A COHOMOLOGY ATTACHED TO A FUNCTION

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Abstract. In this paper, we study a certain cohomology attached to a smooth function, which arose naturally in Poisson geometry. We explain how this cohomology depends on the function, and we prove that it satisfies both the excision and the Mayer-Vietoris axioms. For a regular function we show that the cohomology is related to the de Rham cohomology. Finally, we use it to give a new proof of a well-known result of A. Dimca in complex analytic geometry.

1. Introduction

There are several cohomologies attached to a function that can be defined in terms of differential forms, such as the relative cohomology associated to a singularity, or the cohomology of the complex of logarithmic differential forms associated with the complement of a hyperplane. These cohomologies give, for instance, information on the topology of the complement of the zeros of the function. In this paper, we consider a new cohomology attached to a smooth function on a differentiable manifold.

This new cohomology is also defined in terms of differential forms. More precisely, if M is a differentiable manifold and f is a smooth function on M, we define a coboundary operator

$$\begin{aligned} d_f : \Omega^k(M) &\longrightarrow \Omega^{k+1}(M) \\ \alpha &\longmapsto f d\alpha - k df \wedge \alpha. \end{aligned}$$

where $\Omega^k(M)$ is the space of k-differential forms on M. It is easy to check that $d_f \circ d_f = 0$, and we denote by $H^\bullet_f(M)$ the cohomology associated with the complex $(\Omega^\bullet(M), d_f)$. More generally, for any integer p, we define a coboundary operator

$$\begin{aligned} d^{(p)}_f : \Omega^k(M) &\longrightarrow \Omega^{k+1}(M) \\ \alpha &\longmapsto f d\alpha - (k-p) df \wedge \alpha. \end{aligned}$$

We still have $d^{(p)}_f \circ d^{(p)}_f = 0$ and we denote by $H^\bullet_{f,p}(M)$ the cohomology of this complex. We shall restrict our attention to the cohomology $H^\bullet_f(M)$ but most results readily generalize to the cohomology $H^\bullet_{f,p}(M)$.

This cohomology was considered for the first time in [19] in the context of Poisson geometry, and more generally, Nambu-Poisson geometry. There we have computed this cohomology in the case where f is the germ of a function with an isolated singularity. The aim of this paper is to initiate a systematic study of this cohomology.

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We start, in Section 2, by showing several possible ways of defining this cohomology. First we recall how it arises in Poisson and Nambu-Poisson geometry. Then we construct a certain Lie algebroid attached to a function f for which the Lie algebroid cohomology coincides with $H^\bullet_f(M)$. If 0 is a regular value of a function f, there is another Lie algebroid one can attach to f, namely the Melrose fake tangent bundle of $S = f^{-1}(\{0\})$ (see [4]). This Lie algebroid does not coincide with ours, but they have isomorphic cohomologies. Finally, one can also consider differential forms with a "pole" along S, obtaining a chain complex for which the cohomology is also $H^\bullet_f(M)$.

In Section 3 we study some basic properties of the cohomology. First we discuss how the cohomology varies when the function f changes. In particular, we show that if the function f does not vanish, then the cohomology $H^\bullet_f(M)$ coincides with the de Rham cohomology of M. Then we will show that it is possible to write a Mayer-Vietoris exact sequence, a relative cohomology exact sequence, and an excision theorem, for our cohomology. We also give an appropriate notion of homotopy, but it is an open question whether the cohomology is homotopy invariant in general.

In Section 4 we consider the regular case, i.e., the case where the function f does not have singularities in a neighborhood of $S = f^{-1}(\{0\})$. In this case, we can relate the cohomology with the de Rham cohomology of M and of S, showing that the space $H^k_f(M)$ is isomorphic to $H^k_{dR}(M) \oplus H^{k-1}_{dR}(S)$. As a corollary of this result, one obtains the Poisson cohomology for generic 2-dimensional Poisson structures. In this regular case, we prove homotopy invariance.

Finally, in Section 5, we study the complex case, giving an application of our cohomology to complex algebraic geometry. Namely, we explain how the results we have found in [19] can be applied to give information on the degeneration of a spectral sequence converging to the cohomology of an hypersurface complement. As a corollary, we obtain a new proof of a well-known result of A. Dimca.

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2. Geometrical origins

The following notations will be enforced throughout the paper. We denote by f a smooth function on a n-dimensional manifold M and by $S \subset M$ the level set $f^{-1}(\{0\})$. As usual, $\Omega^k(M)$ denotes the vector space of k-differential forms, and $H^k_{dR}(M)$ the k-th de Rham cohomology group. Dually, $\mathfrak{X}^k(M)$ denotes the vector space of k-vector fields. Also, $[\ ,\ ] : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l-1}(M)$ denotes the Schouten bracket on multi-vector fields. For a cohomology theory, we denote by Z^k (resp. B^k) the space of k-cocycles (resp. k-cobords).

2.1. **The two-dimensional case.** Let M be a Poisson manifold with Poisson 2-vector field $\Pi \in \mathfrak{X}^2(M)$, so that $[\Pi, \Pi] = 0$ (see for instance [4, 14, 27]). If the manifold M has dimension two, this condition is automatically satisfied, so every 2-vector on a 2-dimensional manifold is a Poisson structure.

Assume that (M, Π) is a 2-dimensional orientable Poisson manifold, and fix a volume form $\nu \in \Omega^2(M)$. The contraction $f := i_\Pi \nu$ is a smooth function. We have observed in [19] that the *Poisson cohomology* of (M, Π) is isomorphic to $H^\bullet_f(M)$. Let us recall how this works.

First of all, the Poisson cohomology of (M, Π) is defined to be the cohomology of the following chain complex (see [14]):

$$0 \to \mathfrak{X}^0(M) \xrightarrow{\partial} \mathfrak{X}^1(M) \xrightarrow{\partial} \mathfrak{X}^2(M) \to 0$$

where the boundary map is $\partial(Q) = [Q, \Pi]$. Hence, the map $\partial : \mathfrak{X}^0(M) \to \mathfrak{X}^1(M)$ is the map that associates to a function g its Hamiltonian vector field X_g:

$$\partial(g) = [g, \Pi] \equiv X_g,$$

and $\partial : \mathfrak{X}^1(M) \to \mathfrak{X}^2(M)$ is the map that associates to a vector field X the Lie derivative of Π along X:

$$\partial(X) = [X, \Pi] \equiv \mathcal{L}_X \Pi.$$

This cohomology is an invariant of the Poisson manifold, which has been studied, from different points of view, for instance in [18, 22, 24, 25, 27].

Secondly, we have an isomorphism of chain complexes

$$\phi : (\mathfrak{X}^\bullet(M), \partial) \longrightarrow (\Omega^\bullet(M), d_f),$$

where $\phi^0 : C^\infty(M) \to C^\infty(M)$ is the identity, $\phi^1 : \mathfrak{X}^1(M) \to \Omega^1(M)$ is contraction of ν:

$$\phi^1(X) \equiv -i_X \nu,$$

and $\phi^2 : \mathfrak{X}^2(M) \to \Omega^2(M)$ is the linear application defined by

$$\phi^2(\Gamma) \equiv (i_\Gamma \nu)\nu.$$

The Poisson cohomology of a manifold is, in general, very hard to compute, even in dimension two. Since working with differential forms has many advantages over working with multivectors, one may expect that this isomorphism will lead to actual computations of Poisson cohomology in dimension two. We shall see an example of that in the proof of Theorem 4.11.

2.2. **In higher dimensions.** If M is an orientable manifold of dimension $n > 2$, one generalizes the previous case in a straightforward way. One considers a n-vector $\Lambda \in \mathfrak{X}^n(M)$, and fixes a volume form $\nu \in \Omega^n(M)$, obtaining a smooth function $f := i_\Lambda \nu$. The pair (M, Λ) is no more a Poisson manifold, but it is a *Nambu-Poisson manifold* of degree n, which may be seen as a kind of generalization of Poisson structures (see [23, 26]).

