Categorical representability and intermediate Jacobians of Fano threefolds

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1 Introduction

These notes arise from the attempt to extend the results of [13] to a wider class of complex threefolds with negative Kodaira dimension. If $Y \to S$ is a conic bundle and S is a rational surface, a semiorthogonal decomposition of $\mathrm{D^b}(Y)$ by derived categories of curves and exceptional objects gives a splitting of the intermediate Jacobian as the direct sum of the Jacobians of the curves ([13], Theorem 1.1). This result is based on the relation between fully faithful functors $\mathrm{D^b}(\Gamma) \to \mathrm{D^b}(Y)$ (where Γ is a smooth projective curve) and algebraic cycles on Y. It turns out that the properties needed to prove this result hold true also for certain threefolds other than conic bundles. One of the aims of this article is to describe certain varieties satisfying these representability assumptions. At the same time, semiorthogonal decompositions of rational conic bundles over minimal rational surfaces are described in [13]. These turn out to be the most recent examples in a quite extensive list of varieties (starting with [18]) of dimension 3 with negative Kodaira dimension admitting a semiorthogonal decomposition by exceptional objects and components which should somehow be related to the birational properties. The possible interplays between derived categories and birational geometry have been outlined in [19]. Recently, a challenging conjecture of Kuznetsov [49] has added cubic fourfolds to the list.

In a generalization attempt, we define a new notion of representability based on semiorthogonal decompositions, which we expect to carry useful geometrical insights also in higher dimensions, and which allows to properly write down many of the ideas which have been motivating these researches. Let X be a smooth projective variety of dimension n. We define *categorical representability in (co)dimension* m for X, roughly by requiring that the derived category $\mathrm{D^b}(X)$ admits a semiorthogonal decomposition by categories which can be fully faithfully embedded into derived categories of smooth projective varieties of dimension bounded by m (resp. $n-m$).

The idea of defining categorical representability comes from the classical theory of algebraic cycles: various notions of representability of the group $A^i_{\mathbb{Z}}(X)$ of algebraically trivial cycles of codimension i on X have appeared through the years in the literature, and it seems interesting to understand their interactions with categorical

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representability, as our examples suggest. Roughly speaking (for the actual definitions see Section 2.2), *weak representability* for $A^i_{\mathbb{Z}}(X)$ is given by an algebraic map $J(\Gamma) \to A^i_{\mathbb{Z}}(X)$ whose kernel is an algebraic group, for an algebraic curve Γ. Working with rational coefficients (that is, on $A^i_{\mathbb{Q}}$) gives the notion of *rational representability. Algebraic representability* requires the existence of a quasi-universal regular isomorphism $A^i_{\mathbb{Z}}(X) \to A$ onto an abelian variety A. Finally, if $\dim(X) = 2n+1$ is odd, A is the algebraic representative of $A^n_{\mathbb{Z}}(X)$, and if the principal polarization of A is "well behaved" with respect to this regular isomorphism we say that A carries an *incidence polarization*.

The definition of categorical representability could seem rather disjoint from the classical ones. It is nevertheless clear that rational representability is strongly related to the structure of the motive $h(X)$ of X. Grothendieck (or classical) motives were introduced to give an algebraic universal description of cohomologies and cycles on X. In particular one gets a big amount of information from a Chow–Künneth decomposition of the motive $h(X)$, which is, roughly, a decomposition whose summands are strictly related to algebraic cycles of a given codimension. For example, if X is a threefold, then rational representability of all the $A^i_{\mathbb{Q}}(X)$ is equivalent to the existence of a specific Chow–Künneth decomposition [23]. A first point to note is then that the existence of a fully faithful functor between the derived categories of two smooth projective varieties should be reflected at a motivic level, as stated in the following conjecture by Orlov.

Conjecture 1.1 ([64]). *Let X and Y be smooth projective varieties and $\Phi\colon \mathrm{D}^{\mathrm{b}}(Y) \to \mathrm{D}^{\mathrm{b}}(X)$ be a fully faithful functor. Then the motive $h(Y)$ is a direct summand of the motive $h(X)$.*

A clear link between categorical and rational representability should appear when we consider the former in dimension 1. Note that being categorically representable in dimension 1 is equivalent to the existence of a semiorthogonal decomposition by exceptional objects and derived categories of curves. The motive of a curve splits into two discrete and one abelian motives, the latter corresponding to the Jacobian up to isogeny. Orlov's conjecture would then imply that if X is categorically representable in dimension 1, then its motive is a finite sum of abelian (corresponding to Jacobians of curves) and discrete motives. This would give information about rational representability for $A^i_{\mathbb{Q}}(X)$. Being categorically representable in dimension 1 seems to be in fact a very strong condition. For example a smooth cubic threefold is strongly representable with incidence property but not categorically representable, otherwise we would have the splitting of the intermediate Jacobian (see Corollary 3.10). Notice that in [50] the study of the Abel–Jacobi map for some hypersurfaces and its link with categorical constructions were already treated.

On the other hand, algebraic representability and the incidence property can have deep interactions with categorical representability, and this is indeed the heart of the proof of Theorem 1.1 in [13]. Consider a smooth projective threefold X and assume it to be rationally representable, with $h^1(X) = h^5(X) = 0$ (i.e. X has discrete Picard group), and with $A^2_{\mathbb{Z}}(X)$ algebraically representable with the incidence property. The arguments in [13] show that if X is categorically representable in dimension 1, then the intermediate Jacobian $J(X)$ splits into Jacobians of curves, namely of those curves of positive genus appearing in the semiorthogonal decomposition. This result can then be applied to a large class of complex threefolds with negative Kodaira dimension (see a list in Remark 3.8).

We can then reasonably raise the following question, which also points out how this new definition could be useful: is categorical representability in codimension 2 a necessary condition for rationality? This is true for complex surfaces, since any rational smooth projective complex surface admits a full exceptional sequence. Remark 3.12 shows that this is true for a wide class of complex threefolds with negative Kodaira dimension, but we can only argue so far by a case by case analysis. In dimension 4, Kuznetsov's conjecture about rationality of cubics ([49], Conjecture 1.1) is clearly related to (and indeed stronger than) this question, while in [4] we state a similar conjecture for another class of smooth projective fourfolds explicitly in terms of categorical representability.

Notation. Any triangulated category is assumed to be essentially small. Given a smooth projective variety X, we denote by κ_X its Kodaira dimension, by $\mathrm{D}^{\mathrm{b}}(X)$ the bounded derived category of coherent sheaves on it, by $K_0(X)$ its Grothendieck group, by $\mathrm{CH}^d_{\mathbb{Z}}(X)$ the Chow group of codimension d cycles modulo rational equivalence, and by $A^d_{\mathbb{Z}}(X)$ the subgroup of algebraically trivial cycles in $\mathrm{CH}^d_{\mathbb{Z}}(X)$. If X is pure d-dimensional, and Y any smooth projective variety, we denote by $\mathrm{Corr}^i(X,Y) := \mathrm{CH}^{i+d}_{\mathbb{Q}}(X \times Y)$ the group of correspondences with rational coefficients. If $X = \coprod X_j$, with X_j connected, then $\mathrm{Corr}^i(X,Y) = \bigoplus \mathrm{Corr}^i(X_j,Y)$.

2 Categorical and classical representabilities for smooth projective varieties

2.1 Semiorthogonal decompositions and categorical representability. We start by recalling some categorical definitions which are necessary to define representability. Let K be a field and $\mathbf{T}$ a K-linear triangulated category. A full triangulated subcategory $\mathbf{A}$ of $\mathbf{T}$ is called *admissible* if the embedding functor admits a left and a right adjoint.

Definition 2.1 ([17], [18]). A *semiorthogonal decomposition* of $\mathbf{T}$ is a sequence of admissible[1] subcategories $\mathbf{A}_1, \dots, \mathbf{A}_l$ of $\mathbf{T}$ such that

- $\mathrm{Hom}_{\mathbf{T}}(A_i, A_j) = 0$ for all $i > j$ and any A_i in $\mathbf{A}_i$ and A_j in $\mathbf{A}_j$;

[1] Notice that some authors, as for example [66], do not require admissibility for the subcategories in the definition of semiorthogonal decomposition.

- for all objects A_i in $\mathbf{A}_i$ and A_j in $\mathbf{A}_j$, and for every object T of $\mathbf{T}$, there is a chain of morphisms $0 = T_l \to T_{l-1} \to \cdots \to T_1 \to T_0 = T$ such that the cone of $T_k \to T_{k-1}$ is an object of $\mathbf{A}_k$ for all $k = 1, \ldots, l$.

