-smooth and defined in radius ρ) which satisfies the following properties:

- If $k < k'$, then for any $0 < \rho \leq 1$, $\mathcal{H}_{k',\rho}$ is a linear subspace of $\mathcal{H}_{k,\rho}$: $\mathcal{H}_{k',\rho} \subset \mathcal{H}_{k,\rho}$.
- If $0 < \rho < \rho' \leq 1$, then for each $k \in \mathbb{Z}_+$, there is a given linear map, called the *projection map*, or *radius restriction map*,
$$\pi_{\rho,\rho'} : \mathcal{H}_{k,\rho'} \to \mathcal{H}_{k,\rho}.$$
These projections don't depend on k and satisfy the natural commutativity condition $\pi_{\rho,\rho''} = \pi_{\rho,\rho'} \circ \pi_{\rho',\rho''}$. If $f \in \mathcal{H}_{k,\rho}$ and $\rho' < \rho$, then by abuse of language we will still denote by f its projection to $\mathcal{H}_{k,\rho'}$ (when this notation does not lead to confusions).
- For any f in $\mathcal{H}$ we have
$$\|f\|_{k,\rho} \geq \|f\|_{k',\rho'} \quad \forall\ k \geq k', \rho \geq \rho'. \tag{7.1}$$
In the above inequality, if f is not in $\mathcal{H}_{k,\rho}$ then we put $\|f\|_{k,\rho} = +\infty$, and if f is in $\mathcal{H}_{k,\rho}$ then the right hand side means the norm of the projection of f to $\mathcal{H}_{k',\rho'}$, of course.
- There is a smoothing operator for each ρ, which depends continuously on ρ. More precisely, for each $0 < \rho \leq 1$ and each $t > 1$ there is a linear map, called the *smoothing operator*,
$$S_\rho(t) : \mathcal{H}_{0,\rho} \longrightarrow \mathcal{H}_{\infty,\rho} = \bigcap_{k=0}^{\infty} \mathcal{H}_{k,\rho} \tag{7.2}$$
which satisfies the following inequalities: for any $p, q \in \mathbb{Z}_+$, $p \geq q$ we have
$$\|S_\rho(t)f\|_{p,\rho} \leq C_{\rho,p,q} t^{p-q} \|f\|_{q,\rho} \tag{7.3}$$
$$\|f - S_\rho(t)f\|_{q,\rho} \leq C_{\rho,p,q} t^{q-p} \|f\|_{p,\rho} \tag{7.4}$$
where $C_{\rho,p,q}$ is a positive constant (which does not depend on f nor on t) and which is continuous with respect to ρ.

In the same way as for the Fréchet spaces (see for instance [Ser72]), the two properties (7.3) and (7.4) of the smoothing operator imply the following inequality called *interpolation inequality* : for any positive integers p, q and r with $p \geq q \geq r$ we have

$$(\|f\|_{q,\rho})^{p-r} \leq C_{p,q,r} (\|f\|_{r,\rho})^{p-q} (\|f\|_{p,\rho})^{q-r}\ , \tag{7.5}$$

where $C_{p,q,r}$ is a positive constant which is continuous with respect to ρ and does not depend on f.

Of course, if $\mathcal{H}$ is an SCI-space then each $\mathcal{H}_{\infty,\rho}$ is a tame Fréchet space. The main example that we have in mind is the space of functions in a neighborhood of 0 in the Euclidean space $\mathbb{R}^n$: here ρ is the radius and k is the smoothness class, i.e. $\mathcal{H}_{k,\rho}$ is the space of C^k-functions on the closed ball of radius ρ and centered at 0 in $\mathbb{R}^n$, together with the maximal norm (of each function and its partial derivatives up to order k); the projections are restrictions of functions to balls of smaller radii.

By an *SCI-subspace* of an SCI-space $\mathcal{H}$, we mean a collection $\mathcal{V}$ of subspaces $\mathcal{V}_{k,\rho}$ of $\mathcal{H}_{k,\rho}$, which themselves form an SCI-space (under the induced norms, induced smoothing operators, induced inclusion and radius restriction operators from $\mathcal{H}$ - it is understood that these structural operators preserve $\mathcal{V}$).

By a *subset* of an SCI-space $\mathcal{H}$, we mean a collection $\mathcal{F}$ of subsets $\mathcal{F}_{k,\rho}$ of $\mathcal{H}_{k,\rho}$, which are invariant under the inclusion and radius restriction maps of $\mathcal{H}$.

Remark. The above notion of SCI-spaces generalizes at the same time the notion of tame Fréchet spaces and the notion of scales of Banach spaces [Zeh75]. Evidently, the scale parameter is introduced to treat local problems. When things are globally defined (say on a compact manifold), then the scale parameter is not needed, i.e. $\mathcal{H}_{k,\rho}$ does not depend on ρ and we get back to the situation of tame Fréchet spaces, as studied by Sergeraert [Ser72] and Hamilton [Ham77, Ham82].

SCI-groups. An *SCI-group* $\mathcal{G}$ consists of elements Φ which are written as a (formal) sum

$$\Phi = Id + \chi, \tag{7.6}$$

where χ belongs to an SCI-space $\mathcal{W}$, together with *scaled group laws* to be made more precise below. We will say that $\mathcal{G}$ is modelled on $\mathcal{W}$, if $\chi \in \mathcal{W}_{k,\rho}$ then we say that $\Phi = Id + \chi \in \mathcal{G}_{k,\rho}$ and $\chi = \Phi - Id$ (so as a space, $\mathcal{G}$ is the same as $\mathcal{W}$, but shifted by Id), $Id = Id + 0$ is the neutral element of $\mathcal{G}$.

