Geometric identities, cross ratios and the Hitchin component

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(joint work with Greg McShane)

Identities for lengths of simple closed geodesics in hyperbolic geometry: In [] certain identities for lengths of simple closed geodesics on punctured hyperbolic surfaces were established. To simplify our exposition we state these results for surfaces Σ with a single puncture (and later a single boundary componentd). When C is a closed curve then we denote by $\ell(C)$ the infimum of the set of lengths of curves freely homotopic to C with respect to the hyperbolic metric; this notation extends to C a finite set of curves C_i by $\ell(C) = \sum_i \ell(C_i)$. Let $\mathcal{P}$ be the set of homotopy class of embedded punctured pants (with marked boundary) in Σ, then the McShane identity can be written

$$1 = \sum_{P\in\mathcal{P}} \frac{1}{1+e^{\frac{\ell(\partial P)}{2}}}. \tag{1}$$

Using the same method, M. Mirzakhani [] extends this identity to the case of hyperbolic surfaces with geodesic boundary. Let Σ be a complete hyperbolic surface with one totally geodesic boundary component. Let $\mathcal{P}$ be the set of homotopy class of embedded pants (with marked boundary) in Σ such that first the boundary component of the pair of pants is the boundary of Σ. Then Mirzakhani's version of McShane identity reads

$$\ell(\partial S) = \sum_{P\in\mathcal{P}} \log\big(\frac{e^{\frac{\ell(\partial P)}{2}} + e^{\ell(\partial S)}}{e^{\frac{\ell(\partial P)}{2}}+1}\big). \tag{2}$$

The purpose of the talk is twofold.

- Firstly, we shall explain that the each of the identities above has a natural formulation in terms of (generalized) cross ratios.
- Secondly, using this formulation, we turn our attention to the cross ratios constructed for representations in $SL(n,\mathbb{R})$ by Labourie [].

Let us explain our constructions.

Cross ratio and periods: Let Σ be a closed surface. Let $\partial_\infty\pi_1(\Sigma)$ be the boundary at infinity of the fundamental group $\pi_1(\Sigma)$ of Σ. A *cross ratio* on $\partial_\infty\pi_1(\Sigma)$ is a $\pi_1(\Sigma)$-invariant Hlder function on

$$\partial_\infty\pi_1(\Sigma)^{4*} = \{(x,y,z,t)\in\partial_\infty\pi_1(\Sigma)^4 \ x\neq t, \text{and } y\neq z\},$$

satisfying some rules (see Paragraph **??**, and compare with Otal's original definition in []), the most significant being the "multiplicative cocycle type" identities

$$\begin{aligned} b(x,y,z,t) &= b(x,y,z,w)b(x,w,z,t),\\ b(x,y,z,t) &= b(x,y,w,t)b(w,y,z,t). \end{aligned}$$

Moreover, we can associate to every non trivial element γ of the group $\partial_\infty\pi_1(\Sigma)$ a positive number $\ell_b(\gamma)$ called the *period* of γ (see Equation (**??**)). If γ^+ and γ^-

are respectively the attractive and repulsive fixed points of γ in $\partial_\infty\pi_1(\Sigma)$, then

$$\forall y \in \partial_\infty\pi_1(\Sigma), \quad \ell_b(\gamma) = \log(\gamma^-, \gamma y, \gamma^+, y).$$

The first example of a cross ratio arises in the context of hyperbolic geometry. Indeed, every hyperbolic metric on Σ gives rise to an identification of $\partial_\infty\pi_1(\Sigma)$ with the real projective line. The classical cross ratio on the projective line then gives rise to a cross ratio on $\partial_\infty\pi_1(\Sigma)$, the period of γ being the length of the closed geodesic corresponding to γ.

Pant gap function and the generalised formula: Given a crossratio on $\partial_\infty\pi_1(\Sigma)$, the *pant gap function* associate to every homotopy class of immersed pair of pants with marked boundary a positive number. Every pair of pants P with marked boundary in S correspond to a triple (α, β, γ) of elements of $\pi_1(\Sigma)$, well defined up to conjugation and such that

$$\alpha\gamma\beta = 1.$$

We define the value of pant gap function at P to be the positive number

$$G_b(P) = \log(b(\alpha^+, \gamma^-, \alpha^-, \beta^+).$$

Theorem 1. *Let Σ be closed surface. Let b be a cross ratio on $\partial_\infty\pi_1(\Sigma)$. Let α be a non trivial element of $\pi_1(\Sigma)$. Let $\mathcal{P}$ be the space of homotopy classes of pair of pants with marked boundary in Σ whose first boundary component is α, then*

$$\ell_b(\alpha) = \sum_{P\in\mathcal{P}} G_b(P).$$

Moreover, the theorem generalises to open surfaces of finite type after a suitable extension of the notion of cross ratio in this context.

Cross ratios and hyperbolic geometry: It is specific to the case of hyperbolic geometry that the pant gap function can be computed in terms of the length of the boundary components. Indeed, it is well known that every hyperbolic pair of pants with totally geodesic boundary is completed determined by the length of its boundary. Using Thurston's *shear coordinates* and elementary manipulations involving the classical cross ratio – as opposed to hyperbolic trigonometry in the original proofs – we recover in Section **??**, Mirzakhani-Mc Shane's formulas (1) and (2) for the pant gap function.

