Length series on Teichmuller space

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Introduction

Conventions:

1. For γ an essential closed curve on a surface $l_\gamma(x)$ is the length of the geodesic homotopic to γ where x is the point in the moduli space determined by the metric on the surface.

2. For a homeomorphism $h : M \to M$ and a geodesic γ, $h(\gamma)$ is the *geodesic* homotopic to the image of γ under h.

3. if γ is an oriented curve then $-\gamma$ is the curve with the opposite orientation.

Definitions:

- Let M = *one holed torus*. $\pi_1(M)$ freely generated by two loops γ_1, γ_2 which meet in a single point and such that their commutator is a loop, δ, around the hole.

- M uniformized by a representation
$$\rho : \pi_1 = \langle \gamma_1, \gamma_2 \rangle \to \mathrm{SL}_2(\mathbf{R})$$
$\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ = hyperbolic element of translation length l_δ.

- The mapping class group = the group of orientation preserving diffeomorphisms up to isotopy.
$$\mathcal{MCG} = \pi_0(\mathrm{Diffeo}^+(M))$$
(Nielsen and Mangel)
$$\mathcal{MCG} <_2 \pi_0(\mathrm{Diffeo}(M)) \cong \mathrm{Aut}(\pi_1)/\mathrm{Inn}(\pi_1) \cong \mathrm{GL}_2(\mathbf{Z}).$$
[symplectic representation]

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  - $\mathcal{T}_1(l_\delta)$ = *Teichmuller space of* M
  - $\mathcal{T}_1(l_\delta)/\mathcal{MCG} = \mathcal{M}_1(l_\delta)$ = *moduli space*.

Two questions of Troels

Theorem 1

$$\sum \frac{2}{1+\exp l_\gamma} = 1$$

where the sum extends over all closed simple curves γ on a hyperbolic punctured torus.

Question 1: Proof using *Markoff cubic*?

$$a^2 + b^2 + c^2 - abc = 0,\ a, b, c > 2.$$

Answer: Bowditch summation argument over the edges of the tree, $\mathbf{T}$, of solutions to this equation.

Question 2: Proof using *Wolpert's formula* for variation of length?

Let μ_1, μ_2 be closed simple geodesics on a hyperbolic surface.

Variation of l_{μ_1} along the Fenchel-Nielsen vector field $t(\mu_2)$ associated to μ_2:

$$dl_{\mu_1}.t(\mu_2) = \sum_{z \in \mu_1 \cap \mu_2} \cos(\theta_z),$$

Observation A: Bowditch's "accounting" avoids considering the following divergent series

$$\sum_{\{a,b,c\} \in \mathbf{T}} \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \right) = \sum_{\{a,b,c\} \in \mathbf{T}} 1 = \sharp\text{vertices of } \mathbf{T} = \infty.$$

Observation B: Derivative of this divergent series $= 0$.

Another example

Theorem 2 *For a one holed torus M :*

$$\sum_\gamma \text{arctan}\left(\frac{\cosh(l_\delta/4)}{\sinh(l_\gamma/2)}\right) = \frac{3\pi}{2},$$

where the sum extends over all simple closed geodesics γ on M and l_δ is the length of the boundary geodesic δ.

Another divergent series

Let $\alpha, \beta =$ pair of *oriented* closed simple geodesics on M meeting in exactly one point and such that the *signed angle* $\alpha \vee \beta > 0$

- $([\alpha], [\beta]) =$ basis of $H_1(M, \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$

  $\Rightarrow$ stabiliser (α, β) in $\mathcal{MCG} =< 1 >$ (symplectic representation faithful).

- $g \in \mathcal{MCG} =$ orientation preserving homeomorphism

  $\Rightarrow g(\alpha) \cap g(\beta) =$ single point and $g(\alpha) \vee g(\beta) > 0$

Formal series:

$$\mathcal{Q} = \sum_{g \in \mathcal{MCG}} g(\alpha) \vee g(\beta).$$

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Summation 1 via

- coset decomposition (Dehn twists)
- hyperbolic geometry (limits of iterated Dehn twists)

Step one: Rewrite as a sum over the set of coset representatives $\mathcal{MCG}/\langle T_\gamma \rangle$ where T_γ is the Dehn twist along $\gamma =$ closed simple geodesic.