Scaled composition (product) law. There is a positive constant C (which does not depend on ρ or k) such that if $0 < \rho' < \rho \leq 1$, $k \geq 1$, and $\Phi = Id + \chi \in \mathcal{G}_{k,\rho}$ and $\Psi = Id + \xi \in \mathcal{G}_{k,\rho}$ such that

$$\rho'/\rho \leq 1 - C\|\xi\|_{1,\rho} \tag{7.7}$$

then we can compose Φ and Ψ to get an element $\Phi \circ \Psi$ with $\|\Phi \circ \Psi - Id\|_{k,\rho'} < \infty$, i.e. $\Phi \circ \Psi$ can be considered as an element of $\mathcal{G}_{k,\rho'}$ (if $\rho'' < \rho'$ then of course $\Phi \circ \Psi$ can also be considered as an element of $\mathcal{G}_{k,\rho''}$, by the restriction of radius from ρ' to ρ''). Of course, we require the composition to be *associative* (after appropriate restrictions of radii).

Scaled inversion law. There is a positive constant C (for simplicity, take it to be the same constant as in Inequality (7.7)) such that if $\Phi \in \mathcal{G}_{k,\rho}$ such that

$$\|\Phi - Id\|_{1,\rho} < 1/C \tag{7.8}$$

then we can define an element, denoted by Φ^{-1} and called the inversion of Φ, in $\mathcal{G}_{k,\rho'}$, where $\rho' = (1 - \frac{1}{2}C\|\Phi - Id\|_{1,\rho})\rho$, which satisfies the following condition: the compositions $\Phi\circ\Phi^{-1}$ and $\Phi^{-1}\circ\Phi$ are well-defined in radius $\rho'' = (1-C\|\Phi-Id\|_{1,\rho})\rho$ and coincide with the neutral element Id there.

Continuity conditions. We require that the above scaled group laws satisfy the following continuity conditions i), ii) and iii) in order for $\mathcal{G}$ to be called an SCI-group.

i) For each $k \geq 1$ there is a polynomial $P = P_k$ (of one variable), such that for any $\chi \in \mathcal{W}_{2k-1,\rho}$ with $\|\chi\|_{1,\rho} < 1/C$ we have

$$\|(Id + \chi)^{-1} - Id\|_{k,\rho'} \leq \|\chi\|_{k,\rho} P(\|\chi\|_{k,\rho}) \ , \tag{7.9}$$

where $\rho' = (1 - C\|\chi\|_{1,\rho})\rho$.

ii) If $(\Phi_m)_{m\geq 0}$ is a sequence in $\mathcal{G}_{k,\rho}$ which converges (with respect to $\| \|_{k,\rho}$) to Φ, then the sequence $(\Phi_m^{-1})_{m\geq 0}$ also converges to Φ^{-1} in $\mathcal{G}_{k,\rho'}$, where $\rho' = (1 - C\|\Phi - Id\|_{1,\rho})\rho$.

iii) For each $k \geq 1$ there are polynomials P and Q (of one variable) with vanishing constant term such that if $\Phi = Id + \chi$ and $\Psi = Id + \xi$ are in $\mathcal{G}_{k,\rho}$ and if ρ' and ρ satisfy Relation (7.7), then we have

$$\|\Phi \circ \Psi - \Phi\|_{k,\rho'} \leq P(\|\xi\|_{k,\rho}) + \|\chi\|_{k+1,\rho} Q(\|\xi\|_{k,\rho})\,. \tag{7.10}$$

Remark : As a consequence of the last condition we have, with the same notations, the following inequality:

$$\|\Phi \circ \Psi - Id\|_{k,\rho'} \leq P(\|\xi\|_{k,\rho}) + \|\chi\|_{k+1,\rho}(1 + Q(\|\xi\|_{k,\rho}))\,. \tag{7.11}$$

SCI-actions. We will say that there is a *linear left SCI-action* of an SCI-group $\mathcal{G}$ on an SCI-space $\mathcal{H}$ if there is a positive integer γ (and a positive constant C) such that, for each $\Phi = Id + \chi \in \mathcal{G}_{k,\rho}$ and $f \in \mathcal{H}_{k,\rho'}$ with $\rho' = (1 - C\|\chi\|_{1,\rho})\rho$, the element $\Phi.f$ (the image of the action of Φ on f) is well-defined in $\mathcal{H}_{k,\rho'}$, the usual axioms of a left group action modulo appropriate restrictions of radii (so we have *scaled action laws*) are satisfied, and the following three inequalities i), ii), iii) expressing some continuity conditions are also satisfied:

i) For each k there is a polynomial function $P = P_k$ with vanishing constant term such that

$$\|(Id + \chi) \cdot f\|_{k,\rho'} \leq \|f\|_{k,\rho}(1 + P(\|\chi\|_{k+\gamma,\rho}))\ . \tag{7.12}$$

ii) For each k there are polynomials Q and R (which depend on k) such that

$$\|(Id + \chi) \cdot f\|_{2k-1,\rho'} \leq \|f\|_{2k-1,\rho} Q(\|\chi\|_{k+\gamma,\rho}) + \|\chi\|_{2k-1+\gamma,\rho}\|f\|_{k,\rho} R(\|\chi\|_{k+\gamma,\rho}) \tag{7.13}$$

iii) There is a polynomial function O of 2 variables such that

$$\|(\Phi + \chi) \cdot f - \Phi \cdot f\|_{k,\rho'} \leq \|\chi\|_{k+\gamma,\rho}\|f\|_{k+\gamma,\rho} O(\|\Phi - Id\|_{k+\gamma,\rho}, \|\chi\|_{k+\gamma,\rho}) \tag{7.14}$$

In the above inequalities, ρ' is related to ρ by a formula of the type $\rho' = (1 - C(\|\chi\|_{1,\rho} + \|\Phi - Id\|_{1,\rho}))\,\rho$. ($\Phi = Id$ in the first two inequalities).

