MOSCOW MATHEMATICAL JOURNAL
Volume 16, Number 2, April–June 2016, Pages 205–235

FROM SEMI-ORTHOGONAL DECOMPOSITIONS TO POLARIZED INTERMEDIATE JACOBIANS VIA JACOBIANS OF NONCOMMUTATIVE MOTIVES

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Abstract. Let X and Y be complex smooth projective varieties, and $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$ the associated bounded derived categories of coherent sheaves. Assume the existence of a triangulated category $\mathcal{T}$ which is admissible both in $\mathcal{D}^b(X)$ as in $\mathcal{D}^b(Y)$. Making use of the recent theory of Jacobians of noncommutative motives, we construct out of this categorical data a morphism τ of abelian varieties (up to isogeny) from the product of the intermediate algebraic Jacobians of X to the product of the intermediate algebraic Jacobians of Y. Our construction is conditional on a conjecture of Kuznetsov concerning functors of Fourier–Mukai type and on a conjecture concerning intersection bilinear pairings (which follows from Grothendieck's standard conjecture of Lefschetz type). We describe several examples where these conjectures hold and also some conditional examples. When the orthogonal complement $\mathcal{T}^\perp$ of $\mathcal{T} \subset \mathcal{D}^b(X)$ has a trivial Jacobian (e.g., when $\mathcal{T}^\perp$ is generated by exceptional objects), the morphism τ is split injective. When this also holds for the orthogonal complement $\mathcal{T}^\perp$ of $\mathcal{T} \subset \mathcal{D}^b(Y)$, τ becomes an isomorphism. Furthermore, in the case where X and Y have a unique principally polarized intermediate Jacobian, we prove that τ preserves the principal polarization.

As an application, we obtain categorical Torelli theorems, an incompatibility between two conjectures of Kuznetsov (one concerning functors of Fourier–Mukai type and another one concerning Fano threefolds), and also several new results on quadric fibrations and intersections of quadrics.

2010 Math. Subj. Class. 14A22, 14C34, 14E08, 14J30, 14J45, 14K30, 18E30.

Key words and phrases. Intermediate Jacobians, polarizations, noncommutative motives, semi-orthogonal decompositions, Torelli theorem, Fano threefolds, quadric fibrations, blow-ups.

Received July 5, 2014; in revised form April 11, 2015 and May 26, 2015.

G. Tabuada was supported in part by the National Science Foundation CAREER Award #1350472 and by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project grant UID/MAT/00297/2013 (Centro de Matemática e Aplicações).

Contents

1. Introduction and Statement of Results

In their ICM address [13], Bondal and Orlov suggested that classical birational geometry could be studied using bounded derived categories of coherent sheaves and their semi-orthogonal decompositions. This motivates the general question:

Question: *Does the bounded derived category $\mathcal{D}^b(X)$ of a complex smooth projective variety X carry information about the intermediate Jacobians and their polarizations? If so, how can they be "extracted" from $\mathcal{D}^b(X)$?*

In the case of cubic threefolds, conic bundles, and more generally rationally representable Fano threefolds, some partial answers are known; see [9], [7], [6]. The goal of this article is to show that if one replaces $\mathcal{D}^b(X)$ by its unique differential graded enhancement and considers differential graded semi-orthogonal decompositions, then the above general question admits a precise affirmative answer.

(Polarized) intermediate Jacobians. Given an irreducible smooth projective $\mathbb{C}$-scheme X of dimension d_X, Griffiths introduced in [27] the associated Jacobians $J^i(X)$, $0 \leqslant i \leqslant d_X - 1$. In contrast with the Picard $J^0(X) = \mathrm{Pic}^0(X)$ and the Albanese $J^{d_X-1}(X) = \mathrm{Alb}(X)$ varieties, the intermediate Jacobians are *not* algebraic. Nevertheless, they contain an algebraic torus $J^i_a(X) \subseteq J^i(X)$ defined by the image of the Abel–Jacobi map

$$AJ^i\colon A^{i+1}_{\mathbb{Z}}(X) \to J^i(X), \quad 0 \leqslant i \leqslant d_X - 1, \tag{1.1}$$

where $A^{i+1}_{\mathbb{Z}}(X)$ stands for the group of algebraically trivial cycles of codimension $i+1$; consult Vial [69, Section 2.3] for further details. The map (1.1) is surjective when $i = 0$ and $i = d_X - 1$ and so $J^0_a(X) = \mathrm{Pic}^0(X)$ and $J^{d_X-1}_a(X) = \mathrm{Alb}(X)$.

In general, the abelian varieties $J^i_a(X)$ are only well-defined up to isogeny. However, in the case of curves, Fano threefolds, even-dimensional quadric fibrations over $\mathbb{P}^1$, odd-dimensional quadric fibrations over rational surfaces, and also in the case of the intersection of two (resp. three) quadrics of odd (resp. even) dimension, there is a single non-trivial algebraic Jacobian $J(X) := J_a^{(d_X-1)/2}(X)$, which carries moreover a canonical *principal polarization*; see Clemens and Griffiths [20]. This extra piece of structure is of major importance. For example, in the case of a cubic threefold X the abelian variety $J(X)$ endowed with its canonical principal polarization contains all the information about the birational class of X.

Jacobians of noncommutative motives. Consult André [2, Section 4] for the construction of the functor $M_{\mathbb{Q}}(-)\colon \mathrm{SmProj}(\mathbb{C})^{\mathrm{op}} \to \mathrm{Chow}(\mathbb{C})_{\mathbb{Q}}$, from smooth projective $\mathbb{C}$-schemes to Chow motives, and for the fact that de Rham cohomology $H^*_{\mathrm{dR}}(-)$ factors through $\mathrm{Chow}(\mathbb{C})_{\mathbb{Q}}$. Given a smooth projective $\mathbb{C}$-scheme X of dimension d_X, one can then consider the $\mathbb{Q}$-vector spaces

$$NH^{2i+1}_{\mathrm{dR}}(X)\colon = \sum_{C,\gamma_i} \mathrm{Im}\big(H^1_{\mathrm{dR}}(C) \xrightarrow{H^1_{\mathrm{dR}}(\gamma_i)} H^{2i+1}_{\mathrm{dR}}(X)\big), \quad 0 \leqslant i \leqslant d_X - 1, \tag{1.2}$$

where C is a smooth projective curve and $\gamma_i\colon M_{\mathbb{Q}}(C) \to M_{\mathbb{Q}}(X)(i)$ a morphism in $\mathrm{Chow}(\mathbb{C})_{\mathbb{Q}}$. Intuitively speaking, (1.2) are the odd pieces of de Rham cohomology that are generated by curves. By restricting the classical intersection bilinear pairings on de Rham cohomology (see [2, Section 3.3]) to these pieces one obtains

$$\langle -, - \rangle\colon NH^{2d_X-2i-1}_{\mathrm{dR}}(X) \times NH^{2i+1}_{\mathrm{dR}}(X) \to \mathbb{C}, \quad 0 \leqslant i \leqslant d_X - 1. \tag{1.3}$$

Recall from Section 2 the construction of the category $\mathrm{NChow}(\mathbb{C})_{\mathbb{Q}}$ of noncommutative Chow motives. Examples of noncommutative Chow motives include finite-dimensional $\mathbb{C}$-algebras of finite global dimension as well as the unique dg enhancements $\mathrm{perf}^{\mathrm{dg}}(X)$ (see Lunts–Orlov [51]) of the derived categories $\mathrm{perf}(X)$ of perfect complexes[1]. Now, consult [52] for the construction of the *Jacobian* functor

$$\mathbf{J}(-)\colon \mathrm{NChow}(\mathbb{C})_{\mathbb{Q}} \to \mathrm{Ab}(\mathbb{C})_{\mathbb{Q}}$$

with values in the category of abelian $\mathbb{C}$-varieties up to isogeny. Among other properties, one has an isomorphism $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(X)) \simeq \prod_{i=0}^{d_X-1} J^i_a(X)$ whenever the above pairings (1.3) are non-degenerate. As explained in *loc. cit.*, this is always the case for $i = 0$ and $i = d_X - 1$ and the remaining cases follow from Grothendieck's standard conjecture of Lefschetz type. This latter conjecture holds for curves, surfaces, abelian varieties, complete intersections, uniruled threefolds, rationally connected fourfolds, and for any smooth hypersurface section, product, or finite quotient thereof.

Statement of results. Let X and Y be two irreducible smooth projective $\mathbb{C}$-schemes of dimensions d_X and d_Y, respectively, and $\mathrm{perf}(X)$ and $\mathrm{perf}(Y)$ the associated derived categories of perfect complexes. Assume that X and Y are related by the following *categorical data*:

There exist semi-orthogonal decompositions (see [29, Section 4]) $\mathrm{perf}(X) = \langle \mathcal{T}_X, \mathcal{T}_X^{\perp} \rangle$ *and* $\mathrm{perf}(Y) = \langle \mathcal{T}_Y, \mathcal{T}_Y^{\perp} \rangle$ *and an equivalence* $\phi\colon \mathcal{T}_X \simeq \mathcal{T}_Y$ *of triangulated categories.*

In what follows, Φ denotes the composition $\mathrm{perf}(X) \to \mathcal{T}_X \overset{\phi}{\simeq} \mathcal{T}_Y \hookrightarrow \mathrm{perf}(Y)$, where the first functor is the projection. Our first main result is the following:

[1]Since X is smooth, every complex of coherent sheaves is perfect (up to quasi-isomorphism). Consequently, the canonical inclusion $\mathrm{perf}(X) \hookrightarrow \mathcal{D}^b(X)$ is an equivalence of categories.

Theorem 1.4. *Let X and Y be two $\mathbb{C}$-schemes as above. Assume that Φ is of Fourier–Mukai type, i.e., there exists a perfect complex $\mathcal{E} \in \mathrm{perf}(X \times Y)$ such that Φ is isomorphic to the Fourier–Mukai functor $\Phi_{\mathcal{E}}(-) := Rq_*(p^*(-) \otimes^L \mathcal{E})$, where $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are the projection morphisms. Assume also that the above bilinear pairings* (1.3) (*associated to X and Y*) *are non-degenerate. Under these assumptions, the following holds*:

(i) *There exists a well-defined morphism $\tau\colon \prod_{i=0}^{d_X-1} J_a^i(X) \to \prod_{i=0}^{d_Y-1} J_a^i(Y)$ in* $\mathrm{Ab}(\mathbb{C})_{\mathbb{Q}}$.

(ii) *Assume moreover that $\mathbf{J}(\mathcal{T}_X^{\perp,\mathrm{dg}}) = 0$ (e.g., $\mathcal{T}_X^{\perp}$ admits a full exceptional collection). Under this extra assumption, the morphism τ is split injective.*

(iii) *Assume furthermore that $\mathbf{J}(\mathcal{T}_Y^{\perp,\mathrm{dg}}) = 0$. Under this extra assumption, the morphism τ becomes an isomorphism.*

Remark 1.5. By construction, the morphism τ factors through $\mathbf{J}(\mathcal{T}_X^{\mathrm{dg}})$. Therefore, whenever $\mathbf{J}(\mathcal{T}_X^{\mathrm{dg}}) = 0$ (e.g., $\mathcal{T}_X$ admits a full exceptional collection), we have $\tau = 0$.

Remark 1.6. Theorem 1.4 was used by the authors in [10] to give a new proof of the Beilinson–Bloch type conjectures in the case of a complete intersection of either two quadrics or three odd-dimensional quadrics.

Theorem 1.4 holds (unconditionally) in the following cases:

Example 1.7. Let$X$ be a $\mathbb{C}$-scheme satisfying the Grothendieck's standard conjecture of Lefschetz type, $\mathcal{T}_X := \mathrm{perf}(X)$, and let Y such that $\mathcal{T}_Y := \mathrm{perf}(Y)$ is equivalent to $\mathrm{perf}(X)$. In these cases the orthogonal complements $\mathcal{T}_X^{\perp}$ and $\mathcal{T}_Y^{\perp}$ are trivial and so item (iii) of Theorem 1.4 applies. Thanks to the work of Bondal and Orlov, Bridgeland, and Mukai (see [12], [14], [55] and also [29, Section 11.4]), this holds for example when X is an abelian variety and Y is its dual $\widehat{X}$, when X and Y are two crepant resolutions of a threefold Z with terminal singularities and X is either uniruled or a complete intersection, when X and Y are related by a Mukai flop and X is either a complete intersection, a uniruled threefold, or a rationally connected fourfold, etc.

Example 1.8. Let X be as in Example 1.7 (e.g., a complete intersection, a uniruled threefold, or a rationally connected fourfold), $\mathcal{T}_X := \mathrm{perf}(X)$, and Y the $\mathbb{C}$-scheme obtained from X by a standard flip (resp. a blow-up) along a smooth irreducible subscheme $Z \subseteq X$; see Orlov [58]. In all these cases the orthogonal complement $\mathcal{T}_X^{\perp}$ is trivial. When Y is obtained from X by a standard flip (resp. when $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(Z)) = 0$), then $\mathbf{J}(\mathcal{T}_Y^{\perp}) = 0$. Hence, case (iii) of Theorem 1.4 applies in the case of standard flips or in the case of the blow-up of a Z such that $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(Z)) = 0$. For other blow-ups, case (ii) of Theorem 1.4 applies.

Example 1.9. Let X be a hyperelliptic curve, $\mathcal{T}_X := \mathrm{perf}(X)$, Y a complete intersection of two even-dimensional quadrics, and $\mathcal{T}_Y$ the orthogonal complement of an exceptional collection; see Bondal and Orlov [12]. In the same vein, let X be a hyperelliptic or a trigonal curve, $\mathcal{T}_X := \mathrm{perf}(X)$, Y a rational conic bundle over a Hirzebruch surface or a rational del Pezzo fibration of degree 4 over $\mathbb{P}^1$, and $\mathcal{T}_Y$ the orthogonal complement of an exceptional collection; see [3], [7]. Similarly, let $X \subset$

$\mathbb{P}^3$ be a smooth curve of genus 5 and degree 7 (or a smooth curve of genus 2), $\mathcal{T}_X := \mathrm{perf}(X)$, Y a rational conic bundle over $\mathbb{P}^2$, and $\mathcal{T}_Y$ the orthogonal complement of an exceptional collection; see [7]. In all the above cases the orthogonal complement $\mathcal{T}_X^\perp$ is trivial and $\mathbf{J}(\mathcal{T}_Y^{\perp,\mathrm{dg}}) = 0$. Therefore, item (iii) of Theorem 1.4 applies.

Example 1.10. Let X (resp. Y) be a Fano threefold of index 2 (resp. 1) and degree $d \geqslant 3$ (resp. $4d+2$), and $\mathcal{T}_X$ (resp. $\mathcal{T}_Y$) the orthogonal complement of an exceptional collection of objects in $\mathrm{perf}(X)$ (resp. in $\mathrm{perf}(Y)$); see Kuznetsov [46]. In these cases the orthogonal complements $\mathcal{T}_X^\perp$ and $\mathcal{T}_Y^\perp$ are generated by exceptional objects. Therefore, item (iii) of Theorem 1.4 applies.

Example 1.11. Let X be the intersection of two (resp. three odd-dimensional) quadrics, $\mathcal{T}_X$ the orthogonal complement of an exceptional collection of objects, Y the fibration generated by the family of quadrics, and $\mathcal{T}_Y$ the derived category of perfect complexes over the Clifford algebra associated to the span of the quadrics; see Kuznetsov [45]. In these cases the orthogonal complements $\mathcal{T}_X^\perp$ and $\mathcal{T}_Y^\perp$ are generated by exceptional objects. Therefore, item (iii) of Theorem 1.4 applies.