Such a decomposition will be written

$$\mathbf{T} = \langle \mathbf{A}_1, \ldots, \mathbf{A}_l \rangle.$$

Definition 2.2 ([16]). An object E of $\mathbf{T}$ is called *exceptional* if $\mathrm{Hom}_{\mathbf{T}}(E, E) = K$, and $\mathrm{Hom}_{\mathbf{T}}(E, E[i]) = 0$ for all $i \neq 0$. A collection $\{E_1, \ldots, E_l\}$ of exceptional objects is called *exceptional* if $\mathrm{Hom}_{\mathbf{T}}(E_j, E_k[i]) = 0$ for all $j > k$ and for all integers i.

If E in $\mathbf{T}$ is an exceptional object, the triangulated category generated by E (that is, the smallest full triangulated subcategory of $\mathbf{T}$ containing E) is equivalent to the derived category of a point, seen as a smooth projective variety. The equivalence $\mathrm{D}^{\mathrm{b}}(pt) \to \langle E \rangle \subset \mathbf{T}$ is indeed given by sending $\mathcal{O}_{pt}$ to E. In the case where $\mathbf{T}$ is $\mathrm{D}^{\mathrm{b}}(X)$ for a smooth projective variety X, given an exceptional collection $\{E_1, \ldots, E_l\}$, there is a semiorthogonal decomposition (see [18])

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathbf{A}, E_1, \ldots, E_l \rangle,$$

where $\mathbf{A}$ is the full triangulated subcategory whose objects are all the A satisfying $\mathrm{Hom}(E_i, A) = 0$ for all $i = 1, \ldots, l$, and we denote by E_i the category generated by E_i. We say that the exceptional sequence is *full* if the category $\mathbf{A}$ is trivial. More generally, if $\mathbf{A} \subset \mathbf{T}$ is admissible, we have two semi-orthogonal decompositions

$$\mathbf{T} = \langle \mathbf{A}^{\perp}, \mathbf{A} \rangle = \langle \mathbf{A}, {}^{\perp}\mathbf{A} \rangle,$$

where $\mathbf{A}^{\perp}$ and ${}^{\perp}\mathbf{A}$ are respectively the left and right orthogonal of $\mathbf{A}$ in $\mathbf{T}$ [17].

Definition 2.3. A triangulated category $\mathbf{T}$ is *representable in dimension m* if it admits a semiorthogonal decomposition

$$\mathbf{T} = \langle \mathbf{A}_1, \ldots, \mathbf{A}_l \rangle,$$

and for all $i = 1, \ldots, l$ there exists a smooth projective connected variety Y_i with $\dim Y_i \leq m$, such that $\mathbf{A}_i$ is equivalent to an admissible subcategory of $\mathrm{D}^{\mathrm{b}}(Y_i)$.

Definition 2.4. Let X be a smooth projective variety of dimension n. We say that X is *categorically representable* in dimension m (or equivalently in codimension $n - m$) if $\mathrm{D}^{\mathrm{b}}(X)$ is representable in dimension m.

Remark 2.5. Suppose that X is not smooth. Then to define categorical representability for it, we need to replace in Definition 2.4 the derived category $\mathrm{D}^{\mathrm{b}}(X)$ with another triangulated category $\widetilde{\mathbf{D}}$, enjoying some "smoothness" which would be called a *categorical resolution of singularities.*

In [46], Kuznetsov suggest a definition for which the resolution of singularities $\widetilde{\mathbf{D}}$ of $\mathrm{D}^{\mathrm{b}}(X)$ could be realized as an admissible subcategory of the derived category of geometrical resolution of singularities $\widetilde{X} \to X$, whenever X has rational singularities. The notion of categorical representability would naturally sit inside this kind of approach (for example, all projective varieties would be categorically representable at least in its dimension). Notice that anyway there is in general no unicity or minimality of such a resolution.

Notice that any fully faithful functor $F\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ between the derived categories of two smooth projective varieties X and Y is of Fourier–Mukai type [62], [63], i.e. there is an object $\mathcal{E}$ in $\mathrm{D}^{\mathrm{b}}(X \times Y)$ (called *kernel* of F) and $F(-)$ is given by pulling back a complex to $\mathrm{D}^{\mathrm{b}}(X\times Y)$, tensoring with $\mathcal{E}$ and pushing-forward to $\mathrm{D}^{\mathrm{b}}(Y)$. It is moreover worth noting and recalling the following facts, which are well-known in the derived categorical setting.

Remark 2.6 ([9]). The derived category of $\mathbb{P}^n$ admits a full exceptional sequence.

Remark 2.7 ([59]). If Γ is a smooth connected projective curve of positive genus, then $\mathrm{D}^{\mathrm{b}}(\Gamma)$ has no proper admissible subcategory. Indeed any fully faithful functor $\mathbf{A} \to \mathrm{D}^{\mathrm{b}}(\Gamma)$ is an equivalence, unless $\mathbf{A}$ is trivial. Then being categorically representable in dimension 1 is equivalent to admit a semiorthogonal decomposition by exceptional objects and derived categories of smooth projective curves.

Remark 2.8. If X and Y_i are smooth projective and

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(Y_1), \dots, \mathrm{D}^{\mathrm{b}}(Y_k)\rangle,$$

then

$$K_0(X) = \bigoplus_{i=1}^{k} K_0(Y_i)$$

and the Riemann–Roch Theorem gives an isomorphism of $\mathbb{Q}$-vector spaces

$$\mathrm{CH}^*_{\mathbb{Q}}(X) = \bigoplus_{i=1}^{k} \mathrm{CH}^*_{\mathbb{Q}}(Y_i).$$

Remark that the last isomorphism is in general not compatible with gradings.

Proposition 2.9 ([61]). *Let X be smooth projective and $Z \subset X$ a smooth subvariety of codimension $d > 1$. Denote by $\varepsilon\colon \widetilde{X} \to X$ the blow up of X along Z. Then*

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle \varepsilon^*\mathrm{D}^{\mathrm{b}}(X), \mathrm{D}^{\mathrm{b}}(Z)_1, \dots, \mathrm{D}^{\mathrm{b}}(Z)_{d-1}\rangle,$$

where $\mathrm{D}^{\mathrm{b}}(Z)_i$ is equivalent to $\mathrm{D}^{\mathrm{b}}(Z)$ for all $i = 1, \dots, d-1$.

2.2 Classical representabilities and motives. In general, it is a very deep and interesting geometric problem to understand whether the group $A^i_{\mathbb{Z}}(X)$ of algebraically trivial cycles of codimension i on X can carry a scheme structure. The notion of representability of such groups has been introduced to tackle this problem. In this section we outline a list of definitions of representabilities for the groups $A^i_{\mathbb{Z}}(X)$. This is far for being exhaustive, especially in the referencing. Indeed, giving a faithful list of all contributions to these questions is out of the aim of these notes. Chow motives and their properties could give, through Conjecture 1.1, a way to connect categorical and classical representabilities. We also outline the basic facts needed to stress the possible interplay between new and old definitions.

Let X as usual be a smooth projective variety over a field K. Recall that, if Γ is a curve, then $J(\Gamma) \cong A^1_{\mathbb{Z}}(\Gamma)$.

Definition 2.10. Let T be any nonsingular variety over K. An map $f : T \to A^i_{\mathbb{Z}}(X)$ is an *algebraic map* if there exists a cycle class z in $\mathrm{CH}^i_{\mathbb{Z}}(T \times X)$ such that $f(t)$ is the restriction of z to $\{t\} \times X$. In other words, $f(t) = q_*((p^*t).z)$, where p and q denote, respectively, the projections from $T \times X$ to T and X. In this case, such a map will be denoted by z_*.

Definition 2.11 ([15]). The group $A^i_{\mathbb{Z}}(X)$ is said to be *weakly representable* if there exists a smooth projective curve Γ, a class z of a cycle in $\mathrm{CH}^i_{\mathbb{Z}}(X \times \Gamma)$ and an algebraic subgroup $G \subset J(\Gamma)$ of the Jacobian variety of Γ, such that, for any algebraically closed field $\Omega \supset K$, the induced algebraic map

$$z_* \colon J(\Gamma)(\Omega) \simeq A^1_{\mathbb{Z}}(\Gamma_\Omega) \to A^i_{\mathbb{Z}}(X_\Omega)$$

is surjective with kernel $G(\Omega)$.

When working with coefficients in $\mathbb{Q}$, we have the following definition.