Now we would like to associate a cohomology to the pair (M, Λ), generalizing Poisson cohomology in dimension two. In [13], the authors construct a chain complex (called the Nambu-Poisson complex) associated to any Nambu-Poisson manifold of dimension and of degree larger than 3. This complex is rather difficult to manipulate, but we have shown in [19] that the Nambu-Poisson cohomology of (M, Π) is indeed isomorphic to $H^\bullet_f(M)$.

There is a second complex one can associate to the pair (M, Λ), which also generalizes Poisson cohomology in dimension two, and which is much simpler. One takes

$$0 \to \left(C^\infty(M)\right)^{n-1} \xrightarrow{\partial} \mathfrak{X}^1(M) \xrightarrow{\partial} \mathfrak{X}^n(M) \to 0$$

where the boundary map $\partial : \mathfrak{X}^0(M) \to \mathfrak{X}^1(M)$ is the map that associates to the functions $g_1, \ldots, g_n$ their Hamiltonian vector field $X_{g_1,\ldots,g_{n-1}}$:

$$\partial(g_1, \ldots, g_{n-1}) = i_{dg_1 \wedge \ldots \wedge dg_{n-1}} \Lambda \equiv X_{g_1,\ldots,g_{n-1}},$$

and $\partial : \mathfrak{X}^1(M) \to \mathfrak{X}^n(M)$ is the map that associates to a vector field X the Lie derivative of Λ along X:

$$\partial(X) = [X, \Lambda] \equiv \mathcal{L}_X \Lambda.$$

In the same way as for the 2-dimensional case, one can show that the last two cohomology groups of this chain complex are isomorphic to $H^{n-1}_{f,n-2}(M)$ and $H^n_{f,n-2}(M)$ (see [19]).

2.3. **A Lie algebroid attached to a function.** Recall (see, e.g., [4, 10, 16]) that a *Lie algebroid* over M is a triple $(A, \rho, [\![\,,\,]\!])$ where A is a vector bundle over M, $\rho : A \to TM$ is a bundle map (called the *anchor*), and $[\![\,,]\!]$ is a Lie algebra bracket on the sections $\Gamma(A)$, such that:

- ρ defines a Lie algebra homomorphism $(\Gamma(A), [\![\,,\,]\!]) \to (\mathfrak{X}(M), [\ ,\ ])$;
- for every $u, v \in \Gamma(A)$ and $g \in C^\infty(M)$:

$$[\![u, gv]\!] = g[\![u, v]\!] + \big(\rho(u) \cdot g\big)v.$$

To any Lie algebroid one associates a cohomology $H^\bullet(A)$ by considering the chain complex $(\Omega^\bullet(A), d_A)$, where $\Omega^k(A) \equiv \Gamma(\wedge^k A^*)$ and

$$\begin{aligned} d_A Q(u_0, \dots, u_r) = {}& \frac{1}{r+1} \sum_{k=0}^{r} (-1)^k \rho(u_k) \cdot Q(u_0, \dots, \widehat{u}_k, \dots, u_r) \\ & + \frac{1}{r+1} \sum_{k<l} (-1)^{k+l+1} Q([u_k, u_l], u_0, \dots, \widehat{u}_k, \dots, \widehat{u}_l, \dots, u_r). \end{aligned}$$

Now, for any smooth function f on a manifold M we can attach a Lie algebroid as follows. We take $A = TM$, the anchor $\rho : TM \to TM$ is defined by

$$\rho(X) \equiv fX, \quad X \in \mathfrak{X}(M),$$

and the Lie bracket $[\![\,,\,]\!]$ on $\mathfrak{X}(M)$ is given by

$$[\![X, Y]\!] \equiv \frac{[fX, fY]}{f} = f[X, Y] + (X \cdot f)Y - (Y \cdot f)X, \quad X, Y \in \mathfrak{X}(M).$$

It is easy to check that the triple $(TM, \rho, [\![\,,\,]\!])$ is a Lie algebroid over M and its cohomology is precisely $H^\bullet_f(M)$.

Remark 2.1. The Lie algebroid $(TM, \rho, [\![\,,\,]\!])$ is always integrable to a Lie groupoid since the obstructions to integrability given in [3] vanish.

Remark 2.2. When 0 is a regular value of the function f there is another Lie algebroid attached to f which can be defined as follows (see [17] and [4]). Recall that $S \subset M$ denotes the set $f^{-1}(0)$, which is here an embedded submanifold. It is shown in [17], that the $C^\infty(M)$-module $\mathfrak{X}_S(M)$ of vector fields on M tangent to S is the space of sections of a vector bundle A over M, called the *fake tangent bundle*. On A one has a structure of a Lie algebroid over M, where the bracket is the standard Lie bracket of vector fields, and the anchor may be defined locally as follows. For a point $p \in S$, there exists local coordinates $(U, x, y_2, \dots, y_n)$ such that $U \cap S = \{q \in U : x(q) = 0\}$. If one sets $e_1 = x\frac{\partial}{\partial x}$ and $e_i = \frac{\partial}{\partial y_i}$ for $i > 1$, the e_i's form a local basis of $\mathfrak{X}_S(M)$. The anchor map τ is then defined as $\tau(e_1) = x\frac{\partial}{\partial x}$ and $\tau(e_i) = \frac{\partial}{\partial y_i}$ for $i > 1$. This Lie algebroid does not coincide with the one defined above (the later has points of rank zero, while the first one not), but we will see later (cf. Remark 4.6) that their Lie algebroid cohomologies are isomorphic.

Remark 2.3. For $p \neq 0$ the operator $d_f^{(p)}$ is not a derivation of the exterior algebra, hence the cohomology $H^\bullet_{f,p}(M)$ does not come from a Lie algebroid.

2.4. **Singular k-forms.** Let us call a form $\omega \in \Omega^k(M \setminus S)$ a *singular k-form* if the form $f^k\omega$ can be extended to a smooth form on M. We denote the space of singular k-forms by $\Omega^k_f(M)$.

If $\omega \in \Omega^k_f(M)$ is a singular k-form then $d\omega$ is a singular $(k+1)$-form. In fact, we have

$$f^{k+1}d\omega = d(f^{k+1}\omega) - (k+1)df \wedge (f^k\omega),$$

so $f^{k+1}d\omega$ also extends to a smooth form on M. Therefore we obtain a chain complex $\big(\Omega^\bullet_f(M), d\big)$.

Proposition 2.4. *The cohomology of $\big(\Omega^\bullet_f(M), d\big)$ is isomorphic to $H^\bullet_f(M)$.*

Proof. Define a map of chain complexes $\varphi : \big(\Omega^\bullet_f(M), d\big) \to \big(\Omega^\bullet(M), d_f\big)$ by setting

$$\varphi^k : \Omega^k_f(M) \to \Omega^k(M), \quad \omega \mapsto f^k\omega.$$

It is easy to check that φ induces an isomorphism in cohomology. □

3. Basic Properties

In this section we will study some basic properties of the cohomology defined above.

3.1. **Degree zero cohomology.** If $M \setminus S$ is a dense subset of M (e.g., if f is regular) one can compute the groups $H^0_{f,p}(M)$:

Proposition 3.1. *If $M \setminus S$ is dense in M,*

$$H^0_{f,p}(M) = \begin{cases} 0, & \text{if } p > 0, \\ \\ \mathbb{R}, & \text{if } p \leq 0. \end{cases}$$

Proof. If $p > 0$, note that $d_f^{(p)}(g) = \frac{d(f^p g)}{f^{p-1}}$ for any smooth function g on M. Hence $d_f^{(p)}(g) = 0$ iff $g \equiv 0$, and we obtain $H^0_{f,p}(M) = \{0\}$.

