

1. GROUPS, SETS, ACTIONS

$$\begin{aligned}\mathbb{H} &= \{z : \text{Im}z > 0\} \subset \mathbb{C} \\ \partial\mathbb{H} &= \mathbb{R} \cup \{\infty\}\end{aligned}$$

Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

This metric is conformally equivalent to the Euclidean metric on \mathbb{H} that is angles between two tangent vectors are the same for the Euclidean and Hyperbolic metrics. There are sequences which are Cauchy for the Euclidean metric but which are not Cauchy for the hyperbolic metric. In fact \mathbb{H} with the hyperbolic metric is a complete metric space.

Let γ be a smooth curve parameterised by t i.e. $\gamma(t) = x(t) + iy(t)$ then the *hyperbolic length* of γ

$$\ell(\gamma) := \int \frac{|\frac{d\gamma}{dt}| dt}{y(t)}.$$

If $a, b \in \mathbb{H}$ then

$$d(a, b) := \inf_{\gamma(0)=a, \gamma(1)=b} \ell(\gamma).$$

Definition: f is an isometry of a metric space (X, d) iff $d(f(a), f(b)) = d(a, b)$ for every pair of points a, b of X .

Lemma 1. f is an isometry if and only if $\ell(f(\gamma)) = \ell(\gamma)$ for every smooth curve γ .

1.1. Geometry according to Klein. G, X G a topological group. X a topological space.

G acts on X by homeomorphisms.

G action transitive.

G action effective, faithful.

$K \subset X$ interested in subgroups like $\text{Stab}_K(G)$.

Try to understand the invariants of the G -action.

$X = \mathbb{H}, G = ?$

1.2. Groups of matrices. $GL(2, \mathbb{R})$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Definition: A *topological group* is a group G together with a topology on G such that the group and topological structures are compatible. Specifically, this means that

- Multiplication is continuous

$$G \times G \rightarrow G, (x, y) \mapsto xy,$$

where $G \times G$ has the product topology.

- Inversion is continuous

$$G \rightarrow G, x \mapsto x^{-1}.$$

Topological groups allow one to study the notion of continuous symmetries in the form of continuous group actions.

Theorem 2. $GL(2, \mathbb{R})$ is a topological group.

We give $GL(2, \mathbb{R})$ the topology from its inclusion in the vector space $M_2(\mathbb{R})$.

The two by two matrices with complex coefficients $M_2(\mathbb{C})$ is a four dimensional complex vector space with a (unique) normed topology. In fact, $M_2(\mathbb{C})$ is a *normed algebra* that is there is a norm $|\cdot| : M_2(\mathbb{C}) \rightarrow \mathbb{C}$ in fact, the operator norm, such that.

$$(1) \quad |A + B| \leq |A| + |B|$$

$$(2) \quad |AB| \leq |A||B|$$

The map $M_2(\mathbb{C}) \rightarrow \mathbb{C}, M \mapsto \det M$ is continuous $GL(2, \mathbb{C}) = \det^{-1}(\mathbb{C}^*)$ is an open subset of $M_2(\mathbb{C})$

Multiplication is continuous.

$$(A + \epsilon X)(B + \epsilon Y) = AB + \epsilon(XB + AY) + \epsilon^2 XY$$

$$|(A + \epsilon X)(B + \epsilon Y) - AB| \leq |\epsilon(XB + AY)| + |\epsilon^2 XY| \leq \epsilon(|X||B| + |A||Y|) + \epsilon^2 |X||Y|$$

Inversion is continuous.

Exercise start with:

$$(A + \epsilon X)^{-1} = A(I + \epsilon A^{-1}X)^{-1}$$

□

Lemme 3. *Two conjugacy invariants*

$$\det(M) = ad - bc$$

$$\text{tr}(M) = a + d$$

define continuous functions on $GL(2, \mathbb{R})$.

1.3. **Moebius transformations of \mathbb{H} .** Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define a map

$$\begin{aligned} \mathbb{C} \cup \{\infty\} &\rightarrow \mathbb{C} \cup \{\infty\}, \\ z &\mapsto M(z) \end{aligned}$$

where

$$M(z) = \frac{az + b}{cz + d}$$

One checks that

$$(MN)(z) = M(N(z))$$

so that this is in fact a homomorphism

$$GL(2, \mathbb{R}) \rightarrow \text{Homeo}^+(\mathbb{H}).$$

The kernel is easy to $M(z) = 1_{\mathbb{H}}(z) \Leftrightarrow M = (\lambda I_2)$, $\lambda \in \mathbb{R}^*$.