Definition 2.12. The group $A^i_{\mathbb{Q}}(X)$ is *rationally representable* if there exists a cycle z in $\mathrm{CH}^i_{\mathbb{Q}}(X \times \Gamma)$ giving a surjective algebraic map

$$z_* \colon A^1_{\mathbb{Q}}(\Gamma) \to A^i_{\mathbb{Q}}(X).$$

The variety X is *rationally representable* if $A^i_{\mathbb{Q}}(X)$ is rationally representable for all i.

Rational representability is a name that has been used several times in the literature, so it might lead to some misunderstanding. We underline that Definition 2.12 is exactly the one from ([23], page 5). In the complex case, we have also a stronger notion, which is called the *Abel–Jacobi property* [15], which requires the existence of an isogeny (i.e. a regular surjective morphism) $A^i_{\mathbb{Z}}(X) \to J^i(X)$, induced by a correspondence, onto the i-th intermediate Jacobian $J^i(X) := H^i(X, \mathbb{R})/H^i(X, \mathbb{Z})$. The Abel–Jacobi property implies weak representability for smooth projective varieties defined on $\mathbb{C}$.

An even stronger notion is given by algebraic representability, which requires that the group $A^i_{\mathbb{Z}}(X)$ is isomorphic via a regular map to an abelian variety.

Definition 2.13. Let A be an abelian variety. A group homomorphism $g\colon A^i_{\mathbb{Z}}(X) \to A$ is a *regular map* if for every non-singular variety T and for every algebraic map $f : T \to A^i(X)_{\mathbb{Z}}$, the composite map $g \circ f$ is a morphism.

Definition 2.14 ([7], Definition 3.2.3). An abelian variety A is the *algebraic representative* of $A^i_{\mathbb{Z}}(X)$ if there exists a quasi-universal regular map $G\colon A^i_{\mathbb{Z}}(X) \to A$, i.e. for all regular maps g from $A^i_{\mathbb{Z}}(X)$ to an abelian variety B, there is a unique morphism of abelian varieties $u\colon A \to B$ such that $u \circ G = g$. In this case we say that $A^i_{\mathbb{Z}}(X)$ is *algebraically representable.*

The first examples of algebraic representatives are the Picard variety $\mathrm{Pic}^0(X)$ or the Albanese variety $\mathrm{Alb}(X)$ if $n = 1$ or, respectively, $n = \dim(X)$. We remark that in these two cases the associated correspondences are those induced, respectively, by the first Chern class of the Poincaré bundle on $X \times \mathrm{Pic}^0(X)$ and by the graph of the natural Albanese map.

Definition 2.15 ([7], Definition 3.4.2). Let X be a smooth projective variety of odd dimension $2n+1$ and A the algebraic representative of $A^{n+1}_{\mathbb{Z}}(X)$ via the canonical map $G\colon A^{n+1}_{\mathbb{Z}}(X) \to A$. A polarization of A with class θ_A in $\mathrm{Corr}(A, A)$ is the *incidence polarization* with respect to X if for all algebraic maps $f : T \to A^{n+1}_{\mathbb{Z}}(X)$ defined by a cycle z in $\mathrm{CH}^{n+1}_{\mathbb{Z}}(X \times T)$, we have

$$(G \circ f)^*\theta_A = (-1)^{n+1} I(z),$$

where $I(z)$ in $\mathrm{Corr}(T)$ is the composition of the correspondences $z \in \mathrm{Corr}(T, X)$ and $z \in \mathrm{Corr}(X, T)$.

For example, if C is a smooth projective curve, then the group $A^1_{\mathbb{Z}}(C) \simeq J(C)$ carries an incidence polarization, namely the canonical polarization of the Jacobian. Indeed, it is easy to check that the correspondence associated such polarization is the opposite to the incidence polarization given by a Poincaré line bundle.

There are many complex threefolds X with negative Kodaira dimension, for which $A^2_{\mathbb{Z}}(X)$ is strongly represented by a generalized Prym variety with incidence polarization. For these threefolds, we will show how categorical representability in dimension 1 gives a splitting of the intermediate Jacobian. A list of the main cases will be given in Section 3.2.

A more modern approach to representability questions has to take Chow motives into account. Let us recall their basic definitions and notations. The category $\mathcal{M}_K$ of Chow motives over K with rational coefficients is defined as follows: an object of $\mathcal{M}_K$ is a triple (X, p, m), where X is a variety, m an integer and $p \in \mathrm{Corr}^0(X, X)$ an idempotent, called a *projector*. Morphisms from (X, p, m) to (Y, q, n) are given by elements of $\mathrm{Corr}^{n-m}(X, Y)$ precomposed with p and composed with q.

There is a natural functor h from the category of smooth projective schemes to the category of motives, defined by $h(X) = (X, \mathrm{id}, 0)$, and, for any morphism $\phi\colon X \to Y$, $h(\phi)$ being the correspondence given by the graph of ϕ. We write $\mathbb{1} := (\mathrm{Spec}\, K, \mathrm{id}, 0)$

for the unit motive and $\mathbb{L} := (\operatorname{Spec} K, \mathrm{id}, -1)$ for the Lefschetz motive, and $M(-i) := M \otimes \mathbb{L}^i$. Moreover, we have $\operatorname{Hom}(\mathbb{L}^i, h(X)) = \mathrm{CH}^i_{\mathbb{Q}}(X)$ for all smooth projective schemes X and all integers i.

If X is irreducible of dimension d, the embedding $\alpha\colon pt \to X$ of the point defines a motivic map $\mathbb{1} \to h(X)$. We denote by $h^0(X)$ its image and by $h^{\geq 1}(X)$ the quotient of $h(X)$ via $h^0(X)$. Similarly, $\mathbb{L}^d$ is a quotient of $h(X)$, and we denote it by $h^{2d}(X)$. Notice that both $h^0(X)$ and $h^{2d}(X)$ split off the motive $h(X)$ as direct summands.

In the case of smooth projective curves of positive genus there exists another factor which corresponds to the Jacobian variety of the curve. Let C be a smooth projective connected curve, let us define a motive $h^1(C)$ such that we have a direct sum:

$$h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C).$$

The upshot is that the theory of the motives $h^1(C)$ corresponds to that of Jacobian varieties (up to isogeny), in fact we have

$$\operatorname{Hom}(h^1(C), h^1(C')) = \operatorname{Hom}(J(C), J(C')) \otimes \mathbb{Q}.$$

In particular, the full subcategory of $\mathcal{M}_K$ whose objects are direct summands of the motive $h^1(C)$ is equivalent to the category of abelian subvarieties of $J(C)$ up to isogeny. Such motives can be called *abelian.* We will say that a motive is *discrete* if it is the direct sum of a finite number of Lefschetz motives.

Let S be a surface. Murre constructed [56] the motives $h^i(S)$, defined by projectors p_i in $\mathrm{CH}^i_{\mathbb{Q}}(S \times S)$ for $i = 1, 2, 3$, and described a decomposition

$$h(S) = h^0(S) \oplus h^1(S) \oplus h^2(S) \oplus h^3(S) \oplus h^4(S).$$

We have already remarked that $h^0(S) = \mathbb{1}$ and $h^4(S) = \mathbb{L}^2$. Roughly speaking, the submotive $h^1(S)$ carries the Picard variety, the submotive $h^3(S)$ the Albanese variety and the submotive $h^2(S)$ carries the Néron–Severi group, the Albanese kernel and the transcendental cycles. If S is a smooth rational surface and $K = \bar{K}$, then $h^1(S)$ and $h^3(S)$ are trivial, while $h^2(S) \simeq \mathbb{L}^{\rho}$, where ρ is the rank of the Néron–Severi group. In particular, the motive of S splits in a finite direct sum of (differently twisted) Lefschetz motives.

In general dimension, it is conjectured [57] that if X is a smooth projective variety of dimension d, there exist projectors p_i in $\mathrm{CH}^d_{\mathbb{Q}}(X \times X)$ defining motives $h^i(X)$ such that $h(X) = \bigoplus_{i=0}^{2d} h^i(X)$, and such that (over $\bar{K}$) p_i modulo (co)homological equivalence is the usual Künneth component. For example, if $K = \mathbb{C}$, we require $(p_i)_* H^*(X, \mathbb{Q}) = H^i(X, \mathbb{Q})$. Such a decomposition is called a *Chow–Künneth decomposition* (see [57], Definition 1.3.1) and should be thought of as a universal cohomological theory. We have seen that the motive of any smooth projective curve or surface admits a Chow–Künneth decomposition. This is true also for the motive of a smooth uniruled complex threefold, thanks to [2]. In this case, while h^1 still carries the Picard variety, now it is straightforward to remark that it is h^5 that carries the Albanese kernel. We will call them accordingly.