Example 1.12. Let X be an elliptic curve (resp. a curve of degree 42), $\mathcal{T}_X := \mathrm{perf}(X)$, and Y a 5-dimensional linear section of the Grassmannian $\mathrm{Gr}(2,6)$ (resp. $\mathrm{Gr}(2,7)$). A similar example can be obtained by replacing 42 with 14 and $\mathrm{Gr}(2,7)$ with the Pfaffian $\mathrm{Pf}(4,7)$; see Kuznetsov [43, Section 10–11]. In these cases the orthogonal complement $\mathcal{T}_X^\perp$ (resp. $\mathcal{T}_Y^\perp$) is trivial (resp. is generated by exceptional objects). Therefore, item (iii) of Theorem 1.4 applies.

Remark 1.13 (Homological Projective Duality). The above Examples 1.10 (for $d = 3$) and 1.11–1.12 arise from Kuznetsov's Homological Projective Duality; see [44]. This theory has the potential to provide many more examples in the future.

Remark 1.14 (Conditional examples). If one assumes that the above bilinear pairings (1.3) (associated to X and Y) are non-degenerate, then Theorem 1.4 also holds in the following cases:

(i) Let X be a quadric fibration (over a smooth projective $\mathbb{C}$-scheme S) endowed with a regular section, Y the $\mathbb{C}$-scheme obtained from X by hyperbolic reduction, and $\mathcal{T}_X$ and $\mathcal{T}_Y$ the derived categories of perfect complexes over the sheaves of even parts of Clifford algebras on S; see [3]. In these cases the orthogonal complements $\mathcal{T}_X^\perp$ and $\mathcal{T}_Y^\perp$ are generated by a finite number of copies of $\mathrm{perf}(S)$. Therefore, item (iii) of Theorem 1.4 applies when $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(S)) = 0$.

(ii) Let X be an arbitrary $\mathbb{C}$-scheme, $\mathcal{T}_X := \mathrm{perf}(X)$, $p\colon Y \to X$ a flat fibration for which Y can be embedded in a projective bundle $\mathbb{P}(E) \to X$ such that $\omega_{Y/X} = \mathcal{O}_{\mathbb{P}(E)/X}(-l)_{|X}$ with $l > 0$, and $\mathcal{T}_Y := p^*(\mathrm{perf}(X))$; see [3]. This example is inspired by Orlov's pioneering work [58]. In these cases the orthogonal complement $\mathcal{T}_X^\perp$ is trivial and so item (ii) of Theorem 1.4 applies.

Now, consider the following notion:

Definition 1.15. An irreducible smooth projective $\mathbb{C}$-scheme X of odd dimension $d_X = 2n+1$ is called *verepresentable*[2] if:

[2] The fusion of the words "very" and "representable".

(i) the group of algebraically trivial cycles $A^{i+1}_{\mathbb{Z}}(X)$ is trivial for $i \neq n$;
(ii) the group $A^{n+1}_{\mathbb{Z}}(X)$ admits an *algebraic representative* carrying an *incidence polarization*; see Section 3.
(iii) the Abel–Jacobi map $AJ^n(X)\colon A^{n+1}_{\mathbb{Z}}(X) \twoheadrightarrow J^n_a(X)$ gives rise to an isomorphism $A^{n+1}_{\mathbb{Q}}(X) \simeq J^n_a(X)_{\mathbb{Q}}$.

Example 1.16. Every $\mathbb{C}$-scheme for which $A^i(X) = 0$, $i \geqslant 0$ (e.g., projective spaces and smooth quadrics), is verepresentable. Other examples of verepresentable $\mathbb{C}$-schemes include smooth curves, many Fano threefolds of Picard rank 1, many conic bundles over rational surfaces, many del Pezzo fibrations over $\mathbb{P}^1$, and also smooth complete intersections of two even-dimensional (or three odd-dimensional) quadrics. For a detailed list of Examples please consult Section 3.3.

By combining the above definition (1.1) of $J^i_a(X)$ with Definition 1.15(i) one observes that whenever X is verepresentable, $J^i_a(X) = 0$ for $i \neq n$. Consequently, there is a single non-trivial algebraic Jacobian $J(X) := J^n_a(X)$ which, thanks to Definition 1.15(ii), carries a canonical principal polarization. Moreover, Definition 1.15(iii) implies that this principally polarized abelian variety is isomorphic, up to isogeny, to $A^{n+1}_{\mathbb{Z}}(X)$. Our second main result is the following:

Theorem 1.17. *Let X and Y be two irreducible smooth projective $\mathbb{C}$-schemes as in Theorem 1.4(i)–(ii). Assume that X and Y are verepresentable. Under these assumptions, the split injective morphism $\tau\colon J(X) \to J(Y)$ preserves the principal polarization. When $\mathbf{J}(\mathcal{T}_Y^{\perp,\mathrm{dg}}) = 0$ the morphism τ becomes an isomorphism.*

Thanks to Example 1.16, Theorem 1.17 holds (unconditionally) in the above Examples 1.9–1.11. If one assumes that X and Y (and Z) are verepresentable, then Theorem 1.17 also holds in the above Examples 1.7–1.8.

Remark 1.18 (Conditional example). In the above Example 1.12 the only missing assumption for Theorem 1.17 to hold is the verepresentability of Y. Similar cases occur whenever Homological Projective Duality gives rise to a fully faithful functor $\mathrm{perf}(X) \to \mathrm{perf}(Y)$ with X verepresentable (e.g., a curve) and $\mathcal{T}_Y^{\perp}$ generated by exceptional objects; see [8] for some recent examples.

The assumption that Φ is of Fourier–Mukai type goes back to Kuznetsov:

Conjecture 1.19 (Kuznetsov [44, Conjecture 3.7])**.** *Φ is of Fourier–Mukai type.*

As mentioned above, $\mathrm{perf}(X)$ and $\mathrm{perf}(Y)$ admit (unique) dg enhancements $\mathrm{perf}^{\mathrm{dg}}(X)$ and $\mathrm{perf}^{\mathrm{dg}}(Y)$. Let us denote by $\mathcal{T}_X^{\mathrm{dg}}$, $\mathcal{T}_X^{\perp,\mathrm{dg}}$, $\mathcal{T}_Y^{\mathrm{dg}}$, $\mathcal{T}_Y^{\perp,\mathrm{dg}}$ the inherited dg enhancements. Making use of them, we explain in Lemma 4.4 that Φ is of Fourier–Mukai type if and only if it admits a dg enhancement $\Phi^{\mathrm{dg}}\colon \mathrm{perf}^{\mathrm{dg}}(X) \to \mathrm{perf}^{\mathrm{dg}}(Y)$. If one assumes the existence of this dg enhancement, then Theorems 1.4(i)–(iii) and 1.17 also hold in the following case:

Example 1.20. Let $X = Y$ be a Fano threefold, or a conic bundle over a rational surface, or a del Pezzo fibration over $\mathbb{P}^1$, and $\mathcal{T}_X$ and $\mathcal{T}_Y$ the orthogonal complement of exceptional collections; see [46], [45], and [3], respectively. In the particular cases where there exists a $\mathbb{C}$-scheme Z and a Brauer class α such that $\mathcal{T}_X \simeq \mathrm{perf}(Z, \alpha)$, the equivalence $\phi\colon \mathcal{T}_X \simeq \mathcal{T}_Y$ is known to be of Fourier–Mukai type; see [16].

We conclude this section with some comments on the above Conjecture 1.19. Firstly, it holds in all the cases arising from Homological Projective Duality. Currently, this is the most useful tool to construct $\mathbb{C}$-schemes X and Y satisfying the above categorical data. Secondly, in order to establish the equivalence $\phi\colon \mathcal{T}_X \simeq \mathcal{T}_Y$, one has in some cases to construct first the kernel of Φ. This strongly suggests the correctness of Conjecture 1.19 and also that instead of triangulated categories one should work with their differential graded enhancements. For a survey article on functors of Fourier–Mukai type we invite the reader to consult [17].

Finally, for applications of the above Theorems 1.4 and 1.17 please consult Sections 6–9.

Notations: Throughout the article we will always work over the field $\mathbb{C}$ of complex numbers. All $\mathbb{C}$-schemes will be assumed to be smooth, proper, and irreducible.

Acknowledgments: The authors are very grateful to Thomas Dedieu, Valery Lunts, Alexander Polishchuk, Paolo Stellari, Michel Vaquié, and Charles Vial, for useful discussions, e-mail exchanges, and motivating questions. They are also very grateful to the anonymous referee for his/her comments and suggestions that greatly allowed the improvement of the article. Finally, G. Tabuada would also like to thank the MSRI, Berkeley, for its hospitality and excellent working conditions.

2. Background on dg Categories and Noncommutative Motives

DG categories. Let $\mathcal{C}(\mathbb{C})$ be the category of cochain complexes of $\mathbb{C}$-vector spaces; we use cohomological notation. A *differential graded* (= *dg*) *category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(\mathbb{C})$ (morphism sets $\mathcal{A}(x, y)$ are complexes) in such a way that the composition law satisfies the Leibniz rule $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$. A *dg functor* $F\colon \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(\mathbb{C})$; consult Keller's ICM address [38] for further details. In what follows we will write $\mathsf{dgcat}(\mathbb{C})$ for the category of (small) dg categories and dg functors. The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ of two dg categories $\mathcal{A}$ and $\mathcal{B}$ is defined as follows: the set of objects is the cartesian product of the sets of objects of $\mathcal{A}$ and $\mathcal{B}$ and the complexes of morphisms are given by $(\mathcal{A} \otimes \mathcal{B})((x, w), (y, z)) := \mathcal{A}(x, y) \otimes \mathcal{B}(w, z)$.

(Bi)modules. Let $\mathcal{A}$ be a dg category. Its *opposite* dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects and complexes of morphisms given by $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$. A *right* $\mathcal{A}$-*module* is a dg functor $M\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{dg}}(\mathbb{C})$ with values in the dg category $\mathcal{C}^{\mathrm{dg}}(\mathbb{C})$ of cochain complexes of $\mathbb{C}$-vector spaces; see [38, Section 2.3]. We will write $\mathcal{C}(\mathcal{A})$ for the category of right $\mathcal{A}$-modules. The *derived category* $\mathcal{D}(\mathcal{A})$ *of* $\mathcal{A}$ is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of objectwise quasi-isomorphisms; see [38, Section 3.2]. This is a triangulated category with arbitrary sums. Let us denote by $\mathcal{D}_c(\mathcal{A})$ the full triangulated subcategory of compact objects.

Given dg categories $\mathcal{A}$ and $\mathcal{B}$, an $\mathcal{A}$-$\mathcal{B}$-*bimodule* is a right $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$-module, i.e., a dg functor $B\colon \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{dg}}(\mathbb{C})$. A standard example is the $\mathcal{A}$-$\mathcal{A}$-bimodule

$$\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathrm{dg}}(\mathbb{C}), \quad (x, y) \mapsto \mathcal{A}(y, x). \tag{2.1}$$

Let $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ be the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ consisting of those $\mathcal{A}$-$\mathcal{B}$-bimodules B such that for every object $x \in \mathcal{A}$ the right $\mathcal{B}$-module $B(x, -)$ belongs to $\mathcal{D}_c(\mathcal{B})$. Note that every dg functor $F \colon \mathcal{A} \to \mathcal{B}$ gives rise to an $\mathcal{A}$-$\mathcal{B}$-bimodule ${}_FB \colon \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{dg}}(\mathbb{C})$, $(x, w) \mapsto \mathcal{B}(w, F(x))$, which belongs to $\mathrm{rep}(\mathcal{A}, \mathcal{B})$.

Morita equivalences. A dg functor $F \colon \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if the restriction of scalars functor $\mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ is an equivalence of (triangulated) categories; see [38, Section 4.6]. As proved in [62, Theorem 5.3], the category $\mathsf{dgcat}(\mathbb{C})$ carries a Quillen model category whose weak equivalences are the Morita equivalences. Let us write $\mathsf{Hmo}(\mathbb{C})$ for the associated homotopy category. As proved in *loc. cit.*, the assignment $F \mapsto {}_FB$ gives rise to a bijection

$$\mathrm{Hom}_{\mathsf{Hmo}(\mathbb{C})}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Iso}\,\mathrm{rep}(\mathcal{A}, \mathcal{B}), \tag{2.2}$$

where Iso stands for the set of isomorphism classes. Moreover, under (2.2) the composition law in $\mathsf{Hmo}(\mathbb{C})$ corresponds to the (derived) tensor product of bimodules.

Pretriangulated dg categories. Let $\mathcal{A}$ be a dg category. The $\mathbb{C}$-linear category $\mathsf{H}^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms given by $\mathsf{H}^0(\mathcal{A})(x, y) := H^0(\mathcal{A}(x, y))$, where $H^0(-)$ is the 0^{th} cohomology group functor. The dg category $\mathcal{A}$ is called *pretriangulated* if $\mathsf{H}^0(\mathcal{A})$ is a triangulated category; see [38, Section 4.5].

Noncommutative Chow motives. As explained above, we have the functor

$$\mathsf{dgcat}(\mathbb{C}) \to \mathsf{Hmo}(\mathbb{C}), \quad \mathcal{A} \mapsto \mathcal{A}, \quad F \mapsto {}_FB. \tag{2.3}$$

The *additivization of* $\mathsf{Hmo}(\mathbb{C})$ is the additive category $\mathsf{Hmo}_0(\mathbb{C})$ with the same objects as $\mathsf{Hmo}(\mathbb{C})$ and morphisms $\mathrm{Hom}_{\mathsf{Hmo}_0(\mathbb{C})}(\mathcal{A}, \mathcal{B}) := K_0\,\mathrm{rep}(\mathcal{A}, \mathcal{B})$, where K_0 stands for the Grothendieck group of the triangulated category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. The composition law is induced by the tensor product of bimodules; consult [62, Section 6] for further details. Note that we have the canonical functor

$$\mathsf{Hmo}(\mathbb{C}) \to \mathsf{Hmo}_0(\mathbb{C}), \quad \mathcal{A} \mapsto \mathcal{A}, \quad B \mapsto [B]. \tag{2.4}$$

The $\mathbb{Q}$-*linearization* of $\mathsf{Hmo}_0(\mathbb{C})$ is the $\mathbb{Q}$-linear additive category $\mathsf{Hmo}_0(\mathbb{C})_{\mathbb{Q}}$ obtained by tensoring each abelian group of morphisms of $\mathsf{Hmo}_0(\mathbb{C})$ with $\mathbb{Q}$. By construction, it comes equipped with the following functor

$$\mathsf{Hmo}_0(\mathbb{C}) \to \mathsf{Hmo}_0(\mathbb{C})_{\mathbb{Q}}, \quad \mathcal{A} \mapsto \mathcal{A}, \quad [B] \mapsto [B]_{\mathbb{Q}}. \tag{2.5}$$

Since the above functors (2.3)-(2.5) are the identity on objects we will make no notational distinction between a dg category and its image in $\mathsf{Hmo}_0(\mathbb{C})_{\mathbb{Q}}$.