The main example of a (linear left) SCI-action that we have in mind is the push-forward action of the SCI-group of local diffeomorphisms of $(\mathbb{R}^n, 0)$ on the SCI-space of local tensors of a given type (e.g. 2-vector fields) on $(\mathbb{R}^n, 0)$.

7.2. **Normal form theorem.** Roughly speaking, the following theorem says that whenever we have a "fast normalizing algorithm" in an SCI setting then it will lead to the existence of a smooth normalization. "Fast" means that, putting loss of differentiability aside, one can "quadratize" the error term at each step (going from "ε-small" error to "ε^2-small" error).

In the statement of the following theorem, the polynomials P_k, Q_k, R_k and T_k depend on k and may depend on ρ continuously, but do not depend on f.

Theorem 7.1. *Let $\mathcal{H}$ be a SCI-space, $\mathcal{V}$ a SCI-subspace of $\mathcal{H}$, and $\mathcal{F}$ a subset of $\mathcal{H}$, $\mathcal{F} \ni 0$. Denote $\mathcal{N} = \mathcal{F} \cap \mathcal{V}$. Assume that there is a projection $\pi : \mathcal{H} \longrightarrow \mathcal{V}$ (compatible with restriction and inclusion maps) such that for every f in $\mathcal{H}_{k,\rho}$, the element $\zeta(f) = f - \pi(f)$ satisfies*

$$\|\zeta(f)\|_{k,\rho} \leq \|f\|_{k,\rho} T_k(\|f\|_{[(k+1)/2],\rho}) \tag{7.15}$$

for all $k \in \mathbb{N}$ (or at least for all k sufficiently large), where $[\,]$ is the integer part and T_k a polynomial. Let $\mathcal{G}$ be an SCI-group acting on $\mathcal{H}$ by a linear left SCI-action *which preserves $\mathcal{F}$. Assume that there is $s \in \mathbb{N}$ such that for every f in $\mathcal{F}$ and $0 < \rho \leq 1$, there is an element $\Phi_f = Id + \chi_f \in \mathcal{G}$ (which may depend on ρ but doesn't depend on k) such that for all k in $\mathbb{N}$ (or at least for all k sufficiently large),*

$$\begin{aligned} \|\chi_f\|_{k,\rho} &\leq \|\zeta(f)\|_{k+s,\rho} P_k(\|f\|_{[(k+1)/2]+s,\rho}) \\ &+ \|f\|_{k+s,\rho}\|\zeta(f)\|_{[(k+1)/2]+s,\rho} Q_k(\|f\|_{[(k+1)/2]+s,\rho}) \,, \end{aligned} \tag{7.16}$$

and that the element $f' := \Phi_f \cdot f \in \mathcal{F}$ satisfies the inequality

$$\|\zeta(f')\|_{k,\rho'} \leq \|\zeta(f)\|^2_{k+s,\rho} R_k(\|f\|_{k+s,\rho}, \|\chi_f\|_{k+s,\rho}, \|f\|_{k,\rho}) \tag{7.17}$$

(ρ and ρ' verify Relation (7.7)) where P_k and Q_k (resp. R_k) are some polynomials of 1 variable (resp. 3 variables) and the degree in the first variable of the polynomial R_k does not depend on k. Then there exist $l \in \mathbb{N}$ and two positive constants α and β with the following property: for all $p \in \mathbb{N} \cup \{\infty\}, p \geq l$, and for all $f \in \mathcal{F}_{2p-1,\rho}$ with $\|f\|_{2l-1,\rho} < \alpha$ and $\|f - 0\|_{l,\rho} < \beta$, there exists $\Psi \in \mathcal{G}_{p,\rho/2}$ such that $\Psi \cdot f \in \mathcal{N}_{p,\rho/2}$.

Proof. We construct, by induction, a sequence $(\Psi_d)_{d\geq 1}$ in $\mathcal{G}$, and then a sequence $f^d := \Psi_d \cdot f$ in $\mathcal{F}$, which converges to $\Psi \in \mathcal{G}_{p,\rho/2}$ and such that $f^\infty := \Psi \cdot f \in \mathcal{N}_{p,\rho/2}$.

In order to simplify, we can assume that the constant s of the theorem is the same as the integer γ defined by the SCI-action of $\mathcal{G}$ on $\mathcal{H}$ (see (7.12), (7.12) and (7.12)). We first fix some parameters. Let $A = 6s + 5$ (actually, A just have to be strictly larger than $6s + 4$). We denote by τ the degree in the first variable of the polynomials R_k introduced in Theorem 7.1. We consider a positive real number $\varepsilon < 1$ such that

$$-(1-\varepsilon) + A(1 + \frac{\tau}{2})\varepsilon < -\frac{3}{4} \,. \tag{7.18}$$

Finally, we fix a positive integer $l > 3s + 3$ which satisfies

$$\frac{2s+2}{l-1} < \varepsilon \,. \tag{7.19}$$

The construction of the sequences is the following : Let $t_0 > 1$ be a real constant ; this constant is still not really fixed and will be chosen according to Lemma 7.2. We then define the sequence $(t_d)_{d\geq 1}$ by $t_{d+1} := t_d^{3/2}$. We also define the sequence $r_d := (1 + \frac{1}{d+1})\rho/2$. This is a decreasing sequence such that $\rho/2 \leq r_d \leq \rho$ for all d. Note that we have $r_{d+1} = r_d(1 - \frac{1}{(d+2)^2})$.