The strict interplay between motives and representability for threefolds is shown by Gorchinskiy and Guletskii. In this case, the rational representability of $A^i_{\mathbb{Q}}(X)$ for $i \geq 2$ is known ([55]). In [23] it is proved that $A^3_{\mathbb{Q}}(X)$ is rationally representable if and only if the Chow motive of X has a given Chow–Künneth decomposition.

Theorem 2.16 ([23], Theorem 8). *Let X be a smooth projective threefold. The group $A^3_{\mathbb{Q}}(X)$ is rationally representable if and only if the motive $h(X)$ has the following Chow–Künneth decomposition:*

$$h(X) \cong \mathbb{1} \oplus h^1(X) \oplus \mathbb{L}^{\oplus b} \oplus (h^1(J)(-1)) \oplus (\mathbb{L}^2)^{\oplus b} \oplus h^5(X) \oplus \mathbb{L}^3,$$

where $h^1(X)$ and $h^5(X)$ are the Picard and Albanese motives respectively, $b = b^2(X) = b^4(X)$ is the Betti number, and J is a certain abelian variety, which is isogenous to the intermediate Jacobian $J(X)$ if $K = \mathbb{C}$.

3 Interactions between categorical and classical representabilities

In this section, we will consider varieties defined over the complex numbers. This restriction is not really necessary, since most of the constructions work over any algebraically closed field. Anyway, in the complex case, we can simplify our treatment by dealing with intermediate Jacobians. Moreover, it will be more simple to list examples without the need to make the choice of the base field explicit for any case.

3.1 Fully faithful functors and motives. At the end of the last section we have seen that, in the case of threefolds, rational representability of $A^3_{\mathbb{Q}}(X)$ is equivalent to the existence of some Chow–Künneth decomposition. The first step in relating categorical and rational representability is exploiting an idea of Orlov about the motivic decomposition which should be induced by a fully faithful functor between the derived categories of smooth projective varieties. Assuming this conjecture we get that for threefolds categorical representability in dimension 1 is a stronger notion than rational representability.

Let us sketch Orlov's idea [64]. If X and Y are smooth projective varieties of dimension respectively n and m, and $\Phi\colon \mathrm{D}^{\mathrm{b}}(Y) \to \mathrm{D}^{\mathrm{b}}(X)$ is a fully faithful functor, then it is of Fourier–Mukai type [62], [63]. Let $\mathcal{E}$ in $\mathrm{D}^{\mathrm{b}}(X \times Y)$ be its kernel and $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}(X \times Y)$ the kernel of its right adjoint Ψ, we have $\mathcal{F} \simeq \mathcal{E}^{\vee} \otimes \mathrm{pr}_X^* \omega_X[\dim X]$ (see [53]). Consider $e := \mathrm{ch}(\mathcal{E})\mathrm{Td}(X)$ and $f := \mathrm{ch}(\mathcal{F})\mathrm{Td}(Y)$, two mixed rational cycles in $\mathrm{CH}^*_{\mathbb{Q}}(X \times Y)$. We denote by e_i (resp. f_i) the i-th codimensional component of e (resp. f), that is $e_i, f_i \in \mathrm{CH}^i_{\mathbb{Q}}(X \times Y)$. As correspondences they induce motivic maps $e_i : h(Y) \to h(X)(i-n)$ and $f_j : h(X)(m-j) \to h(Y)$. The Grothendieck–Riemann–Roch Theorem implies that $f.e := \bigoplus_{i=0}^{n+m} f_{n+m-i} e_i = \mathrm{id}_{h(Y)}$. This in turn implies that $h(Y)$ is a direct summand of $\bigoplus_{i \in \mathbb{Z}} h(X)(i)$.

Conjecture 1.1. *Let X and Y be smooth projective varieties and $\Phi\colon \mathrm{D^b}(Y) \to \mathrm{D^b}(X)$ be a fully faithful functor. Then the motive $h(Y)$ is a direct summand of the motive $h(X)$.*

The conjecture is trivially true for Y a smooth point, in which case $\Phi(\mathrm{D^b}(Y))$ is generated by an exceptional object of $\mathrm{D^b}(X)$: then there is an integer d and a split embedding $\mathbb{L}^d \to h(X)$ induced by the exceptional object. In [64], it is proven that the conjecture holds if X and Y have the same dimension n and $\mathcal{E}$ is supported in dimension n. This already covers some interesting examples: if X is a smooth blow-up of Y, or if there is a standard flip from X to Y. Using the same methods as in [13] we will show that if Y is a curve and X a rationally representable threefold with $h^1(X) = h^5(X) = 0$, then $h^1(Y)$ is a direct summand of $h^3(X)(1)$.

But let us first take a look to the simplest case, that is categorical representability in dimension 0. In this case, we have that Chow groups are finite dimensional vector spaces. Over the complex numbers this gives the discreteness of the motive.

Proposition 3.1. *If a smooth projective complex variety X is categorically representable in dimension* 0*, then the group $K_0(X)$ is free of finite type and the motive $h(X)$ is discrete.*

Proof. Being representable in dimension 0 is equivalent to having a full exceptional sequence $\{E_1, \dots, E_l\}$. Then the classes $[E_i]$, for $i = 1, \dots l$, are nontrivial (since E_i is exceptional, we have $\chi(E_i, E_i) = 1$) and give a free system of generators of $K_0(X)$, by the definition of semiorthogonal decomposition. Then $K_0(X) \simeq \mathbb{Z}^l$. From this and Riemann–Roch, we get that $\mathrm{CH}^*_{\mathbb{Q}}(X)$ is a finite dimensional $\mathbb{Q}$-vector space. For a complex smooth projective variety, this is enough to split the motive into Lefschetz motives ([40]). Notice anyway that, since the Riemann–Roch isomorphism $K_0(X) \otimes \mathbb{Q} \simeq \mathrm{CH}^*_{\mathbb{Q}}(X)$ is not compatible with gradings, there is no canonical way to obtain the decomposition of X explicitly from the base of $K_0(X)$, that is, from the exceptional sequence. □

A way more interesting case relates categorical representability in dimension 1 and rational representability for threefolds. In this case, in light of Theorem 2.16, we have a more specific conjecture.

Conjecture 3.2. *If a smooth projective threefold X is categorically representable in dimension* 1*, then it is rationally representable.*

If X is a standard conic bundle over a rational surface and Γ a smooth projective curve, the Chow–Künneth decomposition of $h(X)$ (see [58]) can be used to show that a fully faithful functor $\mathrm{D^b}(\Gamma) \to \mathrm{D^b}(X)$ gives $h^1(\Gamma)(-1)$ as a direct summand of $h(X)$. In particular, this gives an isogeny between $J(\Gamma)$ and an abelian subvariety of $J(X)$, and proves (up to codimensional shift for each direct summand of $h(\Gamma)$) Conjecture 1.1 in this case. The proof in [13] is based on the fact that the motive $h(X)$ splits into a discrete motive and in a unique abelian motive which corresponds to $J(X)$. Let us make a first assumption:

($\star$) X is a smooth projective rationally representable threefold with $h^1(X) = 0$ and $h^5(X) = 0$.

Theorem 3.3. *Suppose X satisfies ($\star$). If there is a smooth projective curve Γ and a fully faithful functor $\mathrm{D^b}(\Gamma) \to \mathrm{D^b}(X)$, then $h^1(\Gamma)(-1)$ is a direct summand of $h(X)$. This gives an injective morphism $J(\Gamma)_{\mathbb{Q}} \to J(X)_{\mathbb{Q}}$, that is an isogeny between $J(\Gamma)$ and an abelian subvariety of $J(X)$.*

Proof. We only need the case where $g(\Gamma) > 0$, and we can use the same argument as in [13], Lemma 4.2: since all but one summand of $h(X)$ are discrete, the map $f.e_{|h^1(\Gamma)} = \mathrm{id}_{h^1(\Gamma)}$ is given by $f_2.e_2$, which proves that $h^1(\Gamma)(-1)$ is a direct summand of $h^3(X) = h^1(J)(-1)$. □

Corollary 3.4. *Suppose X satisfies ($\star$) and let $\{\Gamma_i\}_{i=1}^k$ be smooth projective curves of positive genus. If $\mathrm{D^b}(X)$ is categorically representable in dimension 1 by the categories $\mathrm{D^b}(\Gamma_i)$ and by exceptional objects, then $J(X)$ is isogenous to $\bigoplus_{i=1}^k J(\Gamma_i)$.*

Proof. From Theorem 3.3 together with the semiorthogonality, we get an injective morphism $\phi\colon \bigoplus J(\Gamma_i)_{\mathbb{Q}} \to J(X)_{\mathbb{Q}}$, which has to be surjective by Remark 2.8, as explained in the proof of Theorem 4.1 in [13]. □

Remark 3.5 (Threefolds satisfying ($\star$)). By [23], [58] Fano threefolds, threefolds fibered in Del Pezzo or Enriques surfaces over $\mathbb{P}^1$ with discrete Picard group, and standard conic bundles over rational surfaces satisfy ($\star$).

