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Semiorthogonal decompositions and noncommutative motives in Algebraic Geometry

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Semiorthogonal decompositions and noncommutative motives in algebraic geometry

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Introduction

A straight, fit-in-one-line definition of Algebraic Geometry would say that it deals with the study of Algebraic Varieties. The latter could be defined as the zero loci of polynomial equations over some field of coefficients k. However, since Grothendieck's fundamental work, Algebraic Geometry has grown far beyond this first definition.

In particular, the theory of schemes and algebraic stacks has allowed algebraic geometers to have a much more general description of the geometric framework to their research. One of the greatest effort of algebraic geometers is to create and/or adapt new languages and theories to understand, explain, and define the geometrical objects under examination. The theory of motives is one of the examples that most fit these considerations.

One of the main tasks of Algebraic Geometry is classifying varieties. This amounts to ask questions whose nature is, roughly, "when are two such varieties equivalent?". The most natural equivalence relation is certainly isomorphism. However, the notion of isomorphism is a very strict equivalence relation. A weaker notion is based on the Zariski topology that one can consider on a variety. Roughly speaking, the closed subsets of this topology are (countable unions of) subvarieties of positive codimension; so that the open sets are very "large". Two varieties are said to be birational to each other if they are isomorphic along an open Zariski subset. The classification of birational classes of algebraic varieties is one of the main tasks of Algebraic Geometry since the beginning of last century. A great effort has been produced ever since to construct birational invariants, that is, algebraic structures (cohomologies, Hodge theory, Abelian varieties, Chow groups, just to cite a few) associated to a variety and which are isomorphic for two birationally equivalent varieties. One of the questions one could ask is to determine criteria for a given variety X to be rational, that is, birational to the projective space. In this cases, birational invariants could be used to describe an obstruction to rationality, that is, to exclude the possibility for X to be rational. In this report, we would like to focus on invariants which arise from the study of complexes of coherent sheaves on a given variety, using tools such as homological algebra and category theory. The main constructions considered are noncommutative schemes, their semiorthogonal decompositions, and noncommutative motives.

The theory of exceptional objects and semiorthogonal decompositions started with the Rudakov seminar in the late 80's in Moscow [**hel90**]. One of the main ideas was to use homological algebra, and hence the bounded derived category $\mathrm{D^b}(X)$ of coherent sheaves on a variety X, to study the properties of vector and line bundles and their relation with the geometry of X. These considerations rapidly turned into a much more general theory. Notice that, if X is defined over a field k, the category $\mathrm{D^b}(X)$ has a triangulated k-linear structure. In particular, there is an autoequivalence [1] which correspond to the right shift by one of all the cohomologies of a complex. Considering two line bundles L and M as complexes concentrated in degree 0, we have $\mathrm{Hom}_{\mathrm{D^b}(X)}(L, M[i]) \simeq \mathrm{Ext}^i(L, M)$. The latter is known to be isomorphic to the cohomology $H^i(X, L \otimes M^\vee)$.

For example, consider X a smooth and projective variety. Given any line bundle L on X, we can consider the functor $\Phi : \mathrm{D^b}(k) \to \mathrm{D^b}(X)$ which associates to a bounded complex $C^\bullet$ of finite-dimensional k-vector spaces the bounded complex $\Phi(C^\bullet) := C^\bullet \otimes L$ of vector bundles on L. If we assume $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i \neq 0$, then we have that

$$(1) \qquad \mathrm{Hom}_{\mathrm{D^b}(X)}(L, L) = k, \quad \text{and} \quad \mathrm{Hom}_{\mathrm{D^b}(X)}(L, L[i]) = 0 \text{ for any } i \neq 0.$$

The latter properties ensure that the functor Φ has a right adjoint, namely the functor $\Psi : \mathrm{D^b}(X) \to \mathrm{D^b}(k)$ defined by $\Psi(A^\bullet) = \mathrm{Hom}_{\mathrm{D^b}(X)}(L, A^\bullet)$ for any bounded complex $A^\bullet$ of coherent sheaves. Similarly, there is a left adjoint to Φ. We denote by $\langle L \rangle$ the essential image of Φ in $\mathrm{D^b}(X)$ and note that this is the smallest full triangulated thick subcategory of $\mathrm{D^b}(X)$ containing L.

All the above considerations are purely of homological nature, and can be extended to any variety X. An exceptional object E on X is a bounded complex of coherent sheaves (that is, an object in $\mathrm{D^b}(X)$) satisfying the same homological properties (1) which were satisfied by the line bundle L in the previous example. Recalling that $\mathrm{D^b}(\mathrm{Spec}(k)) \simeq \mathrm{D^b}(k)$, any exceptional object gives a fully faithful functor $\mathrm{D^b}(\mathrm{Spec}(k)) \to \mathrm{D^b}(X)$ admitting right and left adjoints. In general, given a full triangulated thick subcategory $\mathbf{A}$ of $\mathrm{D^b}(X)$, we say that $\mathbf{A}$ is admissible if the embedding functor $\mathbf{A} \to \mathrm{D^b}(X)$ admits right and left adjoints.

An ordered pair of admissible subcategories $(\mathbf{A}, \mathbf{B})$ of $\mathrm{D^b}(X)$ is said to be semiorthogonal if $\mathrm{Hom}_{\mathrm{D^b}(X)}(B, A) = 0$ for any object B of $\mathbf{B}$ and A of $\mathbf{A}$. Similarly we define a semiorthogonal collection of admissible subcategories $(\mathbf{A}_1, \ldots, \mathbf{A}_r)$. For example, consider $X = \mathbb{P}^n$ and the exceptional objects given by two line bundles $\mathcal{O}(i)$ and $\mathcal{O}(j)$. The admissible subcategories $\langle \mathcal{O}(i) \rangle$ and $\langle \mathcal{O}(j) \rangle$ consist of complexes of the form $C^\bullet \otimes \mathcal{O}(i)$ and $C^\bullet \otimes \mathcal{O}(j)$, respectively, where $C^\bullet$ is a bounded complex of finite dimensional k-vector spaces. An easy calculation in cohomology shows that $(\langle \mathcal{O}(i) \rangle, \langle \mathcal{O}(j) \rangle)$ are semiorthogonal if and only if $0 < j - i < n + 1$. It follows for example that $(\mathcal{O}, \ldots, \mathcal{O}(n))$ is a semiorthogonal collection of subcategories, all equivalent to $\mathrm{D^b}(\mathrm{Spec}(k))$, where, by abuse of notations, we dropped the brackets.

Finally, a semiorthogonal collection $(\mathbf{A}_1, \ldots, \mathbf{A}_r)$ in $\mathrm{D^b}(X)$ is a semiorthogonal decomposition, denoted by

$$\mathrm{D^b}(X) = \langle \mathbf{A}_1, \ldots, \mathbf{A}_r \rangle,$$

if, roughly speaking, $\mathrm{D^b}(X)$ is the smallest triangulated full thick subcategory of $\mathrm{D^b}(X)$ containing all the $\mathbf{A}_i$'s. More precisely, we require for any object T of $\mathrm{D^b}(X)$ the existence of a filtration of T by a complex $0 = T_0 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_r} T_r = T$ such that the cone of α_i is in $\mathbf{A}_i$. A first example of a semiorthogonal decomposition is provided by the collection $(\mathcal{O}, \ldots, \mathcal{O}(n))$ on the projective space $\mathbb{P}^n$. To show that this semiorthogonal collection is a semiorthogonal decomposition of $\mathrm{D^b}(\mathbb{P}^n)$ one needs some more delicate calculations, based on a spectral sequence. This amounts to show that if an object A satisfies (right or left) orthogonality to the collection $(\mathcal{O}, \ldots, \mathcal{O}(n))$, then $A = 0$. Indeed, in general, given an exceptional collection $(E_1, \ldots, E_r)$ on a smooth projective variety X, one has a semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathbf{A}, E_1, \ldots, E_r \rangle,$$

where $\mathbf{A}$ is the full triangulated subcategory of all objects satisfying right orthogonality with respect to the E_i's. The previous semiorthogonal decomposition should be thought of as a "noncommutative splitting" of simple categories off X.

The interest on semiorthogonal decompositions grew thanks, among others, to the work of Bondal and Orlov, which is resumed in their 2002 ICM address [**BO02**]. One of the striking features is that semiorthogonal decompositions tend to have a motivic behavior, under two points of view. First of all, if X and Y are smooth and projective and $\Phi : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ is full and faithful, we have that $\Phi(\mathrm{D^b}(X))$ is admissible in $\mathrm{D^b}(Y)$ and that the functor Φ is represented by an object P in $\mathrm{D^b}(X \times Y)$. Such functor is defined indeed as $\Phi(-) = Rp_*(q^*(-) \otimes P)$, where p and q are the projections from $X \times Y$ to Y and X respectively; it deserves the name of Fourier–Mukai functor, since it was first introduced by Mukai [**Muk81**]. One should think of Fourier–Mukai functors, and their composition thereof, as algebro-geometric correspondences between triangulated categories. Secondly, projective bundles and blow-ups have semiorthogonal decompositions which looks like their motivic analog: the derived category of a projective bundle of rank r is decomposed by $r+1$ copies of the derived category of the base, and the blow up of a variety along a codimension c subvariety is decomposed by one copy of the base and $c-1$ copies of the blown-up locus. This behavior, together with a growing amount of examples of decompositions of Fano 3-folds, and the remarkable results of Kuznetsov, lead to think that semiorthogonal decompositions should detect birational properties of a given variety. More than ten years after Bondal and Orlov's ICM address, Kuznetsov's 2014 ICM address witnesses the remarkable growth and richness of this subject [**Kuz14**].

Inspired by this guess, many surprising and interesting phenomena were discovered in the recent years: there exist functors between derived categories of smooth projective varieties which are not of Fourier–Mukai type [**RVdB14**], the Jordan–Hölder property does not hold in general for semiorthogonal decompositions [**BGvBS14**], and there exists nontrivial categories with trivial invariants [**BGvBKS15**], so that we do not know whether semiorthogonal decomposition satisfy to any Noetherianity property. On the other hand, a growing amount of examples and constructions were carried over to study semiorthogonal decompositions. Above all, the theory of Homological Projective Duality developed by Kuznetsov [**Kuz07a**] builds a striking and strong relation between semiorthogonal decompositions of (linear sections of) classically projective dual varieties, showing a very deep interconnection between the (projective) geometry of a variety X and the semiorthogonal decompositions of its derived category.

Inspired, among others, by Kontsevich's ideas [**Kon05**], one should think of $\mathrm{D^b}(X)$ as a *noncommutative scheme* associated to X. This justifies a theory of noncommutative motives, where semiorthogonal decompositions would play the role of motivic decompositions.

However, thinking of $\mathrm{D}^{\mathrm{b}}(X)$ as a triangulated category rises many problems: above all, the fact that a general triangulated functor is not of Fourier–Mukai type. Instead of a k-linear triangulated category, one should consider $\mathrm{D}^{\mathrm{b}}(X)$ as a differential graded (dg) category over k, that is, a category whose morphisms form graded differential complexes of k-vector spaces. In this optic, the k-linear structure of $\mathrm{D}^{\mathrm{b}}(X)$ is obtained by taking zeroth global sections of morphism complexes.

As shown by Lunts and Orlov [**LO10**], in the case where X is quasi separated and quasi compact there is a unique dg structure on the subcategory $\mathrm{perf}(X) \subset \mathrm{D}^{\mathrm{b}}(X)$ of perfect complexes, that is, the category of bounded complexes of vector bundles (which coincides with $\mathrm{D}^{\mathrm{b}}(X)$ if X is smooth and projective). This structure is obtained by giving a *compact generator* E of $\mathrm{perf}(X)$, that is a perfect complex such that any complex right (or left) orthogonal to E is trivial, and taking the endomorphism complex $R := \mathrm{End}(E)$, endowed with a dg algebra structure by the choice of an enhancement (for example, by taking injective resolutions of complexes), so that $\mathrm{perf}(X)$ is equivalent to the dg category of perfect complexes $\mathrm{perf}(R)$. Let us give a first example: suppose that X is smooth and projective and has a full exceptional sequence $(E_1, \ldots, E_r)$, that is this exceptional sequence is a semiorthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(X)$. Then $E := \oplus_{i=1}^r E_i$ is a compact generator.

It is then natural to define, as done by Orlov in [**Orl14**], a noncommutative scheme $\mathcal{A}$ to be a dg category of the form $\mathrm{perf}(R)$ for some cohomologically bounded dg algebra R over k. If X is a scheme (quasi compact and quasi separated) we then have a unique noncommutative scheme associated to it, that is the category $\mathrm{perf}(X)$. Moreover, one can define regularity, smoothness and properness for a noncommutative scheme $\mathcal{A}$, and these properties match those of X in the case $\mathcal{A} = \mathrm{perf}(X)$. One can also define *noncommutative resolution of singularities* of any noncommutative scheme. If X is separated of finite type and k has characteristic zero, Kuznetsov and Lunts show that $\mathrm{perf}(X)$ always have such a resolution [**KL12**]. From this point of view, schemes and (birational) morphisms can be replaced by noncommutative schemes and dg-functors.

Moreover, a triangulated k-linear functor $\Phi : \mathrm{perf}(X) \to \mathrm{perf}(Y)$ is of Fourier–Mukai type if and only if there exists a lifting of Φ into a dg functor, as shown by Toën [**Toë07**]. This means that working with noncommutative schemes and dg functors between them is the natural context to work with derived categories and Fourier–Mukai functors. Moreover, this fact has deep consequences on the construction of noncommutative motives, as expected by Kontsevich. Indeed, thanks to the work of Tabuada (see the recent book [**Tab15**]) one can define the category $\mathrm{NChow}(k)$ of noncommutative Chow motives over k. The objects of such category are the Morita-equivalence classes of smooth and proper dg categories and morphism from $\mathcal{A}$ to $\mathcal{B}$ are, roughly speaking, projections as correspondences in $K_0(\mathcal{A} \otimes \mathcal{B}^{op})$. Considering $\mathrm{perf}(X)$ gives a functor from smooth and projective k-varieties to noncommutative motives. The noncommutative motivic correspondences between two such varieties X and Y are then induced by Fourier–Mukai transforms and semiorthogonal decompositions. Better, one can think to the noncommutative motive to be the motive of the noncommutative scheme $\mathrm{perf}(X)$ in a very deep way: a semiorthogonal decomposition (i.e., a projector in the Grothendieck group) induces a splitting on the motivic level, and such a motive is universal with respect to any additive invariant. Notice that many of these additive invariants, defined in the world of dg categories, are a noncommutative version of well-known cohomological theories, such as Betti, Hodge or de Rham cohomology.

Finally, let us mention that noncommutative Chow motives are related to Grothendieck's Chow motives, at least when one considers rational coefficients, thanks to the Grothendieck-Riemann-Roch theorem. The main difference is that, under a noncommutative point of view, the Lefschetz motive is isomorphic to the unit motive. That is, noncommutative schemes "lose" information on the codimension. On the other hand, there are purely noncommutative construction that are invisible in the commutative world.

We can resume the above considerations on noncommutative schemes in three main points. We let X be a scheme and $\mathcal{A}_X := \mathrm{perf}(X)$ be the associated noncommutative scheme.

- To any k-scheme X we associate $\mathcal{A}_X$ in the category $\mathrm{NSch}(k)$ of noncommutative k-schemes. Morphisms from $\mathcal{A}_X$ to $\mathcal{A}_Y$ are exactly Fourier–Mukai functors, that is, noncommutative geometric correspondences between X and Y. There is a theory of resolution of singularities (which always exist if k has characteristic zero and X is separated of finite type), and we can work in a relative context, replacing the base field k by any noncommutative scheme $\mathcal{B}$.
- Under this point of view, $K_0(\mathcal{A}_X)$ plays the role of the (ungraded) Chow ring. There are noncommutative additive invariants which play the role of known cohomological theories, up to lose the information on the grading. For example, $\mathbb{Z}/2\mathbb{Z}$-graded deRham cohomology and vertically-graded Hodge cohomology of a smooth and projective X are isomorphic to period-cyclic homology and Hochschild homology of $\mathcal{A}_X$ respectively.
- Additive invariants behave well with respect to semiorthogonal decompositions. These decomposition can provide geometric noncommutative schemes not of the form $\mathcal{A}_X$. Moreover, there is a theory of noncommutative Chow, smash-nilpotent, homological and numerical motives which play the noncommutative role of commutative Chow, smash-nilpotent, homological and numerical motives. Noncommutative motives are split by semiorthogonal decompositions.

The aim of this habilitation work is to detail results where the interplay between noncommutative and commutative motives, semiorthogonal decompositions and birational properties reveal some interesting and motivating research direction. This is done first on a purely conceptual level, by introducing some definition and giving some general statement, but, above all, working on explicit examples and questioning coming from Algebraic Geometry, such that the construction of semiorthogonal decompositions, the study of algebraic cycles and the quest for birational invariants.

Homological Projective Duality. As mentioned above, one of the most powerful tool to construct geometrically relevant semiorthogonal decompositions is Kuznetsov's Homological Projective Duality (HPD for short), developed in [**Kuz07a**]. Roughly speaking, if $f : X \to \mathbb{P}(W)$ is a projective variety with a polarization $\mathcal{O}(1) := f^*\mathcal{O}_{\mathbb{P}(W)}(1)$, we have to consider semiorthogonal decompositions of X which are well-behaved with respect to taking hyperplane sections. This leads to the definition of a *Lefschetz decomposition*:

$$\mathrm{D^b}(X) = \langle \mathbf{A}_0, \mathbf{A}_1(1), \ldots, \mathbf{A}_{i-1}(i-1) \rangle,$$

where $\mathbf{A}_{i-1} \subset \ldots \subset \mathbf{A}_0$ is a sequence of admissible subcategories and $\mathbf{A}_j(j) := \mathbf{A}_j \otimes \mathcal{O}(j)$. If X_H is a hyperplane section of X, the Lefschetz decomposition restricts to a decomposition:

$$\mathrm{D^b}(X_H) = \langle \mathbf{C}_H, \mathbf{A}_1(1)_{|X_H}, \ldots, \mathbf{A}_{i-1}(i-1)_{|X_H} \rangle,$$

where $\mathbf{A}_{|X_H}$ denotes the pull-back of a subcategory $\mathbf{A}$ of $\mathrm{D^b}(X)$ to $\mathrm{D^b}(X_H)$ via the embedding $X_H \subset X$. We then obtain a category $\mathbf{C}_H$ for any hyperplane section of X.

These remarks work in a more general framework, which can be easily obtained by performing Kuznetsov's constructions in the language of noncommutative schemes. Indeed, if $\mathcal{A}$ is a noncommutative X-scheme (that is, a noncommutative k-scheme enriched over the dg category $\mathrm{D^b}(X)$), the invertible object $\mathcal{O}(1)$ in $\mathrm{D^b}(X)$ induces a dg-autoequivalence $\otimes\mathcal{O}(1) : \mathcal{A} \to \mathcal{A}$. Then we still can define a Lefschetz decomposition

$$\mathcal{A} = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_{i-1}(i-1) \rangle,$$

with respect to $\mathcal{O}(1)$ and the induced decomposition of $\mathcal{A}_H := \iota^*\mathcal{A}$, where $\iota : X_H \to X$ is the immersion of the hyperplane section. Set $\mathscr{X}_1 \subset X \times \mathbb{P}(W^\vee)$ to be the universal hyperplane section, and $\mathcal{A}_{\mathscr{X}_1} = \mathcal{A} \boxtimes \mathrm{D^b}(\mathbb{P}(W^\vee))$ the base change of $\mathcal{A}$ to $\mathscr{X}_1$. There is a semiorthogonal decomposition:

$$\mathcal{A}_{\mathscr{X}_1} = \langle \mathcal{B}, \mathcal{A}_1(1) \boxtimes \mathrm{D^b}(\mathbb{P}(W^\vee)), \ldots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathrm{D^b}(\mathbb{P}(W^\vee)) \rangle.$$

The orthogonal complement $\mathcal{B}$ is a noncommutative $\mathbb{P}(W^\vee)$-scheme, and has a dual Lefschetz decomposition

$$\mathcal{B} = \langle \mathcal{B}_{j-1}(i-j), \ldots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle,$$

with respect to $\mathcal{O}_{\mathbb{P}(W^\vee)}(1)$. This noncommutative scheme should be thought of as a *homological Lefschetz theory* of $\mathcal{A}$ (see Definition 2.9), in a sense that we are going to describe.

Denote by $N := \dim(W)$ and set $1 \leqslant r \leqslant N-1$. Consider the universal codimension r linear section $\mathscr{X}_r \subset X \times \mathrm{Gr}(r, W)$, parameterizing pairs (X_L, L), for $L \subset W^\vee$ of dimension r and $X_L = X \cap \mathbb{P}(L^\perp)$. Denote by $\mathcal{A}_{\mathscr{X}_r}$ the noncommutative scheme obtained base changing $\mathcal{A}$ to $\mathscr{X}_r$. Using the Lefschetz decomposition of $\mathcal{A}$, Kuznetsov's arguments can be used to prove that the family $\{\mathcal{B}_{\mathrm{Gr}(r,W)}\}_{r=1}^{N-1}$ of noncommutative schemes obtained by base change of $\mathcal{B}$ to $\mathrm{Gr}(r, W)$ comes with dg functors $\Phi_r : \mathcal{A}_{\mathscr{X}_r} \to \mathcal{B}_{\mathrm{Gr}(r,W)}$ inducing semiorthogonal decompositions:

$$\mathcal{A}_{\mathscr{X}_r} = \langle \mathcal{C}_r, \mathcal{A}_r(1) \boxtimes \mathrm{D^b}(\mathrm{Gr}(r, W)), \ldots, \mathcal{A}_{i-r}(i-r) \boxtimes \mathrm{D^b}(\mathrm{Gr}(r, W)) \rangle,$$

$$\mathcal{B}_{\mathrm{Gr}(r,W)} = \langle \mathcal{B}_{j-1}(N-r-j) \boxtimes \mathrm{D^b}(\mathrm{Gr}(r, W)), \ldots, \mathcal{B}_{N-r}(-1) \boxtimes \mathrm{D^b}(\mathrm{Gr}(r, W)), \mathcal{C}_r \rangle,$$

that is, the functor Φ_r realizes the equivalence between the two noncommutative schemes $\mathcal{C}_r$ which arise as orthogonal complements to the restrictions of the Lefschetz decompositions.

Base changing from the universal family to a point in the Grassmannian, that is to a linear subspace $L \subset W$ of codimension r, we obtain:

$$\mathcal{A}_L = \langle \mathcal{C}_L, \mathcal{A}_r(1), \ldots, \mathcal{A}_{i-r}(i-r) \rangle,$$

$$\mathcal{B}_L = \langle \mathcal{B}_{j-1}(N-r-j), \ldots, \mathcal{B}_{N-r}(-1), \mathcal{C}_L \rangle,$$

where $\mathcal{A}_L$ is the restriction of $\mathcal{A}$ to $X_L := X \cap \mathbb{P}(L^\perp)$ and $\mathcal{B}_L$ is the restriction of $\mathcal{B}$ to $\mathbb{P}(L)$. It follows that we have a family of noncommutative schemes and functors $\{\mathcal{B}_{\mathrm{Gr}(r,W)}, \Phi_r\}_{r=1}^{N-1}$ which allow us to describe semiorthogonal decompositions of $\mathcal{A}_L$ as a noncommutative X_L-scheme. This is the reason why we call such a family a homological Lefschetz theory for the noncommutative X-scheme $\mathcal{A}$ with respect to the given Lefschetz decomposition.

Homological Projective Duality is the geometric realization of a homological Lefschetz theory: let $g : Y \to \mathbb{P}(W^\vee)$ be a projective scheme with a line bundle $\mathcal{O}(1) = g^*\mathcal{O}_{\mathbb{P}(W^\vee)}(1)$. We set $\mathscr{Y}_r \subset Y \times \mathrm{Gr}(r,W)$ to be the universal codimension r linear section, and, for a noncommutative Y-scheme $\mathcal{B}$, we denote by $\mathcal{B}_{\mathscr{Y}_r}$ its base change to $\mathscr{Y}_r$.

Definition 1. A noncommutative Y-scheme $\mathcal{B}$ is the *Homological Projective Dual* to $\mathcal{A}$ with respect to the given Lefschetz decomposition, if there are functors $\Phi_r : \mathcal{A}_{\mathscr{X}_r} \to \mathcal{B}_{\mathscr{Y}_r}$ realizing $\{\mathcal{B}_{\mathscr{Y}_r}, \Phi_r\}$ as the homological Lefschetz theory of $\mathcal{A}$.

One of the most interesting cases from the point of view of Algebraic Geometry, is to consider a projective variety $f : X \to \mathbb{P}(W)$ and to take $\mathcal{A}$ to be its (crepant) noncommutative resolution of singularities. Kuznetsov shows in this case that the HP dual $\mathcal{B}$ is the (crepant) noncommutative resolution of singularities of a variety $g : Y \to \mathbb{P}(W^\vee)$ such that the critical locus of the map g coincide with the classical projective dual $X^\vee$ of X. Hence, given an X such that the projective dual is known, and the resolution $\mathcal{A}$ (if X is smooth, just take $\mathrm{D^b}(X)$), the natural problem is to construct a Y and a $\mathcal{B}$ realizing the HPdual. Some geometrically interesting cases are described in Kuznetsov's 2014 ICM address [**Kuz14**].

Let us sketch two examples of HP duality, that were constructed in [**ABB14**] and in [**BBF16**] respectively.

First of all, consider a smooth projective variety S, and a rank n vector bundle E on S. We set $X := \mathbb{P}(E) \to S$. Now consider a family of line bundle valued quadratic forms (E, q_i, L_i) (see Definition 2.18), and the projective bundle $Y := \mathbb{P}(S_2(E)^\vee) \to S$. The family of quadrics $Q_i \subset \mathbb{P}(E)$, defined by the forms q_i, gives a universal family of quadrics $Q \to Y$, to which we can associate a sheaf of Clifford algebras $\mathcal{C}_0$. Hence, we have the noncommutative Y-scheme $\mathrm{D^b}(Y, \mathcal{C}_0)$. Recall that $X = \mathbb{P}(E)$, so that $Q \subset X \times Y$. If we consider the Veronese embedding $ver : X \to \mathbb{P}(S_2(E))$, we have that Q is the universal hyperplane section. Recall we denote by n the rank of E and set $m := \lfloor (n-1)/2 \rfloor$. We have a semiorthogonal decomposition induced by the projective bundle structure $p : X \to S$, giving a Lefschetz decomposition with respect to $\mathcal{O}(2) := ver^*\mathcal{O}_{\mathbb{P}(S_2(E))}(1)$:

$$\mathrm{D^b}(X) = \langle \underbrace{p^*\mathrm{D^b}(S), p^*\mathrm{D^b}(S)(1)}_{\mathcal{A}_0}, \underbrace{p^*\mathrm{D^b}(S)(2), p^*\mathrm{D^b}(S)(3)}_{\mathcal{A}_1(2)}, \ldots, \underbrace{p^*\mathrm{D^b}(S)(n-2)}_{\mathcal{A}_m(2m)} \rangle,$$

where $\mathcal{A}_i$ consist of two copies of $\mathrm{D^b}(S)$ for $i = 1, \ldots, m-1$ and $\mathcal{A}_m$ of either one or two copies of $\mathrm{D^b}(S)$ according to the parity of n.