If $p \leq 0$, let g be a function on M such that $d_f^{(p)}(g) = 0$. We have $d\big(\frac{g}{f^{-p}}\big) = 0$ on $M \setminus S$, so $g = \lambda f^{-p}$ on $M \setminus S$ for some $\lambda \in \mathbb{R}$. It follows that $g = \lambda f^{-p}$ on M, so we obtain $H^0_{f,p}(M) \simeq \mathbb{R}$. □

The higher degree cohomology groups are much harder to compute, even in the case where the function vanishes at a single point.

3.2. **Dependence on the function.** A natural question to ask about the cohomology $H^\bullet_f(M)$ is how it depends on the function f. A first result is the following.

Proposition 3.2. *If $h \in C^\infty(M)$ does not vanish, then the cohomologies $H^\bullet_f(M)$ and $H^\bullet_{fh}(M)$ are isomorphic.*

Proof. For each $k \in \mathbb{N}$, consider the linear isomorphism

$$\phi^k : \Omega^k(M) \to \Omega^k(M), \qquad \alpha \longmapsto \frac{\alpha}{h^k}.$$

If α is a k-form on M, one checks easily that

$$\phi^{k+1}(d_{fh}\alpha) = d_f\big(\phi^k(\alpha)\big),$$

so ϕ induces an isomorphism between the cohomologies $H_f^\bullet(M)$ and $H_{fh}^\bullet(M)$. $\square$

Corollary 3.3. *If the function $f \in C^\infty(M)$ does not vanish, then $H_f^\bullet(M)$ is isomorphic to the de Rham cohomology $H_{dR}^\bullet(M)$.*

It follows also that the cohomology $H_f^\bullet(M)$ depends only on the germ of the function f on its set of zeros:

Corollary 3.4. *If g and f are smooth functions on M such that $S = f^{-1}(0) = g^{-1}(0)$ and $g = f$ on some neighborhood of S, then $H_f^\bullet(M) \simeq H_g^\bullet(M)$.*

3.3. **Relative cohomology.** Let N be a submanifold (eventually with boundary) of M. We assume that N is not included in S and we denote by ι the inclusion $N \hookrightarrow M$. The *relative cohomology* groups $H_f^\bullet(M, N)$ are defined exactly as in the case of the de Rham theory (see, e.g., the construction done in [2]).

As in case of the de Rham cohomology, we have a long exact sequence for the pair (M, N):

Theorem 3.5. *There is a long exact sequence*

$$\cdots \to H_f^{k-1}(N) \to H_f^k(M, N) \to H_f^k(M) \xrightarrow{\iota^*} H_f^k(N) \to \cdots$$

Corollary 3.6. *If $M \setminus S$ is dense in M, we have $H_f^0(M, N) = \{0\}$ (and also $H_{f,p}^0(M, N) = \{0\}$).*

Proof. Apply Proposition 3.1 and Theorem 3.5. $\square$

Now, assume that N is the closure of an open subset of M instead of a manifold. We can still define the relative cohomology $H_f^\bullet(M, N)$. In fact, if we denote by $\Omega_N^k(M)$ the vector space formed by the k-forms which vanish on N, then exterior differentiation $d : \Omega_N^k(M) \to \Omega_N^{k+1}(M)$ is well defined. Indeed, if $\alpha \in \Omega_N^k(M)$, then $\alpha = 0$ on the interior of N, and thus $d\alpha = 0$ on N, i.e., $d\alpha \in \Omega_N^{k+1}(M)$. It follows that the differential operator $d_f : \Omega_N^k(M) \to \Omega_N^{k+1}(M)$ is also well defined. Again, imitating the de Rham case, one obtains:

Proposition 3.7. *If N is the closure of an open subset of M then the cohomology of the complex $(\Omega_N^\bullet(M), d_f)$ is isomorphic to the cohomology $H_f^\bullet(M, N)$.*

3.4. **Excision.** We leave it to the reader to check that the following version of the excision property also holds (again, the proof is similar to the de Rham case):

Theorem 3.8. *Let U be an open subset of M with closure in the interior of N. Then, the inclusion $j : (M \setminus U, N \setminus U) \hookrightarrow (M, N)$ induces an isomorphism*

$$j^* : H_f^\bullet(M, N) \longrightarrow H_f^\bullet(M \setminus U, N \setminus U).$$

3.5. **The Mayer-Vietoris sequence.** Since the differential d_f commutes with the restrictions to open subsets, one can construct, in the same way as for the de Rham cohomology (see [2]), a Mayer-Vietoris exact sequence.

Theorem 3.9. *If $\mathcal{U} = (U, V)$ is an open cover of M, we have the long exact sequence*

$$\ldots \to H_f^{k-1}(U \cap V) \to H_f^k(M) \xrightarrow{R} H_f^k(U) \oplus H_f^k(V) \xrightarrow{J} H_f^k(U \cap V) \to \ldots$$

where for $[\omega] \in H_f^k(M)$ and $([\alpha], [\beta]) \in H_f^k(U) \oplus H_f^k(V)$, we define

$$R([\omega]) = ([\omega|_U], [\omega|_V]), \qquad J([\alpha], [\beta]) = [\alpha|_{U\cap V} - \beta|_{U\cap V}].$$

3.6. **Homotopy invariance.** We need to define an appropriate notion of homotopy. Assuming a more functorial approach, let us think of a pair (M, f) as an *object*. In order to think of $H_f^\bullet(M)$ as a functor, we need a notion of *morphism* between such pairs:

Definition 3.10. Let M and N two differentiable manifolds with smooth functions f and g, respectively. A *morphism* Φ from the pair (M, f) to the pair (N, g) is a pair (ϕ, a) formed by a smooth map $\phi : M \to N$ and smooth function $a : M \to \mathbb{R}$, such a does not vanish on M and $g \circ \phi = af$.

We will say that the pairs (M, f) and (N, g) are *equivalent* if there exists a morphism $\Phi = (\phi, a)$ between these two pairs where ϕ is a diffeomorphism. This notion of equivalence between the pairs is sometimes called "contact equivalence" in singularity theory.

A morphism $\Phi = (\phi, a)$ from the pair (M, f) to the pair (N, g) induces a chain map $\Phi^* : (\Omega^\bullet(N), d_g) \to (\Omega^\bullet(M), d_f)$ defined by:

$$\Phi^* : \Omega^k(N) \to \Omega^k(M), \quad \omega \longmapsto \frac{\phi^*\omega}{a^k}.$$

and this map induces an homomorphism in cohomology $\Phi^* : H_g^\bullet(N) \to H_f^\bullet(M)$. If Φ is an equivalence this map is an isomorphism.

Now, we come back to our problem:

Definition 3.11. A *homotopy* from the pair (M, f) to the pair (N, g) is given by two smooth maps

$$h : M \times [0, 1] \to N, \quad a : M \times [0, 1] \to \mathbb{R},$$

such that for each $t \in [0, 1]$, we have a morphism

$$H_t \equiv (h(\cdot, t), a(\cdot, t)) : (M, f) \to (N, g)$$

(i.e., a does not vanish and $g \circ h(x, t) = a(x, t)f(x)$).

If $H = (h, a)$ is a homotopy from (M, f) to (N, g), we obtain a map at the cohomology level

$$H_t^* : H_g^\bullet(N) \to H_f^\bullet(M).$$

The problem of homotopy invariance is the following: given a homotopy H, from (M, f) to (N, g), is it true that $H_0^* = H_1^*$ at the cohomology level? For general pairs (M, f) and (N, g) this seems to be a hard problem. If the complements of the zero level sets of f and g are dense sets, then in degree zero we do have $H_0^* = H_1^* : H_f^0(M) \to H_g^0(N)$. But for higher degree, this is a much more difficult problem. In the next section, we give some partial results in the regular case.