Restrict to $SL(2, \mathbb{R}) := \{M : \det M = 1\} = \det^{-1}(\{1\}) < GL(2, \mathbb{R})$ is a closed subgroup. Now

$$M(z) = N(z) \Leftrightarrow M = \pm I_2$$

Definition: $\text{PSL}_2(\mathbb{R}) := \text{SL}_2(\mathbb{R}) / \{\pm I_2\}$

Theorem 4. *The action of $\text{PSL}_2(\mathbb{R})$ by Moebius transform preserves \mathbb{H} .*

Proof: For $M \in GL(2, \mathbb{R})$, by computation,

$$\text{Im}M(z) = \frac{\text{Im}(z) \det(M)}{|cz + d|^2}.$$

So if $\text{Im}(z) > 0$ then $\text{Im}M(z) > 0$. \square

Theorem 5. $\text{PSL}_2(\mathbb{R}) \subset \text{Isom}^+(\mathbb{H})$.

$$\gamma(t) = (x(t), y(t)), \quad w(t) = T(z(t)) = (u(t), v(t))$$

$$v = \text{Im}T(\gamma(t)) = \frac{y}{|c\gamma(t) + d|^2}$$

$$\frac{dT}{dz} = \frac{1}{(cz + d)^2} \Rightarrow \frac{dw}{d\gamma} = \frac{1}{(c\gamma + d)^2} \Rightarrow \left| \frac{dw}{d\gamma} \right| = \frac{v}{y}$$

$$\ell(T(\gamma)) = \int \frac{\left| \frac{dw}{dt} dt \right|}{v} = \int \frac{\left| \frac{dw}{d\gamma} \frac{d\gamma}{dt} \right| dt}{v} = \int \frac{\left| \frac{d\gamma}{dt} \right| dt}{y} = \ell(\gamma)$$

\square

Lemme 6.

$$\begin{aligned} SL(2, \mathbb{R}) &\mapsto \mathbb{H} \\ M &\mapsto M(i) \end{aligned}$$

is a continuous map.

Proof:

$$M(i) = \frac{(ai + b)(-ci + d)}{|ci + d|^2} = \frac{ac + bd + i(ad - bc)}{c^2 + d^2}$$

So $\text{Im}M(i)$, $\text{Re}M(i)$ are rational functions in a, b, c, d and the denominator is $c^2 + d^2$ which vanishes iff $c = d = 0$ so that $\det M = 0$. \square

Corollary 7. Let $z \in \mathbb{H}$

$$\begin{aligned} SL(2, \mathbb{R}) &\mapsto \mathbb{H} \\ M &\mapsto M(z) \end{aligned}$$

is a continuous map.

Proof: From the above, given $z \in \mathbb{H}$ there is a $A \in B^+ < SL(2, \mathbb{R})$ such that the Moebius transformation $w \mapsto A(w)$ satisfies

$$z = A(i).$$

For every $M \in SL(2, \mathbb{R})$ $MA(i) = M(z)$. Let $M_n \rightarrow M$ then $M_n A \rightarrow MA$ since $SL(2, \mathbb{R})$ topological group. By the lemma $M_n A(i) \rightarrow MA(i)$ so $M_n(z) \rightarrow M(z)$. \square

1.4. Transitivity of the action. A group action on a space X partitions the space into orbits.

Definition:

- (1) If the space X consists of a single orbit then we say the group acts transitively.
- (2) G is transitive on X iff for all x, y there exists $g \in G$

$$g(x) = y.$$

- (3) G is transitive iff there exists x_0 such that for all $y \in X$ such that

$$g(x_0) = y.$$

G acts simply transitive iff there exists x_0 such that for all $y \in X$ there exists a unique $g \in G$ such that

$$g(x_0) = y.$$

Theorem 8.

$$\text{PSL}_2(\mathbb{R})$$

- (1) *transitive on points in \mathbb{H} .*
- (2) *transitive on triples of points in $\partial\mathbb{H}$*

Proof:

- (1) **CLAIM:**

$B^+ := \{\text{upper triangular matrices}\} < SL(2, \mathbb{R})$
acts simply transitively on points.

$$M = \begin{pmatrix} \lambda & c \\ 0 & \lambda^{-1} \end{pmatrix}$$

If $y > 0$ then the equation

$$M(i) = \lambda^2 i + c = x + iy$$

has a unique solution $\lambda^2 = y > 0$ and $c = x$.