Theorem 1. *The noncommutative Y-scheme $\mathrm{D^b}(Y, \mathcal{C}_0)$ is the HPdual of the noncommutative X-scheme $\mathrm{D^b}(X)$ with respect to the Lefschetz decomposition above.*

Theorem 1 was proved in [**ABB14**] and is a generalization of a result of Kuznetsov's [**Kuz08**] where it was shown for S a point and k algebraically closed of characteristic zero. Alongside with the careful analysis of the Clifford algebras, it is proved that $\mathcal{C}_0$ is Morita-invariant under hyperbolic splitting of the quadratic form defining it.

Theorem 1 gives interesting semiorthogonal decompositions for a large class of varieties. Indeed, it says then that we obtain from $\mathrm{D^b}(Y, \mathcal{C}_0)$ a geometric description of the homological Lefschetz theory of X, relatively over S. In particular, fixing a subbundle $L \subset E$, we have that $X_L \to \mathbb{P}(L)$ is the intersection of the quadrics parametrized by L. It follows that we can describe a semiorthogonal decomposition of any variety with a fibration into intersection of quadrics. For example, fibrations in del Pezzo surfaces of degree 4 fit this setting. An application to the birational geometry of threefolds and fourfolds fibered over $\mathbb{P}^1$ into intersections of two quadrics will be detailed later.

The second example involves determinantal varieties. Let U and V be $\mathbb{C}$-vector spaces of dimension m and n respectively, and assume that $m \leqslant n$. We set $W^\vee := U \otimes V$. Given an integer r with $1 \leqslant r \leqslant m$, we define $f : Z_r^{m,n} \subset \mathbb{P}(W)$ to be the locus in $\mathbb{P}(W)$ of $m \times n$ matrices having rank at most r. Then there exist a noncommutative $Z_r^{m,n}$-scheme $\mathcal{R}^r$, with a Lefschetz decomposition

$$\mathcal{R}^r = \langle \mathcal{A}_0, \ldots, \mathcal{A}_{mr}(mr) \rangle,$$

where $\mathcal{A}_0 \simeq \mathcal{A}_{mr}$ are all equivalent to $\mathrm{D^b}(\mathrm{Gr}(U, r))$. Moreover, the noncommutative scheme $\mathcal{R}^r$ is a categorical resolution of singularities of $Z_r^{m,n}$, which is crepant if $m = n$.

Theorem 2. *The noncommutative $Z_{m-r}^{m,n}$-scheme $\mathcal{R}^{m-r}$ is the HP dual of $\mathcal{R}^r$ with respect to the above Lefschetz decomposition.*

Theorem 2 was proved in [**BBF16**]. It allows to describe semiorthogonal decompositions of (resolution of singularities) of any determinantal variety obtained as a linear section of $Z_r^{m,n}$. On the other hand, notice that a determinantal variety is always locally a base change of a linear section of $Z_r^{m,n}$.

Noncommutative motives in Algebraic Geometry. One of the main interest in studying noncommutative motives in Algebraic Geometry is to try to extract informations about commutative motives and algebraic cycles. In general, the Grothendieck–Riemann–Roch theorem allows one to compare commutative and noncommutative motives, as remarked by Tabuada, see [**Tab13**] for the case of Chow motives, or [**Tab15**] for all the cases.

Consider indeed the category $\mathrm{Chow}(k)_R$ of Chow motives over a field k with coefficients in some ring R (we refer to [**And04**] for an introduction to motives). The Lefschetz motive $\mathbb{L}$ is an invertible object in the additive monoidal category $\mathrm{Chow}(k)_R$. One can hence consider the so-called orbit category $\mathrm{Chow}(k)_R/_{-\otimes\mathbb{L}}$, which has the same objects as $\mathrm{Chow}(k)_R$ and morphisms from M to N are given by the sum of morphism from M to $N\otimes\mathbb{L}^i$ over all integers i. Then there is a fully faithful, monoidal, additive functor $R : \mathrm{Chow}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{L}} \to \mathrm{NChow}(k)_{\mathbb{Q}}$ making the following diagram commute:

$$
\begin{tikzcd}
\mathrm{Var}(k) \arrow[r, "\mathrm{D^b}()"] \arrow[d, "M"'] & \mathsf{dgcat}(k) \arrow[dd, "U"] \\
\mathrm{Chow}(k)_{\mathbb{Q}} \arrow[d] & \\
\mathrm{Chow}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{L}} \arrow[r, "R"] & \mathrm{NChow}(k)_{\mathbb{Q}},
\end{tikzcd} \tag{2}
$$

where $\mathrm{Var}(k)$ is the category of smooth projective varieties. Roughly speaking, the above diagram says that noncommutative Chow motives contain Chow motives when we consider rational coefficients, and we forget the Tate twist. Similar statements are true for smash-nilpotent, homological and numerical motives. In particular, given a variety X, any decomposition of its Chow motive $M(X)_{\mathbb{Q}}$ will induce a decomposition of its noncommutative motive $NC(X)_{\mathbb{Q}} := U(\mathrm{D^b}(X))_{\mathbb{Q}}$. On the other hand, semiorthogonal decompositions of $\mathrm{D^b}(X)$ will give decompositions of $NC(X)_R$ for any ring R. It is then interesting to understand whether we can refine the coefficient ring in the above diagram and which are the differences between $NC(X)_R$ and $M(X)_R$ for those rings R for which the above diagram does not exist. In [**BT15**] these two questions are analyzed: working in dimension d, the above diagram hold replacing $\mathbb{Q}$ with $\mathbb{Z}[1/2d!]$. Moreover, some examples of purely noncommutative decompositions of $NC(X)$ are given for some particular variety X, as, for example, Brauer–Severi varieties.

One of the deepest realizations of the relation between noncommutative motives and algebraic cycles which will be treated in this report is the possibility to reconstruct the algebraic intermediate Jacobians of a smooth complex[1] variety X from noncommutative schemes which are components of $\mathrm{D^b}(X)$. Indeed, even if algebraic equivalence is not defined for noncommutative motives (at least, to the best of the author's knowledge), Marcolli and Tabuada [**MT14a**] define the Jacobian $\mathbf{J}(N)$ of a noncommutative motive N, and hence of a noncommutative scheme, as follows: the category $\mathrm{NNum}(\mathbb{C})_{\mathbb{Q}}$ of noncommutative numerical motives is semisimple, and there is a fully faithful additive and monoidal functor $\mathrm{Num}(\mathbb{C})_{\mathbb{Q}}/_{-\otimes\mathbb{L}} \to \mathrm{NNum}(\mathbb{C})$, where $\mathrm{Num}(\mathbb{C})_{\mathbb{Q}}$ is the category of numerical motives with rational coefficients, which is also semisimple. Moreover, since any Abelian variety is isogenous to an Abelian subvariety of the Jacobian of a smooth projective curve, there is a fully faithful functor $\mathrm{Ab}(\mathbb{C})_{\mathbb{Q}} \to \mathrm{Num}(\mathbb{C})_{\mathbb{Q}}$, where $\mathrm{Ab}(\mathbb{C})_{\mathbb{Q}}$ is the category of Abelian varieties up

[1] More generally, defined over an algebraically closed field $k \subset \mathbb{C}$.

to isogeny. Composing all these functors, $\mathbf{J}(N)$ is defined as the part of the noncommutative numerical motive of N lying in the image of $\mathrm{Ab}(\mathbb{C})_{\mathbb{Q}}$, which is well defined thanks to semi-simplicity of $\mathrm{NNum}(\mathbb{C})_{\mathbb{Q}}$. Hence $\mathbf{J}(N)$ is an Abelian variety well defined up to isogeny.

The most important fact to remark, which was proved by Marcolli and Tabuada [**MT14a**] is that, if X is a smooth projective variety satisfying Grothendieck standard conjectures of Lefschetz type[2] and $\mathbf{J}(X) := \mathbf{J}(NC(X))$, there is an isogeny $\mathbf{J}(X) \simeq \prod_{i=0}^{\dim(X)-1} J_a^i(X)$, where $J_a^i(X)$ is the i-th algebraic intermediate Jacobian of X, which is also well defined up to isogeny. Using the fact that semiorthogonal decompositions induce splittings of noncommutative motives, it is proved in [**BT16b**] that if $\mathrm{D^b}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition[3], such that $\mathbf{J}(U(\mathcal{B})) = 0$, then $\mathbf{J}(U(\mathcal{A}))$ contains all the information about the algebraic Jacobians of X. In particular, if Y is a smooth projective variety and $\mathrm{D^b}(Y) = \langle \mathcal{A}, \mathcal{C} \rangle$ is a semiorthogonal decomposition, then it is possible to construct an isogeny between $\prod_{i=0}^{\dim(X)-1} J_a^i(X)$ and an Abelian subvariety of $\prod_{i=0}^{\dim(Y)-1} J_a^i(Y)$, which is (isogenous to) the whole product in the case where $\mathbf{J}(U(\mathcal{C})) = 0$. Notice that it is not difficult to find examples of noncommutative schemes with trivial Jacobian, for example being generated by an exceptional collection is a sufficient condition for such vanishing.

In some particular cases, among which most Fano threefolds, conic bundles over rational surfaces, del Pezzo fibrations over $\mathbb{P}^1$, a variety X has a unique nontrivial intermediate Jacobian. By Poincaré duality, such an X has an odd dimension $2m+1$, and $J(X) := J_a^m(X)$ is such that $\mathbf{J}(X) \simeq \prod_{i=1}^{2m} J_a^i(X) = J(X)$. A geometrically relevant piece of datum with which $J(X)$ can be endowed in this case is a principal polarization. That is, $J(X)$ is not only well defined up to isogeny, but carries a natural principal polarization. We say that X is *verepresentable* if such a polarization has moreover a universal property with respect to m-cycles on X, which is called the incidence property. Examples of such varieties are smooth and projective curves, most Fano threefolds, conic bundles over rational surfaces, just to cite a few.

Theorem 3. *Let X and Y be verepresentable varieties and assume that $\mathrm{D^b}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ and $\mathrm{D^b}(Y) = \langle \mathcal{A}, \mathcal{C} \rangle$. If $\mathbf{J}(U(\mathcal{B})) = 0$, there exists an injective morphism of principally polarized Abelian varieties $\tau : J(X) \to J(Y)$. If moreover $\mathbf{J}(\mathcal{C}) = 0$, then τ is an isomorphism.*

Theorem 3 was proved in [**BT16b**], and extends previous results for curves and threefolds [**Ber07**,**BMMS12**,**BB13**,**BB12**], which were obtained without using noncommutative motives. Its application to birational geometry, via the fact that the intermediate Jacobian of a threefold provides a birational invariant, will be extensively treated in Chapter 4. Moreover, if a Torelli-type Theorem holds for X, then Theorem 3 allows to prove a categorical Torelli-type Theorem, as follows.

Theorem 4. *Suppose that either:*

- *X and Y are cubic threefolds, or*
- *X and Y are quartic double solids, or*

[2] A weaker condition is sufficient to obtain such isogeny, we will have detailed treatment in Ch. 3, §II.

[3] Working with noncommutative motives, we have to consider the components as dg categories, with the dg structure induced by projection

- *X and Y are intersections of two even dimensional quadrics, or*
- *X and Y are intersections of three odd dimensional quadrics.*

Set

$$\mathcal{A}_X := \langle \mathcal{O}_X, \ldots, \mathcal{O}_X(i-1)\rangle^{\perp}, \quad \mathcal{A}_Y := \langle \mathcal{O}_Y, \ldots, \mathcal{O}_Y(i-1)\rangle^{\perp}.$$

where $i = i(X) = i(Y)$ is the index of X and Y. Then X is isomorphic to Y if and only if $\mathcal{A}_X$ is equivalent to $\mathcal{A}_Y$.

Theorem 4 is proved in [**BT16b**]. Another application of Theorem 3 provides a new proof of a Bloch–Beilinson type conjecture on the Chow groups of complete intersections of very small degree. Let $X \subset \mathbb{P}^n$ a smooth complete intersection of multidegree $(d_1, \ldots, d_r)$, with the convention $d_1 \leqslant \ldots \leqslant d_r$. One has the numerical invariant

$$\kappa := [\frac{n - \sum_{j=2}^{r} d_j}{d_1}],$$

where $[-]$ denotes the integral part of a rational number. A careful analysis of the different Weil cohomology theories of X led to conjecture (explicitly stated by Paranjape in [**Par94**, Conjecture 1.8]) that $CH_i(X)_{\mathbb{Q}} \simeq \mathbb{Q}$ for every $i < \kappa$.

Suppose that $d_i = 2$ for all $i = 1, \ldots, r$. Then a proof of the above conjecture was given by Otwinowska [**Otw99**]. Based on the semiorthogonal decompositions of intersection of quadrics, it is possible to have an alternative proof of some occurrences of the conjecture, as done in [**BT16a**].

Theorem 5. *Suppose that X is*

(i) *either a complete intersection of two quadrics, or*
(ii) *a complete intersection of three odd-dimensional quadrics.*

Then $CH_i(X)_{\mathbb{Q}} \simeq \mathbb{Q}$ for all $i < [\dim(X)/2]$.

We notice that $\kappa = [\dim(X)/2]$ in the previous cases. The proof is based on Theorem 4 and on a dimensional counting for $K_0(X)$.

Another application of the theory of noncommutative motives to Algebraic Geometry allows one to prove Voevodsky's nilpotence conjecture in some new cases, as done in [**BMT14**]. Indeed, smash-nilpotent and numerical noncommutative motives can be defined. From this, one can state a noncommutative version of nilpotence conjecture. Moreover, both for numerical and smash-nilpotent motives, diagrams like (2) hold. It follows that the noncommutative conjecture for $\mathrm{D^b}(X)$ is equivalent to the classical one for any smooth projective variety X, see [**BMT14**]. In particular, using semiorthogonal decompositions, the noncommutative conjecture for X reduces to the noncommutative conjecture for its components. This gives new examples where nilpotence conjecture holds [**BMT14**].

Theorem 6. *Suppose that X is either:*

- *A quadric fibration $Q \to S$ of even dimension over a surface, or a curve; or*
- *A quadric fibration $Q \to S$ of odd dimension over a rational surface of a curve; or*
- *A complete intersection of at most three quadrics.*

Then nilpotence conjecture holds for X.

In other cases of sections of Grassmanians, or determinantal, or Pfaffian varieties, nilpotence conjecture can be proved with similar methods. For other cases of quadric fibrations over higher dimensional varieties or complete intersections of more than 3 quadrics, the noncommutative version allows to reduce nilpotence conjecture of X to nilpotence conjecture of smaller dimensional varieties.

Applications to birational geometry. One of the main aims of this report is to apply the theory of semiorthogonal decompositions, noncommutative schemes and motives to the study of birational property of projective varieties, following the road map started by Bondal and Orlov (see [**BO02**]) and traced more and more clearly by Kuznetsov and many others (see [**Kuz15**]). We focus on motivating, by abstract arguments and low-dimensional examples, the possibility to use semiorthogonal decompositions to detect obstructions to rationality of a given variety X. We introduce then the following definition (see [**BB12**]).

Definition 2. Let X be a k-scheme, and $\mathcal{A}$ a (crepant) noncommutative resolution of singularities of X. We say that X is *categorically representable in dimension* n (or in codimension $\dim(X) - n$)) if there is a semiorthogonal decomposition:

$$\mathcal{A} = \langle \mathcal{A}_1, \dots, \mathcal{A}_r \rangle,$$

and smooth and projective k-schemes $\{Y_i\}_{i=1}^r$, and fully faithful functors $\mathcal{A}_i \to \mathrm{D^b}(Y_i)$ such that $\mathcal{A}_i$ is admissible in $\mathrm{D^b}(Y_i)$ and $\dim(Y_i) \leqslant n$ for all $i = 1, \dots, r$.

We will use the following notations:

$$\mathrm{rdim_{cat}}(X) := \min\{n \mid X \text{ is categorical representable in dimension } n\}$$

$$\mathrm{rcodim_{cat}}(X) := \dim(X) - \mathrm{rdim_{cat}}(X),$$

and notice that $\mathrm{rdim_{cat}}(X) \leqslant n$ if X is smooth of dimension n, and that $\mathrm{rdim_{cat}}(\mathbb{P}^n) = 0$. The motivic behavior of semiorthogonal decompositions leads to formulate the following question:

Question 1. Suppose that X is a k-rational variety. Do we have $\mathrm{rcodim_{cat}}(X) \geqslant 2$?

A positive answer to Question 1 would lead to a necessary criterion of rationality. This criterion is certainly not sufficient, since there are examples of non-rational threefolds having $\mathrm{rdim_{cat}}(X) = 1$, as, e.g., $X \to C$ a projective bundle over a curve of positive genus.

On the other hand, the interest (and the difficulty) of Question 1 rely also on it being independent on the base field and on the dimension of X. Secondly, if k has characteristic zero, so that weak factorization holds [**AKMW02**], there is a well-defined motivic measure $\mu : K_0(\mathrm{Var}(k)) \to PT(k)$, where $PT(k)$ is the Grothendieck ring of noncommutative k-schemes. Moreover, categorical representability induces a ring filtration $PT_0(k) \subset PT_1(k) \subset \dots \subset PT(k)$, and we can show that if X is rational of dimension d then $\mu([X])$ lies in $PT_{d-2}(X)$. This gives a motivic positive (but much weaker) answer to Question 1.

Many examples for low-dimensional varieties of Fano type or carrying a Mori fiber space structure suggest that Question 1 should have a positive answer, or at least be a good way to attack the understanding of the relation between semiorthogonal decompositions and rationality.

For example, let X be a minimal del Pezzo surface over any field k. Then a study of Question 1 provides not only a positive answer, but also a birational invariant, a noncommutative k-scheme $\mathcal{GK}_X$ which we call the *Griffiths–Kuznetsov component.*

Theorem 7. *Let X be a minimal del Pezzo surface of index i, and consider the noncommutative scheme $\mathcal{A}_X := \langle \mathcal{O}, \ldots, \mathcal{O}(i) \rangle^{\perp}$. The following are equivalent:*

i) $\mathrm{rdim}_{\mathrm{cat}}(X) = 0$,
ii) $\mathcal{A}_X$ *is decomposed by derived categories of étale k-algebras,*
iii) X *is k-rational.*

Moreover, we can define, eventually via a semiorthogonal decomposition of $\mathcal{A}_X$, a noncommutative scheme $\mathcal{GK}_X$ such that, if $X' \dashrightarrow X$ is a birational map, then $\mathcal{GK}_{X'} = \mathcal{GK}_X$.

Theorem 7 was proved in [**AB15**]. Its proof is based on the explicit description of vector bundles generating $\mathrm{D^b}(\overline{X})$ and to their descent. In particular, it highlights a dichotomy between surfaces of degree smaller than 5 (where $\mathcal{A}_X$ is indecomposable and a birational invariant) and surfaces of degree bigger or equal to 5 (where $\mathcal{A}_X$ is always decomposable).

In particular, if $\deg(X) \geqslant 5$, there exist two vector bundles V_1 and V_2 generating $\mathcal{A}_X$ whose endomorphism algebras detect the birational class of X, and whose second Chern classes detect the existence of low degree points on X. As an example, if X is a Brauer–Severi surface $X = SB(A)$ associated to an Azumaya algebra A, then V_i is the rank three vector bundle base-changing to $\mathcal{O}(i)^{\oplus 3}$, and the endomorphism algebras of V_1 and V_2 are, respectively, A and A^2. Under this point of view, in the case of high degree del Pezzo surfaces, Theorem 7 can be thought of as an extension of Amitsur's conjecture to other del Pezzo surfaces.

In the case where X is a complex threefold, a necessary condition for rationality is to have a single, principally polarized intermediate Jacobian $J(X)$. Example of such threefolds are Fano threefolds, conic bundles over rational surfaces and del Pezzo fibrations over the projective line. In this cases, Clemens and Griffiths define a natural principally polarized Abelian subvariety $A_X \subset J(X)$, the *Griffiths component*, which is a birational invariant. In particular, if X is rational, then $A_X = 0$.

Theorem 8. *Let X be a verepresentable threefold. If* $\mathrm{rcodim}_{\mathrm{cat}}(X) \geqslant 2$*, then the Griffiths component A_X vanishes.*

Theorem 8 was proved in [**BB13**] and [**BB12**], and can also be seen as a consequence Theorem 3. It shows that categorical representability in codimension 2 is a finer invariant than the Griffths invariant. In particular, it can be applied to conic bundles over minimal surfaces and to del Pezzo fibrations of degree 4 over the projective line.

Theorem 9. *Let X be either a conic bundle over a minimal rational surface, or a del Pezzo fibration of degree 4 on the projective line. Then the following are equivalent:*

i) $\mathrm{rcodim}_{\mathrm{cat}}(X) \geqslant 2$,

ii) *X is rational.*

Theorem 9 is proved in [**BB13**] in the case of conic bundles, and in [**ABB14**] in the case of del Pezzo fibrations. The first case is obtained by using Theorem 8 and Beauville [**Bea77**] and Shokurov [**Sho84**] classification of rational conic bundles, and giving explicit semiorthogonal decompositions of $\mathrm{D^b}(X)$ in the rational cases. The second case is a consequence of the first one and of Homological Projective Duality for relative complete intersections of quadrics, that is, Theorem 1. Indeed, a degree 4 del Pezzo fibration X is always the intersection of two quadric threefold fibrations. The linear span of these two can be reduced by hyperbolic splitting to a conic bundle over a Hirzebruch surface, birational to X. We finally notice that the birational information on X is always contained, in these cases, on the noncommutative k-scheme associated to the Clifford algebras defined by the quadratic forms defining X.

Finally, some examples are also known in dimension four. Let $X \to \mathbb{P}^1$ be a fourfold with a fibration onto the projective line whose fibers are intersections of two quadrics. As above, using Homological Projective Duality for relative quadric fibrations, that is, Theorem 1, in a general case, we obtain a surface $S \to \mathbb{P}^1$ with a hyperelliptic fibration, and a Brauer class α in $\mathrm{Br}(S)$. Indeed, S is the discriminant double cover of a $\mathbb{P}^1$-bundle $T \to \mathbb{P}^1$, over which the linear span of the two quadrics define a four-dimensional quadric fibration. The genericity assumption ensures that the discriminant divisor is smooth (see §1.6 [**ABB14**] for more details). Moreover, we have that the noncommutative scheme $\mathrm{D^b}(S,\alpha)$ is the orthogonal complement of an exceptional collection in $\mathrm{D^b}(X)$. Hence, if $\alpha = 0$, we have that $\mathrm{rcodim_{cat}}(X) \geqslant 2$. In this case, Question 1 has a positive answer

Theorem 10. *Let $X \to \mathbb{P}^1$ be a fibration in intersections of two quadrics, and $S \to \mathbb{P}^1$ the associated hyperelliptic fibration with a Brauer class α in $\mathrm{Br}(S)$. If $\alpha = 0$, we have that X is rational and $\mathrm{rcodim_{cat}}(X) \geqslant 2$.*

Theorem 10 was proved in [**ABB14**]. In particular, it allows to formulate a conjecture on the rationality of X being equivalent to the category $\mathrm{D^b}(S,\alpha)$ being representable in dimension at most 2 (see Conjecture 4.31). A similar conjecture was previously formulated by Kuznetsov for cubic fourfolds [**Kuz10**]. Indeed, if $X \subset \mathbb{P}^5$ is a cubic hypersurface, then one can consider the noncommutative scheme $\mathcal{A}_X = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle^\perp$. Such a scheme is a noncommutative K3 surface, that is, its Serre functor is the shift by 2 in cohomology. Kuznetsov conjectures that X is rational if and only if there exists a K3 surface S and an equivalence $\mathrm{D^b}(S) \simeq \mathcal{A}_X$.

A particular case, similar to the one treated in Theorem 10, is the case where X contains a plane. Indeed, one can project off the plane to obtain a (rational) structure of quadric fibration $X \dashrightarrow \mathbb{P}^2$, degenerating along a sextic curve. This curve is smooth for the general such X and the associated double cover $S \to \mathbb{P}^2$ is a degree 2 K3 surface coming with a Brauer class α in $\mathrm{Br}(S)$ obtained by the Clifford algebra of the quadric bundle. The case treated in Theorem 10 is similar, the surface S there is obtained as a double cover of the Hirzebruch surface parameterizing the family of quadric fibrations whose intersection is the fourfold X.

If X is a cubic fourfold with a plane, Kuznetsov shows that $\mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}(S, \alpha)$. Moreover, the vanishing of α is a sufficient condition both for rationality and for categorical representability in codimension 2. On the other hand, it is natural to wonder whether there are cases where α is not trivial and X is rational and to understand if Kuznetsov conjecture still holds. The cases of cubic fourfolds containing a plane and a rational quintic del Pezzo surface were considered and (generically) completely classified in [**ABBVA14**], where the following statement was proved.

Theorem 11. *There are five irreducible components of the moduli space of cubics containing a plane and a quintic del Pezzo surface, which are indexed by the discriminant $d_X \in \{21, 29, 32, 36, 37\}$ of the intersection form on the algebraic cohomology lattice $A(X) \subset H^4(X, \mathbb{Z})$. The Clifford invariant α in $\mathrm{Br}(S)$ of such a general cubic fourfold X is trivial if and only if d_X is odd. There is a Pfaffian cubic in the $d_X = 32$ locus, and the group of algebraic 2-cycles on this cubic has rank 3.*

As a corollary of Theorem 11, it can be shown that there exists rational cubic fourfolds containing a plane, such that α is nontrivial and such that Question 1 has a positive answer (as well as Kuznetsov's conjecture). Indeed, if $d_X = 32$, it can happen that X is Pfaffian, and an explicit example is constructed in [**ABBVA14**], so that X is rational and, thanks to Homological Projective Duality (see [**Kuz10**]), there is a degree 14 K3 surface S' such that $\mathrm{D}^{\mathrm{b}}(S, \alpha) \simeq \mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}(S')$. Notice that Bolognesi and Russo [**BR16**] have recently shown that any special cubic with an associated K3 surface of degree 14 is rational (see §IV.1 for details), which allows one to extend the previous consideration to all the cases where d_X is even.

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CHAPTER 1

dg categories, semiorthogonal decompositions and noncommutative motives

In this chapter, we introduce the algebraic and categorical objects which play a key role in this report. In particular, we will focus on the notions of triangulated categories, dg categories, and their semiorthogonal decompositions. We will also describe how dg categories define noncommutative schemes and Chow motives. Together with these constructions, many technical points (such as descent or scalar extension) as well as many geometric questions we will introduce in the next Chapters lead us to consider dg categories (in particular, dg enhanced triangulated categories) as the most suitable algebraic structure to work with. We work over an arbitrary field k, and with small categories.

I. Semiorthogonal decompositions and exceptional objects.

In this section, we consider k-linear triangulated categories. These objects were defined by Verdier in his Ph.D. thesis [**Ver96**]. Detailed introductions to homological algebra can be found in the books of Gelfand and Manin [**GM96**] or Weibel [**Wei94**]. A more geometry-oriented introduction can be found in Huybrecht's book [**Huy06**].

I.1. Generators. Let $\mathbf{A}$ be a k-linear triangulated category. A natural question is whether that such a category $\mathbf{A}$ satisfies some finiteness conditions; in particular we would like to define a set of *generators* for $\mathbf{A}$. There are several notions of generation (such as classical or compact, see, e.g., [**BVdB03**], [**LO10**]) for a triangulated category.