Now, recall from Kontsevich [39], [40], [41] that a dg category $\mathcal{A}$ is called *smooth* if the above $\mathcal{A}$-$\mathcal{A}$-bimodule (2.1) belongs to $\mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})$ and *proper* if for each ordered pair of objects (x, y) we have $\sum_i \dim_{\mathbb{C}} H^i\mathcal{A}(x, y) < \infty$. The category $\mathrm{NChow}(\mathbb{C})_{\mathbb{Q}}$ of *noncommutative Chow motives* (with rational coefficients) is defined as the pseudo-abelian envelope of the full subcategory of $\mathsf{Hmo}_0(\mathbb{C})_{\mathbb{Q}}$ consisting of the smooth and proper dg categories. For a survey article on noncommutative (Chow) motives we invite the reader to consult [63].

3. Background on Principal and Incidence Polarizations

In this section we collection the necessary background for Definition 1.15.

3.1. Algebraic and rational representability. Given a $\mathbb{C}$-scheme X of dimension d_X, let $CH^i_{\mathbb{Q}}(X)$ (resp. $CH^i_{\mathbb{Z}}(X)$) denote the rational (resp. integral) Chow group of cycles of codimension i. We will write $A^i_{\mathbb{Q}}(X)$ (resp. $A^i_{\mathbb{Z}}(X)$) for the subgroup of algebraically trivial cycles. Let A be a complex abelian variety. A group homomorphism $g\colon A^i_{\mathbb{Z}}(X) \to A$ is called a *regular map* if for every $\mathbb{C}$-scheme T and for every algebraic map $f\colon T \to A^i_{\mathbb{Z}}(X)$ (i.e., there is an algebraic cycle z on $X \times T$ such that $f(t) = z_t$) the composite $g \circ f\colon T \to A$ is a morphism of $\mathbb{C}$-schemes.

Definition 3.1. (see Beauville [4, Définition 3.2.3]) A complex abelian variety A is called an *algebraic representative* of $A^i_{\mathbb{Z}}(X)$ if there exists a universal regular map $G\colon A^i_{\mathbb{Z}}(X) \to A$, i.e., if for every regular map $g\colon A^i_{\mathbb{Z}}(X) \to B$, there is a unique morphism of abelian varieties $u\colon A \to B$ such that $u \circ G = g$. In this case one says that $A^i_{\mathbb{Z}}(X)$ is *algebraically representable*. As explained by Beauville in *loc. cit.*, whenever it exists the algebraic representative is unique up to isomorphism. The standard examples of algebraic representatives are the Picard variety $\mathrm{Pic}^0(X)$ (when $i = 1$) and the Albanese variety $\mathrm{Alb}(X)$ (when $i = d_X$).

Definition 3.2 (see Vial [69, Corollary 3.6])**.** A $\mathbb{C}$-scheme X is called *rationally representable* if for every integer i there exists a curve Γ and a surjective map $A^1_{\mathbb{Q}}(\Gamma) \twoheadrightarrow A^{i+1}_{\mathbb{Q}}(X)$ induced by a cycle in $\Gamma \times X$. In this case the Abel–Jacobi maps $AJ^i\colon A^{i+1}_{\mathbb{Z}}(X) \to J^i_a(X)$ give rise to isomorphisms $A^{i+1}_{\mathbb{Q}}(X) \simeq J^i_a(X)_{\mathbb{Q}}$ for every $0 \leqslant i \leqslant d_X - 1$.

3.2. Principal and incidence polarizations. From now on, and until the end of Section 3, we will assume that X is of odd dimension $d_X = 2n + 1$.

Definition 3.3. (see Beauville [4, Section 3.4]) Let T be a $\mathbb{C}$-scheme. We denote by $p\colon X \times T \times T \to X \times T$ (resp. $q\colon X \times T \times T \to X \times T$) the projection onto the first (resp. second) copy of T, and $r\colon X \times T \times T \to T \times T$ the projection morphism.

(i) The (*rational*) *divisorial self correspondences* of T are defined as $\mathrm{Corr}_{\mathbb{Q}}(T) := \mathrm{Corr}(T) \otimes \mathbb{Q}$, where $\mathrm{Corr}(T) := \mathrm{Pic}(T \times T)/(\mathrm{Pic}(T) \boxtimes \mathrm{Pic}(T))$, and Pic stands for the Picard group.

(ii) Let z be a cycle in $CH^{n+1}_{\mathbb{Q}}(X \times T)$. The *incidence correspondence* $I(z)$ *associated to* z is the equivalence class of the cycle $r_*(p^*(z) \cdot q^*(z)) \in CH^1_{\mathbb{Q}}(T \times T)$ considered as an element in $\mathrm{Corr}_{\mathbb{Q}}(T)$.

Definition 3.4. Let A be a complex abelian variety.

(i) A *polarization* of A is the first Chern character $\Theta_A := c_1(L)$ of a positive definite line bundle. Equivalently, a polarization of A is a divisorial rational self correspondence $\theta_A \in \mathrm{Corr}_{\mathbb{Q}}(A)$ corresponding to an isogeny $A \to \hat{A}$; see Birkenhake and Lange [48, Section 4.1], Mumford [57, Section 8] and [4, Section 0.2 and 3.4].

(ii) A polarization θ_A is called *principal* if it gives rise to an isomorphism $A \simeq \hat{A}$; see Beauville [4, Section 0.2]. A *principally polarized abelian variety* consists of a pair (A, θ_A), where $\theta_A \in \mathrm{Corr}_{\mathbb{Q}}(A)$ is a principal polarization of A. Every polarization is obtained from a principal one via an isogeny; see [48, Proposition 4.1.2]. Consequently, a principal polarization is unique up to an isomorphism of A.

(iii) A morphism $f\colon A \to B$ of complex tori is a *morphism of principally polarized abelian varieties* if the divisorial self correspondence $f^*\theta_B$ gives rise to an isomorphism $A \simeq \hat{A}$. Indeed, $f^*\theta_B$ is a polarization of A which is principal if and only if it gives rise to such an isomorphism. This is the case if $f^*\theta_B$ equals θ_A up to an isomorphism of A.

Recall that $\mathrm{Ab}(\mathbb{C})_{\mathbb{Q}}$ stands for the category of abelian varieties up to isogeny.

Lemma 3.5. *Let (A, θ_A) and (B, θ_B) be two complex polarized abelian varieties, and $A_{\mathbb{Q}}$ and $B_{\mathbb{Q}}$ the classes of A and B in $\mathrm{Ab}(\mathbb{C})_{\mathbb{Q}}$. Given a morphism $\lambda_{\mathbb{Q}} \colon A_{\mathbb{Q}} \to B_{\mathbb{Q}}$, there exists a morphism of complex abelian varieties $\lambda \colon A \to B$ whose class in $\mathrm{Ab}(\mathbb{C})_{\mathbb{Q}}$ is $\lambda_{\mathbb{Q}}$. Moreover, whenever $\lambda_{\mathbb{Q}}$ is split injective, $\lambda \colon A \to B$ is an isogeny onto a polarized abelian subvariety of B.*

Proof. Without loss of generality one can assume that $\lambda_{\mathbb{Q}}$ is surjective. In fact, since (B, θ_B) is a polarized abelian variety, every algebraic subtorus of B is an abelian variety for which the restriction of θ_B is a polarization; see Birkenhake and Lange [48, Proposition 4.1.1]. Now, note that $\lambda_{\mathbb{Q}}$ is an element of $\mathrm{Hom}_{\mathrm{Ab}(\mathbb{C})}(A, B) \otimes \mathbb{Q}$ and hence can be written as a finite sum $\sum_i f_i \otimes \frac{p_i}{q_i}$. By first choosing a representative A of the class $A_{\mathbb{Q}}$ and then by applying the morphism $\lambda_{\mathbb{Q}}$ to A we obtain an algebraic complex torus B' as $\lambda(A)_{\mathbb{Q}}$. The isogeny class $B'_{\mathbb{Q}}$ of B' is the same as the isogeny class of $\lambda_{\mathbb{Q}}(A_{\mathbb{Q}}) = B_{\mathbb{Q}}$. In particular, B' is an abelian variety isogenous to B. As a consequence, one can take for $\lambda \colon A \to B$ the morphism of complex tori obtained by composing $\lambda_{\mathbb{Q}}$ with the isogeny $B' \to B$. Finally, the last claim follows from the above construction of λ. Whenever $\lambda_{\mathbb{Q}}$ is split injective, the kernel $\ker(\lambda)$ is torsion. $\square$

Definition 3.6. (see Beauville [4, Définition 3.4.2]) Assume that $A^{n+1}_{\mathbb{Z}}(X)$ admits an algebraic representative $G \colon A^{n+1}_{\mathbb{Z}}(X) \to A$. In this case, a principal polarization (A, θ_A) is called the *incidence polarization with respect to X* if for all algebraic maps $f \colon T \to A^{n+1}_{\mathbb{Z}}(X)$ defined by a cycle $z \in CH^{n+1}_{\mathbb{Q}}(T \times X)$ the equality $(G \circ f)^*(\theta_A) = (-1)^{n+1} I(z)$ holds.

3.3. Verepresentability. Recall from Definition 1.15 the notion of verepresentability. Here is a detailed list of examples:

Example 3.7. (Curves) Every curve C is verepresentable. This follows automatically from the fact that there is a single group $A^1_{\mathbb{Z}}(C)$ of algebraically trivial cycles and that $A^1_{\mathbb{Z}}(C) = \mathrm{Pic}^0(C) = J(C)$.

Lemma 3.8. *Let X be a verepresentable threefold. In this case the above bilinear pairings (1.3) are non-degenerate.*

Proof. By definition, one has $A^3_{\mathbb{Q}}(X) = 0$. Hence, $A^3_{\mathbb{Q}}(X)$ is rationally representable, i.e., there exists a curve Γ and a surjective algebraic morphism $A^1_{\mathbb{Q}}(\Gamma) \twoheadrightarrow A^3_{\mathbb{Q}}(X)$. Gorchinskiy and Guletskii [25, Theorem 5.1] proved that the rational representability of $A^3_{\mathbb{Q}}(X)$ is enough to describe a Chow–Künneth decomposition of the Chow motive $M(X)_{\mathbb{Q}}$. As explained by Vial in [69, Theorem 4], this is equivalent to the rational representability of all the $\mathbb{Q}$-vector spaces $A^i_{\mathbb{Q}}(X)$. This implies that X satisfies the standard conjecture of Lefschetz type (see Vial [69, Theorem 4.10]) and consequently that the above bilinear pairings (1.3) are non-degenerate. $\square$

Thanks to the work of Gorchinskiy and Guletskii [25], whenever X is a Fano threefold, a conic bundle over a rational surface, or a del Pezzo fibration over $\mathbb{P}^1$, X is rationally representable and $J_a^i(X)$ is trivial for $i \neq 1$ (we have $\mathrm{Pic}^0(X) = 0$). Hence, a proof similar to the one of Lemma 3.8 allows us to conclude that the bilinear pairings (1.3) are non-degenerate. In all these cases, the verepresentability of X depends only on the existence of an incidence polarization. Therefore, we obtain the following list of verepresentable threefolds:

Example 3.9 (Threefolds)**.** (i) *Trivial Jacobian*: assume that X is $\mathbb{P}^3$, a quadric threefold, a Fano threefold of index 2 and degree 5, or a Fano threefold of index 1 and degree 22. In all these cases the verepresentability follows easily from the triviality of $J(X)$. In fact, $h^{1,2}(X) = 0$; see Iskovskikh and Prokhorov [35, Section 12.2].

(ii) *Fano threefolds of index* 2: Let X be a cubic threefold (see Clemens and Griffiths [20]), a quartic double solid (see Tihomirov [65] and Voisin [70]), the intersection of two quadrics in $\mathbb{P}^5$ (see Donagi[24] and Reid [61]), or a Fano threefold of index 2 and degree 5 (see item (i)). In all these cases there is an incidence polarization.

(iii) *Fano threefolds of index* 1: Let X be a general sextic double solid (see Ceresa and Verra [18]), a quartic in $\mathbb{P}^4$ (see Bloch and Murre [11]), the intersection of a cubic with a quadric in $\mathbb{P}^5$ (see Bloch and Murre [11]), the intersection of three quadrics in $\mathbb{P}^6$ (see Beauville [4] and also Bloch and Murre [11]), a Fano threefold of index 1 and degree 10 (see Iliev [30] and Logachëv [50]) or of degree 14 (see Iliev and Markushevich [33]), a general Fano threefold of index 1 and degree 12 (see Iliev and Markushevich [34]), degree 16 (see Iliev [31] and Mukai [56]), or degree 18 (see Iliev and Manivel [32] and also Iskovskikh and Prokhorov [35]), or a Fano threefold of index 1 and degree 22 (see item (i)). In all these cases there is an incidence polarization.

(iv) *Conic bundles*: Let $X \to S$ be a standard conic bundle over a rational surface. In this case there is an incidence polarization; see Beauville [4] and Beltrametti [5].

(v) *del Pezzo fibrations*: Let $X \to \mathbb{P}^1$ be a del Pezzo fibration of degree d with $d = 2, 3, 4, 5$. In this case there is an incidence polarization; see Kanev [36], [37].

For Fano threefolds of higher Picard rank we invite the interested reader to consult the exhaustive treatment of Iskovskikh and Prokhorov [35].

Example 3.10. (Higher dimensions) When $d_X \geqslant 5$, only a few examples of verepresentable $\mathbb{C}$-schemes are currently known.

(i) If X is the intersection of two even-dimensional quadrics, then X is verepresentable. Thanks to [10, Theorem 1.5] and to the work of Reid [61] and Donagi [24], the only non-trivial Jacobian is the intermediate one. Moreover, this Jacobian carries an incidence polarization.

(i′) If X is the intersection of three odd-dimensional quadrics, then X is also verepresentable. Once again, thanks to [10, Theorem 1.5] and to the work of Beauville [4, Section 6], the only non-trivial Jacobian is the intermediate one. Moreover, this Jacobian carries an incidence polarization.

(ii) If X is an even-dimensional quadric fibration over $\mathbb{P}^1$, then X is verepresentable. This follows from the combination of Vial's motivic description [68, Section 4] with Reid's work on the intermediate Jacobian [61]; see also Donagi [24].

(ii′) If X is a odd-dimensional quadric fibration over $\mathbb{P}^2$, then X is verepresentable. Thanks to the work of Beauville [4, Section 4], the only non-trivial Jacobian is the intermediate one. Moreover, this Jacobian carries an incidence polarization. As in the 3-dimensional case, it is natural to expect that $\mathbb{P}^2$ can be replaced by any rational surface; see Beltrametti [5].

4. Proof of Theorem 1.4

By assumption there exist semi-orthogonal decompositions $\mathrm{perf}(X) = \langle \mathcal{T}_X, \mathcal{T}_X^\perp \rangle$ and $\mathrm{perf}(Y) = \langle \mathcal{T}_Y, \mathcal{T}_Y^\perp \rangle$ and an equivalence $\phi\colon \mathcal{T}_X \simeq \mathcal{T}_Y$ of triangulated categories. Out of this data one constructs the composed functor

$$\Phi\colon \mathrm{perf}(X) \xrightarrow{\pi_X} \mathcal{T}_X \overset{\phi}{\simeq} \mathcal{T}_Y \xrightarrow{i_Y} \mathrm{perf}(Y), \tag{4.1}$$

where π_X stands for the projection and i_Y for the inclusion. Once again by assumption, Φ is of Fourier–Mukai type, i.e., Φ is isomorphic to the Fourier–Mukai functor $\Phi_{\mathcal{E}} := Rq_*(p^*(-) \otimes^L \mathcal{E})$; the complex $\mathcal{E}$ is usually called the *kernel* of $\Phi_{\mathcal{E}}$. As proved by Huybrechts in [29, Proposition 5.9], the right adjoint of $\Phi_{\mathcal{E}}$ is also of Fourier–Mukai type and its kernel is given by $\mathcal{E}_R := \mathcal{E}^\vee \otimes^L p^*\omega_X[d_X]$, where $\mathcal{E}^\vee$ is the dual of $\mathcal{E}$ and $\omega_X[d_X]$ the canonical line bundle of X shifted by the dimension of X. Moreover, the unit $\gamma\colon \mathrm{Id} \Rightarrow \Phi_{\mathcal{E}_R} \circ \Phi_{\mathcal{E}}$ of this adjunction is an isomorphism when evaluated at any object of $\mathcal{T}_X \subset \mathrm{perf}(X)$; see Kuznetsov [44, Theorem 3.3].