- (2) Let M be the matrix

$$\begin{pmatrix} (b-d) & -c(b-d) \\ (b-c) & -d(b-c) \end{pmatrix}$$

then

$$M(z) = \frac{z-c}{z-d} \cdot \frac{b-d}{b-c},$$

and observe that

$$c \mapsto 0, d \mapsto \infty, b \mapsto 1.$$

□

Lemma 9. *For any $a, b \in \mathbb{H}$ there exists T such $\operatorname{Re} T(a) = \operatorname{Re} T(b) = 0$.*

Proof: By (1) in the theorem there is M such that $M(a) = i$.

$$\operatorname{Stab}_i(SL(2, \mathbb{R})) := \{M \in SL(2, \mathbb{R}) : M(i) = i\}$$

$$\frac{ai+b}{ci+d} = i \Rightarrow b = -c, a = d$$

$$\text{and } \det(M) = a^2 + b^2 \Rightarrow a = \cos(\theta), b = \sin(\theta).$$

$$M_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Define a map

$$\mathbb{R} \mapsto \mathbb{H}, \theta \mapsto M_\theta(z).$$

Suppose $\operatorname{Re} M_0(z) = \operatorname{Re} z > 0$, then $\operatorname{Re} M_{\pi/2}(z) = \operatorname{Re}(-1/z) < 0$.
IVT applied to $\theta \mapsto \operatorname{Re} M_\theta(z)$ on $[0, \pi/2]$ yields θ such that $\operatorname{Re} M_\theta(z) = 0$. □

1.5. **Invariants of the action – cross ratio.** The cross ratio of four distinct points $a, b, c, d \in \mathbb{C}$

$$B(a, b, c, d) = \frac{a - c}{a - d} \cdot \frac{b - d}{b - c}.$$

Let $(a, b, c, d) \in \partial\mathbb{H}^4 \setminus \Delta$, Δ is the big diagonal then $B(a, b, c, d) \in \mathbb{R} \setminus \{0, 1, \infty\}$.

Lemma 10. *The cross ratio is invariant by Moebius transformation.*

The cross ratio satisfies the following relations

$$\begin{aligned} B(a, b, c, d) &= B(a, b, c, w)B(a, b, w, d) \\ B(a, b, d, c) &= \frac{1}{B(a, b, c, d)} \end{aligned}$$

$$B(a, b, c, d) = 1 - B(b, c, d, a).$$

Definition: $(a, b, c, d) \in \mathbb{C}^4 \setminus \Delta$ are colinear if they are on a circle or line in $\mathbb{C} \cup \{\infty\}$.

Lemma 11. *Four points are colinear if and only if $B(a, b, c, d)$ is real.*

This is important- we will define lengths etc. in terms of cross ratios.

2. GEODESICS

Theorem 12. *There is a unique geodesic $[a, b]$ between any pair of distinct points in $a, b \in \mathbb{H}$.*

By the Lemma (9) we may suppose that $a = ia', b = ib'$.

Let $\alpha(t) = it$, $a' \leq t \leq b'$. If $\gamma(t) = (u(t), v(t))$ is a path joining a, b then

$$\ell(\gamma) = \int \frac{|\frac{d\gamma}{dt}| dt}{v} \geq \int \frac{\frac{dy}{dt} dt}{y} = \int_{a'}^{b'} \frac{dt}{t} = \ell(\alpha) = \log(b'/a')$$

□

Remark this also shows that

$$d(ia', ib') = |\log(b'/a')|.$$

Corollary 13. *Geodesics of \mathbb{H} are either vertical line or semicircles that meet $\partial\mathbb{H}$ perpendicularly. Conversely every vertical line or semicircle that meet $\partial\mathbb{H}$ perpendicularly is a geodesic.*

Proof: the image of a vertical line, L , by a Moebius transformation is a circle or a line. A vertical line meets $\partial\mathbb{H}$ perpendicularly, so $M(L)$ meets $M(\partial\mathbb{H})$ perpendicularly too since $z \mapsto M(z)$ holomorphic (conformal). □

Corollary 14. *If $x \in \mathbb{H}$ a point and $v \in T_x\mathbb{H}$ a tangent vector then there is a unique (directed) geodesic passing through x tangent to v .*

Definition: *Complete geodesics:* a geodesic is complete if it is not the a proper subset of another geodesic. A complete geodesic $[a, b]$, $a, b \in \partial\mathbb{H}$.

We denote by $\mathcal{G}(\mathbb{H})^+$ the set of all complete directed geodesics of \mathbb{H} .

$$\mathcal{G}(\mathbb{H})^+ = (\partial\mathbb{H} \times \partial\mathbb{H}) \setminus \Delta.$$

Note that the space of directed geodesics is homeomorphic to an annulus and the set of undirected geodesics is a Moebius band.