Definition 1.1. Let $\mathbf{A}$ be a triangulated category. An object A in $\mathbf{A}$ is *compact* if $\mathrm{Hom}_{\mathbf{A}}(A, -)$ commutes with arbitrary direct sums.

Definition 1.2. A set of (compact) objects $\{E_i\}_{i \in I}$ in a k-linear triangulated category $\mathbf{A}$ *(compactly) generates* $\mathbf{A}$ if, for any object of A of $\mathbf{A}$, the vanishing $\mathrm{Hom}_{\mathbf{A}}(E_i, A[n]) = 0$ for all $i \in I$ and all integer n implies $A = 0$.

Definition 1.3. A set of compact objects $\{E_i\}_{i \in I}$ in a k-linear triangulated category $\mathbf{A}$ *classically generates* $\mathbf{A}$ if $\mathbf{A}$ is the smallest full thick triangulated subcategory of $\mathbf{A}$ containing the objects E_i.

Suppose that E is a classical generator for a triangulated category $\mathbf{A}$. One would like to "count" the number of steps which are needed to construct $\mathbf{A}$ from E. More precisely, given two full subcategories $\mathbf{A}_1$ and $\mathbf{A}_2$, we define their product $\mathbf{A}_1 * \mathbf{A}_2$ to be the full subcategory of all the objects A of $\mathbf{A}$ fitting a triangle $A_1 \to A \to A_2$ with A_i in $\mathbf{A}_i$.

For a full subcategory $\mathbf{B}$ of $\mathbf{A}$, denote by $\langle \mathbf{B} \rangle_\oplus$ the smallest full subcategory of $\mathbf{A}$ containing $\mathbf{B}$ and closed under shifts, directs sums and direct summands. Then we denote

$$\mathbf{B}_1 \diamond \mathbf{B}_2 := \langle \mathbf{B}_1 * \mathbf{B}_2 \rangle_\oplus.$$

For an object A of $\mathbf{A}$, set $\langle A \rangle^{\diamond 1} := \langle A \rangle_\oplus$ and $\langle A \rangle^{\diamond n} := \langle A \rangle^{\diamond 1} \diamond \langle A \rangle^{\diamond n-1}$ for any $n > 1$.

Definition 1.4. Let E be a classical generator of a triangulated category $\mathbf{A}$. We say that E is a *strong* generator if there exists n in $\mathbb{N}$ such that

$$\langle E \rangle^{\diamond n} = \mathbf{A}. \tag{3}$$

Let us just mention that, if E is a strong generator, following Ballard-Favero-Katzarkov [**BFK12**], one can define the *generation time* of E to be the smallest integer satisfying (3).

I.2. Semiorthogonal decompositions and their mutations. A full thick triangulated subcategory $\sigma : \mathbf{A}_1 \hookrightarrow \mathbf{A}$ is called *admissible* if the embedding functor σ admits a left adjoint σ^* and a right adjoint $\sigma^!$.

Definition 1.5. A *semiorthogonal decomposition* of $\mathbf{A}$ is a totally ordered set of admissible subcategories $\mathbf{A}_1, \dots, \mathbf{A}_n$ of $\mathbf{A}$ such that

- $\mathrm{Hom}_{\mathbf{A}}(A_i, A_j) = 0$ for all $i > j$ and any A_i in $\mathbf{A}_i$ and A_j in $\mathbf{A}_j$;
- for every object A of $\mathbf{A}$, there is a chain of morphisms $0 = A_n \to A_{n-1} \to \dots \to A_1 \to A_0 = A$ such that the cone of $A_k \to A_{k-1}$ is an object of $\mathbf{A}_k$ for all $k = 1, \dots, n$.

Such a decomposition will be written

$$\mathbf{A} = \langle \mathbf{A}_1, \dots, \mathbf{A}_n \rangle.$$

This notation is justified by the fact that, even if any $\mathbf{A}_i$ is not necessarily generated by a set of objects, the category $\mathbf{A}$ is generated by its subcategories $\mathbf{A}_i$.

For a subcategory $\mathbf{B} \subset \mathbf{A}$, we define the full subcategories

$$\mathbf{B}^\perp = \{ A \in \mathbf{A} \mid \mathrm{Hom}_{\mathbf{A}}(B, A) = 0 \text{ for all } B \in \mathbf{B} \}$$
$${}^\perp\mathbf{B} = \{ A \in \mathbf{A} \mid \mathrm{Hom}_{\mathbf{A}}(A, B) = 0 \text{ for all } B \in \mathbf{B} \}.$$

Assume that $\mathbf{A}$ has finite homological dimension ans is saturated (these properties are satisfied by bounded derived categories of coherent sheaves on a smooth scheme, see Thm. 2.14 [**BK90**]). Then, if $\mathbf{B}$ is admissible, then both $\mathbf{B}^\perp$ and ${}^\perp\mathbf{B}$ are admissible and we have two semiorthogonal decompositions (see [**BK90**, §2])

$$\mathbf{A} = \langle \mathbf{B}^\perp, \mathbf{B} \rangle = \langle \mathbf{B}, {}^\perp \mathbf{B} \rangle.$$

Given an admissible subcategory $\mathbf{B} \hookrightarrow \mathbf{A}$, Bondal [**Bon90**, §3] defines *left* and *right mutation* functors with respect to $\mathbf{B}$:

$$L_{\mathbf{B}} : \mathbf{A} \to \mathbf{A}, \qquad R_{\mathbf{B}} : \mathbf{A} \to \mathbf{A}.$$

Such functors satisfy

$$R_{\mathbf{B}}(\mathbf{B}) = L_{\mathbf{B}}(\mathbf{B}) = 0 \quad R_{\mathbf{B}}\mathbf{B}^\perp = {}^\perp \mathbf{B} \quad L_{\mathbf{B}}({}^\perp\mathbf{B}) = \mathbf{B}^\perp,$$

and $R_{\mathbf{B}|\mathbf{B}^\perp}$ and $L_{\mathbf{B}|^\perp\mathbf{B}}$ are equivalences.

In particular, whenever $\mathbf{A} = \langle \mathbf{B}, \mathbf{C}\rangle$ is a semiorthogonal decomposition, there are semiorthogonal decompositions

$$\mathbf{A} = \langle L_{\mathbf{B}}(\mathbf{C}), \mathbf{B}\rangle, \qquad \mathbf{A} = \langle \mathbf{C}, R_{\mathbf{C}}(\mathbf{B})\rangle,$$

with $L_{\mathbf{B}}(\mathbf{C}) \simeq \mathbf{C}$ and $R_{\mathbf{C}}(\mathbf{B}) \simeq \mathbf{B}$.

Definition 1.6. A triangulated category $\mathbf{A}$ is *indecomposable* if $\mathbf{A}$ is not the disjoint union of two nontrivial triangulated categories. It is *semiorthogonally indecomposable* if it does not admit any nontrivial semiorthogonal decomposition.

Remark 1.7. Notice that a category $\mathbf{A}$ is indecomposable if and only if it does not admit any nontrivial completely orthogonal decomposition, that is a semiorthogonal decomposition $\mathbf{A} = \langle \mathbf{A}_1, \mathbf{A}_2\rangle$ such that $\mathrm{Hom}_{\mathbf{A}}(A_1, A_2) = 0$ for all object A_i in $\mathbf{A}_i$. Hence, indecomposability is much weaker than semiorthogonal indecomposability. For example, if X is a connected smooth projective variety, then its bounded derived category $\mathrm{D}^{\mathrm{b}}(X)$ is indecomposable (see [**Bri99**, Ex. 3.2]), while if X is a Fano variety then $\mathrm{D}^{\mathrm{b}}(X)$ is semiorthogonally decomposable (see [**Kuz09**]).

Example 1.8. There are examples of triangulated categories which are semiorthogonally indecomposable. For example, if X is a connected curve of positive genus (see Okawa [**Oka11**]) or if X is a connected variety with trivial canonical bundle, then the triangulated category $\mathrm{D}^{\mathrm{b}}(X)$ is semiorthogonally indecomposable.

We finally notice that, if $\mathbf{B}$ is a subcategory generated by a given set of compact objects $\{E_i\}_{i\in I}$ of $\mathbf{A}$ then $\mathbf{B}$ is not in general admissible, as the following example shows.

Example 1.9. Let C be an elliptic curve. The subcategory generated by $\mathscr{O}_C$ is a proper subcategory of the triangulated category $\mathrm{D}^{\mathrm{b}}(C)$. Indeed, for any line bundle $\mathscr{L}$ on C, we have $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(C)}(\mathscr{O}_C, \mathscr{L}[n]) = \mathrm{Ext}^n(\mathscr{O}_C, \mathscr{L}) = H^n(C, \mathscr{L})$. Hence, if $\mathscr{L}$ is nontrivial of degree 0, then $\mathscr{L}$ is not in $\langle \mathscr{O}_C\rangle$. On the other hand, we remarked that $\mathrm{D}^{\mathrm{b}}(C)$ is semiorthogonally indecomposable in Example 1.8.

I.3. Splitting functors. Given triangulated categories $\mathbf{A}$ and $\mathbf{B}$ and an exact functor $\Phi : \mathbf{A} \to \mathbf{B}$, one can define the kernel of Φ as the full triangulated category $\ker\Phi = \{A \in \mathbf{A} | \Phi(A) = 0\}$ and the image of Φ as the full category $\mathrm{im}\,\Phi = \{B \simeq \Phi(A) | A \in \mathbf{A}\}$. The latter is not necessarily a triangulated subcategory.

Definition 1.10. A *splitting functor* is an exact functor $\Phi : \mathbf{A} \to \mathbf{B}$ such that $\ker\Phi$ and $\mathrm{im}\,\Phi$ are admissible in $\mathbf{A}$ and $\mathbf{B}$ respectively, and Φ restricted to $\ker\Phi^\perp$ is fully faithful.

Splitting functors are functors which identify admissible subcategories of triangulated categories. By definition, a splitting functor $\Phi : \mathbf{A} \to \mathbf{B}$ gives a semiorthogonal decomposition of $\mathbf{A} = \langle \ker\Phi^\perp, \ker\Phi\rangle$ and of $\mathbf{B} = \langle \mathrm{im}\,\Phi, {}^\perp\mathrm{im}\,\Phi\rangle$.

On the other hand, if $\mathbf{A} = \langle \mathbf{A}_1, \mathbf{A}_2\rangle$ and $\mathbf{B} = \langle \mathbf{B}_1, \mathbf{B}_2\rangle$, an equivalence $\Psi : \mathbf{A}_1 \to \mathbf{B}_1$ will give rise, by precomposition with projection $\mathbf{A} \to \mathbf{A}_1$ and composition with embedding $\mathbf{B}_1 \to \mathbf{B}$, to a splitting functor $\Phi : \mathbf{A} \to \mathbf{B}$ such that $\ker\Phi = \mathbf{A}_2$, and $\mathrm{im}\,\Phi = \mathbf{B}_1$.

I.4. Exceptional objects, blocks, and mutations.

Definition 1.11. Let A be a division (not necessarily central) k-algebra (e.g., A could be a field extension of k). A compact object E of $\mathbf{A}$ is called *A-exceptional* if

$$\mathrm{Hom}_{\mathbf{A}}(E, E) = A \quad \text{and} \quad \mathrm{Hom}_{\mathbf{A}}(E, E[r]) = 0 \quad \text{for} \quad r \neq 0.$$

An *exceptional object* is an A-exceptional object for some algebra A.

Definition 1.12. Let $\{E_i\}_{i=1\dots n}$ be exceptional objects of $\mathbf{A}$. The totally ordered set $\{E_1, \dots, E_n\}$ is called an *exceptional collection* if $\mathrm{Hom}_{\mathbf{A}}(E_j, E_i[r]) = 0$ for all integers r whenever $j > i$. An exceptional collection is *full* if it generates $\mathbf{A}$. Equivalently, a collection is full if for any object T of $\mathbf{A}$, the vanishing $\mathrm{Hom}_{\mathbf{A}}(T, E_i[r]) = 0$ for all i and for all r integers implies $T = 0$. An exceptional collection is *strong* if $\mathrm{Hom}_{\mathbf{A}}(E_i, E_j[r]) = 0$ whenever $r \neq 0$.

Remark 1.13. If k is algebraically closed, any exceptional object is k-exceptional, so that the original definition (see, e.g., [**Bon90**]) matches Definition 1.11 in this case.

Bondal and Kapranov have shown that, for an exceptional collection $\{E_1, \dots, E_n\}$ of $\mathbf{A}$, the subcategory $\mathbf{E} = \langle E_1, \dots, E_n \rangle$ of $\mathbf{A}$ is admissible (see [**BK90**, Prop. 2.6 and Corollary page 530]). In particular, one can check that there is a semiorthogonal decomposition

$$\mathbf{E} = \langle E_1, \dots, E_n \rangle = \langle \langle E_1 \rangle, \dots, \langle E_n \rangle \rangle.$$

Exceptional collections provide an algebraic description of admissible subcategories of $\mathbf{A}$. Indeed, if E is an A-exceptional object in $\mathbf{A}$, the triangulated subcategory $\langle E \rangle \subset \mathbf{A}$ is equivalent to the category $\mathrm{perf}(A)$ of perfect k-linear complexes of A-modules of finite type. Furthermore, as shown by Bondal [**Bon90**], in many cases strong full exceptional collections give an algebraic description of a triangulated category.

Proposition 1.14 ([**Bon90**, Thm. 6.2])**.** *Suppose that $\mathbf{A}$ is the bounded derived category of either a smooth projective k-scheme or of a k-linear Abelian category with enough injective objects and is of finite global dimension. Let $\{E_1, \dots, E_n\}$ be a full strong k-exceptional collection on $\mathbf{A}$, and consider the object $E = \bigoplus_{i=1}^n E_i$ and the k-algebra $R = \mathrm{End}_{\mathbf{A}}(E)$. Then $\mathbf{R}\mathrm{Hom}_{\mathbf{A}}(E, -) : \mathbf{A} \to \mathrm{perf}(R)$ is a k-linear equivalence.*

Remark 1.15. The assumptions on the category $\mathbf{A}$ and on the strongness of the exceptional sequence may seem rather restrictive, and both find a natural solution when triangulated categories are enriched with a dg structure. The first assumption can be indeed replaced by considering a dg enhancement of $\mathbf{A}$ (see [**BK91**, Thm. 1]). When the exceptional collection is not strong, the endomorphisms of E form a dg algebra, see Thm. 1.26.

Example 1.16. The full strong k-exceptional collection $\{\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)\}$ on the bounded derived category of the projective space $\mathrm{D^b}(\mathbb{P}^n_k)$ was described by Beilinson [**Bei78**], [**Bei84**] and Bernšteĭn-Gelfand-Gelfand [**BGG78**]. In this case $R = \mathrm{End}\big(\bigoplus_{i=0}^n \mathcal{O}(i)\big)$ is isomorphic to the path algebra of the Beilinson quiver with $n+1$ vertices, see [**Bon90**, Ex. 6.4].

Given an exceptional pair $\{E_1, E_2\}$ with E_i being A_i-exceptional, consider the admissible subcategories $\langle E_i \rangle$, forming a semiorthogonal pair. We can hence perform right and left mutations, which provide equivalent admissible subcategories. It easily follows that the object $R_{E_2}(E_1)$ is A_1-exceptional, the object $L_{E_1}(E_2)$ is A_2-exceptional, and the pairs $\{L_{E_1}(E_2), E_1\}$ and $\{E_2, R_{E_2}(E_1)\}$ are exceptional. We call $R_{E_2}(E_1)$ the *right mutation* of E_1 through E_2 and $L_{E_1}(E_2)$ the *left mutation* of E_2 through E_1.

A special case of an exceptional pair is a *completely orthogonal* pair $\{E_1, E_2\}$, i.e., an exceptional pair such that $\mathrm{Hom}_{\mathbf{A}}(E_1, E_2[i]) = 0$ for all i. Equivalently, we have that $\{E_2, E_1\}$ is also exceptional. In this case, $R_{E_2}(E_1) = E_1$ and $L_{E_1}(E_2) = E_2$.

Definition 1.17 ([**KN98**], 1.5)**.** An *exceptional block* in a k-linear triangulated category $\mathbf{A}$ is an exceptional collection $\{E_1, \ldots, E_n\}$ such that $\mathrm{Hom}_{\mathbf{A}}(E_i, E_j[r]) = 0$ for every r whenever $i \neq j$. Equivalently, every pair of objects in the collection is completely orthogonal. By abuse of notation, we denote by $\mathbf{E}$ the exceptional block as well as the subcategory that it generates.

If $\mathbf{E}$ is an exceptional block, then $\mathrm{End}_{\mathbf{A}}(\bigoplus_{i=1}^n E_i)$ is isomorphic to the k-algebra $A_1 \times \cdots \times A_n$, where E_i is A_i-exceptional. Proposition 1.14 then yields a k-equivalence $\mathbf{E} \simeq \mathrm{D^b}(A_1 \times \cdots \times A_n)$.

Moreover, given an exceptional block, any internal mutation acts by simply permuting the exceptional objects. Given an exceptional collection $\{E_1, \ldots, E_n, F_1, \ldots, F_m\}$ consisting of two blocks $\mathbf{E}$ and $\mathbf{F}$, the left mutation $L_{\mathbf{E}}(\mathbf{F})$ and the right mutation $R_{\mathbf{F}}(\mathbf{E})$ are obtained by mutating all the objects of one block to the other side of all the objects of the other block, or, equivalently, as mutations of semiorthogonal admissible subcategories.

I.5. Serre functors.

Definition 1.18. Let $\mathbf{A}$ be a triangulated k-linear category with finite dimensional morphism spaces. A functor $S : \mathbf{A} \to \mathbf{A}$ is a *Serre functor* if it is a k-linear equivalence inducing a functorial isomorphism

$$\mathrm{Hom}_{\mathbf{A}}(A, B) \simeq \mathrm{Hom}_{\mathbf{A}}(B, S(A))^{\vee}$$

of k-vector spaces, for any object A and B of $\mathbf{A}$.

Serre functors were defined by Bondal and Kapranov [**BK90**] in order to generalize Serre duality on a smooth projective variety X. Indeed, in this case, if $\dim(X) = n$, the functor $S_X := - \otimes \omega_X[n]$ is a Serre functor for the category $\mathrm{D^b}(X)$. The next proposition collects important properties of Serre functors (see [**BK90**] or [**Huy06**, Ch. 1 and 2]).

Proposition 1.19. *Suppose that* $\mathbf{A}$ *admits a Serre functor* S*. Then* S *is unique, and will be denoted by* $S_{\mathbf{A}}$*.*

Moreover, if $\mathbf{A}'$ *also admits a Serre functor, and* $F : \mathbf{A} \to \mathbf{A}'$ *is a* k*-linear equivalence, then* $S_{\mathbf{A}'} \circ F = F \circ S_{\mathbf{A}}$*. If* $F : \mathbf{A} \to \mathbf{A}'$ *is a functor admitting a left (resp. right) adjoint* G*, then* $H := S_{\mathbf{A}} \circ G \circ S_{\mathbf{A}'}^{-1}$ *(resp.* $H := S_{\mathbf{A}}^{-1} \circ G \circ S_{\mathbf{A}'}$*) is a right (resp. left) adjoint to* F*.*

Definition 1.20. Let $\mathbf{A}$ be a triangulated category with a Serre functor $S_{\mathbf{A}}$. We say that $\mathbf{A}$ is a *Calabi-Yau category* if there exist positive integers r and d such that $S_{\mathbf{A}}^r = [d]$. If

r is the minimal such integer, we say that $\mathbf{A}$ has dimension d/r. Notice that d/r is not a rational number, but rather a quotient of two integer numbers (simplifications of common factors are not allowed).

II. dg categories and enhanced triangulated categories

II.1. dg categories and Morita equivalence. Let $\mathcal{C}(k)$ be the category of cochain complexes of k-vector spaces. A *differential graded (=dg) category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$. This means that morphism sets are complexes of k-vector spaces and that the composition law fulfills the Leibniz rule $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$. A *dg functor* $\Phi : \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$. A *dg algebra* is a dg category with one object; a detailed account can be found in Keller's ICM address [**Kel06**].

Let us denote by $\mathsf{dgcat}(k)$ the category of (small) dg categories and dg functors. Such a category has a monoidal structure given by the *tensor product* $\mathcal{A} \otimes \mathcal{B}$ of two dg categories $\mathcal{A}$ and $\mathcal{B}$, defined as follows: objects of $\mathcal{A} \otimes \mathcal{B}$ are elements of the cartesian product of the sets of objects of $\mathcal{A}$ and $\mathcal{B}$, and the complexes of morphisms are given by

$$\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{B}}((A_1, B_1), (A_2, B_2)) := \mathrm{Hom}_{\mathcal{A}}(A_2, A_2) \otimes \mathrm{Hom}_{\mathcal{B}}(B_1, B_2).$$

For a given dg category $\mathcal{A}$, we define the *opposite* dg category $\mathcal{A}^{\mathrm{op}}$ to be the category having the same objects whose morphisms given by $\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(A, B) := \mathrm{Hom}_{\mathcal{A}}(B, A)$.

Definition 1.21. A *right $\mathcal{A}$-module* M is a dg functor $M : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}^{\mathsf{dg}}(k)$ with values in the dg category $\mathcal{C}^{\mathsf{dg}}(k)$ of cochain complexes of k-vector spaces; see Keller [**Kel06**, §2.3]. We will write $\mathcal{C}(\mathcal{A})$ for the category of right $\mathcal{A}$-modules.

The *derived category* $\mathrm{D}(\mathcal{A})$ *of* $\mathcal{A}$ is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of objectwise quasi-isomorphisms; see Keller [**Kel06**, §3.2]. This category is triangulated. We denote by $\mathrm{D}_c(\mathcal{A})$ the full subcategory of compact objects.

Given dg categories $\mathcal{A}$ and $\mathcal{B}$, an *$\mathcal{A}$-$\mathcal{B}$-bimodule* is a right $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$-module, that is, a dg functor $B : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}^{\mathsf{dg}}(k)$.

For each object A of $\mathcal{A}$, we have the right module represented by A, that is the functor $\mathrm{Hom}_{\mathcal{A}}(-, A)$, which we denote by $A^\wedge$. Another standard example of bimodule is the $\mathcal{A}$-$\mathcal{A}$-bimodule

$$\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathsf{dg}}(k) \qquad (A, B) \mapsto \mathrm{Hom}_{\mathcal{A}}(B, A)\,. \tag{4}$$

Let us denote by $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ the full triangulated subcategory of $\mathrm{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ consisting of those $\mathcal{A}$-$\mathcal{B}$-bimodules M such that for every object $A \in \mathcal{A}$ the right $\mathcal{B}$-module $M(A, -)$ belongs to $\mathrm{D}_c(\mathcal{B})$. Note that every dg functor $\Phi : \mathcal{A} \to \mathcal{B}$ gives rise to an $\mathcal{A}$-$\mathcal{B}$-bimodule

$$\widehat{\Phi} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathsf{dg}}(k) \qquad (A, B) \mapsto \mathcal{B}(B, \Phi(A))$$

which belongs to $\mathrm{rep}(\mathcal{A}, \mathcal{B})$.

Definition 1.22. A dg functor $\Phi : \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if the restriction of scalars functor $\mathrm{D}(\mathcal{B}) \to \mathrm{D}(\mathcal{A})$ is an equivalence of triangulated categories; see Keller [**Kel06**, §4.6].

As proved by Tabuada in [**Tab05**, Theorem 5.3], the category $\mathsf{dgcat}(k)$ carries a structure of Quillen model category whose weak equivalences are the Morita equivalences. Let us write $\mathsf{Hmo}(k)$ for the homotopy category of this model structure. As proved in *loc. cit.*, the assignment $\Phi \mapsto \widehat{\Phi}$ gives rise to a bijection

$$\mathrm{Hom}_{\mathsf{Hmo}(k)}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Iso}\,\mathrm{rep}(\mathcal{A}, \mathcal{B}), \tag{5}$$

where Iso stands for the set of isomorphism classes. Moreover, under (5), the composition law in $\mathsf{Hmo}(k)$ corresponds to the derived tensor product of bimodules.

II.2. Pretriangulated dg categories and enhancements of triangulated categories. Our aim is to consider dg enhancements of a given triangulated category. To this aim, we introduce the notion of pretriangulated dg category. We follow Keller's presentation [**Kel06**, §4.5].

Let $\mathcal{A}$ be a small dg category. The category $Z^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms defined by

$$\mathrm{Hom}_{Z^0(\mathcal{A})}(-,-) = \ker(\mathrm{Hom}^0_{\mathcal{A}}(-,-) \xrightarrow{d} \mathrm{Hom}^1_{\mathcal{A}}(-,-)).$$

The k-linear category $H^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms given by

$$\mathrm{Hom}_{H^0(\mathcal{A})}(-,-) := H^0(\mathrm{Hom}_{\mathcal{A}}(-,-)),$$

where $H^0(-)$ is the 0^{th} cohomology group functor.

Definition 1.23. We say that a dg category $\mathcal{A}$ is pretriangulated if the image of the Yoneda functor

$$Z^0(\mathcal{A}) \longrightarrow \mathcal{C}(\mathcal{A}), \quad A \mapsto A^\wedge$$

is stable under shifts in both directions and extensions. Equivalently, for all objects A and B of $\mathcal{A}$ and all integers n, and for any morphism $f : B \to A$, the object $B^\wedge[n]$ is isomorphic to $(B[n])^\wedge$ and the cone over a morphism $f^\wedge : B^\wedge \to A^\wedge$ is isomorphic to $(\mathrm{Cone}(f))^\wedge$.

For an arbitrary dg category $\mathcal{A}$, one has a unique pretriangulated dg category $\mathrm{pretr}(\mathcal{A})$, called the *pretriangulated hull* of $\mathcal{A}$, satisfying a universal property (see [**Kel06**, §4.5]). For our purpose, we record that if $\mathcal{A}$ is pretriangulated, then $H^0(\mathcal{A})$ has a canonical and functorial structure of k-linear triangulated category.

Definition 1.24. Let $\mathbf{A}$ be a triangulated category. A *dg enhancement* of $\mathbf{A}$ is a pair $(\mathcal{A}, \epsilon)$, where $\mathcal{A}$ is a pretriangulated dg category and $\epsilon : H^0(\mathcal{A}) \to \mathbf{A}$ is an equivalence of triangulated categories.

Let $\mathbf{A}$ be a triangulated category admitting a dg enhancement. Then $\mathbf{A}$ has a *unique* enhancement if for any two enhancements $(\mathcal{A}, \epsilon)$ and $(\mathcal{A}', \epsilon')$ of $\mathbf{A}$ there exists a dg functor

$$\Phi : \mathcal{A} \longrightarrow \mathcal{A}',$$

inducing an equivalence

$$H^0(\Phi) : H^0(\mathcal{A}) \xrightarrow{\simeq} H^0(\mathcal{A}').$$

In this case the enhancements $(\mathcal{A}, \epsilon)$ and $(\mathcal{A}', \epsilon')$ are called *equivalent*.

Enhancements $(\mathcal{A}, \epsilon)$ and $(\mathcal{A}', \epsilon')$ of $\mathbf{A}$ are called *strongly equivalent* if there exists a dg functor Φ as above such that the functors $\epsilon' \circ H^0(\Phi)$ and ϵ are isomorphic.

