Greg McShane Research Statement

I am principally interested in the theory of surfaces and their moduli. The subject, tracing its origins to the foundational work of Fricke and Klein through the work of Teichmueller then the school of Ahlfors and Bers, has been a focus of intense activity since the seminal work of Thurston the 1970's. The central theme of the subject is the interplay between the Teichmueller space and its compactifications, the action of the mapping class group of the surface, and the the moduli space viewed as a quotient of the Teichmueller space by the mapping class group. Starting with these three objects one may choose between several distinct approaches, each with its own language and set of tools, depending on the type of questions one might pose. To illustrate this I cite the following three very important and topical areas of research in surface theory:

- geometry of moduli – systems of coordinates, symplectic/poisson structure, quantization.
- geometry of the curve complex – combinatorics of curves and systems of curves on the surface.
- Teichmueller dynamics – the Teichmueller geodesic flow, Veech surfaces, billiards.

Interaction between the many workers in the field has resulted in a rich rapidly evolving body of knowledge, many new tools and techniques and an ever growing list of fascinating unresolved problems.

My principal contribution to the subject is the discovery of identities among relative Poincaré series for the mapping class group $\mathcal{MCG}$– so called *geometric identities* or *McShane's identities.* The simplest of these identities holds for a once punctured torus,

$$\sum_{\alpha} \frac{2}{1+e^{\ell(\alpha)}} = 1, \qquad (1)$$

where the sum is over all simple closed curves α and $\ell(\alpha)$ is the length of the unique geodesic freely homotopic to α. The mapping class group of the torus $\mathcal{MCG}$ acts transitively on homotopy classes of simple curves, and the stabilizer of a curve is a dihedral group D generated by a Dehn twist and the elliptic involution. One forms a relative Poincaré series by summing the pull-back of the obvious function over the cosets $\mathcal{MCG}/D$ and one obtains an equivalent identity (1)

$$\sum_{f\in\mathcal{MCG}/D} f^*\left(\frac{2}{1+e^{\ell(\alpha)}}\right) = \sum_{f\in\mathcal{MCG}/D} \frac{2}{1+e^{\ell(f_*^{-1}(\alpha))}} = 1.$$

Now, interpreting this as a partition of unity adapted to the action of $\mathcal{MCG}$ and multiplying by an $\mathcal{MCG}$ invariant volume form and unwinding the sum one obtains an effective method for calculating the volume of the moduli space. Relations among the volumes of moduli space for different values of g, n is a problem that has generated much important work since Witten conjectured that the associated generating function satisfied a KdV hierarchy.

Over the past year I have been working on two rather projects, of which I shall now give brief accounts, which address two questions with very different flavours.

1 Hitchin component and geometric identities

In [9] I established a identity for the lengths of simple closed geodesics on a punctured hyperbolic surface, namely

$$\sum_{\alpha,\beta} \frac{2}{1+e^{\frac{\ell(\alpha)+\ell(\beta)}{2}}} = 1,$$

where the sum is over all pairs of geodesics α, β which bound an embedded punctured cylinder. Using the same method, M. Mirzakhani [10] extended this identity to hyperbolic surfaces with geodesic boundary. Although it is possible to state and prove identities for surfaces with multiple boundary components, for the sake of clarity, Σ will denote a complete hyperbolic surface with a single totally geodesic boundary component $\partial\Sigma$. The notion of length extends to a finite set of curves $\{C_i\}_i$ by $\ell(\{C_i\}_i) = \sum_i \ell(C_i)$. With this notation Mirzakhani's version of McShane's identity is

$$\ell(\partial\Sigma) = \sum_{P\in\mathcal{P}} \log\left(\frac{e^{\frac{\ell(\partial P)}{2}} + e^{\ell(\partial\Sigma)}}{e^{\frac{\ell(\partial P)}{2}} + 1}\right), \tag{2}$$

where $\mathcal{P}$ is the set of embedded pants (with marked boundary) up to homotopy such that first the boundary component of the pair of pants is $\partial\Sigma$.

In an article with Francois Labourie [1] we show that the identity above has a natural formulation in terms of (generalized) cross ratios. Then, using this formulation, we study identities arising from the cross ratios constructed for representations in $SL(N,\mathbb{R})$ by Labourie [7]. The associated moduli space is called the *Hitchin component*, for $N = 2$ this is just the Teichmueller space and for $N > 2$ it is homeomorphic to an open ball of dimension

$$(2g-2+n) \times 1/2N(N+1).$$

We give a brief overview of the main ideas.