Let $p \geq l$ and f in $\mathcal{F}_{2p-1,\rho}$. We start with $f_0 := f \in \mathcal{F}_{2p-1,\rho}$. Now, assume that we have constructed $f^d \in \mathcal{F}_{2p-1,r_d}$ for $d \geq 0$. We put $\Phi_{d+1} := \Phi_{f^d} = Id + \chi^{d+1}$ and $\hat{\Phi}_{d+1} := S(t_d)\Phi_{d+1} = Id + \hat{\chi}^{d+1}$. Then, f^{d+1} is defined by

$$f^{d+1} = \hat{\Phi}_{d+1} \cdot f^d \,.$$

Roughly speaking, the idea is that the sequence $(f^d)_{d\geq 0}$ will be such that

$$\|\zeta(f^{d+1})\|_{p,r_{d+1}} \leq \|\zeta(f^d)\|^2_{p,r_d} \,.$$

For every $d \geq 1$, we put $\Psi_d = \hat{\Phi}_d \circ \ldots \circ \hat{\Phi}_1$. We then have to show that we can choose two positive constants α and β such that if $\|f\|_{2l-1,\rho} \leq \alpha$ and $\|f - 0\|_{l,\rho} \leq \beta$ then, the sequence $(\Psi_d)_{d\geq 1}$ converges with respect to $\| \, \|_{p,\rho/2}$. It will follow from these two technical lemmas that we will prove later :

Lemma 7.2. *There exists a real number $t_0 > 1$ such that for any $f \in \mathcal{F}_{2p-1,\rho}$ satisfying the conditions $\|f^0\|_{2l-1,r_0} < t_0^A$, $\|\zeta(f^0)\|_{2l-1,r_0} < t_0^A$ and $\|\zeta(f^0)\|_{l,r_0} < t_0^{-1}$ then, with the construction above, we have for all $d \geq 0$,*

- (1_d) $\|\hat{\chi}^{d+1}\|_{l+s,r_d} < t_d^{-1/2}$
- (2_d) $\|f^d\|_{l,r_d} < C\frac{d+1}{d+2}$ *where C is a positive constant*
- (3_d) $\|f^d\|_{2l-1,r_d} < t_d^A$
- (4_d) $\|\zeta(f^d)\|_{2l-1,r_d} < t_d^A$
- (5_d) $\|\zeta(f^d)\|_{l,r_d} < t_d^{-1}$

Lemma 7.3. *Suppose that for an integer $k \geq l$, there exists a constant C_k and an integer $d_k \geq 0$ such that for any $d \geq d_k$ we have $\|f^d\|_{2k-1,r_d} < C_k t_d^A$, $\|\zeta(f^d)\|_{2k-1,r_d} < C_k t_d^A$, $\|f^d\|_{k,r_d} < C_k\frac{d+1}{d+2}$ and $\|\zeta(f^d)\|_{k,r_d} < C_k t_d^{-1}$. Then, there exists a positive constant C_{k+1} and an integer $d_{k+1} > d_k$ such that for any $d \geq d_{k+1}$ we have*

- (i) $\|\hat{\chi}^{d+1}\|_{k+1+s,r_d} < C_{k+1} t_d^{-1/2}$
- (ii) $\|f^d\|_{k+1,r_d} < C_{k+1}\frac{d+1}{d+2}$
- (iii) $\|f^d\|_{2k+1,r_d} < C_{k+1} t_d^A$
- (iv) $\|\zeta(f^d)\|_{2k+1,r_d} < C_{k+1} t_d^A$
- (v) $\|\zeta(f^d)\|_{k+1,r_d} < C_{k+1} t_d^{-1}$

End of the proof of Theorem 7.1 : We choose t_0 as in Lemma 7.2. Then we fix two positive constants α and β such that $t_0^A \geq \alpha$ and $t_0^{-1} \geq \beta$. Now, if $f \in \mathcal{F}_{2p-1,\rho}$ satisfies $\|f\|_{2l-1,\rho} \leq \alpha$ and $\|f - 0\|_{l,\rho} \leq \beta$ then, since $\|\zeta(f)\|_{l,\rho} \leq \|f - 0\|_{l,\rho}$, using Lemma 7.2 and then applying Lemma 7.3 repetitively, there exists a positive integer d_p such that for all $d \geq d_p$,

$$\|\hat{\chi}^{d+1}\|_{p,r_d} < C_p t_d^{-1/2}\,.$$

Actually it is more convenient to prove the convergence of the sequence $(\Psi_d^{-1})_{d\geq 1}$. The point ii) of the continuity conditions in the definition of SCI-group will then give the convergence of $(\Psi_d)_{d\geq 1}$. For all positive integer d, we have $\Psi_d^{-1} = \hat{\Phi}_1^{-1}\circ\ldots\circ\hat{\Phi}_d^{-1}$ and if we denote $\hat{\Phi}_d^{-1} = Id + \hat{\xi}^d$, the axiom (7.9) implies

$$\|\hat{\xi}^{d+1}\|_{p,r_d} < M_p t_d^{-1/2}\,,$$

for all $d \geq d_p$, where M_p is a positive constant independent of d. Now, by the inequality (7.11), the sequence $(\Psi_d^{-1} - Id)_{d\geq 1}$ is bounded and (7.10) gives then the $\|\ \|_{p,\rho/2}$-convergence of $(\Psi_d^{-1})_{d\geq 1}$.

Proof of Lemma 7.2 : In this proof M denotes a positive constant which does not depend on d and which varies from inequality to inequality. As in the case of Poisson structures, we prove this lemma by induction.