A way to construct a unique enhancement for the derived category $\mathrm{D}(\mathcal{C})$ of a Grothendieck category $\mathcal{C}$ is to have a set of compact generators, as shown by Lunts and Orlov [**LO10**, Thm. 2.7].

Theorem 1.25 (Lunts-Orlov)**.** *Let $\mathcal{C}$ be a Grothendieck category and suppose that there is a small set of compact generators in the derived category $\mathrm{D}(\mathcal{C})$. Then $\mathrm{D}(\mathcal{C})$ has a unique enhancement.*

If $\mathbf{A}$ is a triangulated category with a full exceptional collection $\{E_1, \dots, E_n\}$, then it has a compact generator, namely $E := \oplus_{i=1}^n E_i$. In this case, the dg enhancement $\mathcal{A}$ of $\mathbf{A}$ can be described as the derived category of a dg-algebra, as proved by Bondal and Kapranov [**BK90**]. Notice that the dg algebra is given by a choice of an enhancement of $\mathbf{A}$, for example by injective resolutions if $\mathbf{A}$ has enough injectives. The following result generalizes Prop. 1.14, that is the case where the exceptional collection is strong.

Theorem 1.26 (Bondal-Kapranov)**.** *Let $\mathbf{A}$ be a triangulated category, $(\mathbf{A}, \epsilon)$ an enhancement of $\mathbf{A}$, and $\{E_1, \dots, E_n\}$ be a full exceptional collection on $\mathbf{A}$. Consider the object $E = \bigoplus_{i=1}^n E_i$ and the dg-k-algebra $A = \mathrm{End}_{\mathbf{A}}(E)$, obtained via the enhancement ϵ. Then any dg enhancement $\mathcal{A}$ of $\mathbf{A}$ is equivalent to the dg category $\mathrm{D^b}(A)$.*

Let X be a scheme over the field k, and $\mathrm{D}(\mathbf{Qcoh}(X))$ the derived category of quasi-coherent sheaves on X. Thanks to the work of Bondal-Van den Bergh and Neeman [**BVdB03**, **Nee96**], if X is quasi-compact and quasi-separated, the subcategory of compact objects of $\mathrm{D}(\mathbf{Qcoh}(X))$ coincides with the category $\mathrm{perf}(X)$ of perfect complexes on X, that is, the category of bounded complexes of vector bundles. Lunts and Orlov show that this category admits a unique enhancement (see [**LO10**, Thm. 7.9]).

Theorem 1.27 (Lunts-Orlov)**.** *Let X be a quasi-separated and quasi-compact scheme. The category $\mathrm{perf}(X)$ admits a unique enhancement.*

Remark 1.28. Notice that for any given k-scheme (or even k-stack) X, one can construct enhancements for the categories $\mathrm{perf}(X)$ and $\mathrm{D^b}(X)$, see, e.g., [**LS15**]. The question we are considering here is whether $\mathrm{perf}(X)$ has a unique enhancement in the sense of definition 1.24. Notice that the existence of some dg enhancement of $\mathrm{perf}(X)$ (for example, by complexes of injectives) is necessary to give the endomorphism algebra of the generator a structure of dg algebra.

II.3. dg enhanced semiorthogonal decompositions. We want now to consider semiorthogonal decompositions of pretriangulated dg categories.

Definition 1.29. Let $\mathcal{A}$ be pretriangulated dg-category. A *semiorthogonal decomposition* of $\mathcal{A}$ is a set of pretriangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$ such that

$$H^0(\mathcal{A}) = \langle H^0(\mathcal{A}_1), \dots, H^0(\mathcal{A}_n) \rangle$$

is a semiorthogonal decomposition for the triangulated category $H^0(\mathcal{A})$.

Let $\mathbf{A}$ be a triangulated category, $(\mathbf{A}, \epsilon)$ a dg enhancement of $\mathbf{A}$, and let

$$\mathbf{A} = \langle \mathbf{A}_1, \ldots, \mathbf{A}_n \rangle \tag{6}$$

be a semiorthogonal decomposition. Then the categories $\mathbf{A}_i$ admit enhancements $(\mathbf{A}_i, \epsilon)$ induced by $(\mathbf{A}, \epsilon)$ in such a way that the semiorthogonal decomposition (6) gives a semiorthogonal decomposition of the pretriangulated dg category $(\mathbf{A}, \epsilon)$ as defined in Definition 1.29.

Remark 1.30. [Compare with Conjecture 2.3] Notice that if $\Phi : \mathbf{A} \to \mathbf{B}$ is a splitting functor, and $\mathcal{A}$ and $\mathcal{B}$ are enhancements of $\mathbf{A}$ and $\mathbf{B}$ respectively, there is in general no reason to have have a dg functor $\Phi^{\mathrm{dg}} : \mathcal{A} \to \mathcal{B}$ such that $H^0(\Phi^{\mathrm{dg}}) = \Phi$. In other words, given triangulated semiorthogonal decompositions $\mathbf{A} = \langle \mathbf{A}_1, \mathbf{A}_2 \rangle$ and $\mathbf{B} = \langle \mathbf{B}_1, \mathbf{B}_2 \rangle$, an exact k-linear equivalence $\mathbf{A}_2 \simeq \mathbf{B}_1$ could not come from a dg equivalence.

III. Noncommutative schemes: resolutions of singularities, representability

III.1. Noncommutative schemes. We concluded Section II.2 with the famous Theorem of Lunts and Orlov, stating that if X is a quasi-compact and quasi-separated scheme, then there is a unique enhancement of $\mathrm{perf}(X)$, which is the subcategory of compact objects of $\mathrm{D}(\mathbf{Qcoh}(X))$. We hence have a natural pretriangulated dg category with a strong generator E associated to such a scheme, so that we are lead to replace schemes by dg categories. Following Kontsevich's ideas [**Kon05**, **Kon10**, **Kon09**], some geometrical properties may be rephrased in noncommutative terms (see also Orlov [**Orl14**]). With these definitions in mind, we will detail Orlov's definition [**Orl14**] of a noncommutative k-scheme and compare smoothness, properness and regularity for noncommutative and commutative schemes.

Definition 1.31. A triangulated category $\mathbf{A}$ is *proper* if $\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbf{A}}(A, B[i])$ is finite-dimensional for any pair of objects A and B in $\mathbf{A}$. It is *regular* if it has a strong generator.

Definition 1.32. A dg category $\mathcal{A}$ is *smooth* if it is a compact object in $\mathrm{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})$. It is proper (resp. regular) if $\mathrm{D}_c(\mathcal{A})$ is proper (resp. regular).

We record that smoothness is a stronger notion than regularity: if a small dg category $\mathcal{A}$ is smooth, then it is regular (see [**Lun10**]).

Definition 1.33. A *noncommutative scheme* over k is a pretriangulated dg category $\mathcal{A}$ of the form $\mathrm{perf}(\mathcal{E})$ for some cohomologically bounded dg k-algebra $\mathcal{E}$. It is regular (resp. proper, resp. smooth) if $\mathcal{A}$ is regular (resp. proper, resp. smooth). If $\mathcal{B}$ is an admissible subcategory of $\mathcal{A}$ (that is, if there is a semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{B}, \mathcal{C} \rangle$ for some $\mathcal{C}$), we will say that $\mathcal{B}$ is a *component* of the noncommutative scheme $\mathcal{A}$ (and that $\mathcal{C}$ is its *complement*). A noncommutative k-scheme $\mathcal{A}$ is *geometric* if there exists a smooth and projective scheme X and a fully faithful functor $\mathcal{A} \to \mathrm{perf}(X)$ admitting right and left adjoints. That is, $\mathcal{A}$ is admissible in $\mathrm{perf}(X)$.

For a given geometric noncommutative k-scheme $\mathcal{A}$, we define *noncommutative $\mathcal{A}$-schemes* as noncommutative k-schemes such that $\mathcal{E}$ is a dg-$\mathcal{A}$-algebra. If $\mathcal{A} = \mathrm{perf}(X)$ for some smooth and projective X, a noncommutative X-scheme is a noncommutative $\mathcal{A}$-scheme. Notice that noncommutative $\mathcal{A}$-schemes carry both an $\mathcal{A}$-linear and a k-linear structure. The category of noncommutative k-linear proper and smooth $\mathcal{A}$-schemes will be denoted by $\mathrm{NSch}_{\mathcal{A}}(k)$, and the category of noncommutative $\mathcal{A}$-linear proper and smooth $\mathcal{A}$-schemes will be denoted by $\mathrm{NSch}(\mathcal{A})$.

If K/k is an extension, and $\mathcal{E}$ a dg k-algebra, then we can consider the base change $\mathcal{E}_K = \mathcal{E} \otimes_k K$. For the noncommutative k-scheme $\mathcal{A} = \mathrm{perf}(\mathcal{E})$, we define the noncommutative K-scheme $\mathcal{A}_K := \mathrm{perf}(\mathcal{E}_K)$ to be the *base change* of $\mathcal{A}$ to K.

Remark 1.34. Let Y be a quasi-compact and quasi-separated k-scheme, and L any invertible object for the monoidal structure of $\mathrm{perf}(Y)$, that is, L is a line bundle on Y. Let $\mathcal{A}$ be a noncommutative geometric Y-scheme: as defined above it is a noncommutative k-scheme enriched over $\mathrm{perf}(Y)$, and there exists a smooth and projective Y-variety X such that $\mathcal{A}$ is an admissible subcategory of $\mathrm{perf}(X)$. Then L lifts to an invertible object on $\mathcal{A}$, which we still denote by L.

Remark 1.35. In [**Gai13**], Gaitsgory defines the notion of 1-affineness for a prestack $\mathcal{Y}$ by requiring that the functor of enhanced global sections of sheaves of categories is an equivalence between the ∞-categories of sheaves of categories $\mathrm{ShvCat}(\mathcal{Y})$ over $\mathcal{Y}$ and quasi-coherent dg modules $\mathbf{Qcoh}(\mathcal{Y})$ over $\mathcal{Y}$. The quasi-inverse to global sections is the localization functor in this case.

In particular, if X is a quasi-separated and quasi-compact k-scheme, then X (or, better, the prestack given by the functor of points of X) is 1-affine [**Gai13**, Thm. 2.1.1]. We could then have defined a noncommutative X-scheme $\mathcal{A}$ to be a section of the sheaf of categories $\mathrm{ShvCat}(X)$ over X. Indeed, having such an object is equivalent to giving a section of $\mathbf{Qcoh}(X)$ (where this latter category means dg modules over X, and not quasi-coherent "commutative" sheaves), that is $\mathcal{A}$ is a noncommutative k-scheme enriched over the category $\mathrm{perf}(X)$.

Example 1.36. Thanks to Theorem 1.27, for any smooth and proper k-scheme X, there is a unique smooth and proper noncommutative k-scheme $\mathrm{perf}(X)$. Notice that there exist non-isomorphic k-schemes whose associated noncommutative k-schemes are equivalent. The first such example is provided by an Abelian variety X of dimension $\dim X \geqslant 2$ and its dual $\hat{X}$. Indeed, $\mathrm{perf}(X) \simeq \mathrm{perf}(\hat{X})$, see Mukai [**Muk81**].

In the geometric case, that is for a noncommutative scheme of the form $\mathrm{perf}(X)$ for X a quasi-compact and quasi-separated scheme, the previous properties of properness, regularity and smoothness recover properness, regularity and smoothness of X, as shown by Orlov [**Orl14**] (see also [**LS15**]).

Proposition 1.37 (Orlov)**.** *Let X be a separated regular Noetherian k-scheme. The noncommutative scheme* $\mathrm{perf}(X)$ *is regular. If moreover X is of finite type, then X is smooth and proper if and only if* $\mathrm{perf}(X)$ *is smooth and proper.*

The notion of gluing of dg-categories $\mathcal{A}$ and $\mathcal{B}$ along a $\mathcal{A}-\mathcal{B}$-bimodule S appeared in [**Tab07**, §0], under the name of "catégorie triangulaire supérieure", while the term "gluing" was established by Kuznetsov and Lunts in [**KL12**]. Such a dg-category is denoted by

$\mathcal{C} = \mathcal{A} \oplus_S \mathcal{B}$ in [**Orl14**], and admits a semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$. It is moreover proved in [**Orl14**, Prop. 3.22] that the gluing of regular and proper pretriangulated dg-categories $\mathcal{A}$ and $\mathcal{B}$ via an $\mathcal{A} - \mathcal{B}$-bimodule S is itself regular and proper if and only if $H^*(S(A, B))$ is finite dimensional for any A in $\mathcal{A}$ and B in $\mathcal{B}$.

III.2. Resolutions of singularities. One of the interesting features of noncommutative schemes is the possibility to define *categorical resolution of singularities.* A particularly important question about resolution of singularities is to find the ones that enjoy some minimality property (see e.g. Bondal and Orlov [**BO02**, §5]). The geometrical notion of *crepant* resolution can be described homologically, as done by Kuznetsov [**Kuz07b**]. We first recall Kuznetsov-Lunts definition in terms of dg categories [**KL12**, Def. 1.5] (A definition in the context of big categories was given by Lunts [**Lun10**, Def. 4.1]). In this report, we will use the following definition in the category of noncommutative schemes.

Definition 1.38. Let $\mathcal{A}$ be a geometric noncommutative k-scheme. A *noncommutative resolution* of $\mathcal{A}$ is a smooth noncommutative k-scheme $\mathcal{B}$ with a functor $\Phi : \mathcal{A} \to \mathcal{B}$ inducing a fully faithful functor $H^0(\mathcal{A}) \to H^0(\mathcal{B})$. If $\mathcal{A} = \mathrm{perf}(X)$ for some scheme X, we will say that $\mathcal{B}$ is a noncommutative resolution of X.

Let us recall other definitions in the literature and comment Definition 1.38. The next definition, involving only triangulated categories, was given by Kuznetsov [**Kuz07b**].

Definition 1.39. Let $\mathbf{A}$ be a triangulated category. A *categorical resolution* of $\mathbf{A}$ is a triangulated category $\mathbf{B}$ and a pair of triangulated functors $\pi^* : \mathbf{A}_c \to \mathbf{B}$ and $\pi_* : \mathbf{B} \to \mathbf{A}$ such that

1) There exists a smooth projective variety Y and a fully faithful functor $\mathbf{B} \to \mathrm{perf}(Y)$,
2) The functor π^* is left adjoint to π_* on $\mathbf{A}_c$,
3) The natural transformation $\mathrm{id}_{\mathbf{A}_c} \to \pi_*\pi^*$ is the identity.

A *categorical resolution* of a variety X is a categorical resolution of $\mathrm{D^b}(X)$.

A categorical resolution of singularities $(\mathbf{B}, \pi_*, \pi^*)$ of $\mathbf{A}$ is *weakly crepant* if π^* is right adjoint to π_* on $\mathbf{A}_c$. If $\mathbf{A}$ and $\mathbf{B}$ admit Serre functors, the resolution is called *strongly crepant* if the identity of $\mathbf{B}$ is the relative Serre functor of $\mathbf{B}$ over $\mathbf{A}$.

Remark 1.40. Let $(\mathbf{B}, \pi^*, \pi_*)$ be a categorical resolution of $\mathbf{A}$, and $\mathcal{B}$ and $\mathcal{A}$ dg enhancements of $\mathbf{B}$ and $\mathbf{A}$ respectively, and assume that the functors π^* and π_* lift to dg functors. Then $\mathcal{B}$ is a noncommutative resolution of $\mathcal{A}$. Using this, one can naturally extend the notion of weak crepancy to noncommutative resolutions.

Kuznetsov and Lunts have shown that if X is a k-scheme, then a noncommutative resolution of $\mathrm{D^b}(X)$ induces a categorical resolution of X [**KL12**, Prop. 3.13]. On the other hand, item 1) in Definition 1.39 tells us that $\mathbf{B}$ can be enhanced from the unique enhancement of $\mathrm{perf}(Y)$ and the choice of a semiorthogonal decomposition $\mathrm{perf}(Y) = \langle \mathbf{B}, \mathbf{B}^{\perp} \rangle$, but this enhancement may not be unique.

Kuznetsov and Lunts have shown that categorical resolution of singularities exist for any separated scheme of finite type over a field of characteristic 0 [**KL12**, Thm. 1.4]. On the other hand, Definition 1.38 may seem rather weak: for example, consider a scheme X with rational singularities, and $f : Y \to X$ a (geometric) resolution. Then $\mathrm{perf}(Y)$ and the functors f^* and f_* provide a categorical resolution of X. But notice that composing f the natural projection $Y \times \mathbb{P}^n \to Y$, one obtains $g : Y \times \mathbb{P}^n \to X$ such that $\mathrm{perf}(Y \times \mathbb{P}^n)$ and the functors g^* and g_* also provide a categorical resolution of X.

Remark 1.41. If $\pi : Y \to X$ is a resolution of Gorenstein singularities, then the categorical resolution $\mathrm{perf}(Y)$ of $\mathrm{D^b}(X)$ is weakly (or strongly) crepant if π is crepant, that is $\pi^*\omega_X = \omega_Y$. In particular, strong and weak crepancy are equivalent in this case. Notice however that there cases of weakly but not strongly crepant categorical resolutions. In particular, to define strongly crepant resolutions one needs $\mathbf{B}$ to have a module structure over $\mathbf{A}_c$ (see Kuznetsov for more details [**Kuz07b**]). If $\mathcal{B}$ and $\mathcal{A}$ are their respective enhancements, this amounts to say that $\mathcal{B}$ is a noncommutative $\mathcal{A}$-scheme.

Let us finally mention Van den Bergh's definition of noncommutative crepant resolution for a commutative k-scheme X [**VdB04**, Def. 4.1, Rmk. 4.5].

Definition 1.42. Let R be a normal Gorenstein domain. A *noncommutative crepant resolution* of R is a homologically homogeneous R-algebra of the form $A = \mathrm{End}_R(M)$, where M is a reflexive R-module. If X is a scheme, a non-commutative crepant resolution of X is a stack of Abelian categories $\mathcal{A}$ which is, locally on any affine open subset $\mathrm{Spec}R$ of X, the category of finitely-generated modules over a non-commutative crepant resolution A of R.

Remark 1.43. If $\mathcal{A}$ is a non-commutative crepant resolution of a scheme X, then $\mathcal{A}$ is a noncommutative weakly crepant resolution of X in the sense of Definition 1.38. In particular, $\mathcal{A}$ is a smooth noncommutative X-scheme.

III.3. Representability for noncommutative schemes. Let $\mathcal{A}$ be a smooth and proper noncommutative scheme. The following notion of *representability* was introduced in [**BB12**] for triangulated categories and is motivated by the theory of noncommutative motives which will be introduced in Section IV.

Definition 1.44. A noncommutative scheme $\mathcal{A}$ is *representable in dimension* n if there exists a semiorthogonal decomposition

$$\mathcal{A} = \langle \mathcal{A}_1, \dots, \mathcal{A}_r \rangle$$

and smooth projective k-schemes $Y_1, \dots Y_r$ such that, for all $i = 1, \dots, r$ we have $\dim(Y_i) \leqslant n$ and a fully faithful functor $\mathcal{A}_i \to \mathrm{perf}(Y_i)$ admitting right and left adjoints.

If $\mathcal{A}$ is representable in some dimension m, we use the following notation

$$\mathrm{rdim}\,\mathcal{A} := \min\{m | \,\mathcal{A} \text{ is representable in dimension } m\}$$

Remark 1.45. The categories $\mathcal{A}_i$ in Definition 1.44 are geometric noncommutative schemes. It follows that if a noncommutative scheme $\mathcal{A}$ is representable in dimension n for some integer n then $\mathcal{A}$ is a geometric noncommutative scheme thanks to [**Orl14**]. On the other hand, any geometric noncommutative scheme is representable for some n by definition.

Question 1.46 ([**Orl14**], Question 4.4)**.** All the known examples of noncommutative schemes are geometric, hence representable in some finite dimension n. Do there exists non geometric noncommutative schemes?

Let us give some easy result about representability of noncommutative schemes.

Lemma 1.47. *If a noncommutative scheme $\mathcal{A}$ is representable in dimension n, then it is representable in dimension m for all $m \geqslant n$.*

Proof. Notice that this is straightforward from the definition: suppose that $\mathcal{A}_1 \subset \mathcal{A}$ is an admissible subcategory admitting a full and faithful embedding $\mathcal{A}_1 \to \mathrm{perf}(Y_1)$ for some smooth and projective Y_1 of dimension $\leqslant n$. Then $\dim(Y_1) \leqslant m$ as well. □

Notice that, if $\mathcal{A}_1$ is a noncommutative scheme with a fully faithful embedding $\mathcal{A}_1 \to \mathrm{perf}(Y_1)$ for some smooth projective Y_1 of dimension exactly n, then, for any $m > n$, we can easily construct a smooth projective variety Y of dimension exactly m and a full and faithful embedding $\mathcal{A}_1 \to \mathrm{perf}(Y)$: it is enough to consider $Y := Y_1 \times \mathbb{P}^{m-n}$. More generally, any variety $\pi : Y \to Y_1$ of dimension m and such that π satisfies $R\pi_*\mathcal{O}_Y = \mathcal{O}_{Y_1}$ will admit $\mathcal{A}_1$ as an admissible subcategory (see, e.g., Proposition 4.11).

Lemma 1.48. *Let $\mathcal{A}$ and $\mathcal{B}$ be noncommutative schemes with* $\mathrm{rdim}\,\mathcal{A} = n$ *and* $\mathrm{rdim}\,\mathcal{A} = m$*, and S a perfect $\mathcal{A}-\mathcal{B}$-bimodule. Then the gluing $\mathcal{C} := \mathcal{A} \bigoplus_S \mathcal{B}$ is representable in dimension* $\max(m, n)$*. In particular,* $\mathrm{rdim}\,\mathcal{C} \leqslant \max(m, n)$*.*

Proof. This follows easily from the fact that $\mathcal{C}$ admits a semiorthogonal decomposition $\mathcal{C} := \langle \mathcal{A}, \mathcal{B} \rangle$. □

Notice that we can classify noncommutative schemes with $\mathrm{rep}\mathcal{A} \leqslant 1$.

Proposition 1.49. *Let $\mathcal{A}$ be a smooth and proper noncommutative semiorthogonally indecomposable k-scheme.*

If $\mathrm{rep}\mathcal{A} = 0$*, then there exist a finite field extension l of k and* $\mathcal{A} \simeq \mathrm{perf}(\mathrm{Spec}(l))$*.*

If $\mathrm{rep}\mathcal{A} = 1$*, then either there exists a curve C of positive genus and* $\mathcal{A} \simeq \mathrm{perf}(C)$*, or there exists a class α in* $\mathrm{Br}(k)$ *of a non-k-rational conic such that* $\mathcal{A} = \mathrm{perf}(A)$ *for an Azumaya algebra A with class α.*

If $\mathcal{A}$ is not semiorthogonally indecomposable, and $\mathrm{rep}\mathcal{A} \leqslant 1$*, we have a semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{A}_1, \dots, \mathcal{A}_r \rangle$ with $\mathcal{A}_i$ as above.*

The last statement of Proposition 1.49 is, by definition, just a consequence of the previous ones. The $\mathrm{rep}\mathcal{A} = 0$ case is treated in [**AB15**], via the simple observation that zero dimensional k-schemes are spectra of étale algebras. The second statement is a consequence of Okawa's study of indecomposability of derived categories of curves of positive genus [**Oka11**], and of the description of semiorthogonal decompositions for Brauer-Severi varieties [**Ber09**] together with the observation that all semiorthogonal decompositions of a conic are mutated to that one.

We end this section by remarking that the definition of categorical representability relies on the existence of semiorthogonal decompositions of a noncommutative scheme. In general, Jordan-Hölder property does not hold for semiorthogonal decompositions; see [**BGvBS14**] or [**Kuz13**] (we will recall details in Example 1.52). It follows that if $\mathcal{A}$ and $\mathcal{B}$ are noncommutative schemes such that $\operatorname{rdim} \mathcal{A} = n$, and $\mathcal{B}$ is admissible in $\mathcal{A}$, it is not known how to prove that $\operatorname{rep}\mathcal{B} \leqslant n$, and counterexamples exist, even in the case where $n = 0$: Kuznetsov's example [**Kuz13**] is a rational threefold X with a full exceptional collection, so that $\mathcal{A} := \operatorname{perf}(X)$ has $\operatorname{rdim} \mathcal{A} = 0$, but admits an admissible subcategory $\mathcal{B}$ such that $\operatorname{rep}\mathcal{B} > 0$.

III.4. The Grothendieck ring of noncommutative schemes and its filtration. We sketch the Bondal-Larsen-Lunts construction of the Grothendieck ring of smooth and proper noncommutative schemes [**BLL04**]. Consider the free $\mathbb{Z}$-module generated by smooth and proper noncommutative schemes in $\mathsf{dgcat}(k)$, and introduce the following relation:

$$\mathcal{A} = \mathcal{B} + \mathcal{C} \quad \text{if, up to dg-equivalence, } \mathcal{A} = \langle \mathcal{B}, \mathcal{C} \rangle. \tag{7}$$

We denote the quotient group by $PT(k)$ (see [**BLL04**, §5.1]).

For any noncommutative scheme $\mathcal{A}$ we use either the notation $I(\mathcal{A})$ or the lower case notation a to denote its class in $PT(k)$. More generally, elements of $PT(k)$ will be denoted by a lower case letter. For a smooth projective variety X, we will often use the notation x for the class of $\operatorname{perf}(X)$ in $PT(k)$.

Lemma 1.50. *Let a be any element of $PT(k)$. If $a = \sum_{i=1}^r m_i I(\mathcal{A}_i)$ with $m_i > 0$ for all $i = i, \dots, r$, then there exists a smooth and proper noncommutative scheme $\mathcal{A}$ such that $a = I(\mathcal{A})$.*

Proof. The noncommutative scheme $\mathcal{A}$ can be described as a recursive gluing of the $\mathcal{A}_i$ along the trivial bimodules. Concretely, $\mathcal{A} \simeq \bigoplus_{i=1}^r \mathcal{A}_i^{\oplus m_i}$. □

In the additive commutative group $PT(k)$, define the following associative product:

$$I(\mathcal{A}) \bullet I(\mathcal{B}) = I(\mathcal{A} \otimes \mathcal{B}). \tag{8}$$

Proposition 1.51 ([**BLL04**], Cor. 5.7)**.** *The group $PT(k)$ endowed with the product $\bullet$ is a commutative associative ring with unit $\mathbf{e} = I(\operatorname{perf}(\operatorname{Spec}(k)))$.*

Example 1.52. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be smooth and proper noncommutative schemes. A relation $I(\mathcal{A}) = I(\mathcal{B}) + I(\mathcal{C})$ in $PT(k)$ does not necessarily imply that $\mathcal{A} = \langle \mathcal{B}, \mathcal{C} \rangle$. The simplest example is due to Bondal, and described by Kuznetsov in [**Kuz13**].