Now, recall that the triangulated categories $\mathrm{perf}(X)$ and $\mathrm{perf}(Y)$ admit unique dg enhancements $\mathrm{perf}^{\mathrm{dg}}(X)$ and $\mathrm{perf}^{\mathrm{dg}}(Y)$. As proved by Toën [66, Section 8.3], every perfect complex $\mathcal{E} \in \mathrm{perf}(X \times Y)$ gives rise to a dg functor

$$\Phi^{\mathrm{dg}}_{\mathcal{E}}\colon \mathrm{perf}^{\mathrm{dg}}(X) \to \mathrm{perf}^{\mathrm{dg}}(Y), \quad \mathcal{F} \mapsto Rq_*(p^*(\mathcal{F}) \otimes^L \mathcal{E}) \tag{4.2}$$

such that $\mathsf{H}^0(\Phi^{\mathrm{dg}}_{\mathcal{E}}) \simeq \Phi_{\mathcal{E}}$. Moreover, one has the following bijection

$$\mathrm{Iso}\,\mathrm{perf}(X \times Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{Hmo}(\mathbb{C})}(\mathrm{perf}^{\mathrm{dg}}(X), \mathrm{perf}^{\mathrm{dg}}(Y)), \quad \mathcal{E} \mapsto \Phi^{\mathrm{dg}}_{\mathcal{E}}. \tag{4.3}$$

We record the following easy result:

Lemma 4.4. *The following conditions are equivalent*:

(i) (Kuznetsov's conjecture 1.19) *the functor* (4.1) *is of Fourier–Mukai type*;

(ii) *the functor* (4.1) *admits a* dg enhancement, *i.e., there exists a dg functor* $\Phi^{\mathrm{dg}}\colon \mathrm{perf}^{\mathrm{dg}}(X) \to \mathrm{perf}^{\mathrm{dg}}(Y)$ *such that* $\mathsf{H}^0(\Phi^{\mathrm{dg}}) \simeq \Phi$.

Proof. If Φ is of Fourier–Mukai type, with kernel $\mathcal{E}$, then we can take for Φ^{dg} the dg functor (4.2). This shows the implication (i) ⇒ (ii). Assume now that Φ admits a dg enhancement Φ^{dg}. Thanks to bijection (4.3), Φ^{dg} is of the form $\Phi^{\mathrm{dg}}_{\mathcal{E}}$ for some perfect complex $\mathcal{E} \in \mathrm{perf}(X \times Y)$. Using the equivalence $\mathsf{H}^0(\Phi^{\mathrm{dg}}) \simeq \Phi$, one then concludes that Φ is of the form $\Phi_{\mathcal{E}}$ (and hence of Fourier–Mukai type). This shows the implication (ii) ⇒ (i). □

Let us denote by i_X^{dg} the inclusion of dg categories $\mathcal{T}_X^{\mathrm{dg}} \to \mathrm{perf}^{\mathrm{dg}}(X)$. The above projection functor $\pi_X \colon \mathrm{perf}(X) \to \mathcal{T}_X$ also admits a dg enhancement:

Lemma 4.5. *There exists a well-defined morphism* $\pi_X^{\mathrm{dg}} \colon \mathrm{perf}^{\mathrm{dg}}(X) \to \mathcal{T}_X^{\mathrm{dg}}$ *in* $\mathsf{Hmo}(\mathbb{C})$ *such that* $\pi_X^{\mathrm{dg}} \circ i_X^{\mathrm{dg}} = \mathrm{Id}$.

Proof. Recall first that we have the following inclusions of dg categories

$$i_X^{\mathrm{dg}} \colon \mathcal{T}_X^{\mathrm{dg}} \to \mathrm{perf}^{\mathrm{dg}}(X), \quad i_X^{\perp,\mathrm{dg}} \colon \mathcal{T}_X^{\perp,\mathrm{dg}} \to \mathrm{perf}^{\mathrm{dg}}(X). \tag{4.6}$$

Their intersection in $\mathrm{perf}^{\mathrm{dg}}(X)$ is the zero object. Hence, let us denote by T the full dg subcategory of $\mathrm{perf}^{\mathrm{dg}}(X)$ consisting of those objects that belong to $\mathcal{T}_X^{\mathrm{dg}}$ or to $\mathcal{T}_X^{\perp,\mathrm{dg}}$. Note that the dg functor i_X^{dg} factors through the inclusion $T \subset \mathrm{perf}^{\mathrm{dg}}(X)$. Since by hypothesis one has a semi-orthogonal decomposition $\mathrm{perf}(X) = \langle \mathcal{T}_X, \mathcal{T}_X^{\perp} \rangle$, the objects of $\mathsf{H}^0(T)$ form a set of generators of $\mathrm{perf}(X)$. Consequently, the inclusion of dg categories $T \subset \mathrm{perf}^{\mathrm{dg}}(X)$ is a Morita equivalence (see Keller [38, Lemma 3.10]) and hence an isomorphism in $\mathsf{Hmo}(\mathbb{C})$. Consider then the dg functor $\pi_X^{\mathrm{dg}} \colon T \to \mathcal{T}_X^{\mathrm{dg}}$ that is the identity of $\mathcal{T}_X^{\mathrm{dg}}$ and which sends all the remaining objects to zero. □

Notation 4.7. Given two dg functors $F, G \colon \mathcal{A} \to \mathcal{B}$, the *complex of morphisms* $\underline{\mathrm{Hom}}(F, G)$ has as its n^{th} component the $\mathbb{C}$-vector space formed by the families of morphisms $\phi_x \in \mathcal{B}(F(x), G(x))^n$ of degree n such that $G(f) \circ \phi_x = \phi_x \circ F(f)$ for all $f \in \mathcal{A}(x, y)$ and $x, y \in \mathcal{A}$; see Keller [38, Section 2.3]. The differential is induced by that of $\mathcal{B}(F(x), G(x))$. The set of *morphisms* ν from F to G is by definition in bijection with $Z^0\underline{\mathrm{Hom}}(F, G)$, where $Z^0(-)$ denotes the degree zero cycles functor.

Lemma 4.8. *Let* $F, G \colon \mathcal{A} \to \mathcal{B}$ *be two dg functors between pretriangulated dg categories and* $\nu \colon F \Rightarrow G$ *a morphism from* F *to* G*. This data naturally give rise to a morphism between* $\mathcal{A}$*-*$\mathcal{B}$*-bimodules* ${}_\nu B \colon {}_F B \Rightarrow {}_G B$ *and to a natural transformation* $\mathsf{H}^0(\nu) \colon \mathsf{H}^0(F) \Rightarrow \mathsf{H}^0(G)$ *between triangulated functors. With these notations, whenever* $\mathsf{H}^0(\nu)$ *is an isomorphism,* ${}_\nu B$ *is a quasi-isomorphism.*

Proof. Let $x \in \mathcal{A}$ and $w \in \mathcal{B}$. One needs to prove that the induced homomorphisms

$$(\nu_x)_* \colon H^i\mathcal{B}(w, F(x)) \to H^i\mathcal{B}(w, G(x)), \quad i \in \mathbb{Z}, \tag{4.9}$$

are isomorphisms. By assumption, $\mathcal{A}$ and $\mathcal{B}$ are pretriangulated. Therefore, (4.9) identifies with the induced homomorphisms

$$(\mathsf{H}^0(\nu)_x)_* \colon \mathrm{Hom}_{\mathsf{H}^0(\mathcal{B})}(w[i], F(x)) \to \mathrm{Hom}_{\mathsf{H}^0(\mathcal{B})}(w[i], G(x)), \quad i \in \mathbb{Z}, \tag{4.10}$$

where $w[i]$ is the i^{th} suspension of w in the triangulated category $\mathsf{H}^0(\mathcal{B})$. Since by hypothesis $\mathsf{H}^0(\nu)$ is an isomorphism, one then concludes that (4.10) (and hence (4.9)) is an isomorphism. This completes the proof. □

Proposition 4.11. *Assume that Φ is of Fourier–Mukai type with kernel $\mathcal{E}$. In this case we have the following commutative diagram*

$$\begin{tikzcd} \mathrm{perf}^{\mathrm{dg}}(X) \arrow[r, "\Phi^{\mathrm{dg}}_{\mathcal{E}}"] & \mathrm{perf}^{\mathrm{dg}}(Y) \arrow[r, "\Phi^{\mathrm{dg}}_{\mathcal{E}_R}"] & \mathrm{perf}^{\mathrm{dg}}(X) \\ \mathcal{T}^{\mathrm{dg}}_X \arrow[u, "i^{\mathrm{dg}}_X"] \arrow[rr, equal] & & \mathcal{T}^{\mathrm{dg}}_X \arrow[u, "i^{\mathrm{dg}}_X"'] \end{tikzcd} \tag{4.12}$$

in the homotopy category $\mathsf{Hmo}(\mathbb{C})$.

Proof. Consult Caldararu and Willerton [15] for the construction of the 2-category $\mathsf{Var}(\mathbb{C})$ of integral kernels. The objects are the smooth projective $\mathbb{C}$-schemes, the categories of morphisms are given by $\mathrm{Hom}_{\mathsf{Var}(\mathbb{C})}(X, Y) := \mathrm{perf}(X \times Y)$, the composition law is induced by the convolution of kernels, and the identity of every object X is the structure sheaf $R\Delta_*(\mathcal{O}_X) \in \mathrm{perf}(X \times X)$ of the diagonal $\Delta \subset X \times X$.

Let $\mathcal{E}$ be a perfect complex of $\mathcal{O}_{X\times Y}$-modules, i.e., a morphism in $\mathsf{Var}(\mathbb{C})$ from X to Y. As explained by Caldararu and Willerton [15, Section 3.2 and Appendix], its right adjoint in the 2-category $\mathsf{Var}(\mathbb{C})$ is given by $\Sigma_X \circ \mathcal{E}^\vee$, where $\mathcal{E}^\vee \in \mathrm{perf}(Y \times X)$ is the dual of $\mathcal{E}$ and $\Sigma_X \in \mathrm{perf}(X \times X)$ is the Serre kernel $R\Delta_*(\omega_X[d_X])$. Since the composition of Fourier–Mukai functors corresponds to the convolution of kernels (see Huybrechts [29, Proposition 5.10]), and $R\Delta_*(\omega_X)$ is mapped to $p^*\omega_X$ (via pull-back push-forward), we conclude that $\Sigma_X \circ \mathcal{E}^\vee \simeq \mathcal{E}_R := \mathcal{E}^\vee \otimes^L p^*\omega_X[d_X]$. In particular, we have a well-defined unit morphism $\Gamma\colon R\Delta_*(\mathcal{O}_X) \to \mathcal{E}_R \circ \mathcal{E}$ in $\mathrm{perf}(X\times X)$. Now, as explained by Caldararu and Willerton in [15, Section 1.2], one has a well-defined 2-functor $\mathrm{perf}(-)\colon \mathsf{Var}(\mathbb{C}) \to \mathsf{Cat}$ with values in the 2-category of categories. A $\mathbb{C}$-scheme X is mapped to the derived category $\mathrm{perf}(X)$ of perfect complexes, a kernel $\mathcal{E} \in \mathrm{perf}(X \times Y)$ to the Fourier–Mukai functor $\Phi_{\mathcal{E}}\colon \mathrm{perf}(X) \to \mathrm{perf}(Y)$, and a morphism $\mu\colon \mathcal{E} \Rightarrow \mathcal{E}'$ in $\mathrm{perf}(X \times Y)$ to a natural transformation $\Phi_\mu\colon \Phi_{\mathcal{E}} \Rightarrow \Phi_{\mathcal{E}'}$. In particular, the above adjunction $(\mathcal{E}, \mathcal{E}_R)$ in $\mathsf{Var}(\mathbb{C})$ gives rise to the classical adjunction of categories

$$\begin{tikzcd} \mathrm{perf}(Y) \arrow[d, shift left, "\Phi_{\mathcal{E}_R}"] \\ \mathrm{perf}(X), \arrow[u, shift left, "\Phi_{\mathcal{E}}"] \end{tikzcd}$$

with unit morphism $\gamma = \Phi_\Gamma\colon \mathrm{Id} \Rightarrow \Phi_{\mathcal{E}_R} \circ \Phi_{\mathcal{E}}$. Now, choose a representative $\overline{\Gamma}$ (i.e., a morphism of complexes of $\mathcal{O}_{X\times X}$-modules) for the unit morphism $\Gamma\colon R\Delta_*(\mathcal{O}_X) \to \mathcal{E}_R \circ \mathcal{E}$. As proved by Toën [66, Section 8.3], the composition law in $\mathsf{Hmo}(\mathbb{C})$ corresponds under the above bijection (4.3) to the convolution of kernels. Hence, $\overline{\Gamma}$ naturally gives rise to a morphism

$$\Phi^{\mathrm{dg}}_{\overline{\Gamma}}\colon \mathrm{Id} = \Phi^{\mathrm{dg}}_{R\Delta_*(\mathcal{O}_X)} \Rightarrow \Phi^{\mathrm{dg}}_{\mathcal{E}_R \circ \mathcal{E}} = \Phi^{\mathrm{dg}}_{\mathcal{E}_R} \circ \Phi^{\mathrm{dg}}_{\mathcal{E}}$$

between dg functors. By precomposing all this data with the inclusion $i^{\mathrm{dg}}_X\colon \mathcal{T}^{\mathrm{dg}}_X \to \mathrm{perf}^{\mathrm{dg}}(X)$ one then obtains a well-defined morphism $\Phi^{\mathrm{dg}}_{\overline{\Gamma}} \circ i^{\mathrm{dg}}_X$ from the dg functor

$i_X^{\mathrm{dg}}\colon \mathcal{T}_X^{\mathrm{dg}} \to \mathrm{perf}^{\mathrm{dg}}(X)$ to the composed dg functor

$$\mathcal{T}_X^{\mathrm{dg}} \xrightarrow{i_X^{\mathrm{dg}}} \mathrm{perf}^{\mathrm{dg}}(X) \xrightarrow{\Phi_{\mathcal{E}}^{\mathrm{dg}}} \mathrm{perf}^{\mathrm{dg}}(Y) \xrightarrow{\Phi_{\mathcal{E}_R}^{\mathrm{dg}}} \mathrm{perf}^{\mathrm{dg}}(X). \tag{4.13}$$

