1. Mapping class actions on the character variety

Goldman has studied the induced action of $\mathcal{MCG}$ on the real character variety of the free group on two generators. The function

$$\kappa : (x, y, z) \mapsto x^2 + y^2 + x^2 - xyz$$

is invariant along the $\mathcal{MCG}$ orbits and Goldman gives a complete picture of the dynamics level sets of κ.

Horowitz determined the group of (polynomial) automorphisms of κ showing that it is isomorphic to

$$PGL(2, \mathbb{Z}) \bowtie (\mathbb{Z}/2 \oplus \mathbb{Z}/2).$$

The action of $\mathbb{Z}/2\oplus\mathbb{Z}/2$ is by (double) sign change of the coordinates e.g. $(x, y, z) \mapsto (x, -y, -z)$. The mapping class group $\mathcal{MCG}$ of the holed trus surjects onto the $PGL(2, \mathbb{Z})$ subgroup of the automorphisms of κ.

When $G = PGL(2, \mathbb{R})$, the group of (possibly orientation-reversing) isometries of $\mathbb{H}$, a similar analysis was begun by Stantchev in his thesis. One obtains similar dynamical systems, where $\mathcal{MCG}$ acts now on the space of representations into the group

$$G_{\pm} := SL(2, C) \times GL(2, \mathbb{R}) \times iGL(2, \mathbb{R})$$

doubly covers the two-component group $PGL(2, \mathbb{R})$. These $G_{\pm}$ representations are again parametrized by traces. They comprise four components, one of which is the subset of $\mathbb{R}^3$ parametrizing $SL(2, \mathbb{R})$ representations discussed above. The other three components are

$$\mathbb{R} \times i\mathbb{R} \times i\mathbb{R}, i\mathbb{R} \times \mathbb{R} \times i\mathbb{R}, i\mathbb{R} \times i\mathbb{R} \times \mathbb{R}.$$

respectively. Consider $\mathbb{R} \times i\mathbb{R} \times i\mathbb{R}$. For$-14 \le t < 2$, the $\mathcal{MCG}$ action is ergodic, but when $t < 14$, wandering domains appear. The wandering domains correspond to homotopy-equivalences $\Sigma \to P$, where P is a hyperbolic surface homeomorphic to a two-holed projective plane. The action is ergodic on the complement of the wandering domains.

Problem 3.4. Determine the ergodic behavior of the $\mathcal{MCG}$-action on the level sets

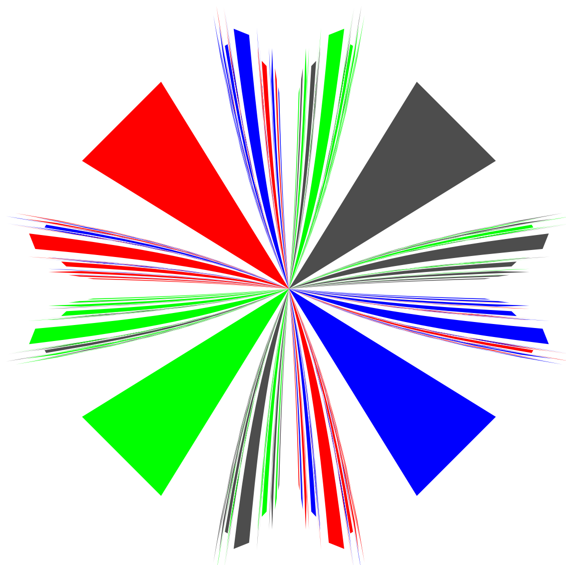
$$\mathbb{R} \times i\mathbb{R} \times i\mathbb{R} \cap \kappa^{-1}(t)$$

where $t > 2$. The level sets for $t > 6$ contains wandering domains corresponding to Fricke spaces of a one-holed Klein bottle.

1.1. A little notation.

The quadratic reflections Q_x, Q_y, Q_z are involutions of $\mathbb{C}^3$ which, together with the sign change automorphisms, generate the group of automorphisms, Γ_{11}, of a certain component of the character variety, denoted $\mathcal{X}_{11}$. The group $\mathcal{X}_{11}$ is commensurable with the group of automorphisms induced by elements of the mapping class group.

$$Q_x : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} yz - x \\ y \\ z \end{pmatrix}, Q_y : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ xz - y \\ z \end{pmatrix}, Q_z : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ -xz - y \end{pmatrix}$$



filled K=3.pdf

FIGURE 1. Part of the orbit of $\mathcal{W}$ for $\kappa = 3$.

1.2. **Two remarks.** The following two observation allow one to completely determine the dynamics solving Goldman's problem.

Observation 1: The first of these is that one can find a linearization of the dynamics at a "global fixed point" for Γ_{11}: for any k the point $(0, 0, \sqrt{k+2}) \in \kappa^{-1}(k)$ is a fixed point of every element of Γ_{11}. The linearization is a homomorphism

$$\begin{array}{rcl} \phi_K : \Gamma_{11} & \rightarrow & GL(T_{(0,0,\sqrt{k+2})}\kappa^{-1}(k)), \\ f^* & \mapsto & D_{(0,0,\sqrt{k+2})} f^* \end{array} .$$

Since the Γ_{11} action preserves an area form on the level set $\kappa^{-1}(k))$ the image of ϕ_K is actually contained in SL.

One might hope that the linearization determines the dynamics on the whole level set. In fact, we shall use the action of this linearization on the tangent plane and the associated projective line to serve as a model for the dynamics on the rest of the level set.

Observation 2: Let Z denote the infinite cyclic subgroup of Γ_{11} generated by $Q_x Q_y$; note that $(Q_x Q_y)^{-1} = Q_y Q_x$.

The second observation is that the subset of $\mathbb{R}^2$

$$\mathcal{W} := \{(x,y) : x > 0,\, y > 0,\, y^2 - (\sqrt{K+2})xy + x^2 < 0\}.$$

$$(y - \frac{1}{2}(\sqrt{K+2} + \sqrt{K-2})x)(y - \frac{1}{2}(\sqrt{K+2} - \sqrt{K-2})x) < 0$$

is precisely invariant under the subgroup Z of Γ_{11}.

Recall that (following Maskit) a set Y is *precisely invariant* under the subgroup H of G iff,

- $H = \mathrm{Stab}_G(Y)$
- $g(Y) \cap Y = \emptyset$, $\forall g \in G1 - H$

1.3. **Results.**

Theorem 1.3.1 (Mc, Tan). *With the notation above*

- *The images of* $\mathcal{W}$ *are disjoint.*
- *The orbit of* $\mathcal{W}$ *is dense in* $\mathbb{R}^2$.
- *The complement*

$$(\cup_{g\in\Gamma_{11}} g(\mathcal{W}))^c$$

is closed and its Hausdorff dimension is $1 + \delta(\phi_K(\Gamma_{11}))$. *Here* $\delta(\phi_K(\Gamma_{11}))$ *is the Hausdorff dimension of the limit set of the linearization of* Γ_{11} *acting on* $\mathbb{P}^1$.

Note that $\delta(\phi_K(\Gamma_{11})) \leq 2$ with equality if and only if $K = 2$

In the Figure, the coloured regions are all images of $\mathcal{W}$ under Γ_{11}. The content of our theorem is that the coloured wedges are all disjoint and that they fill up the plane very snuggly (the word length of the elements is < 6 in the picture).