3.2 Reconstruction of the intermediate Jacobian. The aim of this section is to show how, under appropriate hypotheses, categorical representability in dimension 1 for a threefold X gives a splitting of the intermediate Jacobian $J(X)$. Notice that in the case of curves the derived category carries the information about the principal polarization of the Jacobian [11]. In the case of threefolds, we need first of all the hypotheses of Theorem 3.3. As we will see, the crucial hypothesis that will allow us to recover also the principal polarization is that the polarization on $J(X)$ is an *incidence polarization*.

($\natural$) X is a smooth projective rationally and algebraically representable threefold with $h^1(X) = 0$ and $h^5(X) = 0$ and the algebraic representative of $A^2_{\mathbb{Z}}(X)$ carries an incidence polarization.

Theorem 3.6. *Suppose X satisfies ($\natural$). Let Γ be smooth projective curve and $\mathrm{D^b}(\Gamma) \to \mathrm{D^b}(X)$ fully faithful. Then there is an injective morphism $J(\Gamma) \hookrightarrow J(X)$ preserving the principal polarization, that is $J(X) = J(\Gamma) \oplus A$ for some principally polarized abelian variety A.*

Proof. From Theorem 3.3 we get an isogeny. As in the proof of Proposition 4.4 in [13], the incidence property shows that this isogeny is an injective morphism respecting the principal polarizations. □

Corollary 3.7. *Suppose X satisfies* ($\natural$) *and let* $\{\Gamma_i\}_{i=1}^k$ *be smooth projective curves of positive genus. If* $\mathrm{D}^{\mathrm{b}}(X)$ *is categorically representable in dimension* 1 *by the categories* $\mathrm{D}^{\mathrm{b}}(\Gamma_i)$ *and by exceptional objects, then* $J(X)$ *is isomorphic to* $\bigoplus_{i=1}^k J(\Gamma_i)$ *as principally polarized variety.*

Remark 3.8 (Threefolds satisfying ($\natural$)). The assumptions of Theorem 3.6 seem rather restrictive. Anyway, they are satisfied by a quite big class of smooth projective threefolds with $\kappa_X < 0$. The Chow–Künneth decomposition for the listed varieties is provided by [58] for conic bundles and by [23] in any other case. In the following list the references point out the most general results about strong representability and incidence property. Giving an exhaustive list of all the results and contributors would be out of reach (already in the cubic threefold case). We will consider Fano threefolds with Picard number one only. The interested reader can find an exhaustive treatment in [34].

1) Fano of index > 2: X is either $\mathbb{P}^3$ or a smooth quadric.

2) Fano of index 2: X is a quartic double solid [68] , or a smooth cubic in $\mathbb{P}^4$ [21], or an intersection of two quadrics in $\mathbb{P}^5$ [65], or a V_5 (in the last case $J(X)$ is trivial).

3) Fano of index 1: X is a general sextic double solid [20], or a smooth quartic in $\mathbb{P}^4$ [15], or an intersection of a cubic and a quadric in $\mathbb{P}^5$ [15], or the intersection of three quadrics in $\mathbb{P}^6$ [7], or a V_{10} [52], [26], or a V_{12} [32] ($J(X)$ is the Jacobian of a genus 7 curve), or a V_{14} [31] (in which case the representability is related to the birational map to a smooth cubic threefold), or a general V_{16} [27], [54], or a general V_{18} [29], [34] ($J(X)$ is the Jacobian of a genus 2 curve), or a V_{22} (and the Jacobian is trivial).

4) Conic bundles: $X \to S$ is a standard conic bundle over a rational surface [7], [10], this is the case examined in [13].

5) Del Pezzo fibrations: $X \to \mathbb{P}^1$ is a Del Pezzo fibration with $2 \leq K_X^2 \leq 5$ [35], [36].

Notice that there are still some cases where it is not known (at least, to us) whether a smooth projective threefold of negative Kodaira dimension satisfies ($\natural$), as for example if X is a Fano of index two and degree one, or a Del Pezzo fibration over $\mathbb{P}^1$ of degree one.

From the unicity of the splitting of the intermediate Jacobian we can easily infer the following.

Corollary 3.9. *Suppose X satisfies* ($\natural$) *and is categorically representable in dimension* 1*, with semiorthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(\Gamma_1), \dots, \mathrm{D}^{\mathrm{b}}(\Gamma_k), E_1, \dots, E_l \rangle.$$

Then there is no fully faithful functor $\mathrm{D^b}(\Gamma) \to \mathrm{D^b}(X)$ *unless* $\Gamma \simeq \Gamma_i$ *for some* $i \in \{1, \dots, k\}$*. Moreover, the semiorthogonal decomposition is essentially unique, that is, any semiorthogonal decomposition of* $\mathrm{D^b}(X)$ *by smooth projective curves and exceptional objects is given precisely by the curves* Γ_i *and by* l *exceptional objects.*

Corollary 3.10. *Suppose* X *satisfies* ($\natural$), Γ *is a smooth projective curve of positive genus and there is no splitting* $J(X) = J(\Gamma) \oplus A$*. Then there is no fully faithful functor* $\mathrm{D^b}(\Gamma) \to \mathrm{D^b}(X)$.

The assumptions of Corollary 3.10 are trivially satisfied if the threefold satisfying ($\natural$) has $J(X) = 0$. A way more interesting case is when the intermediate Jacobian is not trivial and there is no injective morphism $J(\Gamma) \to J(X)$ for any curve Γ, in which case the variety is not categorically representable in dimension < 2.

Remark 3.11 (Threefolds not categorically representable in dimension < 2). The assumptions of Corollary 3.10 are satisfied by smooth threefolds with $J(X) \neq 0$ for all curves Γ of positive genus in the following cases:

1) Fano varieties of index 2: X is a smooth cubic [21].

2) Fano varieties of index 1: for instance when X is a generic quartic threefold [51], the intersection of three quadrics in $\mathbb{P}^7$ [7], or a generic complete intersection of type $(3, 2)$ in $\mathbb{P}^5$ [7]. The case of a $(3, 2)$-complete intersection of Fermat polynomials is described in [8].

3) Conic bundles: X is a standard conic bundle $X \to \mathbb{P}^2$ degenerating along a curve of degree ≥ 6 [7], or a non-rational standard conic bundle $X \to S$ on a Hirzebruch surface [67].

4) Del Pezzo fibrations: $X \to \mathbb{P}^1$ non-rational of degree four [1].

There are some other cases of Fano threefolds of specific type satisfying geometric assumptions. For a detailed treatment, see Chapter 8 of [34].

Notice that if X is a smooth cubic threefold, the equivalence class of a notable admissible subcategory $\mathbf{A}_X$ (the orthogonal complement of $\{\mathcal{O}_X, \mathcal{O}_X(1)\}$) corresponds to the isomorphism class of $J(X)$ as principally polarized abelian variety [14]; the proof is based on the reconstruction of the Fano variety and the techniques used there are far away from the subject of this paper.

A natural question is if, under some hypotheses, one can give the inverse statement of Corollaries 3.4 and 3.7, that is, suppose that X is a threefold satisfying either ($\star$) or ($\natural$), such that $J(X) \simeq \bigoplus J(\Gamma_i)$. Can one describe a semiorthogonal decomposition of $\mathrm{D^b}(X)$ by exceptional objects and the categories $\mathrm{D^b}(\Gamma_i)$? Notice that a positive answer for X implies a positive answer for all the smooth blow-ups of X.

Remark 3.12 (Threefolds with $\kappa_X < 0$ categorically representable in dimension ≤ 1). Let X be a threefold satisfying $(\star)$ or $(\natural)$ and with $J(X) = \bigoplus J(\Gamma_i)$. Then if X is in the following list (or is obtained by a finite number of smooth blow-ups from a variety in the list) we have a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(\Gamma_1), \ldots, \mathrm{D}^{\mathrm{b}}(\Gamma_k), E_1, \ldots, E_l \rangle,$$

with E_i exceptional objects.

