1.2 Arclength

It is quite easy to think of our parameter as being time. This fits in with our concept of the curve as being the trace of some particle moving in space. There are times, and in mathematics it is more often, in which it is more convenient to let the parameter express how far along the curve we are. This means that when we write $\alpha(s)$, the parameter s is exactly the distance we have travelled along the curve. This is called *parametrization by arclength*. This will simplify the differential geometry of curve quite substantially. There is small problem. While we can show that every regular curve may be parametrized by arclenght, we usually can NOT write down a parametrization, explicitly. You will see why shortly.

Definition 1.3 Let $\alpha\colon I \to \mathbf{R}^3$ be a curve and let $f\colon J \to I$ be a real valued function. If f is smooth and has a smooth inverse (a *diffeomorphism*), then we say that $\beta = \alpha \circ f$ is a reparameterization of α.

Proposition 1.4 $\beta'(s) = \alpha'(f(s))\, f'(s)$.

To prove this, simply apply the Chain Rule.

Definition 1.4 Suppose $\alpha\colon I \to \mathbf{R}^3$ is a regular, parameterized, differentiable curve and $c \in I$. The **arclength** function $s\colon I \to \mathbf{R}$ is defined by

$$s(t) = \int_c^t |\alpha'(u)|\, du = \int_c^t \sqrt{\alpha'(u)\cdot\alpha'(u)}\, du.$$

Theorem 1.2 *The arclength function $s(t)$ is independent of the reparameterization.*

Proof: Suppose that $f\colon J \to I$ is a reparameterization and suppose that $f(r) = c$. Let $t = f(u)$ and $\beta(u) = \alpha(f(u))$. Then

$$s(u) = \int_r^u |\beta'(v)|\, dv = \int_r^u |\alpha'(g(v)| \left|\frac{df}{dv}\right| dv = \pm \int_c^t |\alpha'(w)|\, dw = s(t),$$

since $\left|\frac{df}{dv}\right| dv = \pm dw$. If f is decreasing, the sign of the change of variables is negative, while if f is increasing the sign is positive. If f is decreasing, then we are integrating in the opposite direction of the integral determined by α and so the negatives cancel. ∎

From the Fundamental Theorem of Calculus, the arclength function has a continuous derivative

$$\frac{ds}{dt} = \frac{d}{dt}\int_c^t |\alpha'(u)|\, du = |\alpha'(t)| > 0.$$

Thus, s is an increasing, continuous function of t on I. Since s is increasing it must be one-to-one on the interval. Let $J = s(I)$ be the image of the interval under s. Thus, s is one-to-one, onto, continuous and therefore has a continuous inverse which is increasing. Let $t\colon J \to I$ be the inverse function to s. Let s denote the variable in the interval J. Thus, we have

$$t(s(t)) = t \text{ and } s(t(s)) = s.$$

It follows then that

$$\begin{aligned} \frac{dt}{ds}\frac{ds}{dt} &= 1 \\ \frac{dt}{ds} &= \frac{1}{ds/dt} > 0 \end{aligned}$$

Let $\beta(s) = \alpha(t(s))$. Then

$$\left|\frac{d\beta}{ds}\right| = \left|\frac{d\alpha}{dt}\right|\left|\frac{dt}{ds}\right| = \frac{|d\alpha/dt|}{|ds/dt|} = \frac{|d\alpha/dt|}{|d\alpha/dt|} = 1.$$

When the curve is parameterized by the arclength, then the tangent vector has unit length everywhere.

We have just proven the following theorem.

Theorem 1.3 *If α is a regular curve, then α can be reparametrized to have unit speed.*

Example 1.2 Let $\alpha(t) = (a\cos(t), a\sin(t), bt)$, then

$$s = \int_0^t |\alpha'(t)|\,dt = \int_0^t \sqrt{a^2+b^2}\,dt = t\sqrt{a^2+b^2}.$$

The inverse function is $t = s/\sqrt{a^2+b^2} = \omega s$, then the natural representation of the curve is

$$\sigma(s) = (a\cos(\omega s), a\sin(\omega s), b\omega s).$$

1.3 Curvature and Plane Curves

We want to be able to associate to a curve a function that measures how much the curve bends at each point.

Let $\alpha\colon (a,b) \to \mathbf{R}^2$ be a curve parameterized by arclength. Now, in the Euclidean plane any three non-collinear points lie on a unique circle, centered at the orthocenter of the triangle defined by the three points.