2.1. Some functions on the space of geodesics.

Lemme 15. • *A pair of distinct complete geodesics are either disjoint or meet in a single point.*

- *The geodesics $[z_1, z_2], [w_1, w_2]$ are meet in \mathbb{H} if and only if*

$$B(z_1, z_2, w_1, w_2) < 0.$$

Proof: By transitivity on pairs of points may suppose $[z_1, z_2] = [0, \infty]$. The geodesic $[w_1, w_2]$ meets $[0, \infty]$ if and only iff w_1, w_2 have different signs. \square

Theorem 16 (Cross ratio and distance between two points). *Let $z_1, z_2 \in \partial\mathbb{H}$ be distinct points and $a', b' \in [z_1, z_2]$ then $d(a', b') = |\log B(z_1, z_2, b', a')|$*

Proof: Invariance of the cross ratio under Moebius transformation and transitivity of the group of Moebius transformations on pairs of distinct points in $\partial\mathbb{H}$ we may suppose, WLOG, that $a, b \in [0, \infty]$

$$\frac{b'}{a'} = \frac{(0 - b')(\infty - a')}{(0 - a')(\infty - b')} = B(0, \infty, b', a').$$

\square

Theorem 17 (Cross ratio length of common perpendicular). *Let $[z_1, z_2], [w_1, w_2]$ be a pair of disjoint geodesics in \mathbb{H} then*

$$d([z_1, z_2], [w_1, w_2]) = |\log B(z_1, z_2, w_1, w_2)|.$$

Let $[z_1, z_2], [w_1, w_2]$ be a pair of geodesics in \mathbb{H} which meet at an angle θ then

$$\cos^2(\theta/2) = B(z_2, w_1, w_2, z_1).$$

Proof: By transitivity of $SL(2, \mathbb{R})$ we may suppose that $[0, \infty]$ is the common perpendicular for $[z_1, z_2], [w_1, w_2]$. We may suppose further that $[z_1, z_2] = [1, -1]$ so that $[w_1, w_2] = [\lambda, -\lambda]$ for some $\lambda > 0$. Then

$$d([1, -1], [\lambda, -\lambda]) = d(i, i\lambda) = |\log \lambda|.$$

$$\begin{aligned}
\tanh^2 \left(\frac{d([1, -1], [\lambda, -\lambda])}{2} \right) &= \frac{(1 - \lambda)^2}{(1 + \lambda)} \\
&= \frac{(1 - \lambda)}{(1 - (-\lambda))} \frac{(-1 - (-\lambda))}{(-1 - \lambda)} \\
&= B(1, -1, \lambda, -\lambda)
\end{aligned}$$

□

2.2. Cross ratio relations and the metric. The cross ratio satisfies the following relations which can be proved directly using from the definition.

$$\begin{aligned}
B(a, b, c, d) &= B(a, w, c, d)B(w, b, c, d) \\
B(a, b, d, c) &= \frac{1}{B(a, b, c, d)}
\end{aligned}$$

$$B(a, b, c, d) = 1 - B(b, c, d, a).$$

Exercise: Find geometric interpretations of these relations.

3. SUBGROUPS AND QUOTIENTS

First study the infinite cyclic subgroups, then give some more examples.

3.1. Classification of elements I: by trace. Set $\text{Tr} : PSL(2, \mathbb{R}) \rightarrow \mathbb{R}_+$ for the function

$$\text{Tr}(A) = |\text{tr}(A)|$$

where A is an arbitrary lift of A in $SL(2, \mathbb{R})$. Then Tr is well-defined. The elements of $PSL(2, \mathbb{R})$ are classified as follows:

- A) elliptic if $\text{Tr}A < 2$, thus conjugate to a unique matrix of the form $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, with $\theta \in (0, 2\pi)$.
- B) hyperbolic if $\text{Tr}A > 2$, thus conjugate to a unique matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda \neq 1$, $\lambda \in \mathbb{R}^+$, hence diagonalizable over \mathbb{R} .
- C) parabolic if $\text{Tr}A = 2$, thus conjugate to either the positive or the negative translation along the real axis. $\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$,

3.2. Classification of elements II: by action on $\mathbb{H} \cup \partial\mathbb{H}$.

Theorem 18. *There are three types of Moebius transformation according to the nature of the set of fixed points.*

- A) *elliptic* $z \mapsto M(z)$ has a (unique) fixed point in the \mathbb{H} .
- B) *hyperbolic* if $\text{Tr}M > 2$, $z \mapsto M(z)$ has a pair of distinct fixed point in the \mathbb{H} .
- C) *parabolic* $\alpha : z \mapsto M(z)$ has a (unique) fixed point α^+ in the $\partial\mathbb{H}$.

Proof: Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and consider the fixed point equation for the corresponding Moebius transformation

$$(3) \quad M(z) = \frac{az + b}{cz + d} = z.$$

One sees that

$$cz^2 + (d - a)z + b = 0.$$

The discriminant is $(a + d)^2 - 4(ad - bc)$ so that if $M \in SL(2, \mathbb{R})$ this has two solutions if and only if $\text{tr}^2(M) \neq 2$. If $\text{tr}^2 < 2$ then there is a pair of conjugate roots, so that (3) has only one solution in \mathbb{H} . If $\text{tr}^2 > 2$ then there is a pair of distinct solutions in $\partial\mathbb{H}$. If $\text{tr}^2 = 4$ then there is a single solution in $\partial\mathbb{H}$.