1) Threefolds with a full exceptional sequence: X is $\mathbb{P}^3$ [9], or a smooth quadric [37], or a $\mathbb{P}^1$-bundle over a rational surface or a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ [61], or a V_5 [60], or a V_{22} Fano threefold [41].

2) Fano threefolds without any full exceptional sequence: X is the complete intersection of two quadrics or a Fano threefold of type V_{18}, and $J(\Gamma) \simeq J(X)$ with Γ a genus 2 curve. The semiorthogonal decompositions are described in [18], [44], and are strikingly related (as in the cases of V_5 and V_{22} and of the cubic and V_{14}) by a correspondence in the moduli spaces, as described in [47]. X is a V_{12} Fano threefold [42], or a V_{16} Fano threefold [44].

3) Conic bundles without any full exceptional sequence: $X \to S$ is a rational conic bundle over a minimal surface [13]. If the degeneration locus of X is either empty or a cubic in $\mathbb{P}^2$, then X is a $\mathbb{P}^1$-bundle and is listed in 1).

4) Del Pezzo fibrations: $X \to \mathbb{P}^1$ is a quadric fibration with at most simple degenerations, in which case the hyperelliptic curve $\Gamma \to \mathbb{P}^1$ ramified along the degeneration appears naturally as the orthogonal complement of an exceptional sequence of $\mathrm{D}^{\mathrm{b}}(X)$ [48]. $X \to \mathbb{P}^1$ is a rational Del Pezzo fibration of degree four. In this case X is birational to a conic bundle over a Hirzebruch surface [1] and the semiorthogonal decomposition is described in [4].

Notice that the first two items cover all classes of Fano threefolds with Picard number one whose members are all rational.

4 Categorical representability and rationality: further developments and open questions

This last section is dedicated to speculations and open questions about categorical representability and rationality. The baby example of curves is full understood. A smooth projective curve X over a field K is categorically representable in dimension 0 if and only if it is rational. Indeed, the only case where $\mathrm{D}^{\mathrm{b}}(X)$ has exceptional objects is $X = \mathbb{P}^1$, and $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{O}_X, \mathcal{O}_X(1) \rangle$.

Let us start with a trivial remark: the projective space $\mathbb{P}^n$ over K is categorically representable in dimension 0. Then if X is given by a finite number of smooth blow-ups of $\mathbb{P}^n$, it is categorically representable in codimension ≥ 2. This is easily obtained from

Orlov's blow-up formula (see Proposition 2.9). More generally, if a smooth projective variety X of dimension ≥ 2 is categorically representable in codimension m, then any finite chain of smooth blow-ups of X is categorically representable in codimension $\geq \min(2, m)$.

One could naively wonder about the inverse statement: if $X \to Y$ is a finite chain of smooth blow-ups and X is categorically representable in codimension m, what can we say about Y? Unfortunately, triangulated categories do not have enough structure to let us compare different semiorthogonal decomposition. For example, the theory of mutations allows to do this only in a few very special cases.

In this section we present some more example to stress how the interaction between categorical representability and rationality can be developed further, and we point out some open question. We deal with surfaces in 4.1 and with threefolds in 4.2. Then we will discuss in 4.3 how categorical representability for noncommutative varieties plays an important role in this frame, to deal with varieties of dimension bigger than 3 in 4.4. Finally, we compare in 4.5 our methods with recent approaches to birationality problems via derived categories. We will work over the field $\mathbb{C}$ for simplicity, even if many problems and arguments do not depend on that.

4.1 Surfaces. If X is a smooth projective rational surface, then it is categorically representable in codimension 2. Indeed, X is the blow-up in a finite number of smooth points of a minimal rational surface, that is either $\mathbb{P}^2$ or $\mathbb{F}_n$. Are there any other example of surfaces categorically representable in codimension 2? Notice that by Proposition 3.1 such a surface would have a discrete motive, and even more: we would have $K_0(X) = \mathbb{Z}^l$. In particular, if $K_0(X)$ is not free, then X is not categorically representable in dimension 0.

In general, an interesting problem is to construct exceptional sequences on surfaces with $p_g = q = 0$, and to study their orthogonal complement. Notice that on such surfaces any line bundle is an exceptional object, so we already have at least a length one exceptional sequence. The main question is then to understand if it is possible to find a somehow "maximal" one, that is, such that the orthogonal complement does not contain exceptional objects. The length of such sequence should be bounded by the rank of maximal free subgroup of $K_0(X)$.

Suppose for example that X is an Enriques surface: a (non-full) exceptional collection of 10 vector bundles on X is described in [69]. Since $K_0(X)$ is not free of finite rank, we do not expect any full exceptional collection. The orthogonal complement $\mathbf{A}_X$ turns then out to be a very interesting object, related also to the geometry of some singular quartic double solid [33]. Using a motivic trick, we can prove that, under some assumption, a surface with $p_g = q = 0$ is either categorically representable in codimension 2 or not categorically representable in positive codimension.

Proposition 4.1. *Let X be a surface with $h(X)$ discrete. Then for any curve Γ of positive genus, there is no fully faithful functor* $\mathrm{D}^{\mathrm{b}}(\Gamma) \to \mathrm{D}^{\mathrm{b}}(X)$.

Proof. Suppose there is such a curve and such a functor $\Phi\colon \mathrm{D}^{\mathrm{b}}(\Gamma) \to \mathrm{D}^{\mathrm{b}}(X)$. Let $\mathcal{E}$ denote the kernel of Φ (which has to be of Fourier–Mukai type) and $\mathcal{F}$ the kernel of its adjoint. Consider the cycles e and f described in Section 3.1, and recall that $f.e = \bigoplus_{i=0}^{3} f_{3-i}.e_i = \mathrm{id}_{h(\Gamma)}$. Restricting now to $h^1(\Gamma)$ we would have that $\mathrm{id}_{h^1(\Gamma)}$ would factor through a discrete motive, which is impossible. □

Corollary 4.2. *Let X be a surface with $h(X)$ discrete and $K_0(X)$ not free of finite rank. Then X is not categorically representable in codimension > 0.*

Remark 4.3 (Surfaces with $p_g = q = 0$ not categorically representable in positive codimension). Proposition 3.1 could be an interesting tool in the study of derived categories of surfaces with $p_g = q = 0$: notice that many of them have torsion in $H_1(X,\mathbb{Z})$ (for an exhaustive treatment and referencing, see [6]). Anyway the discreteness of the motive is a rather strong assumption, which for example implies the Bloch conjecture. There are few cases where this is known.

1) X is an Enriques surface [22].

2) X is a Godeaux surface obtained as a quotient of a quintic by an action of $\mathbb{Z}/5\mathbb{Z}$ [24]. In this case in particular it is shown that the motive decomposes as $\mathbb{1} \oplus 9\mathbb{L} \oplus \mathbb{L}^2$.

These observations lead to state some deep question about categorical representability of surfaces.

Question 4.4. Let X be a smooth projective surface with $p_g = q = 0$.

1) Is there a full exceptional sequence? Equivalently, is X categorically representable in codimension 2? If not, can one describe a non full maximal (i.e. the complement does not contain any exceptional object) exceptional sequence and its complement?

2) If X is representable in codimension 2, is X rational?

4.2 Threefolds. Remark that there are examples of smooth projective non-rational threefolds X which are categorically representable in codimension 2: just consider a rank three vector bundle $\mathcal{E}$ on a curve C of positive genus and take $X := \mathbb{P}(\mathcal{E})$. In Section 6.3 of [13] we provide a conic bundle example. Anyway, Corollary 3.7 somehow suggests that categorical representability in codimension 2 should be a necessary condition for rationality.

A reasonable idea is to restrict our attention to minimal threefolds with $\kappa_X < 0$ (recall that this is a necessary condition for rationality), in particular to the ones we expect to satisfy assumption (♮), in order to have interesting information about the intermediate Jacobian from semiorthogonal decompositions. The three big families of such threefolds are: Fano threefolds, conic bundles over rational surfaces and del Pezzo fibrations over $\mathbb{P}^1$. Remarks 3.8 and 3.12 give a long list of examples either of rational threefolds which are categorically representable in codimension 2 or non-rational ones which cannot be categorically representable in codimension > 1.

Question 4.5. Let X be a smooth projective threefold with $\kappa_X < 0$.

1) If X is rational, is X categorically representable in codimension 2?

2) Is X categorically representable in codimension 2 if and only if X is rational?