Remark 3.12. One can express the notion of homotopy in terms of singular forms. In fact, it is easy to check that under the correspondence between singular k-forms $\omega \in \Omega_f^k(M)$ and k-forms $f^k\omega \in \Omega^k(M)$ (see the the proof of Proposition 2.4), the map $H_t^* : \Omega^k(N) \to \Omega^k(M)$ corresponds to the pullback $h_t^* : \Omega_g^k(N) \to \Omega_f^k(M)$.

4. The regular case

By *regular* case we mean the case of a function f which does not have singularities in a neighborhood of its zero set (i.e. 0 is a regular value). The subset $S = f^{-1}(\{0\})$ in then an embedded submanifold of M. In order to simplify the exposition we assume that S is connected.

4.1. **Computation of the cohomology.** It follows from Proposition 3.1 that $H^0_f(M)=\mathbb{R}$. Our main result is the following:

Theorem 4.1. *If 0 is a regular value of f then, for each $k \geq 1$, there is an isomorphism*

$$H^k_f(M) \simeq H^k_{dR}(M) \oplus H^{k-1}_{dR}(S).$$

Before we start the proof we need to introduce some notation.

Let $U \subset U'$ be tubular neighborhoods of S. We may assume that $U = S\times]-\epsilon,\epsilon[$ and $U' = S\times]-\epsilon',\epsilon'[$, with $\epsilon' > \epsilon$, and that

$$f|_{U'} : S\times]-\epsilon',\epsilon'[\to \mathbb{R}, \quad (x,t) \longmapsto t.$$

We denote by π the projection $U' \to S$.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function which is 1 on $[-\epsilon,\epsilon]$ and has support contained in $[-\epsilon',\epsilon']$. Note that the function $\rho\circ f$ is 1 on U, and we claim that we can assume that the function $\rho \circ f$ vanishes on $M \setminus U'$. Indeed, let $W = \{x \in M : |f(x)| < \varepsilon'\}$. If $W = U'$ there is nothing to prove. If not, we have $W = U' \cup V$ where U' and V are disjoint open sets. Then, there exists a smooth function $\tilde{f}$ which equals f on U' and such that $|f| > \varepsilon'$ on V. By Corollary 3.4, we can replace f by $\tilde{f}$.

If ν is a form on S, we will denote by $\overline{\nu}$ the form $\rho(f)\pi^*\nu$. Notice that

$$d\overline{\nu} = \rho(f)\pi^*(d\nu) + \rho'(f)df \wedge \pi^*\nu,$$

so we conclude that

$$df \wedge d\overline{\nu} = df \wedge \overline{d\nu}. \tag{4.1}$$

Proof of Theorem 4.1. We split the proof into several lemmas.

Lemma 4.2. *Any k-form ω on M can be decomposed, in an unique way, as*

$$\omega = f^k\omega_k + f^{k-1}\omega_{k-1} + \ldots + f\omega_1 + \omega_0, \tag{4.2}$$

where $\omega_i = \overline{\mu_i} + df \wedge \overline{\nu_i}$, with $\mu_i \in \Omega^k(S)$ and $\nu_i \in \Omega^{k-1}(S)$, for $0 \leq i \leq k-1$.

Proof. Let ω be a k-form on M, and write $\omega = (1-\rho(f))\omega + \rho(f)\omega$. We can decompose $\omega|_{U'}$, in an unique way, as

$$\omega|_{U'} = f^k\theta_k + f^{k-1}\theta_{k-1} + \ldots + f\theta_1 + \theta_0,$$

where $\theta_i = \pi^*\mu_i + df \wedge \pi^*\nu_i$, for $0 \leq i \leq k-1$, with $\mu_i \in \Omega^k(S)$ and $\nu_i \in \Omega^{k-1}(S)$. Now, since $\rho \circ f = 0$ on $M \setminus U'$, the k-form $\rho(f)\omega$ may be written as

$$\rho(f)\omega = f^k\rho(f)\theta_k + f^{k-1}(\overline{\mu_i} + df \wedge \overline{\nu_i}) + \ldots + (\overline{\mu_0} + df \wedge \overline{\nu_0}).$$

On the other hand, since $1-\rho(f)$ is 0 on a neighborhood of S, we can write $(1-\rho(f))\omega = f^k\zeta$ for some k-form ζ, and the result follows. □

In the sequel we denote by Φ the linear application

$$\Omega^k(M) \oplus \Omega^{k-1}(S) \to \Omega^k(M), \quad (\alpha,\beta) \longmapsto f^k\alpha + f^{k-1}df \wedge \overline{\beta}.$$

If $(\alpha,\beta) \in \Omega^k(M) \oplus \Omega^{k-1}(S)$, with $d\alpha = 0$ and $d\beta = 0$ then, using (4.1), we find

$$\begin{aligned} d_f\big(\Phi(\alpha,\beta)\big) &= f^{k+1}d\alpha - f^k df \wedge d\overline{\beta}, \\ &= f^{k+1}d\alpha - f^k df \wedge \overline{d\beta} = 0. \end{aligned}$$

Similarly, one checks that if $\mu \in \Omega^{k-1}(M)$ and $\nu \in \Omega^{k-2}(S)$, then

$$\Phi(d\mu, d\nu) = d_f(f^{k-1}\mu - f^{k-2}df \wedge \overline{\nu}).$$

We conclude that Φ induces a map at the level of cohomology

$$\Phi : H^k_{dR}(M) \oplus H^{k-1}_{dR}(S) \to H^k_f(M),$$
$$([\alpha],[\beta]) \longmapsto [f^k\alpha + f^{k-1}df \wedge \overline{\beta}].$$

Lemma 4.3. *If $k > 1$, Φ is surjective.*

Proof. Let ω be a k-form on M with $d_f\omega = 0$. If we decompose ω as in (4.2), we obtain

$$d_f\omega = f^{k+1}d\omega_k + f^k d\omega_{k-1} + f^{k-1}(d\omega_{k-2} - df \wedge \omega_{k-1}) + \cdots + f(d\omega_0 - (k-1)df \wedge \omega_1) - kdf \wedge \omega_0 = 0.$$

If we restrict to U, we get by uniqueness of the decomposition $df \wedge \omega_0|_U = 0$, i.e. $df \wedge \pi^*\mu_0 = 0$ and so, $\mu_0 = 0$. We conclude that $\omega_0 = df \wedge \overline{\nu_0}$.

Now set $\gamma_0 \equiv \frac{1}{k-1}\overline{\nu_0}$. We have

$$\omega + d_f\gamma_0 = f^k\omega_f + f^{k-1}\omega_{k-1} + \ldots + f^2\omega_2 + f(\omega_1 + d\gamma_0).$$

Noting that $d\gamma_0 = \frac{1}{k-1}\overline{d\nu_0} + \frac{\rho'(f)}{k-1}df \wedge \pi^*\nu_0$, writing $d_f(\omega + d_f\gamma_0) = 0$ and restricting to U, we obtain $\mu_1 + \frac{1}{k-1}d\nu_0 = 0$. Therefore:

$$\omega_1 + d\gamma_0 = df \wedge (\overline{\nu_1} + \frac{\rho'(f)}{k-1}\pi^*\nu_0).$$

Thus, if we put $\gamma_1 = \frac{1}{k-2}\overline{\nu_1} + \frac{\rho'(f)}{k-1}\pi^*\nu_0$, we get

$$\omega + d_f(f\gamma_1 + \gamma_0) = f^k\omega_f + f^{k-1}\omega_{k-1} + \ldots + f^3\omega_3 + f^2(\omega_2 + d\gamma_1).$$

This way, we can construct $\gamma_0, \gamma_1, \ldots, \gamma_{k-2}$, with

$$\gamma_{k-2} = \overline{\nu_{k-2}} + \frac{\rho'(f)}{2}\pi^*\nu_{k-3} + \ldots + \frac{\rho^{(k-2)}(f)}{(k-1)!}\pi^*\nu_0,$$

such that

$$\omega + d_f(\gamma_0 + \ldots + f^{k-2}\gamma_{k-2}) = f^k\omega_k + f^{k-1}(\omega_{k-1} + d\gamma_{k-2}).$$