Consider the quiver:

$$Q = \quad \bullet \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \bullet \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \bullet,$$

with the relations $\beta_1\alpha_2 = \beta_2\alpha_1 = 0$. Setting $\mathcal{A} := \operatorname{perf}(Q)$, the perfect complexes on the representation algebra of Q, it is easy to describe a full exceptional sequence $\mathcal{A} = \langle E_1, E_2, E_3 \rangle$ by considering E_i to be the projective module of the i-th vertex. On the other hand, one can consider the exceptional object F give by the module:

$$F = \qquad k \underset{0}{\overset{1}{\rightrightarrows}} k \underset{0}{\overset{1}{\rightrightarrows}} k,$$

and the semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{B}, F \rangle$, so that one easily obtain $\mathcal{B} = 2\mathbf{e}$. But there is no exceptional object in $\mathcal{B}$. Notice that this also gives an example of a noncommutative schemes $\mathcal{B}$ and $\mathcal{A}$, such that $\operatorname{rdim} \mathcal{B} > 0$, $\operatorname{rdim} \mathcal{A} = 0$ and $\mathcal{B}$ admissible in $\mathcal{A}$. Kuznetsov describes moreover how to construct a rational threefold X, with a full exceptional collection, and a fully faithful functor $\mathcal{A} \to \operatorname{perf}(X)$.

Another geometrical example is due to Böhning, Graf von Bothmer and Sosna. For X the classical Godeaux surface, they have constructed two semiorthogonal decompositions

$$\operatorname{perf}(X) = \langle E_1, \ldots, E_{11}, \mathbf{B}_1 \rangle = \langle F_1, \ldots, F_9, \mathbf{B}_2 \rangle,$$

where E_i and F_i are exceptional objects [**BGvBS13**, **BGvBS14**]. Considering the induced dg enhancements, this provides two noncommutative schemes $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $I(\mathcal{B}_2) = I(\mathcal{B}_1) + 2\mathbf{e}$. On the other hand, as proved in [**BGvBS14**], there is no exceptional object in $\mathcal{B}_2$. However, there is no exceptional object in $\mathcal{B}_1$ either, and one can show that $\operatorname{rep}\mathcal{B}_1 = \operatorname{rep}\mathcal{B}_2 = 2$ in this case.

The notion of categorical representability induces a ring filtration on $PT(k)$. Define the following $\mathbb{Z}$-submodules of $PT(k)$:

(9)

$$PT_d(k) := \mathbb{Z}\langle I(\mathcal{A}) \in PT(k) \mid \text{there exists } \mathcal{B} \text{ such that } \operatorname{rdim} \mathcal{B} \leqslant d,\ \mathcal{A} \to \mathcal{B} \text{ is admissible}\rangle.$$

The set $PT_\infty(k) := \bigcup_{i \in \mathbb{N}} PT_i(k)$ is the set of elements in $PT(k)$ generated by geometric noncommutative schemes. We denote by $PT_{>i}(k)$ the complement of $PT_i(k)$ in $PT(k)$. We can restate Question 1.46 as follows

Question 1.53. Is $PT_{>\infty}(k)$ empty?

Proposition 1.54. *The subsets $PT_i(k)$ give a filtration on the ring $PT(k)$. More precisely, suppose that a is in $PT_i(k)$ and b is in $PT_j(k)$. Then*

$$\begin{aligned} a + b \ \ &\textit{is in } PT_{\max(i,j)}(k), \\ a \bullet b \ \ &\textit{is in } PT_{i+j}(k). \end{aligned}$$

In particular, $PT_i(k)$ is an additive subgroup for any i.

Proof. We can test these properties on generators of $PT_i(k)$. First of all, $PT_i(k) \subset PT_{i+1}(k)$ for any $i \geqslant 0$ integer by definition. The second statement follows from Lemmas 1.50 and 1.48.

We are left to show the last statement. To this end, suppose that $a = I(\mathcal{A})$ and $b = I(\mathcal{B})$ such that $\mathcal{A} \subset \operatorname{perf}(X)$ and $\mathcal{B} \subset \operatorname{perf}(Y)$ are admissible with X and Y of dimension i and j respectively. Consider the variety $W := X \times Y$. As shown by Bondal, Larsen and Lunts [**BLL04**], we have $w = x \bullet y$. Using the ring structure, we get $w = a \bullet b + a^\perp \bullet b + a \bullet b^\perp + a^\perp \bullet b^\perp$. In particular, it follows that if $\operatorname{rdim} \mathcal{A} \leqslant i$ and $\operatorname{rdim} \mathcal{B} \leqslant j$, then $a \bullet b$ is in $PT_{i+j}(k)$, and we can conclude. $\square$

Finally notice that there exist noncommutative schemes $\mathcal{B}$ such that rep$\mathcal{B} > 0$, but $I(\mathcal{B}) \in PT_0(k)$, as recalled in Example 1.52. It would be then natural to wonder whether the filtration $PT_i(k)$ is not trivial, that is whether there exist integers i such $PT_i(k) \neq PT_{i+1}(k)$. Using the theory of Jacobians of noncommutative motives and Theorem 3.12, it is possible for example to show that, for k algebraically closed of characteristic zero, if X has a nontrivial intermediate Jacobian, or nontrivial Albanese variety, then $I(\mathrm{D^b}(X))$ cannot lie in $PT_0(k)$. It follows that $PT_0(k) \neq PT_1(k)$.

III.5. Base change and descent of semiorthogonal decompositions. Let K/k be a field extension and $\mathcal{A}$ a noncommutative k-scheme, and $\mathcal{A}_K$ the base change of $\mathcal{A}$, as in Definition 1.33.

The first question one can address is to understand semiorthogonal decompositions of $\mathcal{A}$ under base change. This question was addressed by Kuznetsov in the setting of triangulated categories arising as bounded derived categories of schemes [**Kuz11**], so we restrict to geometric noncommutative schemes. Here we give a sample of results on base change for semiorthogonal decomposition, in the case where K/k is finite.

Proposition 1.55. *If $\mathcal{A} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ is a semiorthogonal decomposition of a geometric noncommutative k-scheme $\mathcal{A}$, then $\mathcal{A}_K = \langle \mathcal{A}_{1K}, \dots, \mathcal{A}_{nK} \rangle$ is a semiorthogonal decomposition of the noncommutative K-scheme $\mathcal{A}_K$.*

If $\{\mathcal{A}_1, \dots, \mathcal{A}_n\}$ is a set of admissible components of noncommutative k-scheme $\mathcal{A}$ and $\mathcal{A}_K = \langle \mathcal{A}_{1K}, \dots, \mathcal{A}_{nK} \rangle$ is a semiorthogonal decomposition of the noncommutative K-scheme $\mathcal{A}_K$, then $\mathcal{A} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ is a semiorthogonal decomposition of the noncommutative k-scheme $\mathcal{A}$.

Proof. We first notice that if $\mathcal{B}$ is a noncommutative k-scheme such that $\mathcal{B}_K = 0$ in NSch(K), then $\mathcal{B} = 0$ in Nsch(k). The proof of the two statements follows then by remarking that semiorthogonality is stable under base change[1]. □

A natural question to consider is to classify, given a noncommutative K-scheme, $\mathcal{A}$, all the noncommutative k-schemes $\mathcal{B}$ such that $\mathcal{B}_K = \mathcal{A}$. We end this section by illustrating descent in the most simple cases. If $\mathcal{A}_K$ is generated by a K-exceptional object, the descent question has been studied by Toën [**Toë12**, Cor. 2.15].

Theorem 1.56 (Toën)**.** *Let $\mathcal{A}$ be a noncommutative k-scheme, K/k a field extension and assume that $\mathcal{A}_K = \mathrm{perf}(K)$. Then there exists a central simple algebra A over k such that $\mathcal{A} = \mathrm{perf}(A)$. In particular, $\mathcal{A}$ is generated by an A-exceptional object which is k-exceptional if and only if A is trivial in the Brauer group.*

Let us denote by k^s the separable closure of k. Recall that L is a finite étale k-algebra if and only if $L \otimes_k k^s \simeq k^s \times \cdots \times k^s$. Equivalently, $L \simeq l_1 \times \cdots \times l_m$ where each l_i/k is a finite separable field extension. The k-dimension of L is called the *degree* of L over k. An Azumaya algebra A over $L \simeq l_1 \times \cdots \times l_m$ is simply a product $A \simeq A_1 \times \cdots \times A_m$ where each

[1]Here we could refer to Sosna's work [**Sos14**] for a different notion of scalar base change of a triangulated category. But, as remarked by the same author, [**Sos14**, page 15], the two definitions coincide.

A_i is a central simple l_i-algebra. A *separable algebra* of dimension nr^2 over k is an Azumaya algebra of degree r over an étale algebra K over k of dimension n.

The following Proposition was proved in [**AB15**] and generalizes the result of Toën to the case of finite products of fields.

Proposition 1.57. *Let $\mathcal{A}$ be a noncommutative k-scheme such that $\mathcal{A}_{k^s}$ is k^s-equivalent to* $\mathrm{perf}((k^s)^n)$*. Then there exists an étale algebra K of degree n over k, an Azumaya algebra A over K, and a k-linear equivalence $\mathcal{A} \simeq \mathrm{perf}(A)$. In this case, $\mathcal{A}$ is an indecomposable category if and only if K is a field extension of k.*

The key idea in the proof of Proposition 1.57 is to extend Toën's construction to study sections over k of the stack F associated to the prestack of dg algebras étale locally Morita equivalent to k^n, which is simply $K(\mathbf{Aut}_{dg/k}(k^n), 1)$, as in [**Toë12**, Cor. 3.12]. The main ingredient is the description of the derived group stack of autoequivalences $\mathbf{Aut}_{dg/k}(k^n)$ of the étale k-algebra k^n (thought of as a dg algebra over k), as the wreath product $(\mathbb{Z} \times K(\mathbb{G}_{\mathrm{m}}, 1)) \wr S_n$, thought of as $n \times n$ generalized permutation matrices filled with shifts of invertible modules.

We conclude this section by remarking that, if $\mathcal{A}$ is a noncommutative k-scheme and $\mathrm{rep}\mathcal{A} = m$, and K/k is a field extension, one easily has $\mathrm{rep}\mathcal{A}_K \leqslant m$, but the strict inequality can hold. For example if A is a nontrivial Azumaya k-algebra with splitting field K, then $\mathrm{rep}\mathcal{A} > 0$ while $\mathrm{rep}\mathcal{A}_K = 0$, thanks to the classification in Proposition 1.49.

IV. Noncommutative Chow motives

IV.1. Definition and basic properties. In this section we recall the construction of the category of noncommutative Chow motives of smooth and proper dg categories. Such category is meant to give a motivic theory for noncommutative schemes which should be parallel to the theory of Chow motives for schemes: perfect bimodules will play the role of correspondences, while a universal property with respect to additive invariants will replace the universal property with respect to Weil cohomologies. For further details we invite the reader to consult the recent book [**Tab15**].

Recall from §II (see, in particular, (5)) that one has a well-defined functor

$$\mathsf{dgcat}(k) \longrightarrow \mathsf{Hmo}(k) \quad \mathcal{A} \mapsto \mathcal{A} \quad F \mapsto \hat{F}\,. \tag{10}$$

The *additivization of* $\mathsf{Hmo}(k)$ is the additive category $\mathsf{Hmo}_0(k)$ which has the same objects as $\mathsf{Hmo}(k)$ and Abelian groups of morphisms given by $\mathrm{Hom}_{\mathsf{Hmo}_0(k)}(\mathcal{A}, \mathcal{B}) := K_0\,\mathrm{rep}(\mathcal{A}, \mathcal{B})$, where K_0 stands for the Grothendieck group of the triangulated category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. The composition law is induced by the tensor product of bimodules; consult [**Tab05**, §6] for further details. Note that we have a canonical functor

$$\mathsf{Hmo}(k) \longrightarrow \mathsf{Hmo}_0(k) \quad \mathcal{A} \mapsto \mathcal{A} \quad B \mapsto [B]\,. \tag{11}$$

For a $\mathbb{Z}$-module R, the *R-linearization* of $\mathsf{Hmo}_0(k)$ is the R-linear additive category $\mathsf{Hmo}_0(k)_R$ obtained by tensoring each Abelian group of morphisms of $\mathsf{Hmo}_0(k)$ with R. In particular

$\mathsf{Hmo}_0(k)_{\mathbb{Z}} = \mathsf{Hmo}_0(k)$. Note that one also has a canonical functor

$$\mathsf{Hmo}_0(k) \longrightarrow \mathsf{Hmo}_0(k)_R \quad \mathcal{A} \mapsto \mathcal{A} \quad [B] \mapsto [B]_R\,. \tag{12}$$

Since the three functors (10)-(12) are the identity on objects we will make no notational distinction between a dg category and its image in $\mathsf{Hmo}_0(k)_R$.

Definition 1.58. The category $\mathrm{NChow}(k)_R$ of *noncommutative Chow motives* (with coefficients in the ring R) is the pseudo-Abelian envelope of the full subcategory of $\mathsf{Hmo}_0(k)_R$ consisting of smooth and proper dg categories.

Let us describe two fundamental properties motivating the fact that such a category should be thought of as the category of motives of noncommutative schemes. Given a dg category $\mathcal{A}$, we denote by $T(\mathcal{A}) :_= \mathcal{A} \oplus_{\mathrm{id}} \mathcal{A}$ the gluing of $\mathcal{A}$ with itself along the identity bimodule. We have two natural inclusion dg functors $i_1 : \mathcal{A} \to T(\mathcal{A})$ $i_2 : \mathcal{A} \to T(\mathcal{A})$.

Definition 1.59. Let $E : \mathsf{dgcat}(k) \to \mathrm{C}$ be a functor with values in an additive category C. The functor E is an *additive invariant* if:

- E sends Morita equivalences to isomorphisms;
- given any dg category $\mathcal{A}$, the inclusion dg functors induce an isomorphism

$$[E(i_1), E(i_2)] : E(\mathcal{A}) \oplus E(\mathcal{A}) \longrightarrow E(T(\mathcal{A})).$$

Thanks to [**Tab05**, Thm 6.3], if $\mathcal{A} = \langle \mathcal{B}, \mathcal{C} \rangle$ is a semiorthogonal decomposition of a dg-category, and E an additive invariant, then $E(\mathcal{A}) = E(\mathcal{B}) \oplus E(\mathcal{B})$.

Example 1.60. Thanks to the work of Blumberg and Mandell, Keller, Schlichting, Tabuada, Thomason and Trobaugh, Waldhausen, and Weibel (see [**BM12**, **Kel99**, **Kel98b**, **Kel98a**, **Sch06**, **Tab12**, **Tab10**, **Tho90**, **Wal85**, **Wei89**]), examples of additive invariants include connective algebraic K-theory, nonconnective algebraic K-theory, homotopy algebraic K-theory, Hochschild homology, cyclic homology, periodic cyclic homology, negative cyclic homology, topological Hochschild homology, and topological cyclic homology.

The following two propositions were proved by Tabuada [**Tab05**]. They show that noncommutative motives are the universal category for additive invariants and that any semiorthogonal decomposition of a dg category $\mathcal{A}$ splits the motive $U(\mathcal{A})$.

Proposition 1.61 (Tabuada)**.** *Every additive invariant $E : \mathsf{dgcat}(k) \to \mathrm{C}$ factors through the natural functor $U : \mathsf{dgcat}(k) \to \mathsf{Hmo}_0(k)$.*

Proposition 1.62 (Tabuada)**.** *Let $\mathcal{A}$ be a smooth proper noncommutative k-scheme and $\mathcal{A} = \langle \mathcal{B}, \mathcal{C} \rangle$ a semiorthogonal decomposition. Then $U(\mathcal{A}) = U(\mathcal{B}) \oplus U(\mathcal{C})$. In particular, $E(\mathcal{A}) = E(\mathcal{B}) \oplus E(\mathcal{C})$ for any additive invariant E.*

Let X be a smooth and projective k-scheme, and $\mathrm{perf}(X)$ the associated smooth and proper noncommutative scheme. One can define the *noncommutative Chow motive* of X (or of $\mathrm{perf}(X)$) as:

$$NC(X)_R := U(\mathrm{perf}(X))_R,$$

where U is the universal functor described above.

IV.2. Noncommutative Smash-nilpotent, Homological, and Numerical motives. As in the commutative case, one can consider categories of Smash-nilpotent, homological and numerical noncommutative motives over k.

Let F be a field. The category $\mathrm{NChow}(k)_F$ is F-linear, additive, idempotent complete and rigid symmetric monoidal. For an F-linear, additive, rigid symmetric monoidal category C, one can define the $\otimes_{\mathrm{nil}}$-ideal by:

$$\otimes_{\mathrm{nil}}(x,y) := \{f \in \mathrm{Hom}_{\mathrm{C}}(x,y) \,|\, f^{\otimes n} = 0 \text{ for some } n > 0\}, \tag{13}$$

and check that is is a $\otimes$-ideal. Similarly to the commutative case, the category $\mathrm{NVoev}(k)_F$ of *$\otimes$-nilpotent noncommutative motives* can be defined as the quotient of $\mathrm{NChow}(k)_F$ by the ideal $\otimes_{nil}$. One can check that the category $\mathrm{NVoev}(k)_F$ is also F-linear, additive and idempotent complete since the quotient functor is F-linear, additive, conservative and idempotent can be lifted along nilpotent ideals [**BMT14**].

Periodic cyclic homology gives a functor $HP : \mathsf{dgcat}(k) \to \mathrm{Vect}_{\mathbb{Z}/2}(k)$ to the category of $\mathbb{Z}/2\mathbb{Z}$-graded k-vector spaces [**Kas87**] (which coincides with odd and even de Rham cohomology for the dg category of perfect complexes over a smooth projective variety X, in the case where k has characteristic zero [**FT87**]). Let F be a field, such that either F is an extension of k or k is an extension of F. Set K to be the bigger field between k and F. Marcolli and Tabuada have proven that HP induces an F-linear symmetric monoidal functor

$$HP^{\pm} : \mathrm{NChow}(k)_F \to \mathrm{Vect}_{\mathbb{Z}/2}(K).$$

The category $\mathrm{NHom}(k)_F$ of *homological commutative motives* can be defined to be the idempotent completion of the quotient of $\mathrm{NChow}(k)_F$ by the kernel of $HP^{\pm}$. This category is F-linear, additive, rigid symmetric monoidal and idempotent complete, see [**MT16**].

Given a proper k-linear dg category $\mathcal{A}$, one can define the bilinear Euler pairing χ on objects of $\mathcal{A}$ as the alternate sum of dimensions of morphism spaces. One can consider its left and right kernels, which coincide whenever $\mathcal{A}$ is smooth, due to the existence of a Serre functor (see [**MT12**, §4]). Moreover χ descends to a bilinear pairing on $K_0(\mathcal{A})$, and Kontsevich [**Kon05**] defines the category $\mathrm{NNum}(k)_F$ of *commutative numerical motives* as the (idempotent completion of the) category whose objects are smooth and proper dg categories and morphisms spaces are

$$\mathrm{Hom}_{\mathrm{NNum}(k)_F}(\mathcal{A},\mathcal{B}) = K_0(\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}})/\ker(\chi). \tag{14}$$

There is an alternative construction of $\mathrm{NNum}(k)_F$ due to Marcolli and Tabuada [**MT14b**] which coincides with the one above [**MT12**]. For an F-linear, additive, rigid symmetric monoidal category C, one can define the $\mathcal{N}$-ideal by:

$$\mathcal{N}(x,y) := \{f \in \mathrm{Hom}_{\mathrm{C}}(x,y) \,|\, \text{ for any } g \in \mathrm{Hom}_{\mathrm{C}}(y,x),\ \mathrm{tr}(g\circ f) = 0\},$$

and check that is a $\otimes$-ideal. The category $\mathrm{NNum}(k)_F$ is equivalent to be the idempotent completion of the quotient of $\mathrm{NChow}(k)_F$ by the ideal $\mathcal{N}$. It is then additive, F-linear, symmetric rigid monoidal and idempotent complete. It is moreover semisimple, see [**MT14b**].

IV.3. Comparison between commutative and noncommutative motives. Given a monoidal category C and a $\otimes$-invertible object $\mathbb{L}$ of C, the *orbit category* of C with respect to $\mathbb{L}$ is the category $\mathrm{C}/-\otimes\mathbb{L}$ with the same objects as C and morphisms

$$\mathrm{Hom}_{\mathrm{C}/-\otimes\mathbb{L}}(A,B) = \oplus_{i\in\mathbb{Z}}\mathrm{Hom}_{\mathrm{C}}(A, B\otimes\mathbb{L}^i).$$

Let R be a $\mathbb{Z}$-module and $\mathrm{Chow}(k)_R$ denote the category of Chow motives of varieties over k with R-coefficients. Consult André's book for the construction and properties of motives [**And04**]. We denote by $\mathbf{L}$ the motive of the affine line in $\mathrm{Chow}(k)_R$. Recall that $\mathbf{L}$ is invertible in the monoidal category $\mathrm{Chow}(k)_R$. As an application of the Grothendieck-Riemann-Roch theorem, Tabuada [**Tab13**] shows that there is a full and faithful embedding

$$\mathrm{Chow}(k)_{\mathbb{Q}}/-\otimes\mathbf{L} \longrightarrow \mathrm{NChow}(k)_{\mathbb{Q}}.$$

One can summarize this result by saying that, in rational coefficients, noncommutative Chow motives encode the same informations as Chow motives, up to codimension. This is no longer true over the integer coefficients, as we will point out in Section I of Chapter 3.

As proved by Marcolli and Tabuada [**MT14a**], similar comparison functors exist also for numerical and homological motives.

IV.4. A filtration by thick subcategories. Define $\mathrm{NChow}_d(k)$ to be the thick subcategory generated by motives $NC(X)$ for X of dimension $\dim(X) \leqslant d$. Then it is easy to see that $\mathrm{NChow}_i(k) \subset \mathrm{NChow}_{i+1}(k)$, and that if M is in $\mathrm{NChow}_i(k)$ and N in $\mathrm{NChow}_j(k)$, then $M\oplus N$ is in $\mathrm{NChow}_{\max(i,j)}(k)$ and $M\otimes N$ is in $\mathrm{NChow}_{i+j}(k)$. Similar filtrations hold for $\mathrm{NHom}(k)$, $\mathrm{NVoev}(k)$, and $\mathrm{NNum}(k)$.

Let $\mathcal{A}$ be a noncommutative scheme. If $\mathcal{A}$ is representable in dimension d, then it is easy to check that its noncommutative motive $U(\mathcal{A})$ lies in $\mathrm{NChow}_d(k)$. This observation justify the terminology of representability inspired by various notions of representability of Chow motives (see Chapter 3 for more details).

We have remarked that if $\mathcal{A}$ is a smooth and proper noncommutative k-scheme such that $\mathrm{rdim}\,\mathcal{A} = d$, then $U(\mathcal{A})$ lies in $\mathrm{NChow}_d(k)$. The converse implication is, of course, highly nontrivial.

CHAPTER 2

Homological Projective Duality

The theory of Homological Projective Duality was developed by Kuznetsov [**Kuz07a**]. The original motivation was to study how semiorthogonal decompositions behave under hyperplane sections (see also [**Kuz05b**]). To this aim, the choice of a polarization is clearly relevant, and forces to consider semiorthogonal decompositions which are compatible with this choice (the so-called Lefschetz decompositions).

Homological Projective Duality is nowadays one of the most powerful tool to describe semiorthogonal decompositions of projective variety, in particular for those obtained as linear sections of a given variety.

Even if developed in the triangulated context, Homological Projective Duality can be adapted to the context of pretriangulated dg-categories, or, better, noncommutative schemes treated in the previous chapter. To this end, we will start recalling some basic properties on Fourier-Mukai functors.

I. Fourier-Mukai functors, splitting functors

I.1. Fourier-Mukai functors and dg functors. Let X be a smooth projective k-scheme. In the previous chapter, we considered the smooth and proper noncommutative scheme $\mathrm{perf}(X)$, that is the canonically dg enhanced triangulated category of perfect complexes on X. In what follows we will consider the bounded derived category $\mathrm{D^b}(X)$, recalling that in this case the smoothness of X implies that the inclusion $\mathrm{perf}(X) \subset \mathrm{D^b}(X)$ is actually an equivalence.

Definition 2.1. Let X and Y be smooth and projective varieties and $\mathcal{P}$ an object of $\mathrm{D^b}(X \times Y)$. The *Fourier-Mukai functor with kernel* $\mathcal{P}$ is the functor $\Phi_{\mathcal{P}} : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ defined by

$$\Phi_{\mathcal{P}}(-) := Rq_*(p^*(-) \otimes \mathcal{P}),$$

where p and q are the projections from $X \times Y$ onto X and Y respectively.

Such functors are exact, and admit right and left adjoint which are also Fourier-Mukai functors. For more details, properties and a deep treatment, we refer to Huybrecht's book [**Huy06**]. Notice that, given any object $\mathcal{P}$ in $\mathrm{D^b}(X \times Y)$, one has two Fourier-Mukai functors, the one defined in 2.1, and the one going in the opposite direction. It is in general clear from the context which are the source and target of the functor, so that no additional notation is required.

Given an exact functor $\Phi : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$, it is natural to wonder under which conditions Φ is of Fourier-Mukai type, that is, if there exists an object $\mathcal{P}$ in $\mathrm{D^b}(X \times Y)$ such that $\Phi \simeq \Phi_{\mathcal{P}}$. Rizzardo and Van den Bergh have constructed an example of functor which does not satisfy this property [**RVdB14**]. However, in many cases an exact functor is of Fourier-Mukai type, for example if Φ is fully faithful [**Orl03**]. The interested reader should consult Canonaco and Stellari's survey [**CS13**].

On the other hand, as recalled in the previous chapter, one can consider $(\mathrm{D}^{\mathrm{b}}(X), \epsilon_X)$ and $(\mathrm{D}^{\mathrm{b}}(Y), \epsilon_Y)$ with their canonical dg enhancements as pretriangulated dg categories. In this case, given an exact functor $\Phi : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$, one can wonder whether Φ can be lifted to a dg functor $\Phi^{\mathsf{dg}} : (\mathrm{D}^{\mathrm{b}}(X), \epsilon_X) \to (\mathrm{D}^{\mathrm{b}}(Y), \epsilon_Y)$. This latter question is equivalent to the previous one: being of Fourier-Mukai type is equivalent to admitting a dg enhancement ([**Toë07**], [**LS15**]).

Lemma 2.2. *Let $\Phi : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ by an exact k-linear functor. The following are equivalent:*

i) *Φ is of Fourier-Mukai type.*
ii) *There exists a dg functor $\Phi^{\mathsf{dg}} : (\mathrm{D}^{\mathrm{b}}(X), \epsilon_X) \to (\mathrm{D}^{\mathrm{b}}(Y), \epsilon_Y)$ such that $H^0(\Phi^{\mathsf{dg}}) = \Phi$.*

Recall the notion of splitting functor from definition 1.10. In Remark 1.30 we noticed how a splitting functor does not necessarily come with a natural enhancement. In the geometric context, this has been conjectured by Kuznetsov in terms of Fourier-Mukai functors.

Conjecture 2.3 (Kuznetsov, [**Kuz07a**] Conj. 3.7)**.** *Any splitting functor $\Phi : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ is of Fourier-Mukai type.*

As we will see in this chapter, splitting functors are one of the main topics in Homological Projective Duality and are in this context always of Fourier-Mukai type, so that they all are dg enhanced.