A positive answer to the second question is provided for standard conic bundles over minimal surfaces [13], but it seems to be quite a strong fact to hold in general: recall that having a splitting $J(X) \simeq \bigoplus J(\Gamma_i)$ is only a necessary condition for rationality, and Corollary 3.7 shows that if X satisfies ($\natural$), categorical representability in codimension 2 would give the splitting of the Jacobian.

Remark 3.12 provides a large list of rational threefolds categorically representable in codimension 2. Is it possible to add examples to this list? In particular in the case of Del Pezzo fibrations over $\mathbb{P}^1$ only the quadric and the degree 4 fibration are described respectively in [48] and [4].

A good way to understand these questions is by studying some special rational or non-rational (that is non generic in their family) threefold. This forces to consider non smooth ones, but we can use Kuznetsov's theory of categorical resolution of singularities [46] and study the categorical resolution of $\mathrm{D^b}(X)$, as we pointed out in Remark 2.5. For example, let $X \subset \mathbb{P}^4$ be nodal cubic threefold with a double point, which is known to be rational.

Proposition 4.6. *Let $X \subset \mathbb{P}^4$ be a cubic threefold with a double point and $\widetilde{X} \to X$ the blow-up of the singular point. There exists a categorical resolution of singularities $\widetilde{\mathbf{D}} \subset \mathrm{D^b}(\widetilde{X})$ of $\mathrm{D^b}(X)$ (in the sense of [46]) which is representable in codimension two. Indeed there is a semiorthogonal decomposition*

$$\widetilde{\mathbf{D}} = \langle \mathrm{D^b}(\Gamma), E_1, \dots, E_3 \rangle,$$

where E_i are exceptional objects and Γ a complete intersection of a quadric and a cubic in $\mathbb{P}^3$.

Proof. This is shown following step by step [49], Section 5, where the four dimensional case is studied. Let us give a sketch of the proof. Let P be the singular point of X, and $\sigma\colon \widetilde{X} \to X$ its blow-up. The exceptional locus $\alpha\colon Q \hookrightarrow \widetilde{X}$ of σ is a quadric surface. The projection of $\mathbb{P}^4$ to $\mathbb{P}^3$ from the point P restricted to X gives the birational map $X \dashrightarrow \mathbb{P}^3$. The induced map $\pi\colon \widetilde{X} \to \mathbb{P}^3$ is the blow-up of a smooth curve Γ of genus 4, given by the complete intersection of a cubic and a quadric surface. If we write $h := \pi^*\mathcal{O}_{\mathbb{P}^3}(1)$ and $H := \sigma^*\mathcal{O}_X(1)$, we have that $Q = 2h - D$, $H = 3h - D$, then $h = H - Q$ and $D = 2H - 3Q$ as in [49], Lemma 5.1. The canonical bundle $\omega_{\widetilde{X}} = -4h + D = -2H + Q$ can be calculated via the blow-up π.

In order to describe a categorical resolution of singularities, we have to provide a Lefschetz decomposition of Q with respect to the conormal bundle (for definitions and details, see [46]). The conormal bundle of Q is $\mathcal{O}_Q(h)$ and the Lefschetz decomposition with respect to it is

$$\langle \mathcal{A}_1(-h), \mathcal{A}_0 \rangle,$$

where $\mathcal{A}_1 = \langle \mathcal{O}_Q \rangle$ and $\mathcal{A}_0 = \langle \mathcal{O}_Q, S_1, S_2 \rangle$, with S_1 and S_2 the two spinor bundles. Indeed, Q is even-dimensional and then has two non-isomorphic spinor bundles giving the previous semiorthogonal decomposition [37]. The case where X is four-dimensional, considered in [49] is slightly different.

We then get, by [46] a categorical resolution of singularities $\widetilde{\mathbf{D}}$ of $\mathrm{D}^{\mathrm{b}}(X)$ in the semiorthogonal decomposition:

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle \alpha_* \mathcal{O}_Q(-h), \widetilde{\mathbf{D}} \rangle.$$

We then get

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle \alpha_* \mathcal{O}_Q(-h), \widetilde{\mathbf{A}}_X, \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(H) \rangle, \tag{4.1}$$

where $\widetilde{\mathbf{A}}_X$ is a categorical resolution of $\mathbf{A}_X$, as in [49], Lemma 5.8. The representability of $\widetilde{\mathbf{D}}$ relies then on the representability of $\widetilde{\mathbf{A}}_X$.

On the other side, applying the blow-up formula to $\pi\colon \widetilde{X} \to \mathbb{P}^3$ (see Proposition 2.9), and choosing $\{\mathcal{O}_{\mathbb{P}^3}(-3), \dots, \mathcal{O}_{\mathbb{P}^3}\}$ as full exceptional sequence for $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3)$, we obtain

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle \Phi \mathrm{D}^{\mathrm{b}}(\Gamma), \mathcal{O}_{\widetilde{X}}(-3h), \mathcal{O}_{\widetilde{X}}(-2h), \mathcal{O}_{\widetilde{X}}(-h), \mathcal{O}_{\widetilde{X}} \rangle,$$

where $\Phi\colon \mathrm{D}^{\mathrm{b}}(\Gamma) \to \mathrm{D}^{\mathrm{b}}(\widetilde{X})$ is fully faithful. Now as in Lemma 5.3 of [49], if we mutate $\mathcal{O}_{\widetilde{X}}(-3h)$ and $\mathcal{O}_{\widetilde{X}}(-2h)$ to the left with respect to $\Phi \mathrm{D}^{\mathrm{b}}(\Gamma)$, and put $\mathbf{B} := \langle \Phi \mathrm{D}^{\mathrm{b}}(\Gamma), \mathcal{O}_{\widetilde{X}}(-h) \rangle$, we get

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle \mathcal{O}_{\widetilde{X}}(-3h + D), \mathcal{O}_{\widetilde{X}}(-2h + D), \mathbf{B}, \mathcal{O}_{\widetilde{X}} \rangle. \tag{4.2}$$

Finally, one can show that $\mathbf{B}$ and $\widetilde{\mathbf{A}}_X$ are equivalent, following exactly the same path of mutations as in Section 5 of [49] to compare the decompositions (4.1) and (4.2). □

Remark 4.7. As noted in [19], we could aim to some kind of minimal resolution of singularities, where minimality has to be taken with respect to full and faithful functors. Then one is lead to suspect that the category $\widetilde{\mathbf{A}}_X$ is not minimal. A natural question to ask for is if it possible to give a categorical resolution of singularities of $\mathbf{A}_X$ equivalent to $\mathrm{D}^{\mathrm{b}}(\Gamma)$.

Another special very interesting example is described in [33]: a singular double solid $X \to \mathbb{P}^3$ ramified along a quartic symmetroid. This threefold is non-rational thanks to [3], because $H^3(X, \mathbb{Z})$ has torsion. A rough account (skipping the details about the resolution of singularities) of Ingalls and Kuznetsov's result is the following: if X' is the small resolution of X, there is an Enriques surface S and a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X') = \langle \mathbf{A}_S, E_1, E_2 \rangle, \tag{4.3}$$

where E_i are exceptional objects and $\mathbf{A}_S$ is the orthogonal complement in $\mathrm{D^b}(S)$ of 10 exceptional vector bundles on S ([69]). By Corollary 4.2, the category $\mathbf{A}_S$ is not representable in dimension < 2. This is anyway not enough to show that X' cannot be categorically representable in codimension > 1, because it does not exclude the existence of other semiorthogonal decompositions.

Remark anyway that the lack of categorical representability of X' (and presumably of X, thinking about the categorical resolution of singularities) should be based on the lack of categorical representability of S, which relies on the presence of torsion in $K_0(S)$ and in particular in $H_1(S, \mathbb{Z})$, and on the structure of the motive. On the other side, the non-rationality of X is due to the presence of torsion in $H^3(X, \mathbb{Z})$. The relation between torsion in $H^3(X, \mathbb{Z})$ and categorical representability needs a further investigation, for example in the case recently described in [28].

4.3 Noncommutative varieties. The previous speculations and partial results give rise to the hope of extending fruitfully the study of categorical representability to higher dimensions and to the noncommutative setting. By the latter we mean, following Kuznetsov ([48], Section 2), an algebraic variety Y with a sheaf $\mathcal{B}$ of $\mathcal{O}_Y$-algebras of finite type. Very roughly, the corresponding noncommutative variety $\overline{Y}$ would have a category of coherent sheaves $\mathbf{Coh}(\overline{Y}) = \mathbf{Coh}(Y, \mathcal{B})$ and a bounded derived category $\mathrm{D^b}(\overline{Y}) = \mathrm{D^b}(Y, \mathcal{B})$. The examples which appear very naturally in our setting are the cases where $\mathcal{B}$ is an Azumaya algebra or the even part of the Clifford algebra associated to some quadratic form over Y.

