Categorical dimension of birational maps
and a filtration of the group of birational automorphisms

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We work with k-linear triangulated categories $\mathbf{T}$, where k is a field of characteristic zero. Such categories will always arise as full subcategories of the bounded derived category $\mathrm{D^b}(X)$ of complexes of coherent sheaves on a smooth projective k-variety X. The dimension of X is always denoted by n.

Semiorthogonal decompositions and a Grothendieck ring

Definition. A *Semiorthogonal decomposition* $\mathbf{T} = \langle \mathbf{T}_1, \ldots, \mathbf{T}_r \rangle$, is an ordered set $\mathbf{T}_1, \ldots \mathbf{T}_r$ of full subcategories of $\mathbf{T}$ such that

✓ the embedding functors $\mathbf{T}_i \subset \mathbf{T}$ admit right and left adjoints;

✓ there is no nontrivial morphism from $\mathbf{T}_j$ to $\mathbf{T}_i$ is $j > i$;

✓ the subcategories $\mathbf{T}_i$ generate $\mathbf{T}$.

Examples

✓ $\mathrm{D^b}(\mathbb{P}^n) = \langle \mathcal{O}, \ldots, \mathcal{O}(n) \rangle$, where each component $\langle \mathcal{O}(i) \rangle$ is equivalent to $\mathrm{D^b}(\mathrm{Spec}(k))$.

✓ If $W \to X$ is the blow up along a smooth Z of codimension c, then $\mathrm{D^b}(W) = \langle \mathrm{D^b}(X), \mathrm{D^b}(Z)_1, \ldots, \mathrm{D^b}(Z)_{c-1} \rangle$.

One can define the *Grothendieck ring* $\mathcal{T}(k)$ of triangulated categories [BLL04]: it is the $\mathbb{Z}$-module generated by derived categories of smooth projective varieties, with the relation $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$ if $\mathbf{T} = \langle \mathbf{T}_1, \mathbf{T}_2 \rangle$. The product is induced by the product of varieties.

We note that the unit of $\mathcal{T}(k)$ is $1 := \mathrm{D^b}(\mathrm{Spec}(k))$. In particular, $\mathrm{D^b}(\mathbb{P}^n) = n+1$ in $\mathcal{T}(k)$.

Definition

A smooth projective X *categorically representable* in dimension m if there is a semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathbf{T}_1, \ldots, \mathbf{T}_r \rangle,$$

and smooth projecitve Y_i of dimension bounded by m such that $\mathbf{T}_i \subset \mathrm{D^b}(Y_i)$ is fully faithful with right and left adjoints. We set:

$$\mathrm{Rep_{cat}}(X) := \min\{m \mid X \text{ is representable in dimension } m\}.$$

Categorical representability induces a filtration on the ring $\mathcal{T}(k)$ by setting $\mathcal{T}_m(k)$ to be the smallest set, closed under direct summands, generated by categories of smooth projective varieties representable in dimension m. In other words, it consists of all the categories which can be embedded in such categories.

We can define the *motivic categorical dimension* of X:

$$\mathrm{mcd}(X) := \min\{m \mid X \in \mathcal{T}_m\}.$$

We notice that $\mathrm{mcd}(X) \leq \mathrm{Cat_{rep}}(X) \leq \dim(X)$, and strcit inequalities can hold.

Notice that $\mathrm{mcd}(\mathbb{P}^n) = \mathrm{Cat_{rep}}(\mathbb{P}^n) = 0$ for any n.

Weak factorization and categorical dimension

Let $\varphi : X \dashrightarrow Y$ be a birational map between smooth and projective varieties.

We say that φ has a weak factorization of type $(b_1, c_1, \ldots, b_r, c_r)$ if there is a diagram:

$$X_0 = X \xleftarrow{b_1} Y_1 \xrightarrow{c_1} X_1 \xleftarrow{b_2} Y_2 \xrightarrow{c_2} \ \ldots \ \xleftarrow{b_r} Y_r \xrightarrow{c_r} X_r = Y, \tag{1}$$

where b_i and c_j are compositions of finite numbers of blow-ups along smooth centers and X_i and Y_i are smooth and projective. We also denote by $\{B_{i,j}\}$ the loci blown-up by the b_i's and by $\{C_{i,j}\}$ the loci blown-up by the c_i's (with an appropriate use of index...). Notice that weak factorization holds for φ [AKMW02].

Definition

Let $\varphi : X \dashrightarrow Y$ be a birational map. We say that φ has *categorical dimension* d, and we write

$$c\dim(\varphi) = d,$$

if there exists a weak factorization of φ of type $(b_1, c_1, \ldots, b_r, c_r)$ such that $\mathrm{mcd}(B_{i,j}) \leq d$ for all $B_{i,j}$ blown-up by the b_i's.