Now, writing $d_f(\omega + d_f(\gamma_0 + \ldots + f^{k-2}\gamma_{k-2})) = 0$ and restricting to U, we obtain $\mu_{k-1} = -d\nu_{k-2}$ (using $\rho \circ f = 1$) and $d\omega_{k-1|U} = 0$. Consequently, we have

$$\omega_{k-1} + d\gamma_{k-2} = df \wedge \overline{\nu_{k-1}} + \eta,$$

where

$$\eta = df \wedge [\rho'(f)\pi^*\nu_{k-2} + \ldots + \frac{\rho^{(k-1)}(f)}{(k-1)!}\pi^*\nu_0] + \frac{\rho'(f)}{2}\pi^*d\nu_{k-3} + \ldots + \frac{\rho^{(k-2)}(f)}{(k-1)!}\pi^*d\nu_0.$$

Since $d\omega_{k-1}|_U = 0$ and $\eta|_U = 0$, we obtain $d\nu_{k-2} = 0$. On the other hand, since η is zero on a neighborhood of S, we can write $\eta = f\xi$. We conclude that

$$\omega = f^k(\omega_k + \xi) + f^{k-1}df \wedge \overline{\nu_{k-2}} + d_f\gamma,$$

where $\gamma = \gamma_0 + \ldots + f^{k-2}\gamma_{k-2}$. We have seen, that $d\nu_{k-2} = 0$. Now, writing $d_f\omega = 0$, we see that $d(\omega_k + \xi) = 0$. This shows that ω is in the image of Φ. $\square$

Lemma 4.4. *If $k > 1$, Φ is injective.*

Proof. Let (α,β) in $\Omega^k(M)\oplus\Omega^{k-1}(S)$ with $d\alpha=0$ and $d\beta=0$. We assume that $f^k\alpha+f^{k-1}df\wedge\overline{\beta}=d_f\gamma$, where $\gamma\in\Omega^{k-1}(M)$.

We decompose γ as in (4.2), i.e.,

$$\gamma=f^{k-1}\gamma_{k-1}+f^{k-2}\gamma_{k-2}+\ldots+f\gamma_1+\gamma_0,$$

with, for $i\leq k-2$, $\gamma_i=\overline{\mu_i}+df\wedge\overline{\nu_i}$, μ_i and ν_i are forms on S. We have

$$\begin{aligned}d_f\gamma=f^kd\gamma_{k-1}+f^{k-1}d\gamma_{k-2}+f^{k-2}(d\gamma_{k-3}-df\wedge\gamma_{k-2})\\+\ldots+f(d\gamma_0-(k-2)df\wedge\gamma_1)-(k-1)df\wedge\gamma_0.\end{aligned}$$

Restricting to U, we obtain

$$\begin{aligned}df\wedge\gamma_0|_U&=0\\(d\gamma_0-(k-2)df\wedge\gamma_1)|_U&=0\\&\vdots\\(d\gamma_{k-3}-df\wedge\gamma_{k-2})|_U&=0\\d\gamma_{k-2}|_U&=df\wedge\pi^*\beta\end{aligned}\tag{4.3}$$

The first relation gives $df\wedge\pi^*\mu_0=0$ and so, $\mu_0=0$. This implies that $\gamma_0=df\wedge\overline{\nu}_0$. Using the second relation, we then get

$$df\wedge\pi^*d\nu_0+(k-2)df\wedge\pi^*\mu_1=0,$$

which implies $\mu_1=-\frac{1}{k-2}d\nu_0$. In this way, we obtain for each $i\leq k-2$,

$$\mu_i=-\frac{1}{k-1-i}d\nu_{i-1}.$$

Now, since $\gamma_{k-2}=\overline{\mu_{k-2}}+df\wedge\overline{\nu_{k-2}}$, the one before the last relation in (4.3) gives $-df\wedge\pi^*d\nu_{k-2}=df\wedge\pi^*\beta$, which implies $\beta=-d\nu_{k-2}$, i.e., β is exact.

On the other hand, we have, for each $1\leq i\leq k-2$,

$$\begin{aligned}d\gamma_{i-1}-(k-1-i)df\wedge\gamma_i&=d\overline{\mu_{i-1}}-df\wedge d\overline{\nu_{i-1}}-(k-1-i)df\wedge\overline{\mu_i},\\&=\overline{d\mu_{i-1}}+\rho'(f)df\wedge\pi^*\mu_{i-1}\\&\qquad-df\wedge[\overline{d\nu_{i-1}}+(k-1-i)\overline{\mu_i}],\\&=-\frac{\rho'(f)}{k-1-i}df\wedge\pi^*d\nu_{i-1},\end{aligned}$$

and

$$d\gamma_{k-2}=df\wedge\overline{\beta}+\rho'(f)df\wedge\pi^*\mu_{k-2}=df\wedge\overline{\beta}-\rho'(f)df\wedge\pi^*d\nu_{k-3}.$$

We conclude that

$$f^k\alpha=f^kd\big(\gamma_{k-1}+\frac{\rho'(f)}{f}df\wedge\pi^*\nu_{k-3}+\frac{\rho'(f)}{f^2}df\wedge\pi^*\nu_{k-3}+\ldots+\frac{\rho'(f)}{(k-2)f^{k-1}}df\wedge\pi^*\nu_0\big).$$

Therefore, α is exact. □

This shows that Φ is bijective for $k>1$. On the other hand, we have

Lemma 4.5. *If $k=1$, Φ is bijective.*

Proof. To prove that Φ is surjective, let ω be a 1-form on M with $d_f\omega = 0$. We write $\omega = f\omega_1 + \omega_0$ with $\omega_0 = \overline{\mu_0} + df \wedge \overline{\nu_0}$ ($\mu_0 \in \Omega^1(S)$, $\nu_0 \in \Omega^0(S)$). We write $d_f\omega = 0$ an we restrict to U. We obtain $\mu_0 = 0$ hence, $\omega_0 = df \wedge \overline{\nu_0}$. Moreover, we have $d\omega_{0|U} = 0$ which gives $d\nu_0 = 0$. It follows that $d\omega_1 = 0$.

Now to prove that Φ is injective, let $\alpha \in \Omega^1(S)$ and $\beta \in \Omega^0(S)$ with $d\alpha = 0$ and $d\beta = 0$. We suppose that

$$f\alpha + df \wedge \overline{\beta} = d_f\gamma = fd\gamma$$

where $\gamma \in \Omega^0(M)$. Restricting to S, we obtain $\beta = 0$. This implies $\alpha = d\gamma$. □

We have establish that Φ is an isomorphism for all $k \geq 1$ so Theorem 4.1 follows. □

Remark 4.6. Comparing this result with Proposition 2.49 in [17], we see that the cohomology of the Lie algebroid attached to a function constructed in Section 2.3 is isomorphic to the cohomology of the Melrose Lie algebroid.

Remark 4.7. For $k - p > 0$ it is possible to adapt this proof in order to compute the cohomology $H^\bullet_{f,p}(M)$. For $k - p < 0$ the decomposition (4.2) is no longer valid. For $k = p$ the expression for $H^p_{f,p}(M)$ is not so nice. For instance, if $p = 1$, and if $H^1_{dR}(M) = \{0\}$, we can show that the space $H^1_{f,1}(M)$ has infinite dimension. In fact, the space $Z^1_{f,1}(M)$ of 1-cocycles is $\{dh \,|\, h \in C^\infty(M)\}$ which is isomorphic, via exterior differentiation d, to the space $C^\infty_0(M)$ of functions which vanish in at least a point of M. Similarly, the space $B^1_{f,1}(M)$ of 1-cobords is isomorphic, via d, to the ideal of $C^\infty_0(M)$ spanned by f. Therefore, the quotient $C^\infty_0(M)/(f)$ has infinite dimension.

