Derived categories and rationality of conic bundles

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Classical setting and the main results

Let \( \pi : X \to S \) be a smooth standard conic bundle over a smooth projective surface, and \( \mathcal{C} \to C \) the associated double cover of the discriminant curve given by connected components of singular conics. A classical question in algebraic geometry is to determine the rationality of \( X \). Necessary conditions: \( S \) is rational, \( C \) is connected and the intermediate Jacobian \( J(X) \) is isomorphic to the direct sum of Jacobians of smooth projective curves.

Consequence [Clemens-Griffiths]. Any smooth cubic threelfold in \( \mathbb{P}^3 \) is not rational.

[Beauville] \( J(Y) \) is isomorphic to the Prym variety \( P(C/C) \).

[Beauville, Shklovskii] If \( S \) is minimal \( X \) is rational if and only if \( J(X) \) splits as the sum of Jacobians of curves. The only possible cases are: \( S = \mathbb{P}^2 \) and \( C = \mathbb{P}^2 \) and \( C \) is a quartic, \( S = \mathbb{P}^3 \) and \( C \) is a quintic. \( \mathbb{P}^3 \) and \( C \) is an even \( \mathcal{C} \)-characteristic, \( S = \mathbb{P}^3 \) and \( C \) is either trigonal or hyperelliptic and the \( \mathcal{C} \) is induced by the ruling.

Question. Can we relate the derived category \( D^b(X) \) and the rationality of \( X \)? The most promising way is looking at semiorthogonal decompositions of \( D^b(X) \).

Example [BMMMS]. If \( X \) is a smooth cubic threefold

\[ D^b(X) = \langle T, C_0, C_1(1) \rangle, \]

and the equivalence class of the category \( \mathcal{C} \) corresponds to the isomorphism class of \( J(X) \).

Main Results. Let \( \pi : X \to S \) be a smooth standard conic bundle over a smooth rational surface and \( D^b(S, R) \) the derived category of \( R \)-sheaves.

Theorem 1. If there are smooth projective curves \( \Gamma_j \) with fully faithful functors \( \Phi_j : D^b(\Gamma_j) \to D^b(S, R) \), exceptional objects \( E_j \) in \( D^b(S, R) \) and semiorthogonal decomposition

\[ D^b(S, R) = \langle \Phi_1(\Gamma_1), \ldots, \Phi_n(\Gamma_n), E_1, \ldots, E_t \rangle, \]

then \( J(X) \cong \oplus_j J(\Gamma_j) \).

Theorem 2. If \( S \) is minimal, then \( X \) is rational and \( J(X) \cong \oplus_j J(\Gamma_j) \) if and only if \( D^b(S, R) \) decomposes like (1).

Remark. We work exclusively with varieties defined over \( \mathbb{C} \).

From semiorthogonal decomposition to rationality

The key of the proof of Theorem 1 is the study of the map induced by a fully faithful functor \( \Phi : D^b(\Gamma) \to D^b(X) \) on the motive \( h(\Gamma) \), where \( \Gamma \) is a smooth projective curve and \( \chi(\Gamma) > 0 \).

If \( \Phi : D^b(\Gamma) \to D^b(X) \) is fully faithful, then it is a Fourier-Mukai functor. Moreover, it admits a right adjoint \( \Psi_0 \), also a FM. Let \( E \) be the kernels of \( \Phi \) and \( \Psi_0 \), respectively. Then \( \Phi \circ \Psi_0 = D_{\mathbb{Q}(\Gamma)} \).

Define \( c \) to be the fixed point \( c_\mathbb{Q}(\Gamma) \). Then, \( c \) is not reduced to \( \mathbb{Q}(\Gamma) \), mixed cycles in \( C(\Gamma) \), and is given by Gotzmann’s Riemann-Roch theorem \( c = c_\mathbb{Q}(\Gamma) \).

By the decomposition of \( b(\Gamma, \{x, x+y, x+2y\}) \) we are reduced to \( D^b(\mathbb{Z}(W)) \). If \( D^b(W) \) is the derived category of \( X \), we have an isomorphism \( c_\mathbb{Q}(\Gamma) \to c_\mathbb{Q}_\mathbb{C}(\Gamma) \), where \( X \) is the algebraic morphism \( J(\Gamma) \to J(X) \) given by the cycle \( c_\mathbb{Q}(\Gamma) \).