II. Homological projective duality for noncommutative schemes

Homological Projective Duality is a duality theory between noncommutative schemes over projective varieties $X \to \mathbb{P}(W)$ and $Y \to \mathbb{P}(W^\vee)$, that is X and Y are endowed with a line bundle, the pullback of the hyperplane section. Notice that here $\mathbb{P}(W)$ denotes the space of 1-dimensional linear subspaces through the origin in W. Kuznetsov's original motivation was to study how semiorthogonal decompositions of $\mathrm{D}^{\mathrm{b}}(X)$ behave under hyperplane sections [**Kuz05b**]. The first step in the theory of Homological Projective Duality is indeed to provide what we will call here a *Homological Lefschetz Theory*: given X with a map $X \to \mathbb{P}(W)$, identify decompositions of *Lefschetz type* of $\mathrm{D}^{\mathrm{b}}(X)$, that are decompositions inducing a semiorthogonal sequence in $\mathrm{D}^{\mathrm{b}}(X_H)$, for X_H the general hyperplane section of X. The orthogonal complement $\mathbf{C}_H$ to the restricted semiorthogonal sequence is the main object of study of Homological Projective Duality. Varying H in $\mathbb{P}(W^\vee)$, the family of these orthogonal complements form a dg category $\mathcal{C}_H$, endowed with a decomposition which can be fully recovered from the chosen decomposition of X, as proved by Kuznetsov [**Kuz07a**]. This category can be thought of as the Homological Lefschetz Theory of X with respect to the chosen decomposition, and its decomposition as a dual Lefschetz decomposition with respect to the hyperplane sections of $\mathbb{P}(W^\vee)$.

Given $Y \to \mathbb{P}(W^\vee)$, we will say that a noncommutative Y-scheme is Homological Projective Dual to X with respect to the chosen decomposition if it is dg equivalent to $\mathcal{C}_H$. It turns out then that the critical locus of $Y \to \mathbb{P}(W^\vee)$ is exactly the dual variety $X^\vee$ and that $\mathrm{D}^{\mathrm{b}}(X)$ is the Homological Lefschetz Theory of its own dual. This motivates the terminology Homological Projective Duality. Notice that one can, and should, start by considering a more general noncommutative X-scheme instead of $\mathrm{D}^{\mathrm{b}}(X)$ only.

II.1. Homological Lefschetz Theory. We recall the basic notions of Homological Projective Duality from [**Kuz07a**], but we consider the setting of noncommutative schemes instead of triangulated categories. Indeed, as pointed in the previous section, considering derived categories of k-schemes with Fourier-Mukai functors is equivalent to considering geometric noncommutative k-schemes, and all formal proofs from [**Kuz07a**] will work in this framework. Notice however that, in order to obtain geometric results (that is, result on the structure of the derived category of some scheme), one should consider noncommutative k-schemes that arise as (crepant) noncommutative resolution of singularities of k-schemes.

Even if stated in the framework of noncommutative schemes, all the results and constructions of this section are a translation of Kuznetsov's results from [**Kuz07a**].

Let X be a projective scheme together with a semi ample line bundle $\mathcal{O}_X(1)$. That is, we fix the hyperplane sections of X with respect to the map $f : X \to \mathbb{P}(W)$, where $W := H^0(X, \mathcal{O}_X(1))^\vee$. We set $N := \dim(W)$.

Let $\mathcal{A}$ be a noncommutative X-scheme. We use the notation $\mathcal{A}(i) := \mathcal{A} \otimes \mathcal{O}_X(i)$. Notice that since $\mathcal{O}_X(i)$ is an invertible object in $\mathrm{perf}(X)$, the functor $\otimes \mathcal{O}_X(i)$ is a dg autoequivalence of $\mathcal{A}$, thanks to Remark 1.34.

Definition 2.4. Let $\mathcal{A}$ be a noncommutative X-scheme. A *Lefschetz decomposition* of $\mathcal{A}$ with respect to $\mathcal{O}_X(1)$ is a semiorthogonal decomposition

$$\mathcal{A} = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{i-1}(i-1) \rangle, \tag{15}$$

with $0 \subset \mathcal{A}_{i-1} \subset \dots \subset \mathcal{A}_0$.

A Lefschetz decomposition is said to be *rectangular* if $\mathcal{A}_0 = \dots = \mathcal{A}_{i-1}$.

A *dual Lefschetz decomposition* of $\mathcal{A}$ with respect to $\mathcal{O}_X(1)$ is a semiorthogonal decomposition

$$\mathcal{A} = \langle \mathcal{B}_{j-1}(1-j), \dots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle, \tag{16}$$

with $0 \subset \mathcal{B}_{j-1} \subset \dots \subset \mathcal{B}_0$. Similarly, one can define rectangular dual Lefschetz decompositions.

For any $0 \leqslant l \leqslant i-1$, the l-th *primitive subcategory* $\mathfrak{a}_l$ associated to the Lefschetz decomposition (15) is the right semiorthogonal complement of $\mathcal{A}_{l+1}$ in $\mathcal{A}_l$, that is $\mathcal{A}_l = \langle \mathfrak{a}_l, \mathcal{A}_{l+1} \rangle$. We similarly define the primitive subcategories $\mathfrak{b}_l$ of a dual Lefschetz decomposition.

Remark 2.5. Notice that $\mathfrak{a}_{i-1} = \mathcal{A}_{i-1}$. For any $0 \leqslant l \leqslant i-1$, we have a semiorthogonal decomposition

$$\mathcal{A}_l = \langle \mathfrak{a}_l, \mathfrak{a}_{l+1}, \dots, \mathfrak{a}_{i-1} \rangle.$$

A Lefschetz decomposition is rectangular if and only if $\mathfrak{a}_l = 0$ for $0 \leqslant l \leqslant i-2$.

Let $f : X \to \mathbb{P}(W)$ be the projective map such $f^*\mathcal{O}_{\mathbb{P}(W)}(1) \simeq \mathcal{O}_X(1)$ and $\iota : \mathscr{X} \subset X \times \mathbb{P}(W^\vee)$ the universal hyperplane section of X. We denote by $\mathcal{A}_{\mathscr{X}}$ the noncommutative X-scheme which is given by the base change of $\mathcal{A}$ via the map $\mathscr{X} \to X$.

One can check that there is a semiorthogonal decomposition

$$\mathcal{A}_{\mathscr{X}} = \langle \mathcal{B}, \mathcal{A}_1(1) \boxtimes \mathrm{D}^{\mathrm{b}}(\mathbb{P}(W^\vee)), \ldots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathrm{D}^{\mathrm{b}}(\mathbb{P}(W^\vee))\rangle, \tag{17}$$

obtained just by taking $\mathcal{B}$ to be the orthogonal complement of the semiorthogonal sequence pulled back from X and restricted to $\mathscr{X}$, where we have omitted ι^* for a more readable notation. We set $j := N - 1 - \max\{l \mid \mathcal{A}_l = \mathcal{A}_0\}$. The first main result in the theory of Homological Projective Duality is the description of a dual Lefschetz semiorthogonal decomposition of $\mathcal{B}$ with respect to the line bundle $\mathcal{O}_{\mathbb{P}(W^\vee)}(1)$, which was originally proved in [**Kuz07a**, Prop. 5.10].

Theorem 2.6 (Kuznetsov)**.** *Let $\mathcal{A}$ be a noncommutative X-scheme, and*

$$\mathcal{A} = \langle \mathcal{A}_0, \ldots, \mathcal{A}_{i-1}(i-1)\rangle \tag{18}$$

a Lefschetz decomposition with respect to $\mathcal{O}_X(1)$, and let $\mathcal{B}$ and j be as above. There is a dual Lefschetz decomposition with respect to the line bundle $\mathcal{O}_{\mathbb{P}(W^\vee)}(1)$:

$$\mathcal{B} = \langle \mathcal{B}_{j-1}(1-j), \ldots, \mathcal{B}_1(-1), \mathcal{B}_0\rangle \tag{19}$$

with the same primitive subcategories as (18)*, namely $\mathcal{B}_l = \langle \mathfrak{a}_0, \ldots, \mathfrak{a}_{N-l-2}\rangle$.*

Moreover, one can consider any integer $1 \leqslant r \leqslant N-1$ and the universal r-codimensional linear section: $\iota_r : \mathscr{X}_r \subset X \times \mathrm{Gr}(r, W^\vee)$, which is defined by the following cartesian diagram:

$$\begin{tikzcd} \mathscr{X}_r \arrow[r, "\iota_r"] \arrow[d] & X \times \mathrm{Gr}(r, W^\vee) \arrow[d] \\ \mathbb{P}_{\mathrm{Gr}(r,W^\vee)}(\mathcal{W}_r^\perp) \arrow[r, "\pi_r"] & \mathbb{P}(W) \times \mathrm{Gr}(r, W^\vee), \end{tikzcd}$$

where $\mathcal{W}_r$ is the tautological rank r bundle on the Grassmannian of r-dimensional subspaces of $W^\vee$, and we denote by $\mathcal{W}_r^\perp := (W^\vee \otimes \mathcal{O}_{\mathrm{Gr(r,W^\vee)}}/\mathcal{W}_r)^\vee$ its orthogonal bundle, which is a subbundle of $W \otimes \mathcal{O}_{\mathrm{Gr}(r,W^\vee)}$, whence the map π_r.

We denote by $\mathcal{A}_{\mathscr{X}_r}$ the noncommutative X-scheme which is given by the base change of $\mathcal{A}$ via the map $\mathscr{X}_r \to X$. We notice that $\mathcal{A}_{\mathscr{X}} = \mathcal{A}_{\mathscr{X}_1}$ in these notations. There is a semiorthogonal decomposition

$$\mathcal{A}_{\mathscr{X}_r} = \langle \mathcal{C}_r, \mathcal{A}_r(1) \boxtimes \mathrm{D}^{\mathrm{b}}(\mathrm{Gr}(r, W^\vee)), \ldots, \mathcal{A}_{i-r}(i-r) \boxtimes \mathrm{D}^{\mathrm{b}}(\mathrm{Gr}(r, W^\vee))\rangle, \tag{20}$$

obtained just by taking $\mathcal{C}_r$ to be the orthogonal complement of the semiorthogonal sequence pulled back from X and restricted to $\mathscr{X}_r$, where we have omitted ι_r^* for a more readable notation. Notice that $\mathcal{C}_1 = \mathcal{B}$ in this notation.

On the other hand, one can consider $\mathcal{B}$ as a smooth and proper noncommutative k-scheme, and the map

$$\rho_r : \mathbb{P}_{\mathrm{Gr}(r,W^\vee)}(\mathcal{W}_r) \to \mathbb{P}(W^\vee) \times \mathrm{Gr}(r, W^\vee).$$

Denote by $\mathcal{B}_{\mathrm{Gr}(r,W^\vee)}$ the dg category obtained by the base-change of $\mathcal{B}$ via ρ_r. Notice that $\mathcal{B}$, and hence $\mathcal{C}_1$, identify naturally with $\mathcal{B}_{\mathrm{Gr}(1,W^\vee)}$.

Proposition 2.7. *For any $1 \leqslant r \leqslant N-1$, there is a dg splitting functor $\Phi_r : \mathcal{A}_{\mathscr{X}_r} \to \mathcal{B}_{\mathrm{Gr}(r,W)}$, inducing a semiorthogonal decompositions*

$$\mathcal{B}_{\mathrm{Gr}(r,W^\vee)} = \langle \mathcal{B}_{j-1}(N-r-j) \boxtimes \mathrm{D^b}(\mathrm{Gr}(r,W^\vee)), \ldots, \mathcal{B}_{N-r}(-1) \boxtimes \mathrm{D^b}(\mathrm{Gr}(r,W^\vee)), \mathcal{C}_r\rangle \tag{21}$$

Proof. The proof of this Proposition is provided in [**Kuz07a**, §6] in geometrical terms. As noticed above, all formal proofs remain valid in the context of dg-categories. □

Remark 2.8. Saying that the functor Φ_r induces the semiorthogonal decomposition (21) amounts to say that

$$\ker \Phi_r = \langle \mathcal{A}_r(1) \boxtimes \mathrm{D^b}(\mathrm{Gr}(r,W^\vee)), \ldots, \mathcal{A}_{i-r}(i-r) \boxtimes \mathrm{D^b}(\mathrm{Gr}(r,W^\vee))\rangle$$

and that

$$\mathrm{im}\, \Phi_r = \langle \mathcal{B}_{j-1}(N-r-j) \boxtimes \mathrm{D^b}(\mathrm{Gr}(r,W^\vee)), \ldots, \mathcal{B}_{N-r}(-1) \boxtimes \mathrm{D^b}(\mathrm{Gr}(r,W^\vee))\rangle^\perp$$

To resume the results collected in this subsection, given a projective scheme $X \to \mathbb{P}(W)$, a noncommutative X-scheme $\mathcal{A}$ with a Lefschetz decomposition with respect to $\mathcal{O}_X(1)$, we have two families of noncommutative $\mathbb{P}(W^\vee)$-schemes $\mathcal{C}_r \subset \mathcal{B}_{\mathrm{Gr}(r,W^\vee)}$, such that:

1. $\mathcal{B} = \mathcal{B}_{\mathrm{Gr}(1,W^\vee)}$ admits a dual Lefschetz decomposition with respect to $\mathcal{O}_{\mathbb{P}(W^\vee)}(1)$ with the same primitive subcategories as the Lefschetz decomposition of $\mathcal{A}$.
2. $\mathcal{A}_{\mathscr{X}_r}$ admits a semiorthogonal decomposition obtained by "restricting" the Lefschetz decomposition of $\mathcal{A}_{\mathscr{X}}$ (obtained by dropping down the biggest component at each hyperplane section) and its orthogonal complement $\mathcal{C}_r$
3. $\mathcal{B}_{\mathrm{Gr}(r,W^\vee)}$ admits a semiorthogonal decomposition obtained by "restricting" the Lefschetz decomposition of $\mathcal{B}$ (obtained by dropping down the biggest component at each hyperplane section) and its orthogonal complement $\mathcal{C}_r$.
4. There is a dg splitting ($\mathrm{Gr}(r,W^\vee)$-linear) functor $\Phi_r : \mathcal{A}_{\mathscr{X}_r} \to \mathcal{B}_{\mathrm{Gr}(r,W^\vee)}$ identifying the two orthogonal complements with $\mathcal{C}_r$.

This motivates the following definition.

Definition 2.9. Given a projective scheme $X \to \mathbb{P}(W)$, a noncommutative X-scheme $\mathcal{A}$ and a Lefschetz decomposition of $\mathcal{A}$ with respect to $\mathcal{O}_X(1)$, the *homological Lefschetz theory* of $\mathcal{A}$ with respect to the Lefschetz decomposition is the family

$$\{\mathcal{B}_{\mathrm{Gr}(r,W^\vee)},\ \Phi_r\}_{r=1}^{N-1}$$

of dg categories with the given splitting functors. The categories $\mathcal{C}_r$ are called the *nonprimitive components* of this homological Lefschetz theory.

II.2. Homological Projective Duality and its consequences. As before, let $X \to \mathbb{P}(W)$ be a projective variety with a fixed hyperplane class $\mathcal{O}_X(1)$. In the previous section, we recalled how Kuznetsov's construction allows one to associate to a smooth and proper noncommutative X-scheme $\mathcal{A}$, with a Lefschetz decomposition with respect to $\mathcal{O}_X(1)$, a homological Lefschetz theory with its nonprimitive components. The second fundamental step in Homological Projective Duality is to have a geometric realization of such a theory.

Given $g : Y \to \mathbb{P}(W^\vee)$ a projective scheme, and $\mathcal{B}$ a noncommutative Y-scheme, we will denote by $\mathscr{Y}_r$ the universal dimension r section, that is the scheme fitting the Cartesian diagram

$$\begin{tikzcd} \mathscr{Y}_r \arrow[r, "j_r"] \arrow[d] & Y \times \mathrm{Gr}(r, W^\vee) \arrow[d] \\ \mathbb{P}_{\mathrm{Gr}(r,W^\vee)}(\mathcal{W}_r) \arrow[r, "\rho_r"] & \mathbb{P}(W^\vee) \times \mathrm{Gr}(r, W^\vee), \end{tikzcd}$$

and by $\mathcal{B}_{\mathscr{Y}_r}$ the base change of $\mathcal{B}$ to $\mathscr{Y}_r$.

Definition 2.10. A noncommutative Y-scheme $\mathcal{B}$ is called the *Homological Projective Dual* (or the *HP dual*) to $\mathcal{A}$ with respect to the given Lefschetz decomposition, if, for any $1 \leqslant r \leqslant N-1$ there exists a $\mathrm{Gr}(r, W^\vee)$-linear dg splitting functor $\Phi_r : \mathcal{B}_{\mathscr{Y}_r} \to \mathcal{A}_{\mathscr{X}_r}$ such that $\{\mathcal{B}_{\mathscr{Y}_r}, \Phi_r\}_{i=1}^{N-1}$ is the homological Lefschetz theory of $\mathcal{A}$.

We denote by $Q \subset \mathbb{P}(W) \times \mathbb{P}(W^\vee)$ the incidence quadric, that is the variety whose points are pairs (x, H) with x in H. Then we consider the incidence quadric of X and Y, that is:

$$Q(X, Y) := (X \times Y) \times_{\mathbb{P}(W) \times \mathbb{P}(W^\vee)} Q,$$

and notice that $Q(X, Y) = \mathscr{X}_1 \times_{\mathbb{P}(W^\vee)} Y$.

More generally, consider the natural map $\pi_r : \mathscr{X}_r \times_{\mathrm{Gr}(r,W^\vee)} \mathscr{Y}_r \to X \times Y$. If $L \subset W^\vee$ is a vector subspace, one sees easily that $\mathbb{P}(L^\perp) \times \mathbb{P}(L) \subset Q \subset \mathbb{P}(W) \times \mathbb{P}(W^\vee)$. It follows that π_r factors through a map $q_r : \mathscr{X}_r \times_{\mathrm{Gr}(r,W^\vee)} \mathscr{Y}_r \to Q(X, Y)$. One of the main insights of Kuznetsov's construction is that the splitting functors Φ_r come from a complex supported on $Q(X, Y)$. In terms of noncommutative schemes, there is a noncommutative $Q(X, Y)$-scheme $\mathcal{Q}$ and a perfect module $\mathcal{E}$ on $\mathcal{Q}$, such that, for any r, the module $q_r^*\mathcal{E}$ is a $\mathcal{A}_{\mathscr{X}_r} \otimes \mathcal{B}_{\mathscr{Y}_r}$-module representing the splitting functor Φ_r.

The main result of Kuznetsov's paper, stated in these terms, is that it is enough to have such a functor for one single r in the range $1, \ldots, N-1$.

Theorem 2.11. *A noncommutative Y-scheme $\mathcal{B}$ is HP dual to $\mathcal{A}$ with respect to the given Lefschetz decomposition if and only if there exists a noncommutative $Q(X, Y)$-scheme $\mathcal{Q}$, a $\mathcal{Q}$-perfect complex $\mathcal{E}$, and an integer r, with $1 \leqslant r \leqslant N-1$, such that $q_r^*\mathcal{E}$ is a $\mathcal{A}_{\mathscr{X}_r} \otimes \mathcal{B}_{\mathscr{Y}_r}$-module representing a $\mathrm{Gr}(r, W^\vee)$-linear splitting functor Φ_r such that $\{\mathcal{B}_{\mathscr{Y}_r}, \Phi_r\}$ is the r-th member of the homological Lefschetz theory of $\mathcal{A}$.*

Proof. Suppose that such a $\mathcal{Q}$-module exists. Kuznetsov's original proof of HPD goes through an induction process, using base change diagrams and change of the value of r. Indeed, once $q_r^*\mathcal{E}$ gives the required kernel, the Homological Lefschetz theory is constructed using the kernels $q_i^*\mathcal{E}$ for $i = 1, \ldots, N-1$, both for $i < r$ and for $i > r$: see [**Kuz07a**, §6], where everything is explained in terms of derived categories and Fourier-Mukai functors.

On the other hand, suppose that $\mathcal{B}$ is HP-dual to $\mathcal{A}$. Then, setting $r = 1$, we are in the case considered by Kuznetsov, and the existence of all the Φ_r is described in the proof of the main Theorem [**Kuz07a**, Thm. 6.3], see [**Kuz07a**, §6]. $\square$

Remark 2.12. Recall that the original definition of Homological Projective Duality [**Kuz07a**, Def. 6.1] requires the existence of a full and faithful functor $\Phi : \mathcal{B} \to \mathcal{A}_{\mathscr{X}}$ such that $\Phi(\mathcal{B}) = \mathcal{C}_1$, with the required kernel. Now, notice that $\mathcal{B} = \mathcal{B}_{\mathrm{Gr}(1,W^\vee)}$, and $\mathcal{A}_{\mathscr{X}} = \mathcal{A}_{\mathscr{X}_1}$ in this case, so that Theorem 2.11 states that Definition 2.10 is equivalent to the original Kuznetsov's definition[1].

For a linear subspace $L \subset W^\vee$, denote its orthogonal by $L^\perp \subset W$, and consider the following linear sections

$$X_L = X \times_{\mathbb{P}(W)} \mathbb{P}(L^\perp), \qquad Y_L = Y \times_{\mathbb{P}(W^\vee)} \mathbb{P}(L).$$

of X and Y respectively. If $\mathcal{A}$ is a noncommutative X-scheme, then denote by $\mathcal{A}_L$ its restriction to X_L, and similarly for $\mathcal{B}$.

Theorem 2.13 ([**Kuz07a**, Thm. 1.1])**.** *Let $X \to \mathbb{P}(W)$ be a projective variety with a semiample line bundle $\mathcal{O}_X(1)$ and $\mathcal{A}$ a smooth and proper noncommutative X-scheme with a Lefschetz decomposition* (15)*. If $\mathcal{B}$ is HP dual to $\mathcal{A}$, then:*

(i) *$\mathcal{B}$ is smooth and proper and admits a dual Lefschetz decomposition*

$$\mathcal{B} = \langle \mathcal{B}_{j-1}(1-j), \dots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle, \qquad 0 \subset \mathcal{B}_{j-1} \subset \cdots \subset \mathcal{B}_1 \subset \mathcal{B}_0$$

(ii) *for any linear subspace $L \subset W^\vee$ with dim$(L) = r$ such that*

$$\dim X_L = \dim X - r, \quad \textit{and} \quad \dim Y_L = \dim Y + r - N,$$

there exists a noncommutative k-scheme $\mathcal{C}_L$ and semiorthogonal decompositions:

$$\mathcal{A}_L = \langle \mathcal{C}_L, \mathcal{A}_r(1), \dots, \mathcal{A}_{i-1}(i-r) \rangle, \tag{22}$$

$$\mathcal{B}_L = \langle \mathcal{B}_{j-1}(N-r-j), \dots, \mathcal{B}_{N-r}(-1), \mathcal{C}_L \rangle. \tag{23}$$

Remark 2.14. Suppose S is a k-scheme, X and Y are S-schemes, $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ are ample relatively over S, so that $\mathcal{O}_X(1)$ maps $X \to \mathbb{P}_S(W)$ and $\mathcal{O}_Y(1)$ maps $Y \to \mathbb{P}_S(W^\vee)$ for a vector bundle W on S. In this relative context we can still define HP dual noncommutative schemes and Theorem 2.13 holds replacing k-linearity with S-linearity.

II.3. Homological Projective duality in Algebraic Geometry. In the geometric case, that is in the case where X is smooth projective and $\mathcal{A} = \mathrm{D^b}(X)$ (without smoothness assumptions, $\mathcal{A} = \mathrm{D^b}(X, \mathcal{R})$ is a noncommutative crepant resolution as in Definition 1.42), there is a strong relation between Homological Projective Duality and classical projective duality. Indeed, as Kuznetsov shows [**Kuz07a**, Thm. 7.9], the critical locus of the map $Y \to \mathbb{P}(W^\vee)$ is the classical dual projective variety $X^\vee$ of X. In general, even in the case where X is smooth, it is well-known that $X^\vee$ is singular, so it is natural to look for a HP dual $\mathcal{B}$ of $\mathcal{A}$ to be a noncommutative crepant resolution of singularities of $X^\vee$.

A detailed treatment of geometric examples and conjectures can be found in [**Kuz14**]. Let us start by giving an example which will be useful later. Let S be a smooth projective

[1] I am grateful to A. Kuznetsov who explained me this equivalence.

k-scheme, and E be a vector bundle of rank r over S. Let $p : X := \mathbb{P}_S(E) \to S$ its projectivization. Orlov's results [**Orl93**] provide a semiorthogonal decomposition:

$$\mathrm{D^b}(X) = \langle p^*\mathrm{D^b}(S), \ldots, p^*\mathrm{D^b}(S) \otimes \mathcal{O}_{X/S}(r-1)\rangle, \tag{24}$$

which should be thought of as a (rectangular) Lefschetz decomposition of X with respect to the line bundle $\mathcal{O}_{X/S}(1)$. Such line bundle gives the projective morphism $f : X \to \mathbb{P}(W)$, where $W^\vee := H^0(X, \mathcal{O}_{X/S}(1)) = H^0(S, E^\vee)$. Let $E^\perp := \ker(W^\vee \otimes \mathcal{O}_S \to E^\vee)$, and $q : Y := \mathbb{P}_S(E^\perp) \to S$ the natural projection. Notice that $H^0(Y, \mathcal{O}_{Y/S}(1)) = H^0(S, (E^\perp)^\vee) = W$, and let $g : Y \to \mathbb{P}(W^\vee)$ be the corresponding projective map. In this case, Kuznetsov shows that HPDuality holds [**Kuz07a**, Cor. 8.3].

Proposition 2.15 (Kuznetsov)**.** *If E is generated by global sections,* $\mathrm{D^b}(Y)$ *is the HP dual of* $\mathrm{D^b}(X)$ *with respect to the Lefschetz decomposition* (24) *relatively over S.*

The great relevance of Homological Projective Duality in Algebraic Geometry relies on the amount of relations between derived categories of (noncommutative crepant resolutions) of projective varieties obtained as dual linear sections of projective dual varieties. From the categorical point of view, we can distinguish two main cases: the case of Calabi-Yau varieties and the case of Fano varieties. In the latter case, Homological Projective Duality allows to construct semiorthogonal decompositions whose admissible components can be described geometrically via the (noncommutative crepant resolution) of a projective dual variety. These examples will be extensively treated in Chapter 4 due to their interaction with birational properties.

III. Homological Projective Duality for quadric fibrations and their intersections

III.1. Generic relative intersections of quadrics. In this Section, we present a generalization of Kuznetsov's Homological Projective dual of intersection of quadrics [**Kuz08**] to the case of fibrations in intersections of quadrics over any field. All the results explained here are taken from [**ABB14**].

Let S be a k-scheme. A *(line-bundle valued) quadratic form* on S is a triple (E, q, L) where E is a vector bundle on S, L a line bundle on S, and q a morphism of $\mathcal{O}_S$-modules $L \to S^2(E^\vee)$. In particular, this is equivalent to the choice of a global section $s_q \in \Gamma(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)/S}(2) \otimes p^*L^\vee)$, where $p : \mathbb{P}(E) \to S$ is the natural projection. The rank of a quadratic form is defined to be the rank of the vector bundle E. For equivalent characterizations and the extension to this case of the definitions of basic notions for quadratic forms (such as primitivity, regularity, discriminant divisor etc.) we refer to [**ABB14**, §1.1]. The geometric objects which will be studied in this Section are quadric fibrations and their generic complete intersections.

