1 Multiplicities in the simple length spectrum

Let $\Sigma_{g,n}$ be a surface of genus g with n boundary components together with a hyperbolic structure σ, that is a metric of constant curvature -1, of finite area such that the boundary curves are totally geodesic. Define the *full length spectrum* to be the collection of all lengths of primitive closed geodesics on the surface counted with multiplicities and the *simple length spectrum* to be the set of lengths of all simple closed geodesics counted with multiplicities. As the hyperbolic structure varies, the length spectrum changes.

In his thesis Margulis gave a solution to counting problem for the full length spectrum for a closed surface. The number of primitive geodesics of length less than L is

$$e^{\delta L}/L,$$

where δ is the entropy of the (recurrent part of) the geodesic flow. In an article [17] with Igor Rivin I gave a complete solution to the counting problem for the simple length spectrum was solved for the once punctured torus with a hyperbolic metric: The number of primitive geodesics of length less than L is

$$A(\sigma)L^2$$

where $A(\sigma)$ is a (non constant) real analytic function on the Teichmueller space of the punctured torus. Building on our work Mirzakhani proved that there is an asymptotic for a surface of genus g with n holes equipped with a hyperbolic structure: The number of primitive geodesics of length less than L on $\Sigma_{g,n}$ is

$$A(\sigma)L^{6g-6+2n}.$$

The significance of $6g-6+2n$ is that it is the dimension of the Teichmueller space. Evidently the simple spectrum is a very different object from the full length spectrum: Whilst the full length spectrum encodes information about the geodesic flow on the unit tangent bundle of the surface the simple length spectrum encodes information about the point in the moduli space determined any the surface.

In an article with Hugo Parlier [16] we study the following three questions: Is there a surface for which all the multiplicities are 1? How big is the set of such surfaces? Is it possible to deform a surface such that the multiplicity stays 1 for all simple geodesics?

It is a surprising result of Randol [18], following work of Horowitz, that given $N > 0$ there are N distinct closed geodesics of the same length. So Randol's theorem gives a negative response to all three of these questions.

We study the (non-empty) subsets $\mathcal{E}(\alpha, \beta)$ of Teichmüller space where a pair of distinct simple closed geodesics α, β have the same length. When the intersection number $i_{alg}(\alpha, \beta)$ is small, the surfaces $\mathcal{E}(\alpha, \beta)$ play an important part in the theory of fundamental domains for the mapping class group in low genus, see for instance [15], but there seems to be little or no work in the literature when $i_{alg}(\alpha, \beta)$ is large. Our main theorem is the following:

Theorem 1 *The set of surfaces with simple simple length spectrum is dense and its complement is Baire meagre.*

Any path in the Teichmüller space of the surface passes through a surface which has at least two distinct simple closed geodesics of the same length.

Let $\mathcal{E}$ denote the set of all surfaces with at least one pair of simple closed geodesics of equal length; $\mathcal{E}$ is the union of a countable family of nowhere dense subsets, namely the sets $\mathcal{E}(\alpha, \beta)$ where α, β vary over all distinct simple closed geodesics. The theorem asserts that $\mathcal{E}$ is dense and moreover that the complement contains no arcs and is thus totally disconnected.

We apply this analysis to give an infinite set of counterexamples to a conjecture of Paul Schmutz which we now describe. A *Markoff number* is an integer such that there is a solution $x, y, z \in \mathbb{Z}\, x \geq y \geq z > 2$ of the equation

$$x^2 + y^2 + z^2 - xyz = 0,$$

and we say that (x, y, z) is a *Markoff triple.* The *Markoff number conjecture*, apparently first stated by Frobenius, states that each Markoff number x determines a unique Markoff triple. More recently this problem had a number of authors [12],[14] have obtained partial results for x having satisfying special arithmetic properties. Following the work of Harvey Cohn [13]and others the Markoff number conjecture is equivalent to the fact that the multiplicity of any number in the simple length spectrum of the modular once punctured torus is at most 6. The *modular once punctured torus* is characterized by the fact that it is a regular 6-fold covering of the modular orbifold and thus it has the maximal possible isometry group, isomorphic to C_6. Schmutz's conjectured [19] that the simple length spectrum of the one holed torus has multiplicity at most 6. Our method yields infinitely many one holed tori with at least one value in the simple spectrum of multiplicity at least 12.

References

[1] J. O. Button. The uniqueness of the prime Markoff numbers. *J. London Math. Soc. (2)*, 58(1):9–17, 1998.

[2] Harvey Cohn. Representation of Markoff's binary quadratic forms by geodesics on a perforated torus. *Acta Arith.*, 18:125–136, 1971.

[3] Mong Lung Lang and Ser Peow Tan. A simple proof of the markoff conjecture for prime powers. *available on the arxiv:math.NT/0508443*, 2005.

[4] Bernard Maskit. A picture of moduli space. *Invent. Math.*, 126(2):341–390, 1996.

[5] Greg McShane and Hugo Parlier. Multiplicities of simple closed geodesics and hypersurfaces in Teichmüller space. Preprint ArXiv/mathgt., 2007.

[6] Greg McShane and Igor Rivin. Simple curves on hyperbolic tori. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(12):1523–1528, 1995.

[7] Burton Randol. The length spectrum of a Riemann surface is always of unbounded multiplicity. *Proc. Amer. Math. Soc.*, 78(3):455–456, 1980.

[8] Paul Schmutz Schaller. Geometry of Riemann surfaces based on closed geodesics. *Bull. Amer. Math. Soc. (N.S.)*, 35(3):193–214, 1998.