The Prym variety \( V(C/C) \) is the algebraic representative of the algebraically trivial part \( A^1(X) \) of the Chow group. The polarization \( b(\Gamma) \) is an equivalence with respect to \( X \). In particular \( \mathbb{Q}(\Gamma) = \mathbb{Q}(\mathbb{C}) \), where \( \mathbb{Q}(\mathbb{C}) \) is an isomorphism between \( J(\Gamma) \) and a principally polarized abelian variety of \( J(X) \).

Consider a semiorthogonal decomposition like (1). Since \( S \) is rational, we get

\[ D^b(X) = \langle \Phi_1(\Gamma_1), \ldots, \Phi_n(\Gamma_n), E_1, \ldots, E_t \rangle. \]

Each \( \Phi_i \) gives a morphism \( c_i \). Moreover

\[ CH^1(X) = \bigoplus_i CH^1(\Gamma_i) \oplus \mathbb{Q} \subset \bigoplus_i CH^1(\Gamma_i) \oplus \mathbb{Q} \]

The cokernel of \( c_i \) is a finite \( \mathbb{Q} \)-vector space. Since \( \mathbb{Q}(\Gamma_1) \to J(X) \) is a morphism of abelian varieties, such cokernel is trivial. Then, \( c_i \) is an isomorphism of principally polarized abelian varieties.

Corollary If \( S \) is minimal and \( D^b(S, R) \) admits a decomposition like (1), then \( X \) is rational and \( J(X) \cong \oplus_j J(\Gamma_j) \).

From rationality to semiorthogonal decomposition

Let \( \pi : X \to S \) be a rational standard conic bundle over a minimal rational surface. \( S \) is isomorphic over the generic point to a quaternion algebra. Since \( Br(S) = 0 \) the double cover \( \mathcal{C} \to C \) determines a unique quaternion algebra in \( Br(K(S)) \) [Artin-Mumford]. Then the category \( D^b(S, R) \) is fixed by \( \mathcal{C} \). Theorem 2 is proved providing an example for each possible case.

In each case we provide an explicit construction as follows:

\[ X \leftarrow S \]

\[ Z \]

\[ \mathcal{C} \] is a smooth projective rational variety with known semiorthogonal decomposition, \( \pi : X \to S \) is induced by an explicit linear system on \( S \), and \( \chi \) is the blow-up of the smooth curve \( S \) in the base locus.

The decompositions are obtained comparing, via mutations, the decompositions induced respectively by the blow-up and by the conic bundle structure:

\[ (A) \]

\[ D^b(X) = \langle \Phi_1(\Gamma_1), \ldots, \Phi_n(\Gamma_n) \rangle. \]

\[ (B) \]

\[ D^b(X) = \langle \Phi_0(S, R), \ast \mathcal{D}(S) \rangle. \]

Here is a table summarizing the different four cases:

<table>
<thead>
<tr>
<th>( C \subset S )</th>
<th>( D^b(S) )</th>
<th>( T )</th>
<th>( D^b(T) )</th>
<th>( D^b(S, R) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plane curve in ( \mathbb{P}^2 )</td>
<td>3 exc.</td>
<td>Quadric genus 2</td>
<td>4 exc.</td>
<td>( D^b(T) ), 1 exc.</td>
</tr>
<tr>
<td>Sm. curve in ( \mathbb{P}^3 )</td>
<td>3exc.</td>
<td>( \mathbb{P}^1 )-bd. over ( \mathbb{P}^1 )</td>
<td>4 exc.</td>
<td>( D^b(T) ), 2 exc.</td>
</tr>
<tr>
<td>Trigonal in ( E )</td>
<td>4 exc.</td>
<td>Quadric bd. over ( \mathbb{P}^1 )</td>
<td>4 exc.</td>
<td>( D^b(T) ), 4 exc.</td>
</tr>
</tbody>
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References


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