Example 4.8. Let $M = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 < 2\}$ be an open ball and $f : M \to \mathbb{R}$ the function $f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2 - 1$, so that $S \subset M$ is the $(n-1)$-sphere. Then,

$$\begin{aligned} &H^0_f(M) = H^1_f(M) = H^n_f(M) = \mathbb{R}, \\ &H^k_f(M) = \{0\}, \quad \text{if } 2 \leq k \leq n-1. \end{aligned}$$

Example 4.9. Let $M = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^n : x_1^2 + \cdots + x_{n+1}^2 = 1\}$ be the n-sphere and $f : M \to \mathbb{R}$ the function $f(x_1, \ldots, x_{n+1}) = x_1$, so that S is the equator. Then,

$$\begin{aligned} &H^k_f(M) = \mathbb{R} \quad \text{if } k = 0, 1, \\ &H^k_f(M) = \{0\} \quad \text{if } 2 \leq k \leq n-1, \\ &H^n_f(M) = \mathbb{R}^2. \end{aligned}$$

Example 4.10. (Poisson geometry) Recall the identification explained in Section 2.1 between the cohomology $H^\bullet_f(M)$ and Poisson cohomology in dimension 2. It leads immediately to the following result, which generalizes a result due to Radko [24] for the compact case:

Theorem 4.11. *Let (M, Π) be an orientable 2-dimensional Poisson manifold with singular set S. Assume that the contraction of the Poisson tensor Π with a volume form on M does not have singularities in a neighborhood of S. Then the Poisson cohomology of (M, Π) is*

$$H^k_\Pi(M) \simeq H^k_{dR}(M) \oplus H^{k-1}_{dR}(S).$$

4.2. **Homotopy invariance in the regular case.** In the regular case we are able to prove homotopy invariance:

Proposition 4.12. *Let U and W be tubular neighborhoods of $S_f = f^{-1}(0)$ and $S_g = g^{-1}(0)$, respectively. We assume that f and g do not have singularities on U and W. If H_t is a homotopy from (U, f) to (W, g). Then the induced linear applications between the cohomology spaces are the same: $H_1^* = H_0^*$.*

Proof. We can assume that $U = S_f \times]-\varepsilon, \varepsilon[$ and $W = S_g \times]-\varepsilon', \varepsilon'[$, with

$$(x, \rho) \xmapsto{f} \rho \quad \text{and} \quad (y, \tau) \xmapsto{g} \tau.$$

By Proposition 3.1 we can take $k \geq 1$. We denote by Ψ_f and Ψ_g the linear maps:

$$\begin{aligned}
\Psi_f : H^k_{dR}(U) \oplus H^{k-1}_{dR}(U) &\to H^k_f(U) \\
([\alpha], [\beta]) &\longmapsto [\rho^k \alpha + \rho^{k-1} d\rho \wedge \beta], \\
\Psi_g : H^k_{dR}(W) \oplus H^{k-1}_{dR}(W) &\to H^k_f(W) \\
([\alpha], [\beta]) &\longmapsto [\tau^k \alpha + \tau^{k-1} d\tau \wedge \beta],
\end{aligned}$$

which, by Theorem 4.1, are isomorphisms.

Now, we set $K_t^* = \Psi_f^{-1} \circ H_t^* \circ \Psi_g$, for every $t \in [0, 1]$. If $([\alpha], [\beta]) \in H^k_{dR}(W) \oplus H^{k-1}_{dR}(W)$, we have

$$\begin{aligned}
H_t^*\big(\Psi_g([\alpha], [\beta])\big) &= \Big[\frac{h_t^*(\tau^k \alpha + \tau^{k-1} d\tau \wedge \beta)}{a_t^k}\Big], \\
&= \Big[\frac{a_t^k \rho^k h_t^* \alpha + a_t^{k-1} \rho^{k-1}(\rho\, da_t \wedge h_t^* \beta + a_t d\rho \wedge h_t^* \beta)}{a_t^k}\Big], \\
&= [\rho^k h_t^* \alpha + \rho^k \frac{da_t}{a_t} \wedge h_t^* \beta + \rho^{k-1} d\rho \wedge h_t^* \beta], \\
&= [\rho^k h_t^* \alpha + \rho^{k-1} d\rho \wedge h_t^* \beta + \rho^k d(log|a_t| h_t^* \beta)].
\end{aligned}$$

We conclude that

$$\begin{aligned}
K_t^*\big([\alpha], [\beta]\big) &= \big([h_t^* \alpha + d(log|a_t| h_t^* \beta)], [h_t^* \beta]\big), \\
&= \big([h_t^* \alpha], [h_t^* \beta]\big).
\end{aligned}$$

Since the de Rham cohomology is homotopy invariant, we have $K_1^* = K_0^*$ and it follows that $H_1^* = H_0^*$. $\square$

Proposition 4.13. *Let H_t be a homotopy from (M, f) to (N, g). We assume that f and g do not have singularities on tubular neighborhoods of S_f and S_g. If $H^{k-1}_{dR}(S)$ is trivial, then the linear maps H_0^* and H_1^* from $H^k_g(N)$ to $H^k_f(M)$ coincide.*

Proof. Note that the assumptions imply that $k \geq 2$.

Let U and W be tubular neighborhoods of S_f and S_g such that f and g are regular on these neighborhoods. We can assume that H sends W onto U, and we set $V = M \setminus S_f$ and $Z = N \setminus S_g$.

Let ω be in $Z^k_g(N)$. According to the previous proposition, we have

$$(H_1^* \omega)|_U = (H_0^* \omega)|_U + d_f \alpha_U, \quad \alpha_U \in \Omega^{k-1}(U).$$

On the other hand, since f and g do not vanish on V and Z and since the de Rham cohomology is homotopy invariant, we have

$$(H_1^*\omega)|_V = (H_0^*\omega)|_V + d_f\alpha_V, \quad \alpha_V \in \Omega^{k-1}(V).$$

Therefore, we obtain

$$d_f(\alpha_U|_{U\cap V} - \alpha_V|_{U\cap V}) = 0,$$

i.e., $\alpha_U|_{U\cap V} - \alpha_V|_{U\cap V} \in Z_f^{k-1}(U\cap V)$.

Now, since $H_f^{k-1}(U\cap V) \simeq H_{dR}^{k-1}(U\cap V) \simeq \left(H_{dR}^{k-1}(S)\right)^2 = \{0\}$, there exists $\beta_{U\cap V} \in \Omega^{k-2}(U\cap V)$ such that

$$\alpha_U|_{U\cap V} - \alpha_V|_{U\cap V} = d_f\beta_{U\cap V}.$$

From the exactness of the Mayer-Vietoris short exact sequence for de Rham cohomology, there exist $\alpha'_U \in \Omega^{k-2}(U)$ and $\alpha'_V \in \Omega^{k-2}(V)$ such that $\beta_{U\cap V} = \alpha'_V|_{U\cap V} - \alpha'_U|_{U\cap V}$. It follows that

$$(\alpha_U + d_f\alpha'_U)|_{U\cap V} = (\alpha_V + d_f\alpha'_V)|_{U\cap V}.$$

Hence, there exists $\eta \in \Omega^{k-1}(M)$ such that

$$\eta|_U = \alpha_U + d_f\alpha'_U \quad \text{and} \quad \eta|_V = \alpha_V + d_f\alpha'_V.$$

This gives

$$(d_f\eta)|_U = (H_1^*\omega - H_0^*\omega)|_U \quad \text{and} \quad (d_f\eta)|_V = (H_1^*\omega - H_0^*\omega)|_V,$$

which shows that

$$H_1^*\omega - H_0^*\omega = d_f\eta.$$

□

5. One step to the complex case

The definition of the cohomology $H_f^\bullet(M)$ readily extends to complex manifolds. In this section we study the local case and give an application of this cohomology to the study of the topology of the complement of a hypersurface.