Lemma 3

$$\mathcal{Q} = \sum_{h \in \mathcal{MCG}/\langle T_\gamma \rangle} \sum_{p \in \langle T_\gamma \rangle} hp(\alpha) \vee hp(\beta) = \sum_{\gamma \in \mathcal{G}_0} \sum_{n \in \mathbf{Z}} T_\gamma^n(\alpha) \vee T_\gamma^{n+1}(\alpha), \tag{1}$$

where the outer sum is over all oriented simple closed geodesics $\mathcal{G}_0$ and γ is the unique simple closed geodesic that meets α, β exactly once.

Notes: for a one holed torus:

- $\mathcal{MCG}$ acts transitively on $\mathcal{G}_0$
- stabiliser of $\gamma \in \mathcal{G}_0$ is precisely $\langle T_\gamma \rangle$
  $\Rightarrow \mathcal{MCG}/\langle T_\gamma \rangle$ is in 1-1 correspondence with $\mathcal{G}_0$.

Step two: evaluating the inner sum over $\mathbf{Z}$:

Lemma 4 *Let γ be a simple closed curve and γ' any simple closed geodesic meeting γ exactly once. Let $A \in \rho(\pi_1)$ be an element such that* a$xis(A)/\langle A\rangle = \gamma$ *and let $\sqrt{A} \in SL_2(\mathbf{R})$ denote the square root of A . Then there exists $c \in H^2$, and $\widehat{\alpha_n} \subset H^2$ such that $\forall n \in \mathbf{Z}$:*

1. *$\widehat{\alpha_n}$ is a lift of $T_\gamma^n(\gamma')$.*

2. *if $a_n := \widehat{\alpha_n} \cap$ a$xis(A)$ then $a_n = (\sqrt{A})^n(a_0)$.*

3. *$c \in \widehat{\alpha_n}$.*

Morever, let γ^+ (resp. γ^-)be the geodesic passing through c and asymptotic to a$xis(A)$ *at the attracting (resp. repelling) fixed point of A. Then:*

$$\widehat{\alpha_n} \to \gamma^{\pm},$$

as $n \to \pm\infty$ and where the convergence is uniform on compact sets.

- lifts of the orbit of γ' under T_γ as per diagram

- angles in the sum are = the angles between consecutive $\widehat{\alpha_n}$ at the point c.

- The sum "telescopes" over n:

$$\begin{aligned}\textstyle\sum_{n\in\mathbf{Z}} T_\gamma^n(\gamma') \vee T_\gamma^{n+1}(\gamma') &= \gamma^- \vee \gamma^+ \\ &= \pi - 2\arctan\left(\frac{\cosh(l_\delta/4)}{\sinh(l_\gamma/2)}\right).\end{aligned}$$

Summation 2 via

- element of finite order
- coset decomposition

Element of order 2, $q \in \mathcal{MCG}$ (note α, β oriented):

$$q(\alpha) = \beta$$
$$q(\beta) = -\alpha,$$

Sum over cosets of $\mathcal{MCG}/\langle q \rangle$:

$$\mathcal{Q} = \sum_{g \in \mathcal{MCG}/\langle q \rangle} g(\alpha) \vee g(\beta) + gq(\alpha) \vee gq(\beta) = \sum_{\mathcal{MCG}/\langle q \rangle} \pi \quad (2)$$

since

$$g(\alpha) \vee g(\beta) + g(\beta) \vee g(-\alpha) = \pi.$$

Observation:

" $\Rightarrow$ " variation of the $\mathcal{Q}$ vanishes when viewed as a 1-form.

Formally :

$$\mathcal{Q}' : x \mapsto \sum_{\gamma} 2 \arctan \left(\frac{\cosh(l_\delta(x)/4)}{\sinh(l_\gamma(x)/2)} \right), \ \mathcal{T}_1(l_\delta) \to \mathbf{R}$$

is constant.