□

- $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. Fixes 0 and ∞ . The associated Moebius transformation is

$$f : z \mapsto \lambda^2 z$$

and

$$0 < \lambda < 1 \Rightarrow f^n(z) \rightarrow 0, \forall z \neq 0, \infty$$

$$\lambda > 1 \Rightarrow f^n(z) \rightarrow \infty, \forall z \neq 0, \infty$$

- $\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$, fixes $\alpha^+ = \infty$.

Definition:

- α^+ – attracting fixed point
- α^- – repelling fixed point
- The complete geodesic $[\alpha^-, \alpha^+]$ – the axis of α .

Lemme 19 (Dynamics of hyperbolic elements). *Let f be a hyperbolic element with attracting fixed point f^+ and repelling fixed point f^- then*

- (1) *for any point $x \in \mathbb{H} \cup \partial\mathbb{H}$, $x \neq f^-$*

$$f^n(x) \rightarrow f^+, n \rightarrow \infty.$$

- (2) *for any closed set $V \subset \mathbb{H} \cup \partial\mathbb{H}$, $f^- \notin V$ and any U open $f^+ \in U$ then there exists $N > 0$ such that $f^n(V) \subset U$ for all $n > N$.*

4. EXAMPLES OF SUBGROUPS

- (1) Infinite cyclic subgroups

For $\lambda > 0$ $\Gamma_\lambda = \langle z \mapsto \lambda z \rangle$. The quotient Riemann surface $\mathbb{H}/\Gamma_\lambda$ is an annulus

- (2) B^+ stabiliser of $+\infty$

B^- stabiliser of

- (3) $SL(2, \mathbb{Z})$

- (4) The principal congruence subgroups (normal)

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\} < SL(2, \mathbb{Z})$$

$$\text{In particular } \Gamma(2) = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

AIM Study the Riemann surfaces (and all their deformations) $\mathbb{H}/\Gamma(n)$ and the like.

We will see that $\Gamma(2) \simeq \mathbb{Z} * \mathbb{Z}$ and that $\mathbb{H}/\Gamma(2) = \mathbb{C} \setminus \{0, 1, \infty\}$.

4.1. **Rank 1 free abelian group.** Choose $\lambda > 1$ and set

$$\Gamma = \langle z \mapsto \lambda z \rangle$$

then the quotient \mathbb{H}/Γ is homeomorphic to an annulus. One can see this by constructing a Γ -invariant function Θ such that

$$\Theta(z) = \Theta(z') \Leftrightarrow \exists f \in \Gamma, f(z) = z'$$

so that the quotient is naturally identified the image of Θ .

Log has a canonical branch on \mathbb{H}

$$\begin{aligned} \log : \mathbb{H} &\mapsto \mathbb{C}^* \\ \log(z) &= \log |z| + i \arg(z) \end{aligned}$$

Two points $z_1, z_2 \in \mathbb{H}$ satisfy

$$\log(z_1) = \log(z_2) \Leftrightarrow z_2 = z_1 + 2\pi ki, k \in \mathbb{Z}.$$

The map

$$\Theta : z \mapsto \exp \frac{2\pi i \log(z)}{|\log(\lambda)|}$$

is a Γ -invariant, holomorphic map onto an annulus

$$\exp \frac{-2\pi}{|\log \lambda|} < |z| < 1.$$

From the invariance of \log we see that

$$\Theta(z) = \Theta(z') \Leftrightarrow \exists f \in \Gamma, f(z) = z'$$

so that the quotient is naturally identified with the annulus (the image of Θ). In particular it is naturally a *Riemann surface*.

Lemme 20. *The quotient is a hyperbolic surface with a unique simple closed geodesic of length $\log |\lambda|$.*

4.2. **Rank 2 free abelian group.**

- (1) Abstractly $\mathbb{Z} + \mathbb{Z}$
- (2) Presentation $\langle f, g : fgf^{-1}g^{-1} \rangle$
- (3) Normal form for words $w = f^m g^n, m, n \in \mathbb{Z}$
- (4) Action on a topological space the complex plane \mathbb{C}

$$f : z \mapsto z + 1, g : z \mapsto z + i$$

- (5) Riemann surface, torus $\Sigma_{1,0} = \mathbb{C}/\langle f, g \rangle$

4.3. Rank 2 free group.

- (1) Abstractly $\mathbb{Z} * \mathbb{Z}$
- (2) Presentation $\langle f, g \rangle$ It is the set of all words in $f, g, 1$ with the operation of concatenation subject to the relations

$$ff^{-1} = f^{-1}f = gg^{-1} = g^{-1}g = 1.$$

- (3) Normal form for words $w = f^{m_1}g^{n_1} \dots f^{m_k}g^{n_k}$, $m_i, n_i \in \mathbb{Z}^*$
- (4) Action on a topological space the complex plane \mathbb{C}

$$f : z \mapsto z + 2, \quad g : z \mapsto z(2z + 1)^{-1}$$

- (5) Riemann surface, triply punctured sphere $\Sigma_{0,3} = \mathbb{H}/\langle f, g \rangle$.