Finally, if a triangulated category $\mathbf{A}$ has Serre functor such that $S_{\mathbf{A}}^m = [n]$, for some integers n and m, with m minimal with this property, we will call it a $\frac{n}{m}$-*Calabi–Yau category*. If $m = 1$, these categories deserve the name of noncommutative Calabi–Yau n-folds, even if they are not a priori given by the derived category of some Calabi–Yau n-fold with a sheaf of algebras.

If S is any smooth projective variety, $X \to S$ a Brauer–Severi variety of relative dimension r, and $\mathcal{A}$ the associated Azumaya algebra in $\mathrm{Br}(S)$, then (see [12])

$$\mathrm{D^b}(X) = \langle \mathrm{D^b}(S), \mathrm{D^b}(S, \mathcal{A}^{-1}), \dots, \mathrm{D^b}(S, \mathcal{A}^{-r+1}) \rangle.$$

The categorical representability of X would then rely on the categorical representability of $(S, \mathcal{A})$, which is an interesting object in itself. For example, if Y is a generic cubic fourfold containing a plane, there are a K3 surface S and an Azumaya algebra $\mathcal{A}$ such that the categorical representability of $(S, \mathcal{A})$ is the subject of Kuznetsov's conjecture [49] about the rationality of cubic fourfolds.

If S is a smooth projective variety and $Q \to S$ a quadric fibration of relative dimension r, we can consider the sheaf $\mathcal{B}_0$ of the even parts of the Clifford algebra associated to the quadratic form defining Q. There is a semiorthogonal decomposition:

$$\mathrm{D^b}(Q) = \langle \mathrm{D^b}(S, \mathcal{B}_0), \mathrm{D^b}(S)_1, \dots, \mathrm{D^b}(S)_r \rangle,$$

where $\mathrm{D}^{\mathrm{b}}(S)_i$ are equivalent to $\mathrm{D}^{\mathrm{b}}(S)$ [48]. The categorical representability of $(S, \mathcal{B}_0)$ should then be a very important tool in studying birational properties of Q. This is indeed the case for conic bundles over rational surfaces [13].

Finally, let $\mathbf{A}$ be an $\frac{n}{m}$-Calabi–Yau category. Such categories appear as orthogonal complements of an exceptional sequence on Fano hypersurfaces in projective spaces, see Corollary 4.3 in [43]. It is then natural to wonder about their representability. For example, if X is a cubic or a quartic threefold, it follows from Remark 3.11 that these orthogonal complements (which are, respectively, $\frac{5}{3}$- and $\frac{10}{4}$-Calabi–Yau) are not representable in dimension 1.

Question 4.8. Let $\mathbf{A}$ be a $\frac{n}{m}$-Calabi–Yau category.

1) Is $\mathbf{A}$ representable in some dimension?

2) If yes, is there an explicit lower bound for this dimension?

3) If $m = 1$, is $\mathbf{A}$ representable in dimension n if and only if there exist a smooth n-dimensional variety X and an equivalence $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathbf{A}$?

4.4 Higher dimensional varieties. Unfortunately, it looks like the techniques used for threefolds in [13] hardly extend to dimensions bigger than 3. The examples and supporting evidences provided so far lead anyway to suppose that categorical representability can give useful information on the birational properties of any projective variety. The main case is a challenging conjecture by Kuznetsov [49]. Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold, then there is a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathbf{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle.$$

The category $\mathbf{A}_X$ is 2-Calabi–Yau.

Conjecture 4.9 (Kuznetsov). *The cubic fourfold X is rational if and only if $\mathbf{A}_X \simeq \mathrm{D}^{\mathrm{b}}(Y)$ for a smooth projective $K3$ surface Y.*

This conjecture has been verified in [49] for singular cubics, Pfaffian cubics and Hassett's [25] examples. When X contains a plane P there is a way more explicit construction: blowing up P we obtain a quadric bundle $\widetilde{X} \to \mathbb{P}^2$ of relative dimension 2, degenerating along a sextic. If the sextic is smooth, let $S \to \mathbb{P}^2$ be the double cover, which is a K3 surface. Then

$$\mathbf{A}_X \simeq \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0) \simeq \mathrm{D}^{\mathrm{b}}(S, \mathcal{A}),$$

where $\mathcal{B}_0$ is associated to the quadric fibration and $\mathcal{A}$ is an Azumaya algebra, obtained lifting $\mathcal{B}_0$ to S. The questions about categorical representability of noncommutative varieties arise then very naturally in this context. Notice that if $\mathbf{A}_X$ is representable in dimension 2, then we know something weaker than Kuznetsov conjecture: we would have a smooth projective surface S' and a fully faithful functor $\mathbf{A}_X \to \mathrm{D}^{\mathrm{b}}(S')$. Point 3) of Question 4.8 appears naturally in this context.

Question 4.10. One can then wonder if the Kuzentsov conjecture may be stated in the following form: the cubic fourfold X is rational if and only if it is categorically representable in codimension 2. An important check in this perspective is to show that the 2-Calabi–Yau category $\mathbf{A}_X$ is representable in dimension 2 if and only if there exist a K3 surface Y and an equivalence $\mathrm{D}^{\mathrm{b}}(Y) \simeq \mathbf{A}_X$.

Notice anyway that there could be a priori other semiorthogonal decompositions not related to the one considered in the conjecture. A very deep question is then to understand if and under which conditions one has a canonical choice for a semiorthogonal decomposition.

We can propose some more examples of fourfolds for which a Kuznetsov-type conjecture seems natural: if X is the complete intersection of three quadrics Q_1, Q_2, Q_3 in $\mathbb{P}^7$, then Homological Projective Duality ([45], [48]) gives an exceptional sequence on X and its complement $\mathbf{A}_X \simeq \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0)$, where $\mathcal{B}_0$ is the even Clifford algebra associated to the family of quadrics generated by Q_1, Q_2, Q_3. Similarly, if we consider two quadric fibrations $Q_1, Q_2 \to \mathbb{P}^1$ of relative dimension 4 and their complete intersection X, there is an exceptional sequence on X, and let $\mathbf{A}_X$ be its orthogonal complement. A relative version of Homological Projective Duality shows that $\mathbf{A}_X$ equivalent to $\mathrm{D}^{\mathrm{b}}(S, \mathcal{B}_0)$, where S is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and $\mathcal{B}_0$ the even Clifford algebra associated to the pencil of quadrics generated by Q_1 and Q_2. It is natural to wonder if representability in dimension 2 of the noncommutative varieties is equivalent or is a necessary condition for rationality of X. A partial answer to the last example is provided in [4].

Other examples in dimension 7 are provided in [30]. If X is a cubic sevenfold, there is a distinguished subcategory $\mathbf{A}_X$ of $\mathrm{D}^{\mathrm{b}}(X)$, namely the orthogonal complement of the exceptional sequence $\{\mathcal{O}_X, \dots, \mathcal{O}_X(5)\}$. This is a 3-Calabi–Yau category. Moreover it can be shown [30] that $\mathbf{A}_X$ cannot be equivalent to the derived category of a 3-dimensional Calabi–Yau variety. It is also conjectured that $\mathbf{A}_X$ is equivalent to the orthogonal complement of an exceptional sequence in the derived category $\mathrm{D}^{\mathrm{b}}(Y)$ of a Fano sevenfold Y of index 3, birationally equivalent to X.

4.5 Other approaches. Of course categorical representability is just one among different approaches to the study of birational geometry of a variety via derived categories. Nevertheless there is some common ground.

First of all, Kuznetsov mentions in [49] the notion of Clemens–Griffiths component of $\mathrm{D}^{\mathrm{b}}(X)$, whose vanishing would be a necessary condition for rationality. It seems reasonable to expect that categorical representability in codimension 2 implies the vanishing of the Clemens–Griffiths component.

Another recent theory is based on Orlov spectra and their gaps [5]. Let us even refrain from sketching a definition of it, but just notice that ([5], Conjecture 2) draws a link between categorical representability and gaps in the Orlov spectrum (see, in particular, [5], Corollary 1.11). Finally, conjectures based on homological mirror symmetry are proposed in [38], [39], but we cannot state a precise relation with our construction. A careful study of the example constructed in [28] would be a good starting point.

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