We now claim that this data satisfies the conditions of the general Lemma 4.8 (with $\nu = \Phi_{\Gamma}^{\mathrm{dg}} \circ i_X^{\mathrm{dg}}$). This follows from the equalities $\mathsf{H}^0(\Phi_{\mathcal{E}_R}^{\mathrm{dg}}) = \Phi_{\mathcal{E}_R}$, $\mathsf{H}^0(\Phi_{\mathcal{E}}^{\mathrm{dg}}) = \Phi_{\mathcal{E}}$, and $\mathsf{H}^0(\Phi_{\Gamma}^{\mathrm{dg}}) = \Phi_\Gamma = \gamma$, and from the fact that the unit morphism $\eta\colon \mathrm{Id} \Rightarrow \Phi_{\mathcal{E}_R} \circ \Phi_{\mathcal{E}}$ of the adjunction $(\Phi_{\mathcal{E}}, \Phi_{\mathcal{E}_R})$ is an isomorphism when evaluated at any object of $\mathcal{T}_X \subset \mathrm{perf}(X)$. Consequently, Lemma 4.8 furnishes us with a quasi-isomorphism

$$_{(\Phi_{\Gamma}^{\mathrm{dg}} \circ i_X^{\mathrm{dg}})}B\colon \ _{i_X^{\mathrm{dg}}}B \Rightarrow {}_{(\Phi_{\mathcal{E}_R}^{\mathrm{dg}} \circ \Phi_{\mathcal{E}}^{\mathrm{dg}} \circ i_X^{\mathrm{dg}})}B \tag{4.14}$$

between two bimodules which belong to $\mathrm{rep}(\mathcal{T}_X^{\mathrm{dg}}, \mathrm{perf}^{\mathrm{dg}}(X))$. Making use of the bijection (2.2), one then concludes that diagram (4.12) is commutative. □

We now have all the necessary ingredients for the conclusion of the proof of Theorem 1.4. By assumption, Kuznetsov's conjecture 1.19 holds. Therefore, item (ii) of Lemma 4.4 furnishes us with a well-defined dg functor $\Phi_{\mathcal{E}}^{\mathrm{dg}}\colon \mathrm{perf}^{\mathrm{dg}}(X) \to \mathrm{perf}^{\mathrm{dg}}(Y)$. By applying to it the Jacobian functor $\mathbf{J}(-)$ one then obtains a morphism of complex abelian varieties up to isogeny

$$\mathbf{J}(\Phi_{\mathcal{E}}^{\mathrm{dg}})\colon \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(X)) \to \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(Y)). \tag{4.15}$$

Since by assumption the bilinear pairings (1.3) (associated to X and Y) are nondegenerate, (4.15) identifies with a well-defined morphism $\tau\colon \prod_{i=0}^{d_X-1} J_a^i(X) \to \prod_{i=0}^{d_Y-1} J_a^i(Y)$ in $\mathrm{Ab}(\mathbb{C})_{\mathbb{Q}}$. This proves item (i).

Let us now prove item (ii). Thanks to Lemma 4.5, there exists a morphism $\pi_X^{\mathrm{dg}}\colon \mathrm{perf}^{\mathrm{dg}}(X) \to \mathcal{T}_X^{\mathrm{dg}}$ in $\mathsf{Hmo}(\mathbb{C})$ such that $\pi_X^{\mathrm{dg}} \circ i_X^{\mathrm{dg}} = \mathrm{Id}$. This implies in particular that $\mathcal{T}_X^{\mathrm{dg}}$ belongs to the category $\mathrm{NChow}(\mathbb{C})_{\mathbb{Q}}$ of noncommutative Chow motives. By applying the Jacobian functor $\mathbf{J}(-)$ to (4.12) one then obtains the following commutative diagram of complex abelian varieties up to isogeny

$$\begin{tikzcd} \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(X)) \arrow[r, "\mathbf{J}(\Phi_{\mathcal{E}}^{\mathrm{dg}})"] & \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(Y)) \arrow[r, "\mathbf{J}(\Phi_{\mathcal{E}_R}^{\mathrm{dg}})"] & \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(X)) \\ \mathbf{J}(\mathcal{T}_X^{\mathrm{dg}}) \arrow[u, "\mathbf{J}(i_X^{\mathrm{dg}})"] \arrow[rr, equal] & & \mathbf{J}(\mathcal{T}_X^{\mathrm{dg}}). \arrow[u, "\mathbf{J}(i_X^{\mathrm{dg}})"'] \end{tikzcd} \tag{4.16}$$

Since by assumption $\mathbf{J}(\mathcal{T}_X^{\perp,\mathrm{dg}}) = 0$, Lemma 4.17 below implies that $\mathbf{J}(i_X^{\mathrm{dg}})$ is an isomorphism. Using the commutativity of diagram (4.16) one then concludes that $\mathbf{J}(\Phi_{\mathcal{E}_R}^{\mathrm{dg}})$ is a retraction of $\mathbf{J}(\Phi_{\mathcal{E}}^{\mathrm{dg}}) \simeq \tau$ or equivalently that τ is split injective. This concludes the proof of item (ii).

Lemma 4.17. *The morphism of complex abelian varieties up to isogeny*

$$\mathbf{J}(i_X^{\mathrm{dg}})\colon \mathbf{J}(\mathcal{T}_X^{\mathrm{dg}}) \to \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(X))$$

is an isomorphism if and only if $\mathbf{J}(\mathcal{T}_X^{\perp,\mathrm{dg}}) = 0$.

Proof. Since by hypothesis one has a semi-orthogonal decomposition $\mathrm{perf}(X) = \langle \mathcal{T}_X, \mathcal{T}_X^\perp \rangle$, the inclusions of dg categories $i_X^{\mathrm{dg}} \colon \mathcal{T}_X^{\mathrm{dg}} \hookrightarrow \mathrm{perf}^{\mathrm{dg}}(X)$ and $i_X^{\perp,\mathrm{dg}} \colon \mathcal{T}_X^{\perp,\mathrm{dg}} \hookrightarrow \mathrm{perf}^{\mathrm{dg}}(X)$ give rise to an isomorphism $\mathcal{T}_X^{\mathrm{dg}} \oplus \mathcal{T}_X^{\perp,\mathrm{dg}} \xrightarrow{\simeq} \mathrm{perf}^{\mathrm{dg}}(X)$ in $\mathsf{Hmo}_0(\mathbb{C})$ (and hence in $\mathrm{NChow}(\mathbb{C})_{\mathbb{Q}}$); see [62, Theorem 6.3]. Therefore, by applying to it the additive Jacobian functor $\mathbf{J}(-)$ one then obtains the following isomorphism

$$[\mathbf{J}(i_X^{\mathrm{dg}}), \mathbf{J}(i_X^{\perp,\mathrm{dg}})] \colon \ \mathbf{J}(\mathcal{T}_X^{\mathrm{dg}}) \oplus \mathbf{J}(\mathcal{T}_X^{\perp,\mathrm{dg}}) \xrightarrow{\simeq} \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(X))$$

in $\mathrm{Ab}(\mathbb{C})_{\mathbb{Q}}$. This isomorphism clearly implies our claim. □

Let us now prove item (iii). From the above arguments and constructions, one has a commutative diagram

$$\begin{tikzcd}
\mathrm{perf}^{\mathrm{dg}}(X) \arrow[r, "\Phi_{\mathcal{E}}^{\mathrm{dg}}"] & \mathrm{perf}^{\mathrm{dg}}(Y) \\
\mathcal{T}_X^{\mathrm{dg}} \arrow[u, "i_X^{\mathrm{dg}}"] \arrow[r, "\simeq"'] & \mathcal{T}_Y^{\mathrm{dg}} \arrow[u, "i_Y^{\mathrm{dg}}"']
\end{tikzcd}$$

in the homotopy category $\mathsf{Hmo}(\mathbb{C})$. By applying to it the Jacobian functor $\mathbf{J}(-)$ one then obtains a commutative diagram of complex abelian varieties up to isogeny

$$\begin{tikzcd}
\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(X)) \arrow[r, "\mathbf{J}(\Phi_{\mathcal{E}}^{\mathrm{dg}})"] & \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(Y)) \\
\mathbf{J}(\mathcal{T}_X^{\mathrm{dg}}) \arrow[u, "\mathbf{J}(i_X^{\mathrm{dg}})"] \arrow[r, "\simeq"'] & \mathbf{J}(\mathcal{T}_Y^{\mathrm{dg}}). \arrow[u, "\mathbf{J}(i_Y^{\mathrm{dg}})"']
\end{tikzcd} \tag{4.18}$$

Since by assumption $\mathbf{J}(\mathcal{T}_Y^{\perp,\mathrm{dg}}) = 0$, one concludes from Lemma 4.17 that the vertical morphisms in (4.18) are isomorphisms. This implies that $\mathbf{J}(\Phi_{\mathcal{E}}^{\mathrm{dg}})$ is an isomorphism and so the proof of Theorem 1.4 is finished.

5. Proof of Theorem 1.17

By assumption, the $\mathbb{C}$-schemes X and Y satisfy all the assumptions of items (i)–(ii) of Theorem 1.4 and are moreover verepresentable. In particular, X (resp. Y) is irreducible of odd dimension $d_X := 2n+1$ (resp. $d_Y := 2m+1$). Moreover, there is a single non-trivial algebraic Jacobian $J(X) := J_a^n(X)$ (resp. $J(Y) := J_a^m(Y)$), which via a universal regular map

$$G_X \colon A_{\mathbb{Z}}^{n+1}(X) \twoheadrightarrow J(X) \quad (\text{resp. } G_Y \colon A_{\mathbb{Z}}^{m+1}(Y) \twoheadrightarrow J(Y)),$$

is the algebraic representative of $A_{\mathbb{Z}}^{n+1}(X)$ (resp. of $A_{\mathbb{Z}}^{m+1}(Y)$); see Section 3.1. A proof of the surjectivity of G_X (resp. G_Y) can be found in Beauville's work [4, Remark 3.2.4(ii)]. Furthermore, we have induced isomorphisms

$$AJ_{\mathbb{Q}}^n \colon A_{\mathbb{Q}}^{n+1}(X) \xrightarrow{\simeq} J(X)_{\mathbb{Q}}, \quad AJ_{\mathbb{Q}}^m \colon A_{\mathbb{Q}}^{m+1}(Y) \xrightarrow{\simeq} J(Y)_{\mathbb{Q}},$$

where $J(X)_{\mathbb{Q}}$ (resp. $J(Y)_{\mathbb{Q}}$) stands for $J(X)$ (resp. $J(Y)$) considered as an abelian variety *up to isogeny*.

Now, recall from the proof of Theorem 1.4 that there exists a perfect complex $\mathcal{E} \in \mathrm{perf}(X \times Y)$ such that the split injective morphism $\tau\colon J(X)_{\mathbb{Q}} \to J(Y)_{\mathbb{Q}}$ is obtained by applying the Jacobian functor $\mathbf{J}(-)$ to the Fourier–Mukai dg functor $\Phi^{\mathrm{dg}}_{\mathcal{E}}\colon \mathrm{perf}^{\mathrm{dg}}(X) \to \mathrm{perf}^{\mathrm{dg}}(Y)$. Consider the following homomorphism

$$e_{m+n+1}\colon CH^{n+1}_{\mathbb{Q}}(X) \to CH^{m+1}_{\mathbb{Q}}(Y), \quad z \mapsto q_*(p^*(z) \cdot \mathrm{ch}(\mathcal{E})_{m+n+1}), \tag{5.1}$$

where $\mathrm{ch}(\mathcal{E})_{m+n+1} \in CH^{m+n+1}_{\mathbb{Q}}(X \times Y)$ is the $(m+n+1)^{\mathrm{th}}$ component of the Chern character $\mathrm{ch}(\mathcal{E})$ of $\mathcal{E}$, and $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are the projection morphisms. Algebraic equivalence is an *adequate* equivalence relation on cycles; see André [2, Définition 3.1.1.1, Section 3.2.1]. This, together with the fact that $q\colon X \times Y \to Y$ is equidimensional, implies that (5.1) gives rise to a map $\overline{\tau}\colon A^{n+1}_{\mathbb{Q}}(X) \to A^{m+1}_{\mathbb{Q}}(Y)$, which is still the correspondence given by $\mathrm{ch}(\mathcal{E})_{m+n+1}$; see André [2, Section 3.1.2].

Lemma 5.2. *One has the following commutative diagram*

$$\begin{tikzcd} A^{n+1}_{\mathbb{Q}}(X) \arrow[r, "\overline{\tau}"] \arrow[d, "AJ^n_{\mathbb{Q}}"', "\simeq"] & A^{m+1}_{\mathbb{Q}}(Y) \arrow[d, "\simeq"', "AJ^m_{\mathbb{Q}}"] \\ J(X)_{\mathbb{Q}} \arrow[r, "\tau"'] & J(Y)_{\mathbb{Q}}. \end{tikzcd}$$

Proof. Let us denote by $\mathrm{Chow}^*(\mathbb{C})_{\mathbb{Q}}$ (resp. by $\mathrm{Num}^*(\mathbb{C})_{\mathbb{Q}}$) the category of Chow (resp. numerical) motives where the morphisms are the graded correspondences. As proved in [53, Theorem 1.9], the category $\mathrm{Num}^*(\mathbb{C})_{\mathbb{Q}}$ (denoted by $\mathrm{Num}(\mathbb{C})_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$ in *loc. cit.*) is abelian semi-simple. As explained in [52, Section 4], one has a well-defined inclusion $\mathrm{Ab}(\mathbb{C})_{\mathbb{Q}} \subset \mathrm{Num}^*(\mathbb{C})_{\mathbb{Q}}$ of abelian semi-simple categories. Now, consider the following graded correspondence

$$e := \mathrm{ch}(\mathcal{E}) \cdot p^*\mathrm{Td}(X) \in \bigoplus_i CH^i_{\mathbb{Q}}(X \times Y),$$

where $\mathrm{Td}(X)$ is the Todd class of X. As proved in [52, Section 4] (see also [64, Section 8]), $J(X)_{\mathbb{Q}}$ (resp. $J(Y)_{\mathbb{Q}}$) is the largest direct summand of $M_{\mathbb{Q}}(X) \in \mathrm{Num}^*(\mathbb{C})_{\mathbb{Q}}$ (resp. of $M_{\mathbb{Q}}(Y)$) which belongs to $\mathrm{Ab}(\mathbb{C})_{\mathbb{Q}}$. Moreover, the morphism $\tau = \mathbf{J}(\Phi^{\mathrm{dg}}_{\mathcal{E}})\colon J(X)_{\mathbb{Q}} \to J(Y)_{\mathbb{Q}}$ is the largest direct summand of the graded correspondence e (considered as a morphism in $\mathrm{Num}^*(\mathbb{C})_{\mathbb{Q}}$ from $M_{\mathbb{Q}}(X)$ to $M_{\mathbb{Q}}(Y)$) which belongs to $\mathrm{Ab}(\mathbb{C})_{\mathbb{Q}}$. The correspondence e is a mixed cycle. Let us now show that the only degree that contributes to τ is e_{m+n+1}; note that this automatically completes the proof. Since the relative dimension of the projection morphism $q\colon X \times Y \to Y$ is equal to the dimension of X, i.e., $2n+1$, we have a well-defined *graded* homomorphism

$$q_*\colon CH^*_{\mathbb{Q}}(X \times Y) \to CH^{*-2n-1}_{\mathbb{Q}}(Y)$$

and consequently we obtain the equality

$$(q_*(p^*(z) \cdot e))_{m+1} = q_*((p^*(z) \cdot \mathrm{ch}(\mathcal{E}) \cdot p^*\mathrm{Td}(X))_{m+2n+2}).$$