At the step $d = 0$ the only thing we have to verify is the point (1_0) (for the point (3_0) we just choose the constant C such that $C > 2\|f^0\|_{l,r_0}$) . We have $\|\hat{\chi}^1\|_{l+s,r_0} = \|S(t_0)\chi^1\|_{l+s,r_0} \leq M\|\chi^1\|_{l+s,r_0}$ by (7.3). Therefore, using (7.16) with the relation $l > 3s + 3$, and the interpolation inequality (7.5), we obtain (P and Q

are polynomial functions) :

$$\begin{aligned}
\|\hat\chi^1\|_{l+s,r_0} &\leq M\|\zeta(f^0)\|_{l+2s,r_0}P(\|f^0\|_{l,r_0}) \\
&+ M\|f^0\|_{l+2s,r_0}\|\zeta(f^0)\|_{l,r_0}Q(\|f^0\|_{l,r_0}) \\
&\leq M\|\zeta(f^0)\|_{l,r_0}^{\frac{l-2s-1}{l-1}}\|\zeta(f^0)\|_{2l-1,r_0}^{\frac{2s}{l-1}} \\
&+ M\|f^0\|_{l,r_0}^{\frac{l-2s-1}{l-1}}\|f^0\|_{2l-1,r_0}^{\frac{2s}{l-1}}\|\zeta(f^0)\|_{l,r_0} \\
&\leq M(t_0^{-\frac{l-2s-1}{l-1}+A\frac{2s}{l-1}} + t_0^{-1+A\frac{2s}{l-1}})
\end{aligned}$$

Then, by (7.19) and (7.18), we obtain $\|\hat\chi^1\|_{l+s,r_0} \leq Mt_0^{-\mu}$ with $-\mu < -3/4 < -1/2$ and, replacing t_0 by a larger number if necessary (independently of f and d), we have $\|\hat\chi^1\|_{l+s,r_0} < t_0^{-1/2}$. Note that we also proved that $\|\chi^1\|_{l+s,r_0} < t_0^{-1/2}$.

Now, we suppose that the conditions $(1_d)\dots(5_d)$ are satisfied and we study the step $d+1$. The point (1_{d+1}) can be proved as above.

Proof of (2_{d+1}) : According to (7.12) we have $\|f^{d+1}\|_{l,r_{d+1}} \leq \|f^d\|_{l,r_d}(1 + P(\|\hat\chi^{d+1}\|_{l+s,r_d}))$ where P is a polynomial with vanishing constant term. Since $\|\hat\chi^{d+1}\|_{l+s,r_d} < t_d^{-1/2}$ we can assume, choosing t_0 large enough, that $P(\|\hat\chi^{d+1}\|_{l+s,r_d}) \leq \frac{1}{(d+1)(d+3)}$ and we get

$$\|f^{d+1}\|_{l,r_{d+1}} < C\frac{d+1}{d+2}(1 + \frac{1}{(d+1)(d+3)}) < C\frac{d+2}{d+3}\,.$$

Proof of (3_{d+1}) : We have $f^{d+1} = \hat\Phi_{d+1}\cdot f^d$ with $\hat\Phi_{d+1} = Id + \hat\chi^{d+1} = Id + S(t_d)\chi^{d+1}$ thus, (7.13) gives

$$\|f^{d+1}\|_{2l-1,r_{d+1}} \leq \|f^d\|_{2l-1,r_d}P_1(\|\hat\chi^{d+1}\|_{l+s,r_d}) + \|\hat\chi^{d+1}\|_{2l-1+s,r_d}\|f^d\|_{l,r_d}P_2(\|\hat\chi^{d+1}\|_{l+s,r_d})$$

where P_1 and P_2 are two polynomials. This gives, by (1_d) and (2_d),

$$\|f^{d+1}\|_{2l-1,r_{d+1}} \leq M(\|f^d\|_{2l-1,r_d} + \|\hat\chi^{d+1}\|_{2l-1+s,r_d})\,.$$

Now, we have

$$\begin{aligned}
\|\hat\chi^{d+1}\|_{2l-1+s,r_d} &\leq Mt_d^{3s}\|\chi^{d+1}\|_{2l-1-2s,r_d} \quad \text{by (7.3)} \\
&\leq Mt_d^{3s}(\|\zeta(f^d)\|_{2l-1-s,r_d}P_3(\|f^d\|_{l,r_d}) \\
&\quad +\|f^d\|_{2l-1-s,r_d}\|\zeta(f^d)\|_{l,r_d}P_4(\|f^d\|_{l,r_d})) \quad \text{by (7.16)}
\end{aligned}$$

where P_3 and P_4 are polynomials. We get $\|\hat\chi^{d+1}\|_{2l-1+s,r_d} \leq Mt_d^{A+3s}$ and, consequently,

$$\|f^{d+1}\|_{2l-1,r_{d+1}} \leq Mt_d^{A+3s}\,.$$

To finish, since $A = 6s+5$, we have that $\|f^{d+1}\|_{2l-1,r_{d+1}} \leq Mt_d^B$ with $0 < B < 3A/2$ thus, replacing t_0 by a larger number if necessary, we get $\|f^{d+1}\|_{2l-1,r_{d+1}} < t_d^{3A/2} = t_{d+1}^A$.

Proof of (4_{d+1}) : We have

$$\|\zeta(f^{d+1})\|_{2l-1,r_{d+1}} \leq \|f^{d+1}\|_{2l-1,r_{d+1}}T(\|f^{d+1}\|_{l,r_{d+1}})$$

where T is a polynomial (see (7.15)). Using the estimate of (3_{d+1}) and (2_{d+1}), we obtain $\|\zeta(f^{d+1})\|_{2l-1,r_{d+1}} \leq Mt_d^{A+3s}$, and we conclude as above.