4.4. Discrete subgroups, proper discontinuous action. Definition:

An action of a group G on a topological space X by homeomorphism is called *properly discontinuous* if every element $x \in X$ has a “nice neighborhood” U_x such that

$$g \neq 1 \Rightarrow g(U_x) \cap U_x = \emptyset.$$

If Γ acts discontinuously then:

- Every subgroup of Γ acts properly discontinuously.
- For every homeomorphism ϕ of X $\phi\Gamma\phi^{-1}$ acts properly discontinuously.
- The orbits of the action have no accumulation points. That is, if Γ_n is a sequence of distinct elements of Γ and then the sequence $\Gamma_n(x)$ has no limit points.
- For $x \in X$, let $\Gamma.x$ be the orbit of x under the action of Γ . The orbit is *locally finite* if every compact set $K \subset X$ contains at most a finite number of points from the orbit $\Gamma.x$.

A metric space is proper if the closure of every distance ball is compact. \mathbb{H} is a proper metric space.

Lemma 21. *If Γ (torsion free) then Γ has locally finite orbits implies Γ acts properly discontinuously.*

Pick a compact $x \in K$. Want to show $\inf_{f \neq 1_\Gamma} d(x, f(x)) > 0$.
Suppose $\inf_{f \neq 1_\Gamma} d(x, f(x)) = 0$.

- Either there is a sequence of distinct $f_n \in \Gamma$ contradicting local finiteness
- Or there exists $f \in \Gamma$ fixing x .

By the classification of isometries f is elliptic. Since f acts by isometry for any $y \neq x$, $d(x, y) = T$ so that $d(f^n(y), x) = d(x, y) = T$. It follows that f must be finite order (irrational rotation on the circle would contradict finite orbits).

Or there exists f_n such that $f_n(x) \rightarrow x$. \square

Definition: Quotient topology

Let $\pi : X \rightarrow Y$ an identification map (associated to an equivalence relation) $U \subset Y$ is open for the quotient topology on Y iff $\pi^{-1}(U)$ open.

Definition: Covering map

A covering map is a surjective local homeomorphism $\pi : X \rightarrow Y$.

Lemma 22. *If X is a locally path-connected Hausdorff space, and the action of Γ is properly discontinuous, then the quotient map*

$$\pi : X \rightarrow X/\Gamma$$

is a covering map.

Definition: Riemann surface, hyperbolic

- Y is a *Riemann surface* if there is an atlas $\{(U_x, \phi_x)\}$

$$\phi_x : U_x \rightarrow \mathbb{C}$$

such that the transition maps $\phi_x \circ \phi_y^{-1}$ are holomorphic.

- Y is a *hyperbolic surface* if there is an atlas $\{(U_x, \phi_x)\}$

$$\phi_x : U_x \rightarrow \mathbb{H}$$

such that the transition maps $\phi_x \circ \phi_y^{-1}$ are hyperbolic isometries (Moebius maps).

Theorem 23. *Γ acts properly discontinuously on \mathbb{H} then \mathbb{H}/Γ is a Riemann surface*

Proof:

We use the axiom of choice to pick a representative for each Γ orbit let Y denote the set of all choices.

Since Γ acts properly discontinuously for each x we choose U_x such that $f(U_x), f \in \Gamma$ are all disjoint.

Take the atlas for the quotient to be the set of all pairs $(\Gamma.U_x, \phi_f)$ where x is a point of \mathbb{H} and $U_x \subset \mathbb{H}$ is a neighborhood of x such that

$$f(U_x) \cap U_x \neq \emptyset \Leftrightarrow f = 1_\Gamma,$$

equivalently, but more useful,

$$f(U_x) \cap g(U_x) \neq \emptyset \Leftrightarrow f = g.$$

Let y be some point of \mathbb{H} note that there exists at most one element of Γ such that, given f, x ,

$$g_{y,f}(y) \in f(U_x),$$

since $g'(y) = g'g^{-1}g(y) \in f(U_x) \cap g'g^{-1}f(U_x) \Rightarrow g'g^{-1} = 1_\Gamma$. So the element $g_{y,f}$ is unique and the following map is well defined

$$\phi_f(\Gamma.y) = g_{y,f}(y)$$

Suppose now that there is an overlap of the of two charts $(\Gamma.U_x, \phi_f)$ and $(\Gamma.V_y, \phi_{f'})$ Let $\Gamma.z, \Gamma.z' \in$ the overlap $\Gamma.U_x \cap \Gamma.V_y$ by the above there exist unique $g_{z,f}, g_{z,f'}, g_{z',f}, g_{z',f'}$ such that

$$\begin{aligned}\phi_f(z) &= g_{z,f}(z) \\ \phi_{f'}(z) &= g_{z,f'}(z) \\ \phi_f(z') &= g_{z',f}(z') \\ \phi_{f'}(z') &= g_{z',f'}(z')\end{aligned}$$