Moreover, since p^* is a morphism of commutative rings, i.e., it respects the commutative intersection pairing (see [28, A1-2, page 426]), we also have the equality

$$p^*(z) \cdot \mathrm{ch}(\mathcal{E}) \cdot p^*\mathrm{Td}(X) = p^*(z \cdot \mathrm{Td}(X)) \cdot \mathrm{ch}(\mathcal{E}).$$

Now, recall from Pappas [60, Section 1] that the Todd class $\mathrm{Td}(X)$ is given by $\sum_{i\geqslant 0} \frac{\mathfrak{D}_i}{T_i}$. Here, $\mathfrak{D}_i$ is a polynomial with integral coefficients in the Chern classes and T_i is the product $\prod_p p^{[\frac{i}{p-1}]}$ taken over all the prime numbers; the symbol $[-]$ stands for the integral part. Note that $\mathfrak{D}_0 = 1$ since the degree zero component of the Todd class of any vector bundle is 1; [28, Appendix A, Section 4]. Moreover, recall that algebraically trivial cycles form an ideal under intersection, see André [2, Définition 3.1.1.1(3)]. Since by assumption X is verepresentable, we have $A^i_{\mathbb{Z}}(X) = 0$ for all $i \neq n+1$. Therefore, in the following intersection product

$$z \cdot \mathrm{Td}(X) = z \cdot \sum_{i\geqslant 0} \frac{\mathfrak{D}_i}{T_i} = \sum_{i\geqslant 0} \left(z \cdot \frac{\mathfrak{D}_i}{T_i} \right),$$

all the components of degree $\neq n+1$ are trivial. Since z has degree $n+1$, $z \cdot \mathrm{Td}(X) = z \cdot \frac{\mathfrak{D}_o}{T_0} = z$, and therefore $p^*(z \cdot \mathrm{Td}(X)) = p^*(z)$ is a purely $n+1$-codimensional cycle. We obtain in this way the following equality

$$(q_*(p^*(z) \cdot e))_{m+1} = q_*(p^*(z) \cdot \mathrm{ch}(\mathcal{E})_{m+n+1}),$$

which completes the proof. □

We now record the following simple result:

Lemma 5.3. *Let X be a verepresentable variety, L a line bundle on X and Ψ^{dg} the associated Morita equivalence $- \otimes L\colon \mathrm{perf}^{\mathrm{dg}}(X) \to \mathrm{perf}^{\mathrm{dg}}(X)$. In this case, Ψ^{dg} induces the identity map $A^{n+1}_{\mathbb{Q}}(X) \to A^{n+1}_{\mathbb{Q}}(X)$.*

Proof. Note first that the Grothendieck–Riemann–Roch theorem implies that the map $CH^*_{\mathbb{Q}}(X) \to CH^*_{\mathbb{Q}}(X)$ induced by Ψ^{dg} is given by $z \mapsto \psi(z) := z \cdot \mathrm{ch}(L)$. Let $z \in A^{n+1}_{\mathbb{Q}}(X)$. Since z is of codimension $n+1$, $\psi(z)$ has components only in codimensions $\geqslant n+1$. Moreover, since algebraically trivial cycles form an ideal under intersection, $\psi(z)$ is algebraically trivial. Now, recall that by assumption X is verepresentable. In particular, $A^i_{\mathbb{Q}}(X) = 0$ for every $i \neq n+1$. This hence implies that $\psi(z)$ is pure of codimension $n+1$ and consequently that $\psi(z) = z \cdot \mathrm{ch}_0(L) = z$ for any $z \in A^{n+1}_{\mathbb{Q}}(X)$. □

Consider the following homomorphism:

$$\overline{\sigma}\colon A^{m+1}_{\mathbb{Q}}(Y) \to A^{n+1}_{\mathbb{Q}}(X), \quad w \mapsto p_*(q^*(w) \cdot (-1)^{m+n}\mathrm{ch}(\mathcal{E})_{m+n+1}). \tag{5.4}$$

Let us denote by $\sigma\colon J(Y)_{\mathbb{Q}} \to J(X)_{\mathbb{Q}}$ the unique morphism induced by $\overline{\sigma}$ via the Abel–Jacobi map $AJ_{\mathbb{Q}}$.

Lemma 5.5. *The map $\sigma\colon J(Y)_{\mathbb{Q}} \to J(X)_{\mathbb{Q}}$ is a retraction of $\tau\colon J(X)_{\mathbb{Q}} \to J(Y)_{\mathbb{Q}}$.*

Proof. As shown in the proof of Theorem 1.4, a retraction of τ is obtained by applying the Jacobian functor $\mathbf{J}(-)$ to the Fourier–Mukai dg functor $\Phi^{\mathrm{dg}}_{\mathcal{E}_R}$, where $\mathcal{E}_R = \mathcal{E}^\vee \otimes p^*\omega_X[d_X]$. Let us denote by $\mathcal{E}_!$ the perfect complex $\mathcal{E}^\vee[d_X]$ and by Ψ^{dg} the Morita equivalence $- \otimes \omega_X\colon \mathrm{perf}^{\mathrm{dg}}(X) \to \mathrm{perf}_{\mathrm{dg}}(X)$. Since $\mathcal{E}_R = \mathcal{E}_! \otimes p^*\omega_X$, we have $\Phi^{\mathrm{dg}}_{\mathcal{E}_R} = \Psi^{\mathrm{dg}} \circ \Phi^{\mathrm{dg}}_{\mathcal{E}_!}$. By applying the Jacobian functor $\mathbf{J}(-)$ to the latter equality, we hence conclude from Lemma 5.3 (with $L = \omega_X$) that $\Phi^{\mathrm{dg}}_{\mathcal{E}_!}$ and $\Phi^{\mathrm{dg}}_{\mathcal{E}_R}$ induce the same map on $A^{n+1}_{\mathbb{Q}}(X)$.

We claim that the map $\overline{\sigma}$ is the unique map induced by $\Phi^{\mathrm{dg}}_{\mathcal{E}_!}$. Indeed, the same calculations as in Lemma 5.2 show that $\overline{\sigma}$ is given by the cycle $\mathrm{ch}_{m+n+1}(\mathcal{E}_!)$ on $X \times Y$. Since $d_X = 2n+1$ is odd we have $\mathrm{ch}(\mathcal{E}^\vee[2n+1]) = -\mathrm{ch}(\mathcal{E}^\vee)$. Moreover, $\mathrm{ch}_i(\mathcal{E}^\vee) = (-1)^i \mathrm{ch}_i(\mathcal{E})$. This follows from the fact that the Chern polynomial $\mathrm{c}_t(\mathcal{E}^\vee)$ is obtained from $\mathrm{c}_t(\mathcal{E})$ by alternating signs; see Hartshorne [28, A.3]. We hence obtain the following equalities:

$$\mathrm{ch}_{m+n+1}(\mathcal{E}^\vee[2n+1]) = -\mathrm{ch}_{m+n+1}(\mathcal{E}^\vee) = (-1)^{m+n}\mathrm{ch}_{m+n+1}(\mathcal{E}),$$

This implies that $\overline{\sigma}$ is a retraction of $\overline{\tau}$. By functoriality of the Abel–Jacobi map, we hence conclude that σ is a retraction of τ. □

We now have all the necessary ingredients for the conclusion of the proof of Theorem 1.17. Since by hypothesis X (resp. Y) is verepresentable, the non-trivial algebraic Jacobian $J(X)$ (resp. $J(Y)$) is an abelian variety endowed with the incidence polarization θ_X (resp. θ_Y) with respect to X (resp. to Y). In particular, this allows one to consider $J(X)$ and $J(Y)$ as canonical representatives of $\mathbf{J}(X) \simeq J(X)_{\mathbb{Q}}$ and $\mathbf{J}(Y) \simeq J(Y)_{\mathbb{Q}}$, respectively. Thanks to Lemma 3.5, having a canonical choice of (principally polarized) abelian varieties representing $J(X)_{\mathbb{Q}}$ and $J(Y)_{\mathbb{Q}}$, the split injective map $\tau\colon J(X)_{\mathbb{Q}} \to J(Y)_{\mathbb{Q}}$ can be seen as an isogeny $\tau\colon J(X) \to J(Y)$. Clearly, this isogeny is described by the algebraic cycle $\mathrm{ch}_{m+n+1}(\mathcal{E})$; see Lemma 5.2. On the other hand, Lemma 3.5 furnishes us with a morphism of algebraic tori $\sigma\colon J(Y) \to J(X)$ and Lemma 5.5 shows that σ is described by the cycle $(-1)^{m+n}\mathrm{ch}_{m+n+1}(\mathcal{E})$.

Let us now show that the isogeny $\tau\colon J(X) \to J(Y)$, considered as a morphism of abelian varieties, pulls-back the principal polarization to a principal polarization. In order to achieve this, we will use the incidence property of θ_Y. By hypothesis, $J(Y)$ is the algebraic representative of $A^{m+1}_{\mathbb{Z}}(Y)$ and the principal polarization θ_Y of $J(Y)$ is the incidence polarization with respect to Y. If $f\colon T \to A^{m+1}_{\mathbb{Z}}(Y)$ is an algebraic map defined by a cycle z in $CH^{m+1}_{\mathbb{Q}}(T \times Y)$, then (as explained in Section 3.2) we have the following equality

$$(G \circ f)^*\theta_Y = (-1)^{m+1} I(z), \tag{5.6}$$

of divisorial self correspondences on T; see Definition 3.3(ii). Note that f is not necessarily surjective. Set $T = J(X)$ and consider the cycle $\mathrm{ch}_{m+n+1}(\mathcal{E})$ as giving rise to an algebraic map $\tilde{\tau}\colon J(X) \to A^{m+1}_{\mathbb{Z}}(Y)$. Indeed, from the surjectivity of $G_X\colon A^{n+1}_{\mathbb{Z}}(X) \twoheadrightarrow J(X)$, for any α in $J(X)$, we can choose a representative in $A^{n+1}_{\mathbb{Z}}(X)$. Since the map τ is induced by such a correspondence, the choice of a representative becomes irrelevant as soon as we consider the map $G_Y\colon A^{m+1}_{\mathbb{Z}}(Y) \twoheadrightarrow J(Y)$. In particular, we have $G_Y \circ \tilde{\tau} = \tau$. By applying the incidence property (5.6) to the algebraic map $\tilde{\tau}$ we then obtain the following equality

$$\tau^*\theta_Y = (G_Y \circ \tilde{\tau})^*\theta_Y = (-1)^{m+1}\mathrm{i}, \tag{5.7}$$

where $\mathfrak{i} := I(\mathrm{ch}_{m+n+1}(\mathcal{E}))$ is the incidence correspondence of $\mathrm{ch}_{m+n+1}(\mathcal{E})$; see Definition 3.3. The principal polarization θ_X allows to identify $J(X)$ with its dual; see Definition 3.4. In order to conclude the proof it remains only to show that $\mathfrak{i}$ is isomorphic, up to a sign, to the identity correspondence (here we identify $J(X)$ with its dual). Thanks to Lemmas 5.2–5.5, we have $\mathrm{id} = \sigma \circ \tau = (-1)^{m+n}\mathfrak{i}$. This concludes the proof of the first claim of Theorem 1.17. The proof of the second claim follows automatically from the arguments above and from item (iii) of Theorem 1.4.

6. Application I: Categorical Torelli

Let X and Y be two verepresentable $\mathbb{C}$-schemes. The (generalized) *Torelli theorem* claims that $X \simeq Y$ if and only if $J(X) \simeq J(Y)$ as principally polarized abelian varieties. The particular case of curves (i.e., the classical Torelli theorem) was proved by Torelli [67] one hundred years ago. Thanks to the work of Clemens and Griffiths, Debarre, Donagi, Laszlo, Mérindol, and Voisin (see [20], [21], [22], [24], [49], [54], [70]), this also holds in the case of cubic threefolds, quartic double solids, and intersections of two (resp. three) quadrics of even (resp. odd) dimension. Theorem 1.17 furnishes us automatically with the following categorification:

Corollary 6.1 (Categorical Torelli)**.** *Let X and Y be two $\mathbb{C}$-schemes as in Theorem 1.17 with $\mathbf{J}(\mathcal{T}_Y^{\perp,\mathrm{dg}}) = 0$. Assuming the (generalized) Torelli theorem holds for X and Y, the $\mathbb{C}$-schemes X and Y are isomorphic.*

Intuitively speaking, Corollary 6.1 shows that the Morita equivalence class of the dg category $\mathcal{T}_X^{\mathrm{dg}}$ determines the isomorphism class of the $\mathbb{C}$-scheme X. Let us now explain how Corollary 6.1 gives rise to the implications (6.3), (6.5), (6.8), and (6.11) below.

Example 6.2 (Intersections of two even-dimensional quadrics)**.** Let X and Y be intersections of two even-dimensional quadrics, and C_X and C_Y the associated hyperelliptic curves. Recall from Examples 1.9 and 3.10 that one has fully faithful functors $\mathcal{T}_X := \mathrm{perf}(C_X) \to \mathrm{perf}(X)$ and $\mathcal{T}_Y := \mathrm{perf}(C_Y) \to \mathrm{perf}(Y)$ whose orthogonal complements are generated by the exceptional objects $\mathcal{O}, \ldots, \mathcal{O}(n-1)$, where $n = \dim(X) = \dim(Y)$. Now, suppose that there exists an equivalence of triangulated categories $\phi\colon \mathrm{perf}(C_X) \simeq \mathrm{perf}(C_Y)$. Under this assumption, all the conditions of Theorem 1.17 are satisfied; see Example 1.9. The associated functor Φ is of Fourier–Mukai type since it is a composition of functors of Fourier–Mukai type; see Kuznetsov [47, Thm. 7.1] for the projection $\mathrm{perf}(X) \to \mathrm{perf}(C_X)$, Orlov [59] for the equivalence ϕ, and Bondal and Orlov [12] for the inclusion $\mathrm{perf}(C_Y) \hookrightarrow \mathrm{perf}(Y)$. By combining Corollary 6.1 with the (generalized) Torelli theorem (see Donagi [24]), we hence obtain the following implication:

$$\mathrm{perf}(C_X) \simeq \mathrm{perf}(C_Y) \Rightarrow X \simeq Y. \tag{6.3}$$

Example 6.4 (Intersections of three odd-dimensional quadrics)**.** Let X and Y be intersections of three odd-dimensional quadrics of Fano type, and $\mathcal{C}_{0,X}$ and $\mathcal{C}_{0,Y}$ the sheaves of even parts of the Clifford algebras associated to the quadric spans $Q_X \to \mathbb{P}^2$ and $Q_Y \to \mathbb{P}^2$. Recall from Examples 1.10 and 3.10 that one has fully faithful functors $\mathcal{T}_X := \mathrm{perf}(\mathbb{P}^2, \mathcal{C}_{0,X}) \to \mathrm{perf}(X)$ and $\mathcal{T}_Y := \mathrm{perf}(\mathbb{P}^2, \mathcal{C}_{0,Y}) \to \mathrm{perf}(Y)$ whose orthogonal complements are generated by the exceptional objects $\mathcal{O}, \ldots, \mathcal{O}(n-2)$, where $n = \dim(X) = \dim(Y)$. Now, suppose that there exists a Fourier–Mukai equivalence of triangulated categories $\phi\colon \mathrm{perf}(\mathbb{P}^2, \mathcal{C}_{0,X}) \simeq \mathrm{perf}(\mathbb{P}^2, \mathcal{C}_{0,Y})$. Under this assumption, all the conditions of Theorem 1.17 are verified; see Example 1.10. The associated functor Φ is of Fourier–Mukai type since the projection $\mathrm{perf}(X) \to \mathrm{perf}(\mathbb{P}^2, \mathcal{C}_{0,X})$ and the embedding $\mathrm{perf}(\mathbb{P}^2, \mathcal{C}_{0,Y}) \hookrightarrow \mathrm{perf}(Y)$ are known to be of Fourier–Mukai type; see Kuznetsov [47, Theorem 7.1][45]. By combining Corollary 6.1 with the (generalized) Torelli theorem (see Laszlo [49]), we hence obtain the following implication:

$$\mathrm{perf}(\mathbb{P}^2, \mathcal{C}_{0,X}) \text{ Fourier–Mukai equivalent to } \mathrm{perf}(\mathbb{P}^2, \mathcal{C}_{0,Y}) \Rightarrow X \simeq Y. \tag{6.5}$$