Proof of (5_{d+1}) : Recall that we have $\Phi_{d+1} = Id + \chi^{d+1}$ and $\hat{\Phi}_{d+1} = Id + S(t_d)\chi^{d+1}$. We can write

$$\begin{aligned} \|\zeta(f^{d+1})\|_{l,r_{d+1}} &= \|\zeta(\hat{\Phi}_{d+1}\cdot f^d)\|_{l,r_{d+1}} \\ &\leq \|\zeta(\hat{\Phi}_{d+1}\cdot f^d - \Phi_{d+1}\cdot f^d)\|_{l,r_{d+1}} + \|\zeta(\Phi_{d+1}\cdot f^d)\|_{l,r_{d+1}} \end{aligned}$$

On the one hand, by (7.17) and using the interpolation inequality (7.5), Point (2_d) and the estimate $\|\chi^{d+1}\|_{l+s,r_d} < t_d^{-1/2}$ (see the proof of (1_0)), we have

$$\begin{aligned} \|\zeta(\Phi_{d+1}\cdot f^d)\|_{l,r_{d+1}} &\leq \|\zeta(f^d)\|^2_{l+s,r_d} R_l(\|f^d\|_{l+s,r_d}, \|\chi^{d+1}\|_{l+s,r_d}, \|f^d\|_{l,r_d}) \\ &\leq M\|\zeta(f^d)\|_{l,r_d}^{2\frac{l-s-1}{l-1}} \|\zeta(f^d)\|_{2l-1,r_d}^{2\frac{s}{l-1}} \times \\ &\quad R_l(\|f^d\|_{l,r_d}^{\frac{l-s-1}{l-1}} \|f^d\|_{2l-1,r_d}^{\frac{s}{l-1}}, \|\chi^{d+1}\|_{l+s,r_d}, \|f^d\|_{l,r_d}) \\ &\leq Mt_d^{-2\frac{l-s-1}{l-1}+2A\frac{s}{l-1}+A\frac{\tau s}{l-1}} \end{aligned}$$

recall that τ is the degree in the first variable of R_l. Then, by (7.19) and (7.18), we have $\|\zeta(\Phi_{d+1}\cdot f^d)\|_{l,r_{d+1}} \leq Mt_d^{-\mu}$ where $-\mu < -3/2$ and, replacing t_0 by a larger number if necessary, we have $\|\zeta(\Phi_{d+1}\cdot f^d)\|_{l,r_{d+1}} < \frac{1}{2}t_d^{-3/2}$.

On the other hand, by (7.15),

$$\begin{aligned} \|\zeta(\hat{\Phi}_{d+1}\cdot f^d - \Phi_{d+1}\cdot f^d)\|_{l,r_{d+1}} &\leq \|\hat{\Phi}_{d+1}\cdot f^d - \Phi_{d+1}\cdot f^d\|_{l,r_{d+1}} \times \\ &\quad \times T(\|\hat{\Phi}_{d+1}\cdot f^d - \Phi_{d+1}\cdot f^d\|_{l,r_{d+1}}) \end{aligned}$$

and since $\hat{\Phi}_{d+1} = \Phi_{d+1} + (\hat{\chi}^{d+1} - \chi^{d+1})$, we have by (7.14),

$$\begin{aligned} \|\hat{\Phi}_{d+1}\cdot f^d - \Phi_{d+1}\cdot f^d\|_{l,r_{d+1}} &\leq \|\hat{\chi}^{d+1} - \chi^{d+1}\|_{l+s,r_d}\|f^d\|_{l+s,r_d} \\ &\quad O(\|\chi^{d+1}\|_{l+s,r_d}, \|\hat{\chi}^{d+1} - \chi^{d+1}\|_{l+s,r_d}) \end{aligned}$$

where O is a polynomial of 2 variables. Since $\|\hat{\chi}^{d+1}\|_{l+s,r_d}$ and $\|\chi^{d+1}\|_{l+s,r_d}$ are both majored by $t_d^{-1/2}$, we can write

$$\|\hat{\Phi}_{d+1}\cdot f^d - \Phi_{d+1}\cdot f^d\|_{l,r_{d+1}} \leq M\|\hat{\chi}^{d+1} - \chi^{d+1}\|_{l+s,r_d}\|f^d\|_{l+s,r_d}\,.$$

By the interpolation inequality we can write $\|f^d\|_{l+s,r_d} \leq M\|f^d\|_{2l-1,r_d}^{\frac{s}{l-1}}$. Moreover, using the property (7.4), the estimate (7.16) with the inequality $l > 3s+3$, and then the interpolation inequality (7.5), we get

$$\begin{aligned} \|\hat{\chi}^{d+1} - \chi^{d+1}\|_{l+s,r_d} &\leq Mt_d^{-1}\|\chi^{d+1}\|_{l+s+1,r_d} \\ &\leq Mt_d^{-1}(\|\zeta(f^d)\|_{l+2s+1,r_d}P(\|f^d\|_{l,r_d}) \\ &\quad +\|f^d\|_{l+2s+1,r_d}\|\zeta(f^d)\|_{l,r_d}Q(\|f^d\|_{l,r_d})) \\ &\leq Mt_d^{-1}(\|\zeta(f^d)\|_{l,r_d}^{\frac{l-2s-2}{l-1}}\|\zeta(f^d)\|_{2l-1,r_d}^{\frac{2s+1}{l-1}} \\ &\quad +\|f^d\|_{l,r_d}^{\frac{l-2s-2}{l-1}}\|f^d\|_{2l-1,r_d}^{\frac{2s+1}{l-1}}\|\zeta(f^d)\|_{l,r_d}) \\ &\leq Mt_d^{-1}(t_d^{-\frac{l-2s-2}{l-1}+A\frac{2s+1}{l-1}} + t_d^{-1+A\frac{2s+1}{l-1}})\,. \end{aligned}$$

Consequently, we have

$$\begin{aligned}\|\hat\Phi_{d+1}\cdot f^d-\Phi_{d+1}\cdot f^d\|_{l,r_{d+1}} &\le Mt_d^{A\frac{s}{l-1}}\|\hat\chi^{d+1}-\chi^{d+1}\|_{l+s,r_d}\\ &\le Mt_d^{-2+A\frac{3s+1}{l-1}},\end{aligned}$$

which implies

$$\|\zeta(\hat\Phi_{d+1}\cdot f^d-\Phi_{d+1}\cdot f^d)\|_{l,r_{d+1}}\le Mt_d^{-2+A\frac{3s+1}{l-1}}T(Mt_d^{-2+A\frac{3s+1}{l-1}}).$$