We feel that this cohomology may have others applications in algebraic geometry or in analytic geometry, and that from it one may be able to obtain more information on the topology of the complement of the zeros of a function f.

5.1. **Cohomology in the the local case.** In this paragraph we give an overview of the results we have found in [19, 20]. There we consider a germified version of the cohomology: we let $\Omega^k(\mathbb{C}^n)$ denote the space of germs at 0 of analytic k-forms, and we let $H_{f,p}^\bullet(\mathbb{C}^n)$ denote the cohomology of the chain complex $(\Omega^k(\mathbb{C}^n), d_f^{(p)})$. We consider only the groups $H_{f,p}^{n-1}(\mathbb{C}^n)$ and $H_{f,p}^n(\mathbb{C}^n)$. The other groups are usually trivial, with the exception of H^0 and H^1 (see [19, 20]).

We will assume that the function f is a quasi-homogeneous polynomial on $\mathbb{C}^n$ of degree N, with respect to the weights $w_1, \ldots, w_n$, and with an isolated singularity at 0. We denote by c the *Milnor number* of the singularity, i.e., the dimension of the vector space $Q_f = \mathcal{O}_n/I_f$ where $\mathcal{O}_n$ is the space of germs of analytic functions and I_f the ideal spanned by the first derivatives of f. Also, for every positive integer q, we denote by $h^{q,n-q}$ the dimension of $(Q_f)_{qN-w_1-\cdots-w_n}$, the quasi-homogeneous part of degree $qN - w_1 - \ldots - w_n$ of the graded space Q_f. These numbers are the mixed Hodge numbers of the quasi-homogeneous singularity f.

Table 1 summarizes the results obtained in [19].

	$\dim H^{n-1}_{f,p}(\mathbb{C}^n)$	$\dim H^{n}_{f,p}(\mathbb{C}^n)$
$0 \leq p \leq n-3$	$\sum_{i=1}^{n-p-1} h^{i,n-i}$	$c + \sum_{i=1}^{n-p-1} h^{i,n-i}$
$p = n-2$	∞	$c + h^{1,n-1}$
$p = n-1$	?	∞
$p \geq n$	0	c

Table 1

Remark 5.1. For $k > 0$ denote by $\Omega^k_{rel}(\mathbb{C}^n, f)$ the quotient $\Omega^k(\mathbb{C}^n)/df \wedge \Omega^{k-1}(\mathbb{C}^n)$. It is easy to check that the de Rham differential d passes to the quotient, so we get a complex $(\Omega^{\bullet}_{rel}(\mathbb{C}^n, f), d)$. The cohomology of this complex is the well-known *relative cohomology* of the singularity f. This cohomology seems to be linked with the cohomology $H^{\bullet}_{f,p}(\mathbb{C}^n)$, but they do not coincide (e.g., compare the table above with the results in [21]). Nevertheless, the computation of the cohomology $H^{\bullet}_{f,p}(\mathbb{C}^n)$ presented in [19], uses the vanishing of certain relative cohomology spaces of f.

5.2. **Cohomology of the complement of a hypersurface.** We shall now explain a method, using the cohomology $H^{\bullet}_{f,p}(\mathbb{C}^n)$, to obtain information on the cohomology of the complement of a hypersurface. More precisely, we apply this cohomology to determine at which stage a certain spectral sequence converging to the cohomology of a hypersurface singularity degenerates. We then use this to give a new proof of a well-known result of A. Dimca ([5]).

5.2.1. *Local case.* Let B be a small open ball at the origin of $\mathbb{C}^n$. We consider a hypersurface singularity $V \subset B$ at the origin. Let $f = 0$ be an equation for V in B and denote by $U = B \setminus V$ the complement. A well-known result of Grothendieck ([12]) states that the cohomology $H^{\bullet}(U, \mathbb{C})$ is isomorphic to the cohomology of the complex $A_0^{\bullet}$ of meromorphic differential forms on B with polar singularity along V.

An element $\omega \in A_0^k$ can be written in the form $\omega = \frac{\alpha}{f^s}$ where α is a holomorphic k-form on B. We consider the decreasing filtration:

$$F^s A_0^j = \begin{cases} \left\{ \frac{\alpha}{f^{j-s}} : \alpha \text{ holomorphic on } B \right\} & \text{if } j - s \geq 0, \\ \{0\} \quad \text{if } j - s < 0. \end{cases}$$

This filtration is exhaustive and bounded above so, it induces a spectral sequence $\big(E_r(V), d_r\big)$ converging to $H^{\bullet}(U, \mathbb{C})$ (see [15]). It is known (see [6]) that this spectral sequence degenerates after a finite number of steps. The problem is to determine this number.

For every p, q, r we set

$$E_r^{p,q}(V) = \frac{Z_r^{p,q}(V)}{Z_{r-1}^{p+1,q-1}(V) + B_{r-1}^{p,q}(V)}$$

where $Z_r^{p,q}(V)$ and $B_r^{p,q}(V)$ are well-known spaces (see [15]) and $d_r^{p,q} : E_r^{p,q}(V) \to E_r^{p+r,q-r+1}(V)$. This spectral sequence degenerates at the step r, i.e., $E_r = E_\infty$, if $d_r^{\cdot,\cdot} = 0$. In order to show that $d_r^{\cdot,\cdot} = 0$ (for some r) it is sufficient to show that for every p, q we have

$$d\big(Z_r^{p,q}(V)\big) \subset B_{r-1}^{p+r,q-r+1}(V). \tag{5.1}$$

If we remark that, for a holomorphic $(p+q)$-form α, we have $d\big(\frac{\alpha}{f^q}\big) = \frac{d_f^{(p)}\alpha}{f^{q+1}}$, then we can rewrite (5.1) as:

- if α is a holomorphic $(p+q)$-form such that f^r divides $d_f^{(p)}\alpha$ then, for some holomorphic $(p+q)$-form ζ, one has $d_f^{(p)}\alpha = d_f^{(p)}(f\zeta)$.

It is known that when the function f does not have singularities one has $E_1 = E_\infty$. Now, we assume that f has an isolated singularity at 0. In this case, it is known (see [6]) that $d_1^{p,q} = 0$ if $p + q < n - 1$. Let us look then at $d_r^{p,q}$ with $p + q = n - 1$. We assume further that f is a W-quasi-homogeneous polynomial of degree N, where $W = w_1 x_1 \frac{\partial}{\partial x_1} + \dots w_n x_n \frac{\partial}{\partial x_n}$, with each w_i a positive integer. This means that:

$$W \cdot f = Nf.$$

In [19], we have computed the spaces $H^n_{f,p}(B)$ under these assumptions, and we recall here our results. We set $Q_f = \mathcal{H}(B)/I_f$, where $\mathcal{H}(B)$ is the algebra of holomorphic functions on B and I_f the ideal spanned by $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$. This vector space has finite dimension (the Milnor number of f) and we let $\mathcal{B}$ denote a monomial basis (for the existence of such a basis, see [1]). Finally, we set $\nu = dx_1 \wedge \dots \wedge dx_n$.

Theorem 5.2 ([19])**.** *Assume that $p < n-1$ and let $\eta \in \Omega^n(B)$. There exist unique polynomials $h_1, \dots, h_{n-p}$ (possibly zero) such that:*

(a) *h_1 is quasi-homogeneous of degree $N - \sum w_i$;*
(b) *h_j for $2 \le j \le n-p-1$ is a linear combination of monomials of $\mathcal{B}$ of degree $jN - \sum w_i$;*
(c) *h_q is a linear combination of monomials of $\mathcal{B}$ and*

$$\eta = (h_{n-p} + f h_{n-p-1} + \dots + f^{n-p} h_1)\nu \pmod{B^n_{f,p}(B)}.$$

This theorem allows us to give a new proof of the following result (see [5]).