Example 6.6 (Quartic double solids)**.** Let X and Y be quartic double solids. As proved by Kuznetsov [46, Section 3], one has the following semi-orthogonal decompositions

$$\mathrm{perf}(X) = \langle \mathcal{T}_X, \langle \mathcal{O}_X, \mathcal{O}_X(1)\rangle\rangle, \quad \mathrm{perf}(Y) = \langle \mathcal{T}_Y, \langle \mathcal{O}_Y, \mathcal{O}_Y(1)\rangle\rangle. \tag{6.7}$$

Suppose that there exists an equivalence of triangulated categories $\phi\colon \mathcal{T}_X \simeq \mathcal{T}_Y$ and that the composed functor $\Phi\colon \mathrm{perf}(X) \to \mathrm{perf}(Y)$ is of Fourier–Mukai type. Under these assumptions, all the conditions of Theorem 1.17 are verified. For the verepresentability of X and Y see Clemens [19] or Tihomirov [65]. By combining Corollary 6.1 with the (generalized) Torelli theorem (see Debarre [22] and Voisin [70]), we hence obtain the following implication:

$$\mathcal{T}_X \simeq \mathcal{T}_Y \text{ and } \Phi \text{ of Fourier–Mukai type} \Rightarrow X \simeq Y. \tag{6.8}$$

Example 6.9 (Cubic threefolds)**.** Let X and Y be cubic threefolds. As proved by Kuznetsov [42], one has the following semi-orthogonal decompositions

$$\mathrm{perf}(X) = \langle \mathcal{T}_X, \langle \mathcal{O}_X, \mathcal{O}_X(1)\rangle\rangle, \quad \mathrm{perf}(Y) = \langle \mathcal{T}_Y, \langle \mathcal{O}_Y, \mathcal{O}_Y(1)\rangle\rangle. \tag{6.10}$$

Similarly to Example 6.6, suppose that there exists an equivalence of triangulated category $\phi\colon \mathcal{T}_X \simeq \mathcal{T}_Y$ and that the composed functor $\Phi\colon \mathrm{perf}(X) \to \mathrm{perf}(Y)$ is of Fourier–Mukai type. Under these assumptions, all the conditions of Theorem 1.17 are verified. For the verepresentability of X and Y see Clemens and Griffiths [20]. By combining Corollary 6.1 with the (generalized) Torelli theorem (see again Clemens and Griffiths [20]), we hence obtain the following implication:

$$\mathcal{T}_X \simeq \mathcal{T}_Y \text{ and } \Phi \text{ of Fourier–Mukai type} \Rightarrow X \simeq Y. \tag{6.11}$$

The above implication (6.3), (6.5), (6.8), and (6.11) give rise to the following categorical Torelli-type result(s):

Proposition 6.12. *Consider the following three cases*:

(i) *let X and Y be complete intersections of two even-dimensional quadrics, $\mathcal{T}_X := \mathrm{perf}(\mathbb{P}^1, \mathcal{C}_{0,X}) \simeq \mathrm{perf}(C_X)$, and $\mathcal{T}_Y := \mathrm{perf}(\mathbb{P}^1, \mathcal{C}_{0,Y}) \simeq \mathrm{perf}(C_Y)$;*

(ii) *let X and Y be complete intersections of three odd-dimensional quadrics of Fano type, $\mathcal{T}_X := \mathrm{perf}(\mathbb{P}^2, \mathcal{C}_{0,X})$, and $\mathcal{T}_Y := \mathrm{perf}(\mathbb{P}^2, \mathcal{C}_{0,Y})$;*

(iii) *let X and Y be quartic double solids* (*resp. cubic threefolds*) *and $\mathcal{T}_X$ and $\mathcal{T}_Y$ as in* (6.7) (*resp. as in* (6.10)).

In each one of the above cases, the $\mathbb{C}$-schemes X and Y are isomorphic if and only if the dg-categories $\mathcal{T}_X^{\mathrm{dg}}$ and $\mathcal{T}_Y^{\mathrm{dg}}$ are Morita equivalent. In case (i), *$\mathcal{T}_X^{\mathrm{dg}}$ is Morita equivalent to $\mathcal{T}_Y^{\mathrm{dg}}$ if and only if $\mathcal{T}_X \simeq \mathcal{T}_Y$. In cases* (ii)–(iii), *assuming Kuznetsov's Conjecture 1.19, $\mathcal{T}_X^{\mathrm{dg}}$ is Morita equivalent to $\mathcal{T}_Y^{\mathrm{dg}}$ if and only if $\mathcal{T}_X \simeq \mathcal{T}_Y$.*

Proof. As explained in Examples 6.2–6.9, the (generalized) Torelli theorem holds in each one of the cases (i)–(iii). Therefore, the implication ($\Leftarrow$) follows automatically from Corollary 6.1. Let us now prove the converse implication. Suppose that the $\mathbb{C}$-schemes X and Y are isomorphic via an isomorphism $f\colon X \overset{\simeq}{\to} Y$. Clearly, the inverse image functor gives rise to an equivalence $f^*\colon \mathrm{perf}(Y) \overset{\simeq}{\to} \mathrm{perf}(X)$ which is of Fourier–Mukai type with kernel $\mathcal{O}_{\Gamma_f}$, where Γ_f is the graph of f; see Huybrechts [29, Example 5.4], It remains then only to show that f^* restricts to an equivalence $\mathcal{T}_X \simeq \mathcal{T}_Y$. Note that since f^* is of Fourier–Mukai type, the equivalence $\mathcal{T}_X \simeq \mathcal{T}_Y$ will admit a dg enhancement $\mathcal{T}_X^{\mathrm{dg}} \simeq \mathcal{T}_Y^{\mathrm{dg}}$.

In the above case (i) with $d_X = 1$, the $\mathbb{C}$-schemes X and Y are elliptic curves and $C_X \simeq X$ (resp. $C_Y \simeq Y$). It is clear then that f^* yields an equivalence of categories $\mathcal{T}_Y = \mathrm{perf}(C_Y) \simeq \mathrm{perf}(C_X) = \mathcal{T}_X$. In all the remaining cases, X and Y are Fano varieties of Picard rank one and of the same index i. Let $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ be the effective generators of $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$, respectively. Using the fact that $f^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is an isomorphism, one observes that $f^*\mathcal{O}_Y(1)$ is an effective generator and consequently that $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(1)$. Since the triangulated category $\mathcal{T}_X$ (resp. $\mathcal{T}_Y$) is defined as the left orthogonal complement of $\langle \mathcal{O}_X, \dots, \mathcal{O}_X(i-1)\rangle$ (resp. $\langle \mathcal{O}_Y, \dots, \mathcal{O}_Y(i-1)\rangle$, one hence concludes that $\mathcal{T}_X \simeq \mathcal{T}_Y$. □

Remark 6.13. Assuming Kuznetsov's Conjecture 1.19, the above Proposition 6.12 generalizes both implications of the (unconditional) main result of [9].

7. Application II: Kuznetsov's Conjecture on Fano Threefolds

Let $X_{d'}$ and Y_d be two Fano threefolds of Picard number one, indexes 1 and 2, and degrees d' and d, respectively. As proved by Kuznetsov [46, Corollary 3.5 and Lemma 3.6], whenever $d' \equiv 2$ modulo 4, one has the following semi-orthogonal decompositions

$$\mathrm{perf}(X_{d'}) = \langle \mathcal{T}_{X_{d'}}, \langle \mathcal{E}_{X_{d'}}, \mathcal{O}_{X_{d'}}\rangle\rangle, \quad \mathrm{perf}(Y_d) = \langle \mathcal{T}_{Y_d}, \langle \mathcal{O}_{Y_d}, \mathcal{O}_{Y_d}(1)\rangle\rangle,$$

where $\mathcal{E}_{X_{d'}}$ is an exceptional vector bundle of rank 2.

Conjecture 7.1 (Kuznetsov [46, Conjecture 3.7])**.** *Let $\mathcal{MF}_d^i$ be the moduli space of Fano threefolds with Picard number one, index i and degree d. Under these notations, there exists a correspondence $Z_d \subset \mathcal{MF}_d^2 \times \mathcal{MF}_{4d+2}^1$ which is dominant over each factor. Moreover, at each point (Y_d, X_{4d+2}) of this correspondence Z_d one has an equivalence $\mathcal{T}_{X_{4d+2}} \simeq \mathcal{T}_{Y_d}$ of triangulated categories.*

Kuznetsov proved in *loc. cit.* this conjecture for $d = 3, 4, 5$. Making use of Theorem 1.17, we show that the case $d = 2$ is *not* true in the differential graded setting:

Theorem 7.2. *The case $d = 2$ of Conjecture 7.1 does not hold if instead of the equivalence $\mathcal{T}_{X_{4d+2}} \simeq \mathcal{T}_{Y_d}$ one requires that the dg categories $\mathcal{T}^{\mathrm{dg}}_{X_{4d+2}}$ and $\mathcal{T}^{\mathrm{dg}}_{Y_d}$ are Morita equivalent. This implies that Conjecture 7.1 is* not *compatible with Conjecture 1.19.*

Proof. The second claim follows from the combination of Conjecture 1.19 with Lemma 4.4. Let us then prove the first claim.

Recall from Debarre, Iliev and Manivel [23, Section 4.1] that when $d = 2$, Y_2 is a quartic double solid and that the generic X_{10} is a linear section of a quadric hypersurface inside $\mathrm{Gr}(2, \mathbb{C}^5)$. Thanks to the work of Clemens, Iliev, Logachëv and Tihomirov (see [19], [30], [50], [65]), Y_2 and X_{10} have only one non-trivial principally polarized intermediate Jacobian carrying an incidence polarization. Hence, under the assumption that the dg categories $\mathcal{T}^{\mathrm{dg}}_{X_{4d+2}}$ and $\mathcal{T}^{\mathrm{dg}}_{Y_d}$ are Morita equivalent, these schemes satisfy all the assumptions of Theorem 1.17; see Lemma 3.8 and Example 3.9. One would then conclude that $J(X_{10}) \simeq J(Y_2)$ as principally polarized abelian varieties. The key point now is that this isomorphism *does not exist* in full generality. As explained by Debarre, Iliev and Manivel [23, Corollary 5.4], varying X_{10} in $\mathcal{MF}^1_{10}$ gives rise to a 20-dimensional family of intermediate Jacobians $J(X_{10})$; consult also [23, Theorem 5.1]. However, $\mathcal{MF}^2_2$ is only 19-dimensional. Therefore, under the assumption that the dg categories $\mathcal{T}^{\mathrm{dg}}_{X_{4d+2}}$ and $\mathcal{T}^{\mathrm{dg}}_{Y_d}$ are Morita equivalent, one would then conclude that the correspondence Z_2 (whose existence is part of Conjecture 7.1) cannot be dominant onto $\mathcal{MF}^1_{10}$. □

Remark 7.3. As the proof of Theorem 7.2 suggests, the assumption in Conjecture 7.1 that the correspondence Z_d is dominant over each factor should be removed.

8. Application III: Quadric Fibrations and Intersection of Quadrics

Let S be a $\mathbb{C}$-scheme and $Q \to S$ a flat quadric fibration of relative dimension n. Out of this data, we can construct the sheaf $\mathcal{C}_0$ on S of the even parts of Clifford algebras and the derived category $\mathrm{perf}(S, \mathcal{C}_0)$ of perfect complexes of $\mathcal{C}_0$-algebras; consult Kuznetsov [45, Section 3] for details. As proved by Kuznetsov [45, Theorem 4.2], we have the following semi-orthogonal decomposition

$$\mathrm{perf}(Q) = \langle \mathrm{perf}(S, \mathcal{C}_0), \langle \underbrace{\mathrm{perf}(S), \ldots, \mathrm{perf}(S)}_{n\text{-factors}} \rangle \rangle. \tag{8.1}$$

Now, let T be a $\mathbb{C}$-scheme, $X \to T$ a generic relative complete intersection of $r+1$ quadric hypersurfaces of dimension n, and $Q \to S$ the associated linear span; consult [3]. As proved in *loc. cit.*, the following holds:

(i) when $2r < n$, the fibers of X are of Fano type and, thanks to homological projective duality, we have the following semi-orthogonal decomposition
$$\mathrm{perf}(X) = \langle \mathrm{perf}(S, \mathcal{C}_0), \langle \underbrace{\mathrm{perf}(T), \ldots, \mathrm{perf}(T)}_{(n-2r)\text{-factors}} \rangle \rangle\,; \tag{8.2}$$

(ii) when $2r = n$, the fibers of X are generically of Calabi–Yau type and, thanks once again to homological projective duality, we have $\mathrm{perf}(X) \simeq \mathrm{perf}(S, \mathcal{C}_0)$;

(iii) when $2r > n$, the fibers of X are generically of general type and there exists a fully faithful functor $\mathrm{perf}(X) \hookrightarrow \mathrm{perf}(S, \mathcal{C}_0)$.

In what follows, we will treat separately the cases (i)–(ii) from (iii). In cases (i)–(ii), the orthogonal complement of $\mathrm{perf}(S, \mathcal{C}_0)$ is well understood. Moreover, all the known examples where X and Q are verepresentable are of this form; see Section 8.3. In case (iii), the orthogonal complement of $\mathrm{perf}(S, \mathcal{C}_0)$ in $\mathrm{perf}(X)$ is less well-understood, and we will only briefly study it in Section 8.4.