Definition 2.16. The *quadric fibration* $\pi : Q \to S$ associated to a nonzero quadratic form (E, q, L) of rank $n \geqslant 2$ is the restriction of the projection $p : \mathbb{P}(E) \to S$ via the closed embedding $j : Q \to \mathbb{P}(E)$ defined by the vanishing of the global section $s_q \in \Gamma(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)/S}(2) \otimes p^*L^\vee)$. We Write $\mathcal{O}_{Q/S}(1) = j^*\mathcal{O}_{\mathbb{P}(E)/S}(1)$.

Remark 2.17. The form (E, q, L) is primitive if and only if $\pi : Q \to S$ is flat of relative dimension $n-2$, see [**Mat86**, 8 Thm. 22.5]. The fiber Q_y is a smooth projective quadric (resp. a quadric cone with isolated singularity) over any point y where (E, q, L) is (semi)regular (resp. has simple degeneration).

Definition 2.18. We say that a finite set of generically (semi)regular primitive quadratic forms (E, q_i, L_i) (or quadric fibrations $Q_i \to S$) for $1 \leqslant i \leqslant m$ is *generic* if the following properties hold:

(1) the images of $L_i \to S^2(E^\vee)$ span an $\mathcal{O}_S$-submodule $L \subset S^2(E^\vee)$ of rank m,
(2) the associated linear span quadric fibration $Q \to \mathbb{P}_S(L_1 \oplus \ldots, \oplus L_m)$, has simple degeneration with regular discriminant divisor,
(3) the associated intersection $X \to S$ of the quadric fibrations $Q_i \subset \mathbb{P}_S(E)$ is a relative complete intersection.

By a *generic relative intersection of quadrics* we mean any intersection $X \to S$ of a generic set of quadric fibrations.

III.2. The Clifford algebra and the HP dual. In this section, we recall the tensorial construction of the even Clifford algebra of a line bundle-valued quadratic form from [**ABB14**]. In *loc.cit.*, we show that it extends to any field the one in [**Kuz08**, §3.3].

Let (E, q, L) be a (line bundle-valued) quadratic form of rank n on a scheme S. We define ideals J_1 and J_2 of the tensor algebra $T(E \otimes E \otimes L)$, generated by

$$v \otimes v \otimes f - f(q(v)), \quad \text{and} \quad u \otimes v \otimes f \otimes v \otimes w \otimes g - f(q(v))\, u \otimes w \otimes g,$$

respectively, for sections u, v, w of E and f, g of L. We define the *even Clifford algebra* of (E, q, L) as the quotient algebra

$$\mathcal{C}_0(E, q, L) = T(E \otimes E \otimes L)/(J_1 + J_2).$$

It is not difficult to see that if the quadratic form q is nondegenerate, the noncommutative S-scheme $\mathrm{perf}(\mathcal{C}_0)$ is smooth and proper.

Now, let S be a smooth scheme and E be a vector bundle of rank n on S. Consider the projective bundle $p : X := \mathbb{P}(E) \to S$, the relative ample line bundle $\mathcal{O}_{X/S}(1)$, and the semiorthogonal decomposition (see [**Orl93**, Thm. 2.6])

$$\mathrm{D^b}(X) = \langle p^*\mathrm{D^b}(S)(-1), p^*\mathrm{D^b}(S), \ldots, p^*\mathrm{D^b}(S)(n-2) \rangle. \tag{25}$$

Let us denote by $m = \lfloor (n-1)/2 \rfloor$ and put

$$\mathcal{A}_0 = \mathcal{A}_1 = \ldots = \mathcal{A}_{m-1} = \langle p^*\mathrm{D^b}(S)(-1), p^*\mathrm{D^b}(S) \rangle,$$

$$\mathcal{A}_m = \begin{cases} \langle p^*\mathrm{D^b}(S)(-1), p^*\mathrm{D^b}(S) \rangle & \text{if } n \text{ is even} \\ \langle p^*\mathrm{D^b}(S)(-1) \rangle & \text{if } n \text{ is odd.} \end{cases}$$

Then the decomposition (25) is a Lefschetz decomposition

$$\mathrm{D^b}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(2), \ldots, \mathcal{A}_m(2m)\rangle, \tag{26}$$

with respect to the relative double Veronese embedding $f : X = \mathbb{P}(E) \to \mathbb{P}(S_2E) =: \mathbb{P}(V)$, as $f^*\mathcal{O}_{\mathbb{P}(S^2E)/S}(1) \simeq \mathcal{O}_{\mathbb{P}(E)/S}(2)$. We set $Y := \mathbb{P}((S^2E)^\vee)$.

Definition 2.19. Let $\mathscr{Q} \subset \mathbb{P}(E) \times_S \mathbb{P}((S_2E)^\vee) = X \times Y$ be the universal hyperplane section with respect to f, then we will refer to the projection

$$\pi : \mathscr{Q} \to Y,$$

as the *universal relative quadric fibration* in $\mathbb{P}(E)$.

Let $\mathcal{C}_0$ be the even Clifford algebra of the universal relative quadric fibration. The following Theorem is one of the main results of [**ABB14**].

Theorem 2.20. *The noncommutative Y-scheme $\mathrm{D^b}(Y, \mathcal{C}_0)$ is the HP dual of $\mathrm{D^b}(X)$ over S with respect to the Lefschetz decomposition* (26)*.*

The proof of Theorem 2.20 is based on the description of a semiorthogonal decomposition of the universal quadric. This was described in [**ABB14**] extending Kuznetsov's [**Kuz08**] original result to a more general context.

Applying Theorem 2.13 to the HP dual pair $\mathrm{D^b}(X)$, $\mathrm{D^b}(Y, \mathcal{C}_0)$ we obtain semiorthogonal decompositions of generic relative intersections of quadrics over S (obtained as linear sections of X) which can be described via the restriction of the Clifford algebra $\mathcal{C}_0$ to the orthogonal section of Y.

Now we describe some consequences of Theorem 2.13 in the case of flat quadric fibrations. Let (E_i, q_i, L_i) be a finite set of primitive generically (semi)regular quadratic forms. Denote by $L \to S^2E^\vee$ the $\mathcal{O}_Y$-submodule generated by the line subbundles L_i. Then the linear section X_L is a relative intersection of the quadric fibrations $Q_i \to S$ in $\mathbb{P}(E)$. Indeed, the projection map $\pi : X \to S$ has fibers the intersection of the fibers of $Q_i \to S$ in the projective space given by the fibers of $\mathbb{P}(E)$. On the other hand, the linear section Y_L is precisely $\mathbb{P}(L) \subset \mathbb{P}(S^2E^\vee)$. Then the restriction $\mathcal{C}_0|_{\mathbb{P}(L)} = \mathcal{C}_0 \boxtimes_{\mathcal{O}_{\mathbb{P}(S^2E^\vee)}} \mathcal{O}_{\mathbb{P}(L)}$ (which we shamelessly denote by $\mathcal{C}_0$) to $\mathbb{P}(L)$ is then isomorphic to the even Clifford algebra of the corresponding linear span quadric fibration $Q \to \mathbb{P}(L)$ associated to the $Q_i \to S$. We assume that $L \simeq \oplus_i L_i$ and that this relative intersection is complete. Here is a sample of consequences of HPD, which will be applied to explicit geometric cases in Chapter 4.

Theorem 2.21. *Let S be a smooth scheme, $Q \to S$ a linear span of m quadric fibrations of relative dimension $n-2$ over S, and $X_L \to S$ their relative complete intersection. Let $\mathcal{C}_0$ be the even Clifford algebra of $Q \to \mathbb{P}(L) = Y_L$.*

(1) *If $2m < n$, then the fibers of $X_L \to S$ are Fano and relative homological projective duality yields*

$$\mathrm{D^b}(X_L) = \langle \mathrm{D^b}(Y_L, \mathcal{C}_0), \pi^*\mathrm{D^b}(S)(1) \ldots \pi^*\mathrm{D^b}(S)(n-2m)\rangle.$$

(2) *If $2m = n$ then the fibers of $X_L \to S$ are generically Calabi-Yau and relative homological projective duality yields*

$$\mathrm{D^b}(X_L) \simeq \mathrm{D^b}(Y_L, \mathcal{C}_0).$$

(3) *If $2m > n$, then the fibers of $X_L \to S$ are generically of general type and there exists a fully faithful functor $\mathrm{D^b}(X_L) \to \mathrm{D^b}(Y_L, \mathcal{C}_0)$ with explicitly describable orthogonal complement.*

III.3. Hyperbolic splitting. Let us end this Section with a result from [**ABB14**] which is not directly related to Homological Projective Duality, but which gives a useful tool to study sheaves of even parts of Clifford algebras under Morita equivalence.

Let $Q \to S$ be the quadric fibration associated to a quadratic form (E, q, L). A section $s : S \to Q$ of $Q \to S$ is called *smooth* if the image of s only consists of smooth points of the fibers of $Q \to S$. An isotropic line subbundle $N \subset E$ of (E, q, L) is called *smooth* if the associated section of its quadric fibration is smooth. We notice that in particular, if Q and S are smooth schemes over a field k, then any section of π is smooth, see [**ABB14**, Lemma 1.3.2].

If $N \to E$ be a smooth isotropic subbundle, then Then $q|_{N^\perp} : N^\perp \to L^\vee$ vanishes on N, hence defines a quadratic form $q' : N^\perp/N \to L^\vee$ on $E' = N^\perp/N$. We call (E', q', L) the *reduced quadratic form* associated to N. Notice that $\mathrm{rk}E' = \mathrm{rk}E - 2$. If $Q' \to S$ is the associated quadric fibration, then $\dim(Q') = \dim(Q) - 2$.

A careful analysis of the quadratic forms (E, q, L) and (E', q', L) show that the form q splits, over an open Zariski subset of S, as $q' \perp u$, for $(u, N^\vee, L)$ a hyperbolic quadratic form. We hence say that $Q' \to S$ is obtained from $Q \to S$ by *hyperbolic splitting* along a (smooth) section. The Morita-invariance of the sheaf of even parts of the Clifford algebra under hyperbolic splitting was proved in [**ABB14**].

Theorem 2.22. *Let (E, q, L) be a quadratic form over a regular integral scheme S, with simple degeneration along a regular divisor D. In the case of odd rank, assume that 2 is invertible on S. Let (E', q', L') be the reduced quadratic form associated to a smooth isotropic subbundle $N \to E$. Then the even Clifford algebras $\mathcal{C}_0(E, q, L)$ and $\mathcal{C}_0(E', q', L)$ are Morita S-equivalent.*

IV. Homological Projective Duality for determinantal varieties

In this Section, we present another example of HP dual varieties, which was considered in [**BBF16**]. In this case X and Y are the so-called generalized determinantal varieties, that is, varieties that are defined by vanishing of r-minors of a $n \times m$ matrix with linear entries. Both varieties $X \to \mathbb{P}(W)$ and $Y \to \mathbb{P}(W^\vee)$ are singular and hence we have to deal with noncommutative resolution of singularities in order to work with smooth and proper noncommutative schemes.

IV.1. Desingularizations of the space of matrices of bounded ranks. Let U, V be k-vector spaces, with $\dim U = m$, $\dim V = n$, and assume $n \geqslant m$. Let r be an integer in the range $1 \leqslant r \leqslant m - 1$. We define $Z^r_{m,n}$ to be the variety of $m \times n$ matrices $M : V \to U^\vee$

having rank at most r, i.e. the locus in $\mathbb{P}W = \mathbb{P}(U^\vee \otimes V^\vee)$ cut by the minors of size $r+1$ of the matrix of indeterminates:

$$\psi = \begin{pmatrix} x_{1,1} & \dots & x_{m,1} \\ \vdots & \ddots & \vdots \\ x_{1,n} & \dots & x_{m,n} \end{pmatrix}$$

Consider the Grassmann variety $\mathrm{Gr}(U,r)$ of r-dimensional quotient spaces of U, the tautological subbundle and the quotient bundle over $\mathrm{Gr}(U,r)$ are denoted respectively by $\mathcal{U}$ and $\mathcal{Q}$ and have respectively rank $m-r$ and r. The tautological (or Euler) exact sequence reads:

$$0 \to \mathcal{U} \to U \otimes \mathcal{O}_{\mathrm{Gr}(U,r)} \to \mathcal{Q} \to 0. \tag{27}$$

We will use the following notation:

$$p : X^r_{m,n} = \mathbb{P}(V^\vee \otimes \mathcal{Q}^\vee) \to \mathrm{Gr}(U,r),$$

and $\mathcal{O}_{X^r_{m,n}}(H)$ for the relatively ample tautological line bundle. The manifold $X^r_{m,n}$ has dimension $r(n+m-r)-1$. It is the resolution of singularities of the variety of $m \times n$ matrices of rank at most r, as follows. The space $H^0(\mathrm{Gr}(U,r),\mathcal{Q})^\vee$ is naturally identified with U. Hence we get natural isomorphisms:

$$H^0(\mathrm{Gr}(U,r), V \otimes \mathcal{Q}) \simeq H^0(X^r_{m,n}, \mathcal{O}_{X^r_{m,n}}(H)) \simeq W = U \otimes V.$$

Setting $W := U^\vee \otimes V^\vee$, the map f associated to the linear system $\mathcal{O}_{X^r_{m,n}}(H)$ maps $X^r_{m,n}$ to $\mathbb{P}(W)$.

A rank-1 quotient of $W^\vee = U \otimes V$ corresponds to the choice of a linear map $M : V \to U^\vee$, so an element of $\mathbb{P}(W)$ can be considered as (the proportionality class of) the linear map M. On the other hand, f sends a rank-1 quotient of $V^\vee \otimes \mathcal{Q}^\vee$ over a point $\lambda \in \mathrm{Gr}(U,r)$ to the quotient of $U^\vee \otimes V^\vee$ obtained by composition with the obvious quotient $U^\vee \to \mathcal{Q}^\vee_\lambda$. The matrix M lies in the image of f if and only if M factors through $V^\vee \to \mathcal{Q}_\lambda$, for some $\lambda \in \mathrm{Gr}(U,r)$, i.e., if and only if $\mathrm{rk}(M) \leqslant r$. The map $f : X^r_{m,n} \to Z^r_{m,n}$ is a desingularization, called the *Springer resolution*, of $Z^r_{m,n}$. It is an isomorphism above the locus of matrices of rank exactly r.

Consider now the projective bundle:

$$q : Y^r_{m,n} = \mathbb{P}(V \otimes \mathcal{U}) \to \mathrm{Gr}(U,r),$$

and denote $\mathcal{O}_{Y^r_{m,n}}(H)$ the tautological ample line bundle on $Y^r_{m,n}$. The linear system associated to $\mathcal{O}_{Y^r_{m,n}}(H)$ sends $Y^r_{m,n}$ to $\mathbb{P}(W^\vee) \simeq \mathbb{P}(V \otimes U)$ via a map that we call g. By the same argument as above, g is a desingularization of the variety $Z^{m-r}_{m,n}$ of matrices of corank at least r.

On the other hand, one can consider the following rectangular Lefschetz decomposition for $\mathrm{D^b}(X^r_{m,n})$ with respect to $\mathcal{O}_{X^r_{m,n}}(H)$:

$$\mathrm{D^b}(X^r_{m,n}) = \langle p^*\mathrm{D^b}(\mathrm{Gr}(U,r)), \dots, p^*\mathrm{D^b}(\mathrm{Gr}(U,r))(mr) \rangle, \tag{28}$$

and similarly for $\mathrm{D^b}(Y^r_{m,n})$. Using the Euler exact sequence on $\mathrm{Gr}(U,r)$ and Proposition 2.15 one can show that $X^r_{m,n}$ and $Y^r_{m,n}$ are HP dual relatively over $\mathrm{Gr}(U,r)$.

Proposition 2.23. *The smooth projective variety $Y^r_{m,n}$ is the HP dual of $X^r_{m,n}$ with respect to the Lefschetz decomposition* (28)*.*

IV.2. Noncommutative resolutions and HP duality. Consider $X^r_{m,n}$ as a projective bundle over $\mathrm{Gr}(U,r)$. Kapranov shows that $\mathrm{Gr}(U,r)$ has a full strong exceptional collection consisting of vector bundles [**Kap85**]. From the semiorthogonal decomposition (28), we then have a strong exceptional collection on $X^r_{m,n}$ consisting of vector bundles. Set E to be the direct sum of the bundles from this full exceptional collection. Let us consider $M := Rf_*E$, and let $\mathcal{S}^r_{m,n} := \mathcal{E}nd(E)$ and $\mathcal{R}^r_{m,n} := \mathcal{E}nd(M)$ (where $\mathcal{E}nd$ denotes the sheaf of endomorphisms). To simplify notations, we will drop the indexes from $\mathcal{S}$ and $\mathcal{R}$ if their choice is clear. The following result from [**BBF16**] describes a noncommutative resolution of singularities of $Z^r_{m,n}$ and can be thought of as the projective version of Buchweitz, Leuschke and Van den Bergh's results in the affine case [**BLVdB10**, **BLVdB11**].

Proposition 2.24. *The endomorphism algebra $\mathcal{R}$ is a coherent $\mathcal{O}_{Z^r_{m,n}}$-algebra Morita-equivalent to $\mathcal{S}$, so that $\mathrm{D^b}(Z^r_{m,n},\mathcal{R}) \simeq \mathrm{D^b}(X^r_{m,n})$. The noncommutative $Z^r_{m,n}$-scheme $\mathcal{R}$ is a noncommutative resolution of singularities, which is crepant if $m = n$, in the sense of Definition 1.42.*

The main tool in the proof of Proposition 2.24 is calculating the higher direct images $R^if_*(E_i)$ for the E_i the exceptional vector bundles which are direct summands of E, and to show that $\mathcal{R}^r_{m,n}$ is maximally Cohen-Macaulay basing on a similar result for the affine case proved in [**BLVdB10**].

Combining Proposition 2.23 and Proposition 2.24, one gets HP duality for the noncommutative resolution of singularities of determinantal varieties.

Corollary 2.25. *The noncommutative $Z^{m-r}_{m,n}$-scheme $\mathcal{R}^{m-r}_{m,n}$ is HP dual to the noncommutative $Z^r_{m,n}$-scheme $\mathcal{R}^r_{m,n}$ relatively over $\mathrm{Gr}(U,r)$, with respect to the Lefschetz decomposition* (28)*.*

CHAPTER 3

Noncommutative motives and algebraic cycles

Recall from Chapter 1 the definitions of noncommutative k-schemes and noncommutative motives. In this Chapter, we describe some application of the theory of noncommutative motives to Algebraic Geometry. First of all, we compare the theory of noncommutative and commutative Chow motives of smooth projective k-varieties, with a particular attention to motivic decompositions. The main applications to geometry, in the case where $k \subset \mathbb{C}$ is algebraically closed, will be given by the notion of *Jacobian* of (the noncommutative motive of) a noncommutative k-scheme, introduced by Marcolli and Tabuada [**MT14a**]. This is a functor which assigns to any noncommutative k-scheme $\mathcal{A}$ (via its noncommutative motive) an Abelian variety $\mathbf{J}(\mathcal{A})$ well-defined up to isogeny.

In the case where X is a smooth projective k-scheme with a unique principally polarized intermediate Jacobian, we will show that the noncommutative Jacobian also carries a principal polarization, and that this allows to prove categorical Torelli-type theorems, which state which noncommutative subschemes of X identify the isomorphism class of X. These results have some geometrical applications, in particular they provide a way to describe the Chow groups of intersections of quadrics of very low degree and any dimension, giving a noncommutative proof to a motivic Beilinson-Bloch type conjecture (a "commutative" proof was given by Otwinowska [**Otw99**]).

Finally, one can define smash-nilpotence and numerical equivalence for noncommutative motives and formulate a noncommutative version of Voevodsky's smash-nilpotence conjecture. If X is a smooth projective variety over an algebraically closed field of characteristic zero, then such a conjecture is equivalent to the commutative smash-nilpotence conjecture. This allows one to use semiorthogonal decompositions to prove new cases of the commutative smash-nilpotence conjecture.

We will assume the reader to be familiar with the theory of (commutative) motives. An exhaustive treatment can be found in Andre's book [**And04**].

I. Comparing commutative and noncommutative Chow motives

In this section, we mainly recall results from [**BT15**] where the relation between the Chow and the noncommutative motive of a smooth projective k-scheme are studied. The most important tool is the Grothendieck-Riemann-Roch Theorem and the analysis of its denominators, which will allow to describe the rings where these motives have similar decompositions. On the other hand, looking to integer coefficients, interesting phenomena arise, such as purely noncommutative motivic decompositions, probably related to the arithmetic properties of the variety.

I.1. motives of Lefschetz and trivial type. Let R be a $\mathbb{Z}$-module and $\mathrm{Chow}(k)_R$ denote the category of Chow motives of varieties over k with R-coefficients. We denote by $\mathbf{L}_R$ (or simply by $\mathbf{L}$ if no confusion on the coefficient ring is possible) the *Lefschetz motive*, that is the motive of the affine line in $\mathrm{Chow}(k)_R$. Recall moreover from Chapter 1, Section IV, the universal functor $U : \mathsf{dgcat}(k) \to \mathsf{Hmo}_0(k)$, associating to any noncommutative k-scheme its noncommutative motive. Consider the noncommutative scheme $\underline{k} := \mathrm{perf}(\mathrm{Spec}(k))$, that is the dg category of bounded complexes of k-vector spaces. We denote by $\mathbf{1}_R := U(\underline{k})_R$ (also denoted by $\mathbf{1}$ if no confusion on the coefficient ring is possible) its noncommutative motive, which is the $\otimes$-unit of $\mathsf{Hmo}_0(k)_R$.

Following Gorchinskiy-Orlov [**GO13**], a Chow motive is called of *Lefschetz type* if it is isomorphic to $\mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m}$ for some non-negative integers $l_1, \ldots, l_m$. In the same vein, a noncommutative motive is called of *trivial type* if it is isomorphic to $\oplus_{i=1}^m \mathbf{1}$ for some integer m. The following implication was established by Gorchinskiy-Orlov in [**GO13**, §4] (assuming that $\mathbb{Z} \subseteq R$):

$$M(X)_R \text{ Lefschetz type} \Rightarrow NC(X)_R \text{ trivial type}\,. \tag{29}$$

In the particular case where $R = \mathbb{Q}$, (29) becomes an equivalence (see [**MT15**, §1]):

$$M(X)_{\mathbb{Q}} \text{ Lefschetz type} \Leftrightarrow NC(X)_{\mathbb{Q}} \text{ trivial type}\,. \tag{30}$$

The following result was proved in [**BT15**] as an application of Grothendieck-Riemann-Roch theorem and establishes a partial converse of the above implication (29):

Theorem 3.1. *Let X be an irreducible smooth projective k-scheme of dimension d. Assume that $\mathbb{Z} \subseteq R$ and that every finitely generated projective $R[1/(2d)!]$-module is free (e.g. R a principal ideal domain). Assume also that $NC(X)_R \simeq \oplus_{i=1}^m \mathbf{1}$ for some integer m. Under these assumptions, there is a choice of integers (up to permutation) $l_1, \ldots, l_m \in \{0, \ldots, d\}$ giving rise to an isomorphism*

$$M(X)_{R[1/(2d)!]} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m}\,. \tag{31}$$

Intuitively speaking, Theorem 3.1 shows that the converse of the above implication (29) holds as soon as one inverts the integer $(2d)!$ (or equivalently its prime factors). By combining this result with (29), one obtains a refinement of (30):

Corollary 3.2. *Given X and R as in Theorem 3.1, we have the equivalence*

$$M(X)_{R[1/(2d)!]} \text{ Lefschetz type} \Leftrightarrow NC(X)_{R[1/(2d)!]} \text{ trivial type}\,.$$

However the (strict) converse of implication (29) is false in general, as in the following example from [**BT15**].

Proposition 3.3. *Let q be a non-singular quadratic form and Q_q the associated smooth projective quadric. Assume that q is even dimensional, anisotropic, and has trivial discriminant and trivial Clifford invariant (see Lam [**Lam**, § V.2]).*

(i) *The noncommutative motive $NC(Q_q)_{\mathbb{Z}}$ is of trivial type.*
(ii) *The Chow motive $M(Q_q)_{\mathbb{Z}}$ is not of Lefschetz type.*

The proof of proposition 3.3 relies on Rost's description of the motive of quadrics [**Ros**] (point (i)) and on Kapranov and Kuznetsov's description of the derived category (see [**ABB14**] for a description over any field) of a quadric together with the fact that the discriminant extension of such a quadric splits and the even Clifford algebra is trivial (point (ii)).

As an application of Theorem 3.1, we obtain the following sharpening of the main result of [**MT15**] (which was obtained only with rational coefficients).

Corollary 3.4. *Let X be an irreducible smooth projective k-scheme of dimension d. Assume that $\mathrm{D^b}(X)$ admits a full exceptional collection $(E_1, \dots, E_m)$ of length m. Under these assumptions, there is a choice of integers (up to permutation) $l_1, \dots, l_m \in \{0, \dots, d\}$ giving rise to an isomorphism*

$$M(X)_{\mathbb{Z}[1/(2d)!]} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m} . \tag{32}$$

Notice that in the case where X is a surface, Ch. Vial has shown a stronger version of the previous result, that is, that $M(X)_{\mathbb{Z}}$ is of Lefschetz type [**Via15**].

I.2. Decomposability. More in general, decomposability of the commutative Chow motive holds decomposability of the noncommutative Chow motive, up to inverting enough coefficients.

Theorem 3.5. *Let X be an irreducible smooth projective k-scheme of dimension d. Under the assumption $\mathbb{Z} \subseteq R$, the following implication holds:*

$$M(X)_{R[1/(2d)!]} \text{ decomposable} \Rightarrow NC(X)_{R[1/(2d)!]} \text{ decomposable} . \tag{33}$$

One can fairly say then that, inverting enough coefficients, noncommutative motives contain both strictly noncommutative decompositions (coming from semiorthogonal decompositions) and strictly commutative ones (coming from algebraic correspondences). As the following proposition shows, if one does not invert the dimension of X, the converse of implication (33) is false.

Proposition 3.6. *(Cf. [**BT15**]) Let A be a central simple k-algebra of degree $\sqrt{\dim(A)} = d$ and $X = \mathrm{SB}(A)$ the associated Severi-Brauer variety.*

(i) *For every commutative ring R one has the following motivic decomposition*

$$NC(X)_R \simeq \mathbf{1} \oplus U(\underline{A})_R \oplus U(\underline{A})_R^{\otimes 2} \oplus \cdots \oplus U(\underline{A})_R^{\otimes d-1} . \tag{34}$$

In particular, the noncommutative motive $NC(X)_R$ is decomposable.

(ii) *When A is a division algebra and $d = p^s$ some prime power, the Chow motive $M(X)_{\mathbb{Z}}$ (and also $M(X)_{\mathbb{Z}/p\mathbb{Z}}$) is indecomposable.*

Remark 3.7. Item (ii) holds also for $M(X)_{\mathbb{Z}_p}$; see De Clercq [**DC10**, Rmq. 2.3].

Roughly speaking, Proposition 3.6 shows that the decomposition (34) is "truly noncommutative". The proof of proposition 3.6 relies on Karpenko's incompressibility results [**Kar95**] and on the description of the derived category of a Brauer-Severi variety [**Ber09**].