There are now two choices for $\phi_{f'} \circ \phi_f^{-1}$

- $g_{z,f'} \circ g_{z,f}^{-1}$
- $g_{z',f'} \circ g_{z',f}^{-1}$

There exists h such that

$$h(g_{z,f}(z)), h(g_{z',f}(z')) \in h \circ f(U_x) \cap f'(V_y)$$

Furthermore h is unique since

$$h(g_{z,f}(z)), h'(g_{z,f}(z)) \in f'(V_y) \Rightarrow h^{-1} \circ f'(V_y) \cap (h')^{-1} \circ f'(V_y) \neq \emptyset$$

Follows that

$$g_{z,f'} \circ g_{z,f}^{-1} = g_{z',f'} \circ g_{z',f}^{-1} = h.$$

Theorem 24. *If Γ is a torsion free discrete subgroup of $\text{PSL}_2(\mathbb{R})$ if and only if it acts properly discontinuously on \mathbb{H} by Moebius transformations.*

\mathbb{H} is a locally compact metric space with isometry group transitive on points. Understand compacts in \mathbb{H} .

After conjugating by a ϕ Moebius such that $\phi(x) = i$ we get another discrete group $\phi\Gamma\phi^{-1}$. Only have to check $\inf_{f \neq 1_\Gamma} d(i, f(i)) > 0$.

In fact we only have to show that there some compact $K \subset \mathbb{H}$ containing i such that $\Gamma.i \cap K$ is a finite set.

Any compact $K \subset \mathbb{H}$ is contained in some closed rectangle

$$\{\epsilon \leq \text{Im}z \leq N\} \times \{M_1 \leq \text{Re}z \leq M_2\}$$

For $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ we have,

$$\begin{aligned} \operatorname{Im} f(i) &= |ci + d|^{-2} = \frac{1}{c^2 + d^2} \\ \operatorname{Re} f(i) &= \frac{ac + bd}{c^2 + d^2} = \frac{ac + (ad - 1)/cd}{c^2 + d^2}. \end{aligned}$$

Proof: (\Leftarrow) If Γ isn't discrete then there exists f_{n_k} that converges. Easy to see that $f_k(i)$ converges.

(\Rightarrow) Since Γ is discrete it is countable. If $f_k(i)$ converges then there exists a compact rectangle K such that for all k , $f_k(i) \in K$. Thus $c_k^2 + d_k^2 < 1/\epsilon$ for all k . Passing to a subsequence if necessary we may suppose that the sequences c_k and d_k converge.

Note also that as $a_k c_k + (1 - a_k d_k)/c_k d_k$ is bounded for all k . So that a_k is bounded ($x \mapsto x(x - d)/c$ is a proper function) and so has a convergent subsequence. Finally $b_k = (1 - a_k d_k)/c_k$. \square

In fact the theorem is a corollary of the following.

Definition: A map between topological spaces is *proper* iff the preimage of every compact set is compact.

Theorem 25. *If $z \in \mathbb{H}$ then the orbit map*

$$SL(2, \mathbb{R}) \mapsto \mathbb{H}, M \mapsto M(z)$$

is a proper map.

CLAIM: Let G be a (Hausdorff) topological group $K_1, K_2 \subset G$ compact sets then $K_1.K_2$ is compact.

Proof:

$$\begin{aligned} K_1 \times K_2 &\rightarrow K_1.K_2 \\ (x, y) &\mapsto x.y \end{aligned}$$

is continuous and the image of a compact set is compact. \square

Proof: The rectangle is the orbit of

$$K_0 = \left\{ \begin{pmatrix} \lambda & p/\lambda \\ 0 & \lambda^{-1} \end{pmatrix}, \epsilon \leq \lambda^2 \leq N, M_1 \leq p \leq M_2 \right\}$$

this is closed, bounded in $M_2(\mathbb{R})$ so compact.

$$M(i) = M'(i) = z \Rightarrow M^{-1}M' \in \operatorname{Stab}_i.$$

It follows that the preimage of the rectangle in \mathbb{H} is just $K_0 \operatorname{Stab}_i$, which we have just seen is compact. \square

5. LIMIT SETS

The action on the limit set will not be properly discontinuous, in fact it should be as far from being properly discontinuous as possible.