Main idea

$\mathcal{Q}'$ is constant i.e.:

$$0 = d\mathcal{Q}' = d\mathcal{Q} = \sum_g d(g(\alpha) \vee g(\beta)).$$

Justify the rearrangements used in summation 2 i.e. show that RHS *converges absolutely*.

Use "Fenchel-Nielsen geometry" of the cotangent bundle avoid metric. Consider pairing of $d\mathcal{Q}$ with the *Fenchel-Nielsen vector fields* $t(\mu)$ associated to a simple closed geodesic μ.

Absolute convergence = Absolute convergence for numerical series.

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Theorem 5 *Let μ be a simple closed geodesic $t(\mu)$ the associated Fenchel-Nielsen vector field then the series*

$$\sum_{g \in \mathcal{MCG}} d(g(\alpha) \vee g(\beta)).t(\mu)$$

converges absolutely and its sum vanishes.

- $\exists$ finitely many simple closed geodesics μ_i such that the associated t_{μ_i} generate the tangent space at every point in the Teichmuller space of the surface (finite type).

- Wolpert's formula for variation of lengths

  Let μ_1, μ_2 be closed simple geodesics on a hyperbolic surface.

$$dl_{\mu_1}.t(\mu_2) = \sum_{z \in \mu_1 \cap \mu_2} \cos(\theta_z),$$

$\Rightarrow$

$$|dl_{\mu_1}.t(\mu_2)| \leq \sum_{x \in \mu_1 \cap \mu_2} 1 = \sharp(\mu_1 \cap \mu_2) := i(\mu_1, \mu_2).$$

- "elementary" estimates for the lengths of simple.

Angles and lengths

Computation: $\alpha \vee \beta$ is a function of $l_\alpha, l_\beta, l_\delta$.

Let γ : $[\gamma] + [\alpha] = [\beta]$ in the homology.

Embedded triangle in M sides of length $l_\alpha/2, l_\beta/2, l_\gamma/2$:

$$\cosh(l_\gamma/2) = \cosh(l_\alpha/2)\cosh(l_\beta/2) - \sinh(l_\alpha/2)\sinh(l_\beta/2)\cos(\alpha\vee\beta),$$

Replacing in the trace relation/Markoff cubic one obtains:

$$\sinh^2(l_\alpha/2)\sinh^2(l_\beta/2)\sin^2(\alpha\vee\beta) = \cosh^2(l_\delta/4), \quad (3)$$

where δ is the boundary geodesic.

Remark: Hyperbolic version of the usual formula for the area of a Euclidean torus:

$$2 l_\alpha l_\beta \sin(\alpha \vee \beta) = \text{area of torus},$$

where α, β are closed Euclidean geodesics meeting in a single point at angle $\alpha \vee \beta$.

Differentiability:

$$d(\alpha\vee\beta) = \cosh(l_\delta)/4)\frac{\coth(l_\alpha/2)dl_\alpha + \coth(l_\beta/2)dl_\beta}{\sinh^2(l_\alpha/2)\sinh^2(l_\beta/2))\cos(\alpha\vee\beta)},$$

provided $\alpha \vee \beta \neq \pi/2$ (by equation (3)) , that is off the subset where $\alpha \vee \beta$ attains its maximum.

Extends continuously to the whole of Teichmuller space by 0 on this exceptional set.

The length spectrum of simple geodesics

Notation: the mapping class group will not figure explicitly

$$B^+ := \mathcal{MCG}.(\alpha, \beta).$$

Def: (simple) length spectrum

- $\mathcal{G}_0$ be the set of all closed simple geodesics $\gamma \neq \delta$.

- $\sigma_0(x) \subset \mathbf{R}^+ =$ *simple length spectrum* $= \{l_\gamma, \gamma \in \mathcal{G}_0\}$ counted *with* multiplicities.