8.1. Reduction by hyperbolic splitting. Let S be a $\mathbb{C}$-scheme and $Q \to S$ a flat quadric fibration of relative dimension $n \geqslant 2$. Whenever such fibration admits a *regular section*, i.e., a section cutting a regular point at each fiber, one can perform its *reduction by hyperbolic splitting* (along this regular section); consult [3, Section 1] for details. Roughly speaking, one takes the base of the cone given (fiberwise) by the intersection of the quadric with the tangent space at the section. We obtain in this way another quadric fibration $Q' \to S$ of relative dimension $n-2$, and another sheaf $\mathcal{C}'_0$ on S of the even parts of Clifford algebras. As proved in [3, Theorem 1.26], one has an equivalence of categories $\mathrm{perf}(S, \mathcal{C}_0) \simeq \mathrm{perf}(S, \mathcal{C}'_0)$ (given by a Fourier–Mukai functor) whenever $Q \to S$ is a generic quadric fibration and Q is smooth. Making use of Theorem 1.4 one obtains the following (conditional) result:

Corollary 8.3. *Let $Q \to S$ and $Q' \to S$ be quadric fibrations as above. Assume that the bilinear pairings (1.3) (associated to Q and Q') are non-degenerate and that $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(S)) = 0$. Under these assumptions, there exists an isomorphism $\tau\colon \prod_{i=0}^{d_Q-1} J_a^i(Q) \to \prod_{i=0}^{d_{Q'}-1} J_a^i(Q')$ of abelian varieties up to isogeny.*

Proof. Thanks to the work of Kuznetsov and others (see [47, Theorem 7.1][45] and [3]), the composed functor $\Phi\colon \mathrm{perf}(Q) \to \mathrm{perf}(S, \mathcal{C}_0) \simeq \mathrm{perf}(S, \mathcal{C}'_0) \hookrightarrow \mathrm{perf}(Q')$ is of Fourier–Mukai type. Moreover, we have the semi-orthogonal decompositions

$$\mathrm{perf}(Q) = \langle \mathrm{perf}(S, \mathcal{C}_0), \underbrace{\langle \mathrm{perf}(S), \ldots, \mathrm{perf}(S)\rangle}_{n\text{-factors}} \rangle,$$

$$\mathrm{perf}(Q') = \langle \mathrm{perf}(S, \mathcal{C}'_0), \underbrace{\langle \mathrm{perf}(S), \ldots, \mathrm{perf}(S)\rangle}_{(n-2)\text{-factors}} \rangle.$$

Therefore, the proof follows from the combination of Theorem 1.4 with the assumption $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(S)) = 0$. ☐

Example 8.4. The bilinear pairings (1.3) (associated to Q) are known to be non-degenerate whenever $\dim(S) \leqslant 2$; see Vial [68, Theorem 7.4]. When $\dim(S) \leqslant 2$ and S is the projective space or a rational surface, the triangulated category $\mathrm{perf}(S)$ admits a full exceptional collection; see [6, Section 4-4.1]. Consequently, $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(S)) = 0$.

By combining Theorem 1.17 and Corollary 8.3, one then obtains the following result (whose applications will be discussed in Section 8.3):

Corollary 8.5. *Let $Q \to S$ and $Q' \to S$ be quadric fibrations as in Corollary 8.3 with $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(S)) = 0$. Assume moreover that Q and Q' are verepresentable. Under these extra assumptions, $J(Q) \simeq J(Q')$ as principally polarized abelian varieties.*

8.2. Complete intersections of Fano and Calabi–Yau type. Let T be a $\mathbb{C}$-scheme, $X \to T$ a generic relative complete intersection of $r+1$ quadric hypersurfaces of dimension n, and $Q \to S$ the associated linear span; see [3]. Suppose that $2r \leqslant n$. This implies that the fibers of X are either of Fano ($2r < n$) or Calabi–Yau ($2r = n$) type. By combining (relative) homological projective duality with Theorem 1.4 one obtains the following (conditional) result:

Corollary 8.6. *Let $X \to T$ be a relative complete intersection as above* (*with* $2r \leqslant n$) *and $Q \to S$ the associated linear span. Assume that the bilinear pairings* (1.3) (*associated to X and Q*) *are non-degenerate, and that $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(T)) = 0$. Under these assumptions, there exists an isomorphism $\tau\colon \prod_{i=0}^{d_X-1} J_a^i(X) \to \prod_{i=0}^{d_Q-1} J_a^i(Q)$ of abelian varieties up to isogeny.*

Proof. Thanks to relative homological projective duality (see [3] and Kuznetsov [44, Section 6.1]), the composed functor $\Phi\colon \mathrm{perf}(X) \to \mathrm{perf}(S, \mathcal{C}_0) \simeq \mathrm{perf}(S, \mathcal{C}_0) \hookrightarrow \mathrm{perf}(Q)$ is of Fourier–Mukai type. By applying the Jacobian functor $\mathbf{J}(-)$ to the (unique) differential graded enhancement of the above semi-orthogonal decompositions (8.1)-(8.2), one obtains the following isomorphisms of abelian varieties up to isogeny:

$$\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(Q)) \simeq \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(S, \mathcal{C}_0)) \times \prod_{i=0}^{n} \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(S)),$$
$$\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(X)) \simeq \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(S, \mathcal{C}_0)) \times \prod_{i=0}^{n-2r} \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(T)).$$

Since $S \to T$ is a $\mathbb{P}^r$-bundle, the triangulated category $\mathrm{perf}(S)$ admits a semi-orthogonal decomposition $\langle \mathrm{perf}(T), \ldots, \mathrm{perf}(T)\rangle$; see Orlov [58]. Since by hypothesis $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(T)) = 0$, we conclude that $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(Q)) \simeq \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(X))$. The proof follows now from Theorem 1.4. ∎

Example 8.7. The bilinear pairings (1.3) (associated to X and Q) are known to be non-degenerate when $r = 0$ and $\dim(S) \leqslant 2$ (see Vial [68, Theorem 7.4]), when $r = 1$, $\dim(S) \leqslant 2$, and $d_X \leqslant 6$ (see Vial [68, Theorem 7.6]), and when r is arbitrary and S is a point. Note that since $S \to T$ is a projective bundle, $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(T)) = 0 \Rightarrow \mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(S)) = 0$.

8.3. Verepresentability of quadrics, intersections, and their fibrations. Let $X \to T$ be a fibration in complete intersections of $r+1$ quadrics of relative dimension n. In the following three cases X is verepresentable; see Examples 3.9–3.10 also.

(i) ($T = $ point). In this case X is the intersection of $r+1$ quadric hypersurfaces in $\mathbb{P}^{n+1}$. When $r = 0$, X is just a quadric. Hence, it is known to be verepresentable for all n. When $r = 1$, X is verepresentable for n even (see [10, Theorem 1.5] and Reid [61]). When $r = 2$, X is verepresentable for $n > 3$ odd (see [10, Theorem 1.5] and Beauville [4, Section 6]). When $r = n \geqslant 3$, X is a curve of genus $g > 1$ in $\mathbb{P}^{n+1}$. Therefore, it is verepresentable.

(ii) ($T = \mathbb{P}^1$). When $r = 0$, $X \to \mathbb{P}^1$ is a quadric fibration. If n is odd, then X is verepresentable; see Vial [68, Section 4]. If n is even, then

X is also verepresentable. This follows from the combination of Vial's motivic description [68, Section 4] with Reid's work [61] on the intermediate Jacobian. When $r = 1$ and $n = 3$, $X \to \mathbb{P}^1$ is a fibration in del Pezzo surfaces of degree 4. Thanks to the work of Gorchinskiy and Guletskii [25] and Kanev [36], X is also verepresentable.

(iii) (T = rational surface). When $r = 0$ and $n = 1$, X is a conic bundle over T. Hence, it is verepresentable; see Beauville and others [4], [5], [7]. When $r = 0$, n is odd, and $T = \mathbb{P}^2$, X is also verepresentable; see Beauville [4]. One could fairly expect that a similar proof should work for every rational surface.

Now, let $Q \to S$ be the linear span associated to the above fibration $X \to T$. Assume that $r \neq 0$ (otherwise $X \simeq Q$). In the following cases Q is verepresentable:

(i′) (T = point, $r = 1$, n even). In this case, X is the intersection of two even-dimensional quadrics and $Q \to \mathbb{P}^1$ is an even-dimensional fibration. Therefore, making use of the above item (ii), one concludes that Q is verepresentable.

(ii′) (T = point, $r = 2$, n odd). In this case, X is the intersection of three odd-dimensional quadrics and $Q \to \mathbb{P}^2$ is an odd-dimensional fibration. Therefore, making use of the above item (iii), one concludes that Q is verepresentable.

(iii′) ($T = \mathbb{P}^1$, $r = 1$, $n = 3$). In this case $X \to \mathbb{P}^1$ is a del Pezzo fibration of degree 4 and $Q \to S$ is a quadric fibration over a Hirzebruch surface. Therefore, making use of the above (conditional) item (iii), one concludes that Q is verepresentable.

By combining Corollary 8.6 with Theorem 1.17 one obtains the following (conditional) result:

Corollary 8.8. *Let $X \to T$ and $Q \to S$ be as in Corollary 8.6 with $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(T)) = 0$. Assume moreover that X and Q are verepresentable. This holds for example in the above cases* (i′)–(iii′). *Under these assumptions, $J(X) \simeq J(Q)$ as principally polarized abelian varieties.*

Suppose now that $Q \to S$ admits a regular section. Let us write $Q' \to S$ for its reduction by hyperbolic splitting. If Q belongs to one of the above cases (i′)–(iii′), then so does Q'. Moreover, when $S = \mathbb{P}^1$ and the fiber is of Fano type, such a section exists thanks to the work of Graber, Harris and Starr; see [26].

Corollary 8.9. *Let $X \to T$ and $Q \to S$ be as in Corollary 8.6 with $\mathbf{J}(\mathrm{perf}^{\mathrm{dg}}(T)) = 0$. Assume that X and Q are verepresentable. Assume moreover that $Q \to S$ admits a regular section, and let $Q' \to S$ be its reduction by hyperbolic splitting. Under these assumptions, $J(X) \simeq J(Q')$ as principally polarized abelian varieties.*

Example 8.10. When $T = \mathbb{P}^1$, consider $X \to \mathbb{P}^1$ as a del Pezzo fibration of degree 4, i.e., $r = 1$ and $n = 3$. Let $Y \to S$ be the conic bundle (birational to X) introduced by Alexeev [1]. Using Corollary 8.9 we hence recover the isomorphism $J(X) \simeq J(Q')$ of principally polarized abelian varieties described by Alexeev in [1].

8.4. Complete intersections of general type. Let T be a $\mathbb{C}$-scheme, $X \to T$ a generic relative complete intersection of $r+1$ quadric hypersurfaces of dimension n, and $Q \to S$ the associated linear span; consult [3] for details. Assume that $2r > n$. In this case the fibers of X are of general type. Recall from above that we have a fully faithful functor $\mathrm{perf}(X) \to \mathrm{perf}(S, \mathcal{C}_{0,X})$ of Fourier–Mukai type. Thanks to Kuznetsov and others (see [3], [45]) the orthogonal complement can be described in terms of higher Clifford modules. By combining (relative) homological projective duality results with Theorem 1.4 one obtains the following (conditional) result:

Corollary 8.11. *Let $X \to T$ be a relative complete intersection as above (with $2r > n$) and $Q \to S$ the associated linear span. Assume that the bilinear pairings (1.3) (associated to X and Q) are non-degenerate. Under these assumptions, there exists a split injective morphism $\tau\colon \prod_{i=0}^{d_X-1} J_a^i(X) \to \prod_{i=0}^{d_Q-1} J_a^i(Q)$ of abelian varieties up to isogeny.*

Proof. The composition $\Phi\colon \mathrm{perf}(X) \simeq \mathrm{perf}(S, \mathcal{C}_0) \hookrightarrow \mathrm{perf}(Q)$ is fully faithful. Therefore, it is of Fourier–Mukai type; see Orlov [59]. Let $\mathcal{T}_X := \mathrm{perf}(X)$. Since the orthogonal complement of $\mathcal{T}_X$ is trivial, the proof follows then from items (i)–(ii) of Theorem 1.4. □

To the best of the authors' knowledge, X is known to be verepresentable only when T is a point and $r = n - 1 \geqslant 2$. In this particular case, X is curve (obtained as a complete intersection of $r+1$ quadrics in $\mathbb{P}^{r+2}$) and $Q \to \mathbb{P}^r$ is a quadric fibration of relative dimension $r+1$. In contrast with X, very little is known about Q. For instance, it is neither known if Q is verepresentable nor whether the bilinear pairings (1.3) are non-degenerate. Nevertheless, by combining Kuznetsov's results [45, Theorems 4.2 and 5.5] with the fact that $\mathrm{perf}(\mathbb{P}^r)$ is generated by exceptional objects, we obtain the following semi-orthogonal decomposition

$$\mathrm{perf}(Q) = \langle \mathcal{C}_{-r+1}, \ldots \mathcal{C}_{-1}, \mathrm{perf}(X), E_1, \ldots, E_{(r+1)^2} \rangle,$$

where the E_i's are exceptional objects and the $\mathcal{C}_{-i}$'s are the Clifford modules introduced by Kuznetsov in [45, Section 3]. By combining Corollary 8.11 with Theorem 1.17 one obtains the following (conditional) result:

Corollary 8.12. *Let X be a curve, complete intersection of $r+1$ quadrics in $\mathbb{P}^{r+1}$, and $Q \to \mathbb{P}^r$ the associated linear span. Assume that the pairings (1.3) (associated to Q) are non-degenerate and that the modules $\mathcal{C}_{-i}$ are exceptional objects in $\mathrm{perf}(\mathbb{P}^r, \mathcal{C}_0)$. Under these assumptions, there is a morphism $\tau\colon J(X) \to \prod_{i=0}^{d_Q-1} J_a^i(Q)$ of abelian varieties up to isogeny. Assume moreover that Q is verepresentable. Under this extra assumption, the morphism $\tau\colon J(X) \to J(Q)$ preserves the principal polarization. Furthermore, if the modules $\mathcal{C}_{-i}$ are exceptional objects in $\mathrm{perf}(\mathbb{P}^r, \mathcal{C}_0)$, then τ becomes an isomorphism $J(X) \simeq J(Q)$ of principally polarized abelian varieties.*

9. Application IV: Blow-Ups

Theorem 1.17 gives rise to the following (conditional) result:

Corollary 9.1. *Let X be a verepresentable $\mathbb{C}$-scheme for which the bilinear pairings (1.3) are nondegenerate, $Z \subset X$ a verepresentable subscheme of codimension 2 (resp. a smooth subscheme such that $\mathbf{J}(Z) = 0$), and $Y \to X$ be the blow-up of X along Z. If Y is verepresentable, then $J(Y) \simeq J(X) \oplus J(Z)$ (resp. $J(Y) \simeq J(X)$) as principally polarized abelian varieties.*

Proof. If $Y \to X$ is the blow-up of X along a smooth subscheme $Z \subset X$ of codimension d, then we have a semi-orthogonal decomposition (see Orlov [58])

$$\mathrm{perf}(Y) = \langle \mathrm{perf}(X), \mathrm{perf}(Z)_1, \ldots, \mathrm{perf}(Z)_{d-1} \rangle, \tag{9.2}$$

where $\mathrm{perf}(Z)_i$ is equivalent to $\mathrm{perf}(Z)$ for all i. Let $\mathcal{T}_X := \mathrm{perf}(X)$. If $\mathbf{J}(Z) = 0$, then the isomorphism $J(Y) \simeq J(X)$ of principally polarized abelian varieties follows from Theorem 1.17. Let us then assume that Z is verepresentable of codimension 2. By applying Theorem 1.17 first to $\mathcal{T}_X := \mathrm{perf}(X)$ and then to $\mathcal{T}_Z := \mathrm{perf}(Z)$, one obtains an injective morphism $J(X) \oplus J(Z) \hookrightarrow J(Y)$ preserving the principal polarizations. This gives rise to a decomposition $J(Y) = J(X) \oplus J(Z) \oplus A$ for some abelian variety A. We need to show that $A = 0$. By applying the functor $\mathbf{J}(-)$ to the semi-orthogonal decomposition (9.2), we obtain an isogeny $\prod_{i=0}^{d_Y -1} J_a^i(Y) = J(Y) \simeq J(X) \oplus J(Z)$. This implies that A is isogenous to 0 and consequently that $A = 0$. ☐

Remark 9.3. Let X be a verepresentable threefold and Z either a smooth curve or a set of points. If the blow-up Y is verepresentable, then Corollary 9.1 give an alternative (conditional) proof of a classical result of Clemens and Griffiths [20, Section 3]. This result allowed Clemens and Griffiths to construct a birational invariant of X, namely the "Clemens–Griffiths component".

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