For example given any two open sets U, V not equal to the whole limit set we want to find g such that $U \subset g(V)$.

Definition: $\Gamma < \text{PSL}_2(\mathbb{R})$ a not necessarily discrete group of Mobius transformations $z \in \mathbb{H}$ a point

$$\Lambda(\Gamma) := \overline{\Gamma.z} \cap \partial\mathbb{H}$$

Note that if Γ is discrete then it has locally finite orbits so that the only points of the orbit $\Gamma.z$ are in $\partial\mathbb{H}$.

Lemme 26. *If $d_{\mathbb{H}}(x, y) < K$ then for every $f \in \text{SL}(2, \mathbb{R})$*

$$e^{-K} < \frac{\text{Im}f(x)}{\text{Im}f(y)} < e^K.$$

Proof: Triangle inequality gives

$$K > d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(f(x), f(y)) > |\log(\text{Im}f(x)) - \log(\text{Im}f(y))|.$$

□

Corollary 27. *Let $x \in \mathbb{H}$ and $f_n \in \Gamma$ then*

$$f_n(z) \rightarrow x \Leftrightarrow f_n(w) \rightarrow x.$$

So Λ doesn't depend on the choice of z .

Proof: The hyperbolic ball $f_n(B_k(x))$ is a euclidean ball not centered at but containing $f_n(x)$. By the above $f_n(B_k(x))$ is contained in the strip bounded by $\partial\mathbb{H}$ and below $e^k \text{Im}f_n(x)$.

so

$$|f_n(x) - f_n(y)| < 2e^k \text{Im}f_n(x)$$

□

Definition: $\Gamma < \text{PSL}_2(\mathbb{R})$ is elementary if there is a point $x \in \mathbb{H} \cup \partial\mathbb{H}$ such that $f(x) = x$ for every element f of Γ .

Exercise find examples of elementary subgroups $\sharp\Lambda(\Gamma) = 0, 1, 2$.

Lemme 28. *Every two generator subgroup of B^+ has a relation*

Proof: Let $f, g \in B^+$ After conjugating we can suppose that

$$f : z \mapsto kz, g : z \mapsto az + b$$

Then, by direct computation,

$$[g, f] = gfg^{-1}f^{-1}(z) = z + (1 - k)b$$

(when $k = 1$ or $b = 0$, f, g commute). The group of all transformations is an abelian subgroup of B^+ so

$$[[g^2, f], [g, f]] = 1$$

is a relation in $\langle f, g \rangle$. \square

Theorem 29 (Baby Tits). *Every non elementary group contains a free group of rank 2.*

Lemme 30. *If Γ is not elementary then Γ contains a pair of hyperbolic elements which have no common fixed point.*

CLAIM: There is at least one hyperbolic. Suppose only contains parabolics and elliptics. Conjugate so that $f : z \mapsto z + 1$ is in the group. Take

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$\text{tr}^2 f^n g = (a + d + nc)^2 \leq 4$$

since no hyperbolics. Thus $c = 0$, g, f commute and the group is elementary.

CLAIM: There is a pair of hyperbolics with disjoint axes.

Let g be hyperbolic then, since Γ not elementary, there is f such that $\{g^\pm\}$ is not stable under f .

If $f(\{g^\pm\}) \cap \{g^\pm\} = \emptyset$ then we are finished.

If $\{g^+\} = f(\{g^\pm\}) \cap \{g^\pm\}$ then we must continue. $f = \alpha z, g = \beta z + c$

CLAIM: $p = [g, fgf^{-1}]$ is parabolic and fixes ∞ There exists $h \in \Gamma$ that doesn't fix α $hp^n h^{-1} g h p^n h^{-1}$ works for some n (so that $hp^n h^{-1}(\{g^\pm\}) \cap \{g^\pm\} = \emptyset$). \square

Corollary 31. *A non elementary group contains infinitely many hyperbolics no two of which have a common fixed point*

Proof: Let f, g be as in the theorem above then for any $n \in \mathbb{N}$ $f^n g f^{-n}$ is hyperbolic and all the fixed points are different. \square

Theorem 32 (Ping-pong lemma). *Let Γ be a group of permutations on a set X , let f, g be elements of Γ of order at least 3. If A, B are disjoint subsets of X and if, for all $n \neq 0$*

$$f^n(A) \subset B, g^n(B) \subset A$$

*then f, g freely generate the free group $\mathbb{Z} * \mathbb{Z}$.*

We have to show that no reduced word is trivial. If w is trivial then $w(x) = x, \forall x \in X$ so to show that w is non trivial it suffices to find x such that $w(x) \neq x$. Let $w \in \Gamma \setminus \{1\}$ be a reduced word then, without loss of generality, $w = f^k w'$ for a reduced word w' which is