Let Σ be a closed surface. and $\partial_\infty\pi_1(\Sigma)$ be the *boundary at infinity* of the fundamental group $\pi_1(\Sigma)$ of Σ. A *cross ratio* on $\partial_\infty\pi_1(\Sigma)$ is a $\pi_1(\Sigma)$-invariant Hölder function on

$$\partial_\infty\pi_1(\Sigma)^{4*} = \{(x,y,z,t) \in \partial_\infty\pi_1(\Sigma)^4 \ x \neq t, \text{ and } y \neq z\},$$

satisfying certain rules, the most significant being the *multiplicative cocycle identities*. To every non trivial element γ of the group $\partial_\infty \pi_1(\Sigma)$ we associate a positive number, $\ell_b(\gamma)$, called the *period* of γ

$$\ell_b(\gamma) = \log b(\gamma^-, \gamma y, \gamma^+, y),$$

where γ^+ and γ^- are respectively the attractive and repulsive fixed points of γ in $\partial_\infty \pi_1(\Sigma)$ and where y is any point of $\partial_\infty \pi_1(\Sigma)$ such that $\gamma(y) \neq y$. A complete hyperbolic metric on Σ gives rise to an identification of $\partial_\infty \pi_1(\Sigma)$ with the real projective line, the classical cross ratio on the projective line then gives rise to a cross ratio on $\partial_\infty \pi_1(\Sigma)$ The period of γ is just the hyperbolic length of the closed geodesic freely homotopic to γ.

A pair of pants P with marked boundary in Σ corresponds to a triple (α, β, γ) of elements of $\pi_1(\Sigma)$, unique up to conjugation, such that $\alpha\gamma\beta = 1$. Tthe *pant gap function* G_b at P to be the positive number

$$G_b(P) = \log(b(\alpha^+, \gamma^-, \alpha^-, \beta^+),$$

where b a cross ratio on $\partial_\infty \pi_1(\Sigma)$.

The general form of the McShane identity is:

Theorem 1 *Let Σ be closed surface. Let b be a cross ratio on $\partial_\infty \pi_1(\Sigma)$. Let α be a non trivial element of $\pi_1(\Sigma)$. Let $\mathcal{P}$ be the set of homotopy classes of pair of pants with marked boundary in Σ whose first boundary component is α, then*

$$\ell_b(\alpha) = \sum_{P \in \mathcal{P}} G_b(P).$$

For $SL(2, \mathbb{R})$ the pant gap function can be computed in terms of the length of the boundary components of pants using hyperbolic trigonometry [10]. We show in [1] how to determine G_b using just Thurston's *shear coordinates* and elementary manipulations involving the classical cross ratio. In [3] Fock and Goncharov introduced a far reaching generalisations of Thurston's shear coordinates, which we call FGT coordinates, on the (augmented) moduli space of positive representations. We compute the gap functions for Hitchin representations using FGT coordinates for the moduli of pants; since the augmented moduli space is a "covering" of the space of Hitchin representations, we obtain $(n!)^3$ different answers because of the action of the Weyl group. It turns out that, for a suitable choice of FGT coordinates, the pant gap function has a nice expression. However, using the explicit description of the holonomies [4] in the case of $n = 3$, we see the pant gap function has in general a very complicated expression for some choices of coordinates.

2 Multiplicities in the simple length spectrum

Let $\Sigma_{g,n}$ be a surface of genus g with n boundary components together with a hyperbolic structure σ, that is a metric of constant curvature -1, of finite area such that the boundary curves are totally geodesic. Define the *full length spectrum* to be the collection of all lengths of primitive closed geodesics on the surface counted with multiplicities and the *simple length spectrum* to be the set of lengths of all simple closed geodesics counted with multiplicities. As the hyperbolic structure varies, the length spectrum changes.

In his thesis Margulis gave a solution to counting problem for the full length spectrum for a closed surface. The number of primitive geodesics of length less than L is

$$e^{\delta L}/L,$$

where δ is the entropy of the (recurrent part of) the geodesic flow. In an article [17] with Igor Rivin I gave a complete solution to the counting problem for the simple length spectrum was solved for the once punctured torus with a hyperbolic metric: The number of primitive geodesics of length less than L is

$$A(\sigma)L^2$$

where $A(\sigma)$ is a (non constant) real analytic function on the Teichmueller space of the punctured torus. Building on our work Mirzakhani proved that there is an asymptotic for a surface of genus g with n holes equipped with a hyperbolic structure: The number of primitive geodesics of length less than L on $\Sigma_{g,n}$ is

$$A(\sigma)L^{6g-6+2n}.$$

The significance of $6g-6+2n$ is that it is the dimension of the Teichmueller space. Evidently the simple spectrum is a very different object from the full length spectrum: Whilst the full length spectrum encodes information about the geodesic flow on the unit tangent bundle of the surface the simple length spectrum encodes information about the point in the moduli space determined any the surface.

In an article with Hugo Parlier [16] we study the following three questions: Is there a surface for which all the multiplicities are 1? How big is the set of such surfaces? Is it possible to deform a surface such that the multiplicity stays 1 for all simple geodesics?

It is a surprising result of Randol [18], following work of Horowitz, that given $N > 0$ there are N distinct closed geodesics of the same length. So Randol's theorem gives a negative response to all three of these questions.

We study the (non-empty) subsets $\mathcal{E}(\alpha, \beta)$ of Teichmüller space where a pair of distinct simple closed geodesics α, β have the same length. When the intersection number $i_{alg}(\alpha, \beta)$ is small, the surfaces $\mathcal{E}(\alpha, \beta)$ play an important part in the theory of fundamental domains for the mapping class group in low genus, see for instance [15], but there seems to be little or no work in the literature when $i_{alg}(\alpha, \beta)$ is large. Our main theorem is the following:

Theorem 2 *The set of surfaces with simple simple length spectrum is dense and its complement is Baire meagre.*

Any path in the Teichmüller space of the surface passes through a surface which has at least two distinct simple closed geodesics of the same length.