- *counting function*:

$$N(\mathcal{G}_0, t) := \sharp\{\gamma \in \mathcal{G}_0, l_\gamma(x) < t\}.$$

3 facts about the simple length spectrum:

1. *the systole* $l_\gamma(x) \geq \text{sys}(x) > 0, \forall\gamma$.

2. $\sigma_0(M)$ is *discrete*

$$N(\mathcal{G}_0, t) < \infty, \forall t \geq 0$$

3. *polynomial growth*

$$N(\mathcal{G}_0, t) \leq At^{6g-6+2n},$$

   for some $A = A(x) > 0$.

Lemma 6 *Let* $x \in \mathcal{M}_1(l_\delta)$ *then* $\forall t > 0$ *there exists* $N = N(t,x) > 0$ *such that the inequality:*

$$\sinh(l_\alpha(x)/2)\sinh(l_\beta(x)/2) \geq t,$$

for all but N *pairs* $(\alpha,\beta) \in B^+$*.*

Proof:

$$\begin{aligned} LHS &\geq \tfrac{1}{2}(\sinh(l_\alpha/2)\sinh(\mathrm{sys}(x)/2) + \sinh(\mathrm{sys}(x)/2)\sinh(l_\beta/2)) \\ &\geq \tfrac{1}{2}(l_\alpha + l_\beta)\sinh(\mathrm{sys}(x)/2), \end{aligned}$$

+ discreteness of the length spectrum.

The Collar lemma: closed simple geodesic μ

- embedded collar (= regular tubular neighbourhood)

  $$\text{(width of collar round } \mu) \geq w(l_\mu),$$

  for $w(s) := 2\mathrm{arcsinh}(1/\sinh(s/2))$.

- $$i(\gamma,\mu) \leq \frac{l_\gamma}{w(l_\mu)}, \tag{4}$$

  $i(\gamma,\mu) := \sharp\gamma \cap \mu =$ *geometric intersection number.*

A proof

- Fix a metric on M. $x \in \mathcal{M}_1(l_\delta)$ = point in the moduli space.

- Fix μ closed simple.

CLAIM: $\exists K = K(\mathrm{sys}(x), l_\mu, l_\delta)$

$$\sum_{B^+} |d(\alpha \vee \beta).t(\mu)| \leq K \left(\sum_{\mathcal{G}_0} l_\gamma e^{-\frac{l_\gamma}{2}} \right)^2,$$

Convergence follows since the simple length spectrum grows polynomially.

Proof of claim: Start with

$$d(\alpha \vee \beta) = \cosh(r)\frac{\coth(a)da + \coth(b)db}{(\sinh^2(a)\sinh^2(b) - \cosh(r))^{1/2}}$$

where to simplify notation:

$$a = l_\alpha/2,\ b = l_\beta/2,\ r = l_\delta/4.$$

By Wolpert:

$$|d(\alpha \vee \beta).t(\mu)| \leq \left| \frac{\cosh(r)(\coth(a)i(\alpha,\mu) + \coth(b)i(\beta,\mu))}{\sqrt{(\sinh^2(a)\sinh^2(b) - \cosh(r))}} \right|.$$

- $\coth(a), \coth(b) \leq \coth(\mathrm{sys}(x)/2)$ since $a, b \geq \mathrm{sys}(x)/2$.
- Replace for $i(\alpha,\mu), i(\beta,\mu)$ using collar lemma:

$$\leq \left(\frac{\cosh(r)\coth(\mathrm{sys}(x)/2)}{w(l_\mu)} \right) \cdot \left(\frac{l_\alpha + l_\beta}{\sqrt{(\sinh^2(a)\sinh^2(b) - \cosh^2(r))}} \right),$$

Note that the leading factor does not depend on l_α, l_β.

- By lemma 6 for all but finitely many pairs (α, β) in B^+ one has:

$$\sinh^2(a)\sinh^2(b) - \cosh^2(r) \geq \frac{1}{2}\sinh^2(a)\sinh^2(b.$$

- $$l_\alpha + l_\beta \leq \frac{2}{\mathrm{sys}(x)} l_\alpha l_\beta.$$