As above, we can conclude that

$$\|\zeta(\hat\Phi_{d+1}\cdot f^{d+1}-\Phi_{d+1}\cdot f^{d+1})\|_{l,r_{d+1}}<\frac12 t_d^{-3/2}.$$

Finally, we obtain

$$\begin{aligned}\|\zeta(f^{d+1})\|_{l,r_{d+1}} &\le \|\zeta(\hat\Phi_{d+1}\cdot f^d-\Phi_{d+1}\cdot f^d)\|_{l,r_{d+1}}+\|\zeta(\Phi_{d+1}\cdot f^d)\|_{l,r_{d+1}}\\ &< \frac12 t_d^{-3/2}+\frac12 t_d^{-3/2}=t_{d+1}^{-1}\end{aligned}$$

Lemma 7.2 is proved. □

Proof of Lemma 7.3 : As in the proof of the previous lemma, the letter M_k is a positive constant which does not depend on d and which varies from inequality to inequality.

Proof of (i) : In the same way as in the proof of the point (1_0) of the previous lemma, we can show that for all $d\ge d_k$, we have

$$\begin{aligned}\|\hat\chi^{d+1}\|_{k+1+s,r_d} &\le M_k\|\zeta(f^d)\|_{k+1+2s,r_d}P(\|f^d\|_{k,r_d})\\ &+ M_k\|f^d\|_{k+1+2s,r_d}\|\zeta(f^d)\|_{k,r_d}Q(\|f^d\|_{k,r_d})\\ &\le M_k\|\zeta(f^d)\|_{k,r_d}^{\frac{k-2s-2}{k-1}}\|\zeta(f^d)\|_{2k-1,r_d}^{\frac{2s+1}{k-1}}\\ &+ M_k\|f^d\|_{k,r_d}^{\frac{k-2s-2}{k-1}}\|f^d\|_{2k-1,r_d}^{\frac{2s+1}{k-1}}\|\zeta(f^d)\|_{k,r_d}\\ &\le M_k(t_d^{-\frac{k-2s-2}{k-1}+A\frac{2s+1}{k-1}}+t_d^{-1+A\frac{2s+1}{k-1}})\\ &\le M_k t_d^{-\mu}\end{aligned}$$

where $-\mu<-1/2$. Thus, there exists $d_{k+1}>d_k$ such that for all $d\ge d_{k+1}$ we have $\|\hat\chi^{d+1}\|_{k+1+s,r_d}<t_d^{-1/2}$. Note that we also have $\|\chi^{d+1}\|_{k+1+s,r_d}<t_d^{-1/2}$.

Proof of (ii) : For $d\ge d_{k+1}$, we have by (7.12)

$$\|f^{d+1}\|_{k+1,r_{d+1}}\le\|f^d\|_{k+1,r_d}(1+P(\|\hat\chi^{d+1}\|_{k+1+s,r_d}))$$

where P is a polynomial with vanishing constant term. In Point (i) we saw that $\|\hat\chi^{d+1}\|_{k+1+s,r_d}<t_d^{-1/2}$ then, we can assume, replacing d_{k+1} by a larger integer if necessary, that $P(\|\hat\chi^{d+1}\|_{k+1+s,r_d})\le\frac{1}{(d+1)(d+3)}$. Now we choose a positive constant $\tilde C_{k+1}$ (independent on d) such that $\|f^{d_{k+1}}\|_{k+1,r_{d_{k+1}}}<\tilde C_{k+1}\frac{d_{k+1}+1}{d_{k+1}+2}$. We then obtain, as in the proof of Point (2) of the previous lemma, that $\|f^d\|_{k+1,r_{d+1}}<\tilde C_{k+1}\frac{d+1}{d+2}$ for any $d\ge d_{k+1}$. Note that $\tilde C_{k+1}$ is a priori not the constant of statement of the lemma. Later in the proof (see the proof of the point (iii)), we will replace it by a larger one.

Proof of (v) : The proof follows the same idea as the proof of Point (5) in the previous lemma. Let d be an integer such that $d \geq d_{k+1} - 1 \geq d_k$.

We have

$$\|\zeta(f^{d+1})\|_{k+1,r_{d+1}} \leq \|\zeta(\hat{\Phi}_{d+1} \cdot f^d - \Phi_{d+1} \cdot f^d)\|_{k+1,r_{d+1}} + \|\zeta(\Phi_{d+1} \cdot f^d)\|_{k+1,r_{d+1}}$$

Writing (7.17) with Point (i) and the estimate $\|f^d\|_{k+1,r_d} < \tilde{C}_{k+1}$, and the interpolation inequality (7.5), we get

$$\begin{aligned} \|\zeta(\Phi_{d+1} \cdot f^d)\|_{k+1,r_{d+1}} &\leq \|\zeta(f^d)\|^2_{k+1+s,r_d} R_k(\|f^d\|_{k+s+1,r_d}, \|\chi^{d+1}\|_{k+1+s,r_d}, \|f^d\|_{k+1,r_d}) \\ &\leq M_k \|\zeta(f^d)\|^{2\frac{k-s-2}{k-1}}_{k,r_d} \|\zeta(f^d)\|^{2\frac{s+1}{k-1}}_{2k-1,r_d} \times \\ &\quad R_k(\|f^d\|^{\frac{k-s-2}{k-1}}_{k,r_d} \|f^d\|^{\frac{s+1}{k-1}}_{2k-1,r_d}, \|\chi^{d+1}\|_{k+1+s,r_d}, \|f^d\|_{k+1,r_d}) \\ &\leq M_k t_d^{-2\frac{k-s-2}{k-1} + 2A\frac{s+1}{k-1} + A\tau\frac{s+1}{k-1}} \end{aligned}$$

(τ is the degree in the first variable of the polynomials R_k introduced in Theorem 7.1). Then, by (7.19) and (7.18), we have $\|\zeta(\Phi_{d+1} \cdot f^d)\|_{k+1,r_{d+1}} \leq M_k t_d^{-\mu}$ where $-\mu < -3/2$ and, replacing d_{k+1} by a larger integer if necessary, we have $\|\zeta(\Phi_{d+1} \cdot f^d)\|_{k+1,r_{d+1}} < \frac{1}{2} t_d^{-3/2}$.