Let $\mathcal{E}$ denote the set of all surfaces with at least one pair of simple closed geodesics of equal length; $\mathcal{E}$ is the union of a countable family of nowhere dense subsets, namely the sets $\mathcal{E}(\alpha, \beta)$ where α, β vary over all distinct simple closed geodesics. The theorem asserts that $\mathcal{E}$ is dense and moreover that the complement contains no arcs and is thus totally disconnected.

We apply this analysis to give an infinite set of counterexamples to a conjecture of Paul Schmutz which we now describe. A *Markoff number* is an integer such that there is a solution $x, y, z \in \mathbb{Z}\, x \geq y \geq z > 2$ of the equation

$$x^2 + y^2 + z^2 - xyz = 0,$$

and we say that (x, y, z) is a *Markoff triple*. The *Markoff number conjecture*, apparently first stated by Frobenius, states that each Markoff number x determines a unique Markoff triple. More recently this problem had a number of authors [12],[14] have obtained partial results for x having satisfying special arithmetic properties. Following the work of Harvey Cohn [13]and others the Markoff number conjecture is equivalent to the fact that the multiplicity of any number in the simple length spectrum of the modular once punctured torus is at most 6. The *modular once punctured torus* is characterized by the fact that it is a regular 6-fold covering of the modular orbifold and thus it has the maximal possible isometry group, isomorphic to C_6. Schmutz's conjectured [19] that the simple length spectrum of the one holed torus has multiplicity at most 6. Our method yields infinitely many one holed tori with at least one value in the simple spectrum of multiplicity at least 12.

3 Perspectives

There are many interesting questions and projects related to the work discussed above. I shall give brief statements of just two of my current projects each of which is an extension of one the above.

3.1 Analytic continuation on the character variety

The first project concerns the relation between geometric identities and the Baker-Hausdorff-Campbell formula. Recall that the classical Baker-Campbell-Hausdorff formula gives a recursive way to compute the Hausdorff series $H = ln(e^X e^Y)$ for non-commuting X, Y. Formally H lives in the graded completion of the free Lie algebra L generated by X, Y.

Tan, Wong and Zhang [11] have proven variations of McShane's identity for surfaces with cone angles less than or equal to π. For example on the torus with a single cone point of angle θ they obtain

$$\sum_{\alpha} \frac{e^{i\theta/2} + e^{\ell(\alpha)}}{e^{-i\theta/2} + e^{\ell(\alpha)}} = \theta/2.$$

The standard philosophy is that cone angles are boundary components of "imaginary length" i.e. the monodromy round a boundary loop is elliptic. Do and

Norbury have indicated that a by proving identities for surfaces with cone angles up to 2π one can obtain important new relations between the volumes of moduli space. In note in preparation I give a proof of their result using the Baker-Campbell-Hausdorff formula to prove absolute convergence of the series defining the identity on the $SL(2,\mathbb{C})$ character variety and then using analytic continuation. This method, in principle, allows one to prove the formulae that Do and Norbury have obtained by another methods but there are still some subtle points concerning the action of the mapping class group on the character variety to be clarified.

Vadim Schectman has often asked the following questions: why does the term in the geometric identities take the form it does and what are the possible relations, if any with the generating function, for the Bernoulli numbers? Recall that the term has the form

$$\frac{e^{\frac{\ell(\partial P)}{2}} + e^{\ell(\partial\Sigma)}}{e^{\frac{\ell(\partial P)}{2}} + 1}. \tag{3}$$

where $\ell(\partial P)$ is the sum of the lengths of an embedded pair of pants in a surface Σ with a single boundary component $\partial\Sigma$. In a remarkable preprint V. Kurlin gives a closed explicit formula for $H = ln(e^X e^Y)$ in a linear basis of the graded completion of the free metabelian Lie algebra $L/[[L,L],[L,L]]$. At a purely formal level Kurlin's solutions resemble (3) but his moduli are not simply lengths or periods but the adjoint operators $\operatorname{ad} X, \operatorname{ad} Y$. In addition, his exposition makes clear the role the Bernoulli numbers play in the derivation of his formulae. It seems that the geometric identities we have found may avatars of a deeper relation at work between the adjoint operators. Such relations might help to clarify the relations between the many different identities obtained with Labourie due to the action of the Weyl group.

3.2 Lengths

The procedure which we use in [16] is non-constructive. Essentially it is based on applying the intermediate value theorem to certain a function $\ell(\alpha) - \ell(\beta)$ where α, β are carefully chosen simple curves. The related function $\cosh(\ell(\alpha)) - \cosh(\ell(\beta))$ is polynomial on the character variety of the fundamental group of the holed torus, By a more careful analysis of these polynomials we hope to make the procedure more 'èffective", in particular deciding whether or not the roots can occur at certain integers. This problem seems well adapted to "computer assisted proof" and I have already begun a program of computer experiments on the large sets of polynomials .

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