Cross ratios and $SL(n,\mathbb{R})$: Let us now explain the relation with representations of $\pi_1(\Sigma)$ in $SL(n,\mathbb{R})$ for a closed surface Σ. In [], the first author gives an interpretation of a connected component of the space of representations of the fundamental group of Σ in $SL(n,\mathbb{R})$ as the space of cross ratios satisfying an extra functional identity depending on n. The corresponding representations are called *Hitchin representations*. As an example consider $SL(2,\mathbb{R})$ where the associated cross ratio, i.e. the classical cross ratio on the projective line, satisfies an additional relation

$$b(t,y,z,x) = 1 - b(x,y,z,t). \tag{3}$$

Conversely, if we have a cross ratio b on a set A satisfying Relation (3), it is well known that the set injects in the projective line in such a way that the cross ratio b is induced by the classical cross ratio. Therefore, if furthermore the cross ratio is invariant under the action of a group, we obtain a representation of this group in $PSL(2,\mathbb{R})$. Working a little more, one obtains a bijection between Teichmller space and the set of cross ratios on $\partial_\infty\pi_1(\Sigma)$ satisfying the extra Relation (3). The article [] generalises this situation to $SL(n,\mathbb{R})$.

Therefore, every Hitchin representation of $\pi_1(\Sigma)$ in $SL(n,\mathbb{R})$ can be represented by a cross ratio on $\partial_\infty\pi_1(\Sigma)$. Unfortunately, as opposed to the case of hyperbolic geometry, it is not true anymore that the pant gap function of an embedded pair of pants can be computed only from the monodromies of the boundary components. The gap function depends on some "internal parameters" that we now describe.
Hitchin representation for open surfaces. Our aim is now to describe a "good" set of representations of the fundamental group of an open surface – in particular a pair of pants – and to describe coordinates, generalising shear coordinates, for the corresponding moduli space. For short, we say an element in $SL(n,\mathbb{R})$ is *purely loxodromic* if it is real split with eigenvalues of multiplicity one. Let S be a compact surface with or without boundary components. We define a *Fuchsian representation* of $\pi_1(\Sigma)$ in $SL(n,\mathbb{R})$ to be a representation which is the composition of

- a discrete faithful representation without parabolics in $SL(2,\mathbb{R})$,
- the irreducible representation of $SL(2,\mathbb{R})$ in $SL(n,\mathbb{R})$.

More generally, a *Hitchin representation* is a representation of $\pi_1(\Sigma)$ in $SL(n,\mathbb{R})$ such that

- the boundary components have purely loxodromic images by the representation,
- it can be deformed to a Fuchsian representation in such a way the images of the boundary components stay purely loxodromic.

Hitchin representations for closed surfaces have been studied in [], [] and []. It is shown in theses articles, that they are discrete and faithful, that every non trivial element is purely loxodromic and that the mapping class group acts properly on the moduli space of Hitchin representations. We prove [] a "doubling" theorem which in particular implies that we can always find a closed surface Σ containing S such that every Hitchin representation of $\pi_1(S)$ is the restriction of a Hitchin representation of $\pi_1(\Sigma)$. Conversely, by Theorem **??**, the restriction of a Hitchin representation to a surface embedded in another one is a Hitchin representation. It then follows that these representations are *positive* in the sense of V. Fock and A. Goncharov.
Pant gap functions in Fock-Goncharov coordinates: From our previous discussion, we obtain coordinates on the space of Hitchin representations of the fundamental group of a pair of pants. In their article [] Fock and Goncharov introduced a far reaching generalisations of Thurston's shear coordinates, which we call Fock-Goncharov coordinates, on the (augmented) moduli space of positive representations. To compute the gap functions for Hitchin representations we use their coordinates for the moduli of pants. Actually, since the augmented moduli space is a "covering" of the space of Hitchin representations, we obtain $(n!)^3$ different sets of coordinates. It turns out that for a suitable choice of coordinates, the

pant gap function has a nice expression. On the other hand , using a computer assisted proof and the explicit description of the holonomies given by V. Fock and A. Goncharov in [] even in the case of $n = 3$, the pant gap function has a very complicated expression for other choices of coordinates.
Possible applications and conclusion: Using her identities, M. Mirzakhani gives a recursive formula for the volume of moduli space of hyperbolic structure, *i.e* the quotient of Teichmller space by the mapping class group. From the work of the first author in [], it follows that the mapping class group acts properly on the moduli space of Hitchin representations. The formulae we obtain, combined with the use of Fock-Goncharov coordinates should enable one to compute geometric quantities associated to the corresponding quotient. However the volume is not the quite the right thing to compute since for $n \geq 3$, one can show it is infinite. We conclude by remarking that so many of the features of hyperbolic geometry translate (via generalized cross ratios) to the setting of Hitchin representations, we might therefore say that this is in fact a *higher (rank) Thurston theory*.

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