On the other hand, exactly in the same way as in the previous lemma (using the interpolation inequality with k and $2k-1$), we can show that

$$\begin{aligned} \|\hat{\Phi}_{d+1} \cdot f^d - \Phi_{d+1} \cdot f^d\|_{k+1,r_{d+1}} &\leq M_k \|f^d\|_{k+1+s,r_d} \|\hat{\chi}^{d+1} - \chi^{d+1}\|_{k+1+s,r_d} \\ &\leq M_k \|f^d\|^{\frac{s+1}{k-1}}_{2k-1,r_d} \|\hat{\chi}^{d+1} - \chi^{d+1}\|_{k+1+s,r_d} \\ &\leq M_k t_d^{A\frac{s+1}{k-1}} t_d^{-1} (t_d^{-\frac{k-2s-3}{k-1} + A\frac{2s+2}{k-1}} + t_d^{-1+A\frac{2s+2}{k-1}}) \\ &< M_k t_d^{-\mu} \end{aligned}$$

with $-\mu < -3/2$. Then, using (7.15) and replacing d_{k+1} by a larger integer if necessary, we can write

$$\|\zeta(\hat{\Phi}_{d+1} \cdot f^d - \Phi_{d+1} \cdot f^d)\|_{k+1,r_{d+1}} < \frac{1}{2} t_d^{-3/2}.$$

We then obtain for all $d \geq d_{k+1} - 1$,

$$\|\zeta(f^{d+1})\|_{k+1,r_{d+1}} < t_{d+1}^{-1}.$$

Proof of (iii) and (iv) : We first write, using the inequality (7.15), for all $d \geq d_{k+1}$, $\|\zeta(f^d)\|_{2k+1,r_d} \leq \|f^d\|_{2k+1,r_d} T_{2k+1}(\|f^d\|_{k+1,r_d})$ where T_{k+1} is a polynomial. Putting $V_{k+1} := \max(1, T_{k+1}(\tilde{C}_{k+1}))$, we obtain by Point (ii),

$$\|\zeta(f^d)\|_{2k+1,r_d} \leq V_{k+1} \|f^d\|_{2k+1,r_d}. \tag{7.20}$$

We will use this inequality at the end of the proof.

In the same way as in the proof of (3_d) of the previous lemma, we can show that for all $d \geq d_{k+1}$ we have

$$\begin{aligned} \|f^{d+1}\|_{2k+1,r_{d+1}} &\leq M_k(\|f^d\|_{2k+1,r_d} + \|\hat{\chi}^{d+1}\|_{2k+1+s,r_d}\|f^d\|_{k+1,r_d}) \\ &\leq M_k(\|f^d\|_{2k+1,r_d} + \|\hat{\chi}^{d+1}\|_{2k+1+s,r_d}) \quad \text{by (ii)}\,. \end{aligned}$$

By (7.3) and (7.16), we write

$$\begin{aligned} \|\hat{\chi}^{d+1}\|_{2k+1+s,r_d} &\leq M_k t_d^{3s+2}\|\chi^{d+1}\|_{2k-1-2s,r_d} \\ &\leq M_k t_d^{3s+2}\|\zeta(f^d)\|_{2k-1-s,r_d} P(\|f^d\|_{k,r_d}) \\ &\quad +\|f^d\|_{2k-1-s,r_d}\|\zeta(f^d)\|_{k,r_d} Q(\|f^d\|_{k,r_d}) \\ &\leq M_k t_d^{3s+2}(\|\zeta(f^d)\|_{2k-1,r_d} + \|f^d\|_{2k-1,r_d}\|\zeta(f^d)\|_{k,r_d}) \\ &\leq M_k t_d^{A+3s+2}\,. \end{aligned}$$

Now, since $A = 6s+5 > 6s+4$, replacing d_{k+1} by a larger integer if necessary, we can assume that for any $d \geq d_{k+1}$, we have $M_k t_d^{A+3s+2} < \frac{1}{2V_{k+1}} t_d^{3A/2}$ (note that it also implies $M_k < \frac{1}{2V_{k+1}} t_d^{A/2}$). This gives

$$\|f^{d+1}\|_{2k+1,r_{d+1}} \leq \frac{1}{2V_{k+1}} t_d^{A/2}\|f^d\|_{2k+1,r_d} + \frac{1}{2V_{k+1}} t_d^{3A/2}\,. \tag{7.21}$$

We choose a positive constant C_{k+1} such that

$$C_{k+1} > \max\left(1, \tilde{C}_{k+1}, \frac{\|f^{d_{k+1}}\|_{2k+1,r_{d_{k+1}}}}{t^A_{d_{k+1}}}\right).$$

We then have $\|f^{d_{k+1}}\|_{2k+1,r_{d_{k+1}}} < C_{k+1} t^A_{d_{k+1}}$ and, using (7.21) we obtain by induction :

$$\|f^d\|_{2k+1,r_d} < \frac{C_{k+1}}{V_{k+1}} t_d^A < C_{k+1} t_d^A\,,$$

for all $d \geq d_{k+1}$.

Now, by (7.20), we have

$$\|\zeta(f^d)\|_{2k+1,r_d} \leq V_{k+1}\frac{C_{k+1}}{V_{k+1}} t_d^A\,,$$

for all $d \geq d_{k+1}$.

Moreover, the definition of C_{k+1} completes the proof of the point (i), (ii) and (v).

Lemma 7.3 is proved. □

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