Corollary 5.3. *If f is a quasi-homogeneous polynomial with an isolated singularity at 0, then the spectral sequence degenerates after the second step, i.e., $E_2 = E_\infty$.*

Proof. We only need to consider $d_2^{p,q}$ with $p+q = n-1$. Also, if $q = 0$ it is easy to see that $d_2^{n-1,0} = 0$, so we assume $q > 0$, i.e., $p < n-1$.

Let α be an $(n-1)$-form on B. We will show that if for some n-form θ on B one has $f^2\theta = d_f^{(p)}\alpha$, then there exists an $(n-1)$-form ζ such that $d_f^{(p)}\alpha = d_f^{(p)}(f\zeta)$.

By Theorem 5.2, if θ is a holomorphic n-form on B, we have

$$f\theta = (h_{n-p-1} + f h_{n-p-2} + \dots + f^{n-p-1} h_1)\nu + d_f^{(p+1)}\zeta,$$

where ζ is a holomorphic $(n-1)$-form and the h_i are as in the theorem. It follows from Lemma 3.12 in [19] that $d_f^{(p+1)}\zeta \in I_f$. Since f is also in I_f we must have $h_{n-p-1} = 0$. Since $f d_f^{(p+1)}\zeta = d_f^{(p)}(f\zeta)$, we see that

$$f^2\theta = (f^2 h_{n-p-2} + \dots + f^{n-p} h_1)\nu + d_f^{(p)}(f\zeta).$$

Hence, if $f^2\theta = d_f^{(p)}\alpha \in B^n_{f,p}(B)$, we have $(f^2 h_{n-p-2} + \ldots + f^{n-p}h_1)\nu \in B^n_{f,p}(B)$. The previous relation then implies that $h_{n-p-2} = \cdots = h_1 = 0$. Therefore, we conclude that $d_f^{(p)}\alpha = f^2\theta = d_f^{(p)}(f\zeta)$. $\square$

5.2.2. *Global projective case.* Let W be the vector field $w_1 x_1 \frac{\partial}{\partial x_1} + \ldots w_n x_n \frac{\partial}{\partial x_n}$, with $w_1, \ldots, w_n$ positive integers. We denote by $\mathbb{P}^n(W)$ the weighted projective space associated to W (see [8]). We consider a quasi-homogeneous polynomial (with respect to W) $f \in \mathbb{C}[x_0, \ldots, x_n]$ of degree N and we denote by V the hypersurface in $\mathbb{P}^n(W)$ with equation $f = 0$. Again, the cohomology $H^\bullet(U, \mathbb{C})$, where $U = \mathbb{P}^n(W) \setminus V$), is isomorphic to the cohomology of the complex $A^\bullet$ of algebraic differential forms (see [12]).

An element ω in A^k can be written as $\omega = \frac{\alpha}{f^s}$ where α is a quasi-homogeneous k-form of degree sN. This means that $i_W\alpha = 0$ and the Lie derivative satisfies $\mathcal{L}_W\alpha = (sN)\alpha$. In this case we can consider the following decreasing filtration

$$\tilde{F}^s A^j = \begin{cases} \left\{ \begin{array}{c} \frac{\alpha}{f^{j-s}} : \alpha \text{ quasi-homogeneous of degree } (j-s)N \\ \text{and } i_W\alpha = 0 \end{array} \right\} & \text{if } j - s > 0, \\ \{0\} \quad \text{if } j - s \leq 0. & \end{cases}$$

This filtration induces a spectral sequence $(\tilde{E}_r(V), \tilde{d}_r)$ converging to $H^\bullet(U, \mathbb{C})$. If f does not have singularities, this spectral sequence degenerates after the first step (see [11]). Now, we assume that f has an isolated singularity at 0. In this case, one knows that $\tilde{d}_1^{p,q} = 0$ if $p + q < n - 1$ (see [6]).

Proposition 5.4. *If f is a quasi-homogeneous polynomial with an isolated singularity at 0, then the spectral sequence degenerates after the second step, i.e., $\tilde{E}_2 = \tilde{E}_\infty$.*

Proof. According to [6], we only need to consider $\tilde{d}_2^{p,q}$ with $p + q = n - 1$. As for the local case, we need to show that if α is a quasi-homogeneous $(n-1)$-form on $\mathbb{C}^{n+1}$, of degree qN ($q = n - 1 - p$) such that $i_W\alpha = 0$ and f^2 divides $d_f^{(p)}\alpha$, then there exists a quasi-homogeneous $(n-1)$-form ζ which satisfies $i_W\zeta = 0$ and $d_f^{(p)}\alpha = d_f^{(p)}(f\zeta)$.

Let us denote by η the n-form $d_f^{(p)}\alpha$. It is easy to check that $i_W\eta = 0$. Therefore, we have $\eta = i_W(g\nu)$, where g is some quasi-homogeneous polynomial of degree $(q+1)N - \sum w_i$. Set $\sigma = i_W\nu$, so that $\eta = g\sigma$. By Lemma 3.26 in [19], we have

$$\eta \in B^n_{f,p}(\mathbb{C}^{n+1}) \Longleftrightarrow g\nu \in B^{n+1}_{f,p+1}(\mathbb{C}^{n+1}).$$

Since f^2 divides g, we can write $g\nu = f^2\xi$, where ξ is some quasi-homogeneous $(n+1)$-form on $\mathbb{C}^{n+1}$. We then have $f^2\xi \in B^{n+1}_{f,p+1}(\mathbb{C}^{n+1})$. Now, it is possible to adapt the argument we gave above in the local case (theorem 5.2 is still valid in a polynomial version because of the homogeneity of the operators $d_f^{(p)}$) to obtain $f^2\xi = d_f^{(p+1)}(f\mu)$, where μ is a quasi-homogeneous n-form. We conclude that

$$\eta = i_W\big(d_f^{(p+1)}(f\mu)\big) = -d_f^{(p)}\big(f(i_W\mu)\big).$$

$\square$

6. A link with the Witten complex ?

In order to give an analytic proof of the Morse inequalities, E. Witten defined in [28], a deformed differential on the complex of smooth differential forms on a compact manifold M. If $t \in \mathbb{R}$ and f is a Morse function on M, it is defined by

$$\delta_{tf} = d + tdf \wedge \cdot$$

The idea is that the cohomology of this complex is isomorphic to the de Rham cohomology for every t and to use Hodge theory and properties of the laplacien operator associated to the differential δ_{tf}. For a short presentation, see for instance [29].

This Witten complex looks similar to the one defined here even if it is clear that they are not the same (the cohomology defined here is not always isomorphic to the de Rham cohomology). However, there might be an important difference between these two cohomology : the definition of the Witten differential does not involves the zero set of the function. Whereas, as we saw, the great feature of the cohomology we defined in this paper is that it deals with the singularities of the function on its zero set : the singularities at points which are not zeros of the function do not matter.

The cohomology of the Witten deformed differential may be, once more, related to the topology of the fiber $f^{-1}(c)$. In [9], the "polynomial" case on a noncompact manifold with a cylindrical end is studied and the cohomology is related to the relative cohomology of the couple $(M, f^{-1}(-c))$ (where $c > 0$ is sufficiently large).

In a complex context, A. Dimca and M. Saito used the differential $\delta_f = d - df \wedge \cdot$, where f is a polynomial, on the complex of global algebraic differential form in order to compute the cohomology of a generic fiber $f^{-1}(c)$ (see [7]). More precisely, they showed that for any k, we have $H^{k+1}(\Omega^\bullet, D_f) \simeq \tilde{H}^k(f^{-1}(c), \mathbb{C})$ where $\tilde{H}$ denotes the reduced cohomology. In the isolated singularities case, the reduced cohomology of the fiber is closely related to the relative cohomology of the singularity. We saw in [19] that this cohomology plays an important part in the computation of the spaces $H_f^\bullet$ in the local case and might be closely linked with it.

At this moment, we are not able to give a precise link (if there is one) between the Witten complex and ours but we hope that the techniques used on the Witten complex may be applied to the complex defined here.

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