For $s \in (a,b)$ choose s_1, s_2, and s_3 near s so that $\alpha(s_1)$, $\alpha(s_2)$, and $\alpha(s_3)$ are non-collinear. This is possible as long as α is not linear near $\alpha(s)$. Let $C = C(s_1,s_2,s_3)$ be the center of the circle through $\alpha(s_1)$, $\alpha(s_2)$, and $\alpha(s_3)$. The radius of this circle is approximately $|\alpha(s) - C|$. A better function to consider is the square of the radius:

$$\rho(s) = (\alpha(s) - C)\cdot(\alpha(s) - C).$$

Since α is smooth, so is ρ. Now, $\alpha(s_1)$, $\alpha(s_2)$, and $\alpha(s_3)$ lie on the circle so $\rho(s_1) = \rho(s_2) = \rho(s_3)$. By Rolle's Theorem there are points $t_1 \in (s_1,s_2)$ and $t_2 \in (s_2,s_3)$ so that

$\rho'(t_1) = \rho'(t_2) = 0$. Then, using Rolle's Theorem again on these points, there is a point $u \in (t_1, t_2)$ so that $\rho''(u) = 0$. Using Leibnitz' Rule we have $\rho'(s) = 2\alpha'(s) \cdot (\alpha(s) - C)$ and

$$\rho''(s) = 2[\alpha''(s) \cdot (\alpha(s) - C) + \alpha'(s) \cdot \alpha'(s)].$$

Since $\rho''(u) = 0$, we get

$$\alpha''(u) \cdot (\alpha(u) - C) = -\alpha'(s) \cdot \alpha(s) = -1.$$

Now, as s_1, s_2, and s_3 get closer to s, then the center of the circles will converge to a value $C_\alpha(s)$. Then t_1 and t_2 go to s, so $\rho'(s) = 0$ which forces $\alpha'(s) \cdot (\alpha(s) - C_\alpha(s)) = 0$. Furthermore, $\alpha''(s) \cdot (\alpha(s) - C_\alpha(s)) = -1$.

This says that the circle centered at $C_\alpha(s)$ with radius $\alpha(s) - C_\alpha(s)$ shares the point $\alpha(s)$ with the curve α. Furthermore, from the above the tangent to the circle at $\alpha(s)$ is a multiple of $\alpha'(s)$. Thus, this circle, called the **osculating circle**, is tangent to the curve at $\alpha(s)$. The point $C_\alpha(s)$ is called the **center of curvature** of α at s, and the curve given by the function $C_\alpha(s)$ is called the **curve of centers of curvature**.

Definition 1.5 The (unsigned) **plane curvature** of α at s is the reciprocal of the radius of the osculating circle:

$$\kappa_\pm(s) = \frac{1}{|\alpha(s) - C_\alpha s}|.$$

Theorem 1.4 $\kappa_\pm(s) = |\alpha''(s)|$.

PROOF: Since $\alpha'(s) \cdot \alpha'(s) = 1$, differentiating gives $\alpha'(s) \cdot \alpha''(s) = 0$. This means that $\alpha''(s)$ is perpendicular to $\alpha'(s)$. Since we have seen that $\alpha(s) - C_\alpha(s)$ is also perpendicular to $\alpha'(s)$, there exists a $k \in \mathbf{R}$ so that

$$\alpha(s) - C_\alpha(s) = k\alpha''(s).$$

From above we have

$$\begin{aligned} -1 &= \alpha''(s) \cdot (\alpha(s) - C_\alpha(s)) \\ &= \alpha''(s) \cdot k\alpha''(s) \\ &= k|\alpha''(s)|^2. \end{aligned}$$

Thus,

$$|\alpha(s) - C_\alpha(s)| = |k|\,|\alpha''(s)| = \frac{1}{|\alpha''(s)|^2}|\alpha''(s)| = \frac{1}{|\alpha''(s)|}.$$

∎

We rarely can symbolically represent a curve as parameterized by arclength. Quite often, a different parameterization is more reasonable. To find the curvature, though, would require that we parameterize by arclength and then differentiate. There is an easier way.

Theorem 1.5 *The plane curvature of a regular plane curve* $\sigma(t) = (x(t), y(t))$ *is given by*

$$\kappa_{\pm}(t) = \left| \frac{x''y' - y''x'}{((x')^2 + (y')^2)^{3/2}} \right|.$$

Let $\alpha\colon (a,b) \to \mathbf{R}^2$ be a curve. The **reverse curve** is $\hat{\alpha}\colon (a,b) \to \mathbf{R}^2$ is given by $\hat{\alpha}(t) = \alpha(b-t)$. We wish to distinguish these two curves.