The filtration

Let $\varphi : X \dashrightarrow Y$ be a birational map. We use the relations iduced by the blow-up formula in $\mathcal{T}(k)$: if $W \to X$ is the blow-up of Z as above, then

$$\mathrm{D^b}(W) = \mathrm{D^b}(X) + (c-1)\mathrm{D^b}(Z) \quad \text{in } \mathcal{T}(k)$$

Applying this to a weak factorization (1), we obtain the following relation in $\mathcal{T}(k)$.

$$\mathrm{D^b}(X) - \sum_{i=1}^{r} \sum_{j=1}^{s_i} (\alpha_j - 1)\mathrm{D^b}(B_{i,j}) = \mathrm{D^b}(Y) - \sum_{i=1}^{r} \sum_{j=1}^{t_i} (\beta_j - 1)\mathrm{D^b}(C_{i,j}). \tag{2}$$

Suppose that $\mathrm{D^b}(X) = \mathrm{D^b}(Y)$ in $\mathcal{T}(k)$, for example $X = Y$. Then we can simplify (2) as

$$\sum_{i=1}^{r} \sum_{j=1}^{s_i} (\alpha_j - 1)\mathrm{D^b}(B_{i,j}) = \sum_{i=1}^{r} \sum_{j=1}^{t_i} (\beta_j - 1)\mathrm{D^b}(C_{i,j}).$$

In particular, the LRS and the RHS have to belong to the same subgroup $\mathcal{T}_d(k)$ of $\mathcal{T}(k)$.

Main Theorem

Let X be a smooth projective variety. There is a group filtration of $\mathrm{Bir}(X)$ given by the subsets

$$\mathrm{Bir}_d(X) := \{\varphi : X \dashrightarrow X \mid c\dim(\varphi) \leq d\}$$

(Note: under a more formal point of view, the proof of the main Theorem relies on the existence of a *motivic measure*, *i.e.* a ring homomorphism $\mu : K_0(\mathrm{Var}(k)) \to \mathcal{T}(k)$)

Intermediate Jacobians: recovering the genus of a birational map

Given any variety X, one of the main problems is to calculate $\mathrm{mcd}(X)$ and/or $\mathrm{Rep_{cat}}(X)$, and hence to calculate categorical dimensions of birational maps, once a weak factorization is known. One of the most powerful tool is given by *Noncommutative motives* [Tab15].

If $\mathbf{T}$ is a triangulated category, noncommutative motives can define a noncommutative Jacobian $\mathbb{J}(\mathbf{T})$ as an Abelian variety (well defined up to isogeny), such that $\mathbb{J}(\mathbf{T}) = \mathbb{J}(\mathbf{T}_1) \oplus \mathbb{J}(\mathbf{T}_2)$ if $\mathbf{T} = \langle \mathbf{T}_1, \mathbf{T}_2 \rangle$.

Given a variety X, the Jacobian $\mathbb{J}(\mathrm{D^b}(X))$ coincides with the sum of all the algebraic Jacobians of X. Moreover, if X has a unique principally polarized intermediate Jacobian $J(X)$, then $\mathbb{J}(\mathrm{D^b}(X)) = J(X)$ as a ppav [BT].

In particular semiorthogonal decompositions keep track of the intermediate Jacobian as ppav.

Theorem

Let X be a smooth projective threefold. We say that a birational map $\varphi : X \dashrightarrow X$ has genus g if there is a weak factorization of type $(b_1, c_1, \ldots, b_r, c_r)$ such that $\dim(\mathbb{J}(B_{i,j})) \leq g$ for all i, j and such g is minimal.

The subsets $\mathrm{Bir}^g(X)$ of genus g birational maps of $\mathrm{Bir}(X)$ form a group filtration, coinciding with the one defined by Frumkin [Fru73].

We also have that $\mathrm{Bir}^0(X) = \mathrm{Bir}_0(X)$. In particular, $\mathrm{Bir}_0(X) \neq \mathrm{Bir}_1(X) = \mathrm{Bir}(X)$ for such an X.

For any X, we can define an *Abelian type* of φ whenever the $B_{i,j}$ and the $C_{i,j}$ in the weak factorization have pp Jacobians, and maps of fixed Abelian type form a subgroup. Similarly, we can define the *total genus* of φ to be the sum of the dimensions of the $\mathbb{J}(B_{i,j})$.

Relations to rationality and examples

Rationality. Suppose $\varphi : X \dashrightarrow \mathbb{P}^n$ is a birational map. Using (2) and $\mathrm{mcd}(\mathbb{P}^n) = 0$, we obtain:

$$X \text{ rational} \implies \mathrm{mcd}(X) \leq n-2$$

In particular, is categorical representability in dimension $n-2$ necessary for rationality? The inverse to the above implication is conjectured to be true for $n = 2$.

Question. Let $\dim(X) = 4$ Is $\mathrm{Bir}_0(X)$ made of birational maps whose indeterminacy locus is union of rational varieties?

Birational maps with toric centers. We say that φ has *toric centers* if there is a weak factorization such that all the $B_{i,j}$ are toric. We know that $\mathrm{mcd}(T) = 0$ for any toric variety [Kaw06], so that $c\dim(\varphi) = 0$ if φ has toric centers.

Cremona transformations of categorical dimension at least 2. If X is a cubic or a V_{14} threefold, then $\mathrm{Rep_{cat}}(X) > 1$, but $\mathrm{mcd}(X)$ is not known. Consider the special Cremona transformation $\varphi : \mathbb{P}^8 \dashrightarrow \mathbb{P}^8$ from [HKS92] which is resolved by blowing up a V_{14} threefold. We expect that $c\dim(\varphi) > 1$.

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