Definition 1.6 Let $\mathbf{e}_1, \mathbf{e}_2$ denote the standard basis vectors in $\mathbf{R}^2$. An ordered pair of vectors $[\mathbf{u}, \mathbf{v}]$, $\mathbf{u}, \mathbf{v} \in \mathbf{R}^2$ is said to be in **standard orientation** if the matrix representing the transformation from $[\mathbf{u}, \mathbf{v}]$ to $[\mathbf{e}_1, \mathbf{e}_2]$ has a positive determinant.

If $\alpha(s)$ is a regular curve parameterized by arclength, then the unit tangent vector is $\mathbf{T}(s) = \alpha'(s)$. Let $\mathbf{N}(s)$ denote the unique unit vector perpendicular to $\mathbf{T}(s)$ with standard orientation $[\mathbf{T}(s), \mathbf{N}(s)]$. $\mathbf{N}(s)$ is the **unit normal vector** to α at s. Since $\mathbf{T}(s)$ is a unit vector, we see that $\mathbf{T}(s) \cdot \mathbf{T}'(s) = 0$. Thus, $\alpha''(s) = \mathbf{T}'(s)$ must be a multiple of $\mathbf{N}(s)$.

Definition 1.7 The **directed curvature** $\kappa(s)$ of a unit-speed curve α is given by the identity

$$\alpha''(s) = \kappa(s)\mathbf{N}(s).$$

Note that since $\mathbf{N}(s)$ is a unit vector, we see that $|\kappa(s)| = |\alpha(s)| = \kappa_{\pm}(s)$.

Theorem 1.6 (Fundamental Theorem for Plane Curves) *Given any continuous function* $\kappa\colon (a,b) \to \mathbf{R}$, *there is a curve* $\sigma\colon (a,b) \to \mathbf{R}^2$, *which is parameterized by arclength, such that* $\kappa(s)$ *is the directed curvature of* σ *at* s *for all* $s \in (a,b)$. *Furthermore, any other curve* $\bar{\sigma}\colon (a,b) \to \mathbf{R}^2$ *satisfying these conditions differs from* σ *by a rotation followed by a translation.*

The proof of this is a very neat, simple proof which uses differential equations.

Proof: From the theorem, we have a function $f\colon (a,b) \to \mathbf{R}^2$ written as $f(s) = (f_1(s), f_2(s))$ satisfying the following system of differential equations:

$$\begin{gathered}(f_1'(s), f_2'(s)) = \kappa(s)(-f_2(s), f_1(s)),\\ \text{subject to } f(c) = \mathbf{u} \text{ and } |\mathbf{u}| = 1\end{gathered}$$

Note that if f is a solution to this differential equation, then it is a unit-speed curve because

$$\begin{aligned}\frac{d}{ds}(f_1^2(s) + f_2^2(s)) &= 2f_1(s)f_1'(s) + 2f_2(s)f_2'(s)\\ &= 2(f_1(s), f_2(s)) \cdot (f_1'(s), f_2'(s))\\ &= 2\kappa(s)(f_1(s), f_2(s)) \cdot (-f_2(s), f_1(s)) = 0\end{aligned}$$

Thus, $|f(s)|$ is a constant and since $|f(c)| = 1$, $|f(s)| = 1$ for all $s \in (a,b)$.

Lemma 1.1 *If* $\mathfrak{g}(t)$ *is a continuous* $(n \times n)$*-matrix-valued function on an interval, then there exist solutions,* $F\colon (a,b) \to \mathbf{R}^n$, *to the differential equation* $F'(t) = \mathfrak{g}(t)F(t)$.

Applying this lemma, we have a function $\mathfrak{g}(s)$ given by

$$\mathfrak{g}(s) = \begin{pmatrix} 0 & -\kappa(s) \\ \kappa(s) & 0 \end{pmatrix}$$

The equation $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$ becomes $\mathbf{T}'(s) = \mathfrak{g}(s)\mathbf{T}(s)$. Thus, the above lemma gives us the function $\mathbf{T}(s)$ for the curve $\sigma(s)$ with the correct curvature. To find the curve $\sigma(s)$ we only need to integrate $\mathbf{T}(s)$. We can choose $\sigma(c)$ to be any point in $\mathbf{R}^2$ and we can choose $\mathbf{u}$ to be any unit vector in $\mathbf{R}^2$. Changing $\mathbf{u}$ at $\sigma(c)$ involves a rotation. That rotation passes through the differential equation so that another solution would appear as $\overline{\mathbf{T}}(s) = \rho_\theta \mathbf{T}(s)$, where ρ_θ is a rotation matrix. A translation resets the point $\sigma(c)$ to be any point in $\mathbf{R}^2$. Thus a second solution $\overline{\sigma}(s)$ must satisfy

$$\overline{\sigma}(s) = \rho_\theta \sigma(s) + \omega_0.$$

This proves the theorem. ∎