Finally, we can lift motivic isomorphisms from Chow motive to noncommutative motives, up to inverting enough coefficients.

Theorem 3.8. *Let $\{X_i\}_{1\leqslant i\leqslant n}$ (resp. $\{Y_j\}_{1\leqslant j\leqslant m}$) be irreducible smooth projective k-schemes of dimension d_{X_i} (resp. d_{Y_j}), $d := \max\{d_{X_i}, d_{Y_j} \mid i,j\}$, and $\{l_i\}_{1\leqslant i\leqslant n}$ (resp. $\{l_j\}_{1\leqslant j\leqslant m}$) arbitrary integers. Assume that $\mathbb{Z} \subseteq R$ and $1/(2d)! \in R$. Under these assumptions, we have the following implication*

$$\oplus_i M(X_i)_R \otimes \mathbf{L}^{\otimes l_i} \simeq \oplus_j M(Y_j)_R \otimes \mathbf{L}^{\otimes l_j} \Rightarrow \oplus_i NC(X_i)_R \simeq \oplus_j NC(Y_j)_R\,.$$

As the following example shows, if one does not inverts the maximum of the dimensions, the converse of the implication of Theorem 3.8 is false.

Example 3.9. The Chow motives $M(X)_\mathbb{Z}$ and $M(\widehat{X})_\mathbb{Z}$ of an Abelian variety X and of its dual $\widehat{X}$ are in general not isomorphic. However, thanks to the work of Mukai [**Muk81**], we have $NC(X)_R \simeq NC(\widehat{X})_R$ for every commutative ring R.

II. From semiorthogonal decompositions to intermediate Jacobians

Let k be an algebraically closed of characteristic zero. In this Section, we recall the definition of the Jacobian of a noncommutative motive as an Abelian variety, well defined up to isogeny. In the case where X is a smooth projective variety, the noncommutative Jacobian of $\mathrm{D^b}(X)$ is isomorphic to the product of all the algebraic Jacobians of X, as defined by Griffiths. If X has only such a Jacobian (the intermediate one) $J(X)$ with a natural principal polarization, one can identify noncommutative schemes $\mathcal{A}_X$ components of $\mathrm{D^b}(X)$ whose dg-equivalence class identifies the Abelian variety $J(X)$ with its principal polarization. This result gives also a noncommutative analog to the famous Torelli theorem: if the the isomorphism class of $J(X)$ as principally polarized Abelian variety identifies the isomorphism class of X, then so does the equivalence class of $\mathcal{A}_X$ as a noncommutative k-scheme.

II.1. (Polarized) intermediate Jacobians. Given an irreducible smooth projective k-scheme X of dimension d_X, Griffiths introduced in [**Gri69**] the associated Jacobians $J^i(X), 0 \leqslant i \leqslant d_X - 1$. In contrast with the Picard $J^0(X) = \mathrm{Pic}^0(X)$ and the Albanese $J^{d_X-1}(X) = \mathrm{Alb}(X)$ varieties, the intermediate Jacobians are in general not algebraic, see [**Voi02**, §12] for a detailed treatment. Nevertheless, they contain an algebraic torus $J^i_a(X) \subseteq J^i(X)$ defined by the image of the Abel-Jacobi map

$$AJ^i : A^{i+1}_\mathbb{Z}(X) \to J^i(X) \qquad 0 \leqslant i \leqslant d_X - 1\,, \tag{35}$$

where $A^{i+1}_\mathbb{Z}(X)$ stands for the group of algebraically trivial cycles of codimension $i+1$; consult, e.g., Vial [**Via13**, §2.3] for further details.

In general, the Abelian varieties $J^i_a(X)$ are only well-defined up to isogeny. However, in the case of curves, Fano threefolds, even dimensional quadric fibrations over $\mathbb{P}^1$, odd dimensional quadric fibrations over rational surfaces, and also in the case of the intersection of two (resp. three) quadrics of odd (resp. even) dimension, there is a single non-trivial algebraic Jacobian $J(X) := J^{(d_X-1)/2}_a(X)$ which carries moreover a canonical *principal polarization*;

see Clemens and Griffiths [**CG72**]. This extra piece of structure is of major importance. For example, in the case of a Fano threefold X the Abelian variety $J(X)$ endowed with its canonical principal polarization contains all the information about the birational class of X. We distinguish these particular cases by the following definition.

Definition 3.10. An irreducible smooth projective k-scheme X of odd dimension $d_X = 2n+1$ is called *verepresentable*[1] if:

(i) the group of algebraically trivial cycles $A^{i+1}_{\mathbb{Z}}(X)$ is trivial for $i \neq n$;
(ii) the group $A^{n+1}_{\mathbb{Z}}(X)$ admits an *algebraic representative* carrying an *incidence polarization*; see [**Bea77**, §3.4].
(iii) the Abel-Jacobi map $AJ^n(X) : A^{n+1}_{\mathbb{Z}}(X) \twoheadrightarrow J^n_a(X)$ gives rise to an isomorphism $A^{n+1}_{\mathbb{Q}}(X) \simeq J^n_a(X)_{\mathbb{Q}}$.

Apart of trivial examples of schemes with $J^i(X) = 0$ (such as projective spaces, smooth quadrics, Grassmannians and other homogeneous spaces), examples of verepresentable schemes include smooth projective curves of any genus and a large amount of Fano threefolds, and complete intersection of two even (resp. three odd) dimensional quadrics, see [**BT16b**] for an exhaustive list.

II.2. Jacobians of noncommutative schemes. Given a smooth projective k-scheme X of dimension d_X, one can then consider the $\mathbb{Q}$-vector spaces

$$NH^{2i+1}_{dR}(X) := \sum_{C,\gamma_i} \mathrm{Im}\big(H^1_{dR}(C) \xrightarrow{H^1_{dR}(\gamma_i)} H^{2i+1}_{dR}(X)\big) \qquad 0 \leqslant i \leqslant d_X - 1\,, \tag{36}$$

where C is a smooth projective curve and $\gamma_i : M_{\mathbb{Q}}(C) \to M_{\mathbb{Q}}(X)(i)$ a morphism in $\mathrm{Chow}(k)_{\mathbb{Q}}$. Intuitively speaking, (36) are the odd pieces of de Rham cohomology that are generated by curves. By restricting the classical intersection bilinear pairings on de Rham cohomology (see [**And04**, §3.3]) to these pieces one obtains

$$\langle -,- \rangle : NH^{2d_X-2i-1}_{dR}(X) \times NH^{2i+1}_{dR}(X) \longrightarrow k \qquad 0 \leqslant i \leqslant d_X - 1\,. \tag{37}$$

Marcolli and Tabuada [**MT14a**] constructed the *Jacobian* functor

$$\mathbf{J}(-) : \mathrm{NChow}(k)_{\mathbb{Q}} \longrightarrow \mathrm{Ab}(k)_{\mathbb{Q}}$$

with values in the category of Abelian k-varieties up to isogeny. Given a noncommutative Chow motive N, the Abelian variety $\mathbf{J}(N)$ is constructed as follows:

(i) firstly, the category of Abelian varieties up to isogeny $\mathrm{Ab}(k)_{\mathbb{Q}}$ can be identified with an Abelian semi-simple full subcategory of $\mathrm{NNum}(k)_{\mathbb{Q}}$, via fully faithful functors $\mathrm{Ab}(k)_{\mathbb{Q}} \to \mathrm{Num}(k)_{\mathbb{Q}}$ and $\mathrm{Num}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)} \to \mathrm{NNum}(k)_{\mathbb{Q}}$, checking that passing to the orbit category does preserve the fully faithfulness.
(ii) secondly, the semi-simplicity of $\mathrm{NNum}(k)_{\mathbb{Q}}$ implies that the noncommutative numerical motive N admits a unique finite direct sum decomposition $S_1 \oplus \cdots \oplus S_n$ into simple objects;

[1]The fusion of the words "very" and "representable".

(iii) finally, one defines $\mathbf{J}(N)$ as the smallest piece of the noncommutative numerical motive $N \simeq S_1 \oplus \cdots \oplus S_n$ which contains all the simple objects belonging to the Abelian semi-simple full subcategory $\mathrm{Ab}(k)_{\mathbb{Q}}$.

By abuse of notations, if $\mathcal{A}$ is a smooth and proper noncommutative k-scheme, then we will denote by $\mathbf{J}(\mathcal{A}) := \mathbf{J}(U(\mathcal{A}))$, and call this Abelian variety the *Jacobian of the noncommutative scheme* $\mathcal{A}$. Even more abusive, but justified by the next results, if X is a smooth and proper k-scheme, we will denote by $\mathbf{J}(X) := \mathbf{J}(\mathrm{D}^{\mathrm{b}}(X))$.

As proved in [**MT14a**, Theorem 1.7], whenever the above pairings (37) are non-degenerate for all i, one has an isomorphism $\mathbf{J}(X) \simeq \prod_{i=0}^{d-1} J_a^i(X)$ in $\mathrm{Ab}(k)_{\mathbb{Q}}$. As explained in *loc. cit.*, (37) is always non-degenerate for $i = 0$ and $i = d - 1$. Moreover, if Grothendieck's standard conjecture of Lefschetz type is true for X, then (37) is non-degenerate for all i; see Vial [**Via13**, Lemma 2.1]. This latter conjecture holds for curves, surfaces, Abelian varieties, complete intersections, uniruled threefolds, rationally connected fourfolds, and for any smooth hypersurface section, product, or finite quotient thereof. In particular, it holds for all known examples of verepresentable varieties.

Example 3.11. Suppose that $\mathcal{A}$ is a noncommutative k-scheme with a full exceptional sequence. Then $\mathbf{J}(\mathcal{A}) = 0$.

II.3. From semiorthogonal decompositions to intermediate Jacobians. Let X and Y be two irreducible smooth projective k-schemes of dimensions d_X and d_Y. Assume that X and Y are related by the following **categorical data:**

There exist dg enhanced semi-orthogonal decompositions $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, \mathcal{A}_X^{\perp} \rangle$ *and* $\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathcal{A}_Y, \mathcal{A}_Y^{\perp} \rangle$ *and an equivalence* $\phi : \mathcal{A}_X \simeq \mathcal{A}_Y$ *of noncommutative k-schemes.*

In what follows, Φ denotes the splitting functor $\mathrm{D}^{\mathrm{b}}(X) \to \mathcal{A}_X \overset{\phi}{\simeq} \mathcal{A}_Y \hookrightarrow \mathrm{D}^{\mathrm{b}}(Y)$. As recalled in Lemma 2.2, the assumption of Φ being a dg equivalence is equivalent to the assumption of Φ being a Fourier-Mukai functor.

Theorem 3.12 ([**BT16b**]). *Let X and Y be two k-schemes as above. Assume also that the above bilinear pairings* (37) *(associated to X and Y) are non-degenerate. Under these assumptions, the following holds:*

(i) *There is a well-defined morphism* $\tau : \prod_{i=0}^{d_X-1} J_a^i(X) \to \prod_{i=0}^{d_Y-1} J_a^i(Y)$ *in* $\mathrm{Ab}(k)_{\mathbb{Q}}$.

(ii) *Assume moreover that* $\mathbf{J}(\mathcal{A}_X^{\perp}) = 0$. *Under this extra assumption, the morphism τ is split injective.*

(iii) *Assume furthermore that* $\mathbf{J}(\mathcal{A}_Y^{\perp}) = 0$. *Under this extra assumption, the morphism τ becomes an isomorphism.*

Theorem 3.12 states how one can find noncommutative schemes inside $\mathrm{D}^{\mathrm{b}}(X)$, from which it is possible to reconstruct the algebraic intermediate Jacobians of X.

From Definition 3.10(i), one observes that whenever X is verepresentable, $J_a^i(X) = 0$ for $i \neq n$. Consequently, there is a single non-trivial algebraic Jacobian $J(X) := J_a^n(X)$ which, thanks to Definition 3.10(ii), carries a canonical principal polarization. Moreover, Definition 3.10(iii) implies that this principally polarized Abelian variety is isomorphic, up to isogeny, to $A_{\mathbb{Z}}^{n+1}(X)$. In these cases, Theorem 3.12 can be strengthened as follows.

Theorem 3.13. *Let X and Y be two irreducible smooth projective k-schemes as in Theorem 3.1(i)-(ii). Assume that X and Y are verepresentable. Under these assumptions, the split injective morphism $\tau : J(X) \to J(Y)$ preserves the principal polarization. When $\mathbf{J}(\mathcal{A}_Y^{\perp}) = 0$ the morphism τ becomes an isomorphism.*

Notice that if X is a curve, $\mathcal{A}_X = \mathrm{D}^{\mathrm{b}}(X)$, and Y is a verepresentable threefold, an analog of Theorem 3.13 was proved in [**BB13**,**BB12**], without using the theory of noncommutative motives but rather Fourier-Mukai functors and Chow motives. The basic case where Y is also curve and $\mathcal{A}_Y = \mathrm{D}^{\mathrm{b}}(Y)$ was treated in [**Ber07**] still via Fourier-Mukai functors.

Recall that Clemens and Griffiths [**CG72**] have shown that the intermediate Jacobian of a Fano threefold is a birational invariant in the following sense: if X is a smooth projective rational threefold then there exist a finite number of smooth projective curves C_i such that $J(X) = \oplus J(C_i)$ as a principally polarized Abelian variety. We can see then how Theorem 3.13 provides an evidence to the fact that there should be a noncommutative scheme inside $\mathrm{D}^{\mathrm{b}}(X)$ which contains informations on the birational equivalence class of X. This is indeed true in the case of conic bundles, which will be treated in Chapter 4.

II.4. Categorical Torelli Theorems. One of the most striking applications of Theorem 3.13 is the possibility to establish categorical Torelli-type Theorems for verepresentable varieties. Namely, to establish whether a given noncommutative scheme, component of $\mathrm{D}^{\mathrm{b}}(X)$, identifies the isomorphism class of X. These theorems occur indeed whenever the classical Torelli theorem holds, and are based on the possibility to reconstruct the intermediate Jacobian with its principal incidence polarization. Notice that Bondal and Orlov have shown that if X is smooth and projective with ample or antiample canonical bundle, then the dg equivalence class of $\mathrm{D}^{\mathrm{b}}(X)$ determines its isomorphism class, namely if $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(Y)$, then Y is isomorphic to X. Hence, categorical Torelli theorems are interesting whenever they provide a noncommutative scheme $\mathcal{A}_X \subset \mathrm{D}^{\mathrm{b}}(X)$ identifying the isomorphism class of X. The following Theorem is based on Theorem 3.13.

Theorem 3.14. *Suppose that either:*

- *X and Y are cubic threefolds, or*
- *X and Y are quartic double solids, or*
- *X and Y are intersections of two even dimensional quadrics, or*
- *X and Y are intersections of three odd dimensional quadrics.*

Set

$$\mathcal{A}_X := \langle \mathcal{O}_X, \dots, \mathcal{O}_X(i-1)\rangle^{\perp}, \quad \mathcal{A}_Y := \langle \mathcal{O}_Y, \dots, \mathcal{O}_Y(i-1)\rangle^{\perp}.$$

where $i = i(X) = i(Y)$ is the index of X and Y. Then X is isomorphic to Y if and only if $\mathcal{A}_X$ is equivalent to $\mathcal{A}_Y$.

In the case of cubic threefolds, Theorem 3.14 was proved in [**BMMS12**] using stability conditions, and showing also that any splitting triangulated functor Φ of that type is a Fourier-Mukai functor.

II.5. Chow groups of intersections of quadrics. Another application of Theorem 3.12 is a description of Chow groups of intersections of quadrics of very low degree. As explained in [**BT16a**], this gives an alternative proof of a Bloch-Beilinson type conjecture for such varieties, for which a proof was already provided by Otwinowska [**Otw99**].

In general, let k be a field and X a complete intersection of multidegree $(d_1, \dots, d_r)$ in $\mathbb{P}^n$, with $d_1 \leqslant \dots \leqslant d_r$. One has the numerical invariant

$$\kappa := [\frac{n - \sum_{j=2}^r d_j}{d_1}],$$

where $[-]$ denotes the integral part of a rational number. A careful analysis of the different Weil cohomology theories of X led to the following conjecture of Beilinson-Bloch type (explicitly stated by Paranjape in [**Par94**, Conjecture 1.8]):

Conjecture 3.15. *There is an isomorphism $CH_i(X)_{\mathbb{Q}} \simeq \mathbb{Q}$ for every $i < \kappa$.*

Otwinowska proved Conjecture 3.15 in the case where X is a complete intersection of quadrics, i.e., when $d_1 = \dots = d_r = 2$; see [**Otw99**, Cor. 1]. Otwinowska's proof is based on a geometric recursive argument. First, one establishes the induction step: if Conjecture 3.15 holds for complete intersections of multidegree $(d_1, \dots, d_r)$, then it also holds for complete intersections of multidegree $(d_1, \dots, d_r, d_r)$; see [**Otw99**, Theorem 1]. Then, one uses the fact that Conjecture 3.15 is known in the case of quadric hypersurfaces. One should also mention the work of Esnault-Levine-Viehweg [**ELV97**]. In *loc. cit.*, a geometric proof of Conjecture 3.15 for very small values of i was obtained via a generalization of Roitman's techniques.

If one further assumes that $\kappa = [d_X/2]$, and that $k \subseteq \mathbb{C}$ is algebraically closed, then Conjecture 3.15 admits an alternative proof, based on Theorem 3.12. The interest of such proof is the clear interplay between semiorthogonal decompositions and algebraic cycles via the theory of noncommutative motives.

Theorem 3.16. *Conjecture 3.15 holds when:*

(i) *X is a complete intersection of two quadrics;*
(ii) *X is a complete intersection of three odd-dimensional quadrics.*

The proof Theorem 3.16 is mainly based on Theorem 3.12 and on the semiorthogonal decomposition of $\mathrm{D^b}(X)$ described by Kuznetsov as a consequence of Homological Projective Duality [**Kuz08**] (see also Theorem 2.21). In the cases of Theorem 3.16 we have that

$$\mathrm{D^b}(X) = \langle \mathrm{D^b}(\mathbb{P}^r, \mathcal{C}_0), \mathcal{O}_X, \dots, \mathcal{O}_X(i-1) \rangle,$$

where $r = 1$ in case (i) and $r = 2$ in case (ii), we denote by i is the index of X, and $\mathcal{C}_0$ is the sheaf of even parts of the Clifford algebra associated to the span of the quadrics defining X. We notice moreover that all involved functors are of Fourier-Mukai type, so we can consider $\mathrm{D^b}(\mathbb{P}^2, \mathcal{C}_0)$ as a noncommutative scheme (i.e. it comes with a natural dg enhancement).

Theorem 3.12 implies then that $\mathbf{J}(\mathrm{D^b}(\mathbb{P}^r, \mathcal{C}_0)) \simeq \mathbf{J}(X) \simeq \prod_{i=0}^{d_X} J_a^i(X)$ are isogenous Abelian varieties. Then one can proceed by a case by case analysis (see [**BT16a**]).

If X is the complete intersection of two quadrics and d_X is even, then $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1, \mathcal{C}_0)$ is generated by exceptional objects, so that $CH^*(X)_{\mathbb{Q}}$ is a finite $\mathbb{Q}$-vector space by Grothendieck-Riemann-Roch. A dimension count and a comparison with cohomologies described in [**Rei72**] gives the proof.

If X is the complete intersection of two quadrics and $d_X = 2n+1$ is odd, then $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1, \mathcal{C}_0) \simeq \mathrm{D}^{\mathrm{b}}(C)$, where $C \to \mathbb{P}^1$ is the hyperelliptic curve ramified along the degeneration divisor. In particular, $J^n(X) \simeq J(C)$ is the only nontrivial Jacobian of X, and this isomorphism is recovered via Theorem 3.12. Then $A^*(X)_{\mathbb{Q}} \simeq J(X) \simeq J(C)$ and $CH^*(X)_{\mathbb{Q}}/A^*(X)_{\mathbb{Q}}$ is a finite vector space. The proof follows then from a dimension count.

If X is the complete intersection of three quadrics and $d_X = 2n+1$ is odd, then consider the associated quadric fibration $Q \to \mathbb{P}^2$ with even Clifford algebra $\mathcal{C}_0$ and degeneration divisor C with its double cover $\widetilde{C} \to C$. Beauville describes of the intermediate Jacobian of X [**Bea77**], as $J^n(X) = \mathrm{Prym}(\widetilde{C}/C)$, where the latter denotes the Prym variety associated to the discriminant double cover, as the only nontrivial Jacobian. From Theorem 3.12 we then get $\mathbf{J}(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{C}_0)) \simeq \mathrm{Prym}(\widetilde{C}/C)$. To complete the proof, it is enough to calculate $K_0(\mathbb{P}^2, \mathcal{C}_0)_{\mathbb{Q}}$ and count the dimension of $CH^*(X)_{\mathbb{Q}}/A^*(X)_{\mathbb{Q}}$ using the semiorthogonal decomposition and Grothendieck-Riemann-Roch.

III. A noncommutative version of smash nilpotence conjecture

Let k be a base field and F a field of coefficients of characteristic zero. In a foundational work [**Voe95**], Voevodsky introduced the smash-nilpotence equivalence relation $\sim_{\otimes \mathrm{nil}}$ on algebraic cycles and conjectured its agreement with the classical numerical equivalence relation $\sim_{\mathrm{num}}$. Concretely, given a smooth projective k-scheme X, he stated the following:

Conjecture $V(X)$: $\mathcal{Z}^*(X)_F/\!\sim_{\otimes \mathrm{nil}} = \mathcal{Z}^*(X)_F/\!\sim_{\mathrm{num}}$.

Thanks to the work of Kahn-Sebastian, Matsusaka, Voevodsky, and Voisin (see [**KS09**, **Mat57**, **Voe95**, **Voi96**] and also [**And04**, §11.5.2.3]), the above conjecture holds in the case of curves, surfaces, and Abelian 3-folds (when k is of characteristic zero).

III.1. Noncommutative nilpotence conjecture. Recall from IV.2 the construction of Noncommutative Numerical and smash-nilpotent motives. For a smooth and proper k-scheme $\mathcal{A}$, one can identify $\mathrm{Hom}(\mathbf{1}, \mathcal{A})$ in each of the categories $\mathrm{NNum}(k)_F$ and $\mathrm{NVoev}(k)_F$ with a quotient of $K_0(\mathcal{A})$ by an appropriate equivalence relation. Indeed, the ideals defined by (13) and (14) respectively endow the Grothendieck group $K_0(\mathcal{A})$ with a $\otimes$-nilpotence equivalence relation $\sim_{\otimes \mathrm{nil}}$ and with a numerical equivalence relation $\sim_{\mathrm{num}}$. Motivated by Conjecture $V(X)$, we stated in [**BMT14**] the following conjecture:

Conjecture $V_{NC}(\mathcal{A})$: $K_0(\mathcal{A})_F/\!\sim_{\otimes \mathrm{nil}} = K_0(\mathcal{A})_F/\!\sim_{\mathrm{num}}$.

If $\mathcal{A} = \mathrm{D}^{\mathrm{b}}(X)$ for some smooth projective k-scheme X, we will abuse by notation by setting $V_{NC}(X) := V_{NC}(\mathrm{D}^{\mathrm{b}}(X))$. In [**BMT14**] it is proved that $V(X)$ and $V_{NC}(X)$ coincide.

Theorem 3.17. *Conjecture $V(X)$ is equivalent to conjecture $V_{NC}(X)$.*

The proof of Theorem 3.17 relies on the existence of the following diagram:

$$\begin{tikzcd}
\mathrm{Chow}(k)_F \arrow[r, "\pi"] \arrow[d] & \mathrm{Chow}(k)_F/_{-\otimes F(1)} \arrow[r, "R"] \arrow[d] & \mathrm{NChow}(k)_F \arrow[d] \\
\mathrm{Voev}(k)_F \arrow[r, "\pi"] \arrow[d] & \mathrm{Voev}(k)_F/_{-\otimes F(1)} \arrow[r, "R_{\otimes\mathrm{nil}}"] \arrow[d] & \mathrm{NVoev}(k)_F \arrow[d] \\
\mathrm{Num}(k)_F \arrow[r, "\pi"] & \mathrm{Num}(k)_F/_{-\otimes F(1)} \arrow[r, "R_{\mathrm{num}}"'] & \mathrm{NNum}(k)_F
\end{tikzcd} \tag{38}$$

where π are the natural functors and R, $R_{\otimes\mathrm{nil}}$ and R_{num} are fully faithful (see [**BMT14**, Prop. 4.2]), and the fact that:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Voev}(k)_F/_{-\otimes F(1)}}(\mathrm{Spec}(k), X) &\simeq \mathcal{Z}^*(X)_F/\!\sim_{\otimes\mathrm{nil}} \\
\mathrm{Hom}_{\mathrm{Num}(k)_F/_{-\otimes F(1)}}(\mathrm{Spec}(k), X) &\simeq \mathcal{Z}^*(X)_F/\!\sim_{\mathrm{num}} .
\end{aligned}$$

The identification of $V(X)$ and $V_{NC}(X)$ is then obtain via the fully faithful functors $R_{\otimes\mathrm{nil}}$ and R_{num}.

III.2. From semiorthogonal decomposition to a proof of Voevodsky's conjecture. Theorem 3.17 can be applied to give new examples of varieties X such that $V(X)$ holds. The general idea is to consider varieties X with a semiorthogonal decomposition $\mathrm{D^b}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n\rangle$ and check $V_{NC}(\mathcal{A}_i)$ for the noncommutative schemes $\mathcal{A}_i$. Indeed, if $V_{NC}(\mathcal{A}_i)$ holds for all $i = 1, \ldots, n$, then $V_{NC}(X)$ holds, so that $V(X)$ holds by Theorem 3.17. Here is a list of applications (see [**BMT14**]).

Quadric fibrations. Let S be a smooth projective k-scheme and $q : Q \to S$ a flat quadric fibration of relative dimension n with Q smooth. Let $\mathcal{C}_0$ be the associated sheaf of even parts of the Clifford algebra. Then we have the following semiorthogonal decomposition [**Kuz08**]:

$$\mathrm{D^b}(X) = \langle \mathrm{D^b}(S, \mathcal{C}_0), \mathrm{D^b}(S)_1 \ldots, \mathrm{D^b}(S)_{n-1}\rangle,$$

where $\mathrm{D^b}(S)_i \simeq \mathrm{D^b}(S)$ and all the involved functors are Fourier-Mukai and hence naturally dg enhanced (compare with Lemma 2.2). The noncommutative scheme $\mathrm{D^b}(S, \mathcal{C}_0)$ has a geometrical description in the case where the discriminant divisor Δ is smooth: if n is odd, we denote by $\widehat{S}$ the root stack of S with $\mathbb{Z}/2\mathbb{Z}$-structure along Δ. If n is even, we denote $\widetilde{S} \to S$ the discriminant double cover ramified along Δ. In both cases, there is an Azumaya algebra $\mathcal{B}_0$ and an equivalence of noncommutative schemes $\mathrm{D^b}(S, \mathcal{C}_0) \simeq \mathrm{D^b}(\widetilde{S}, \mathcal{B}_0)$ (resp. $\mathrm{D^b}(\widehat{S}, \mathcal{B}_0)$), see [**Kuz08**] for more details.

Theorem 3.18. *The following holds:*

(i) *We have* $V(Q) \Leftrightarrow V_{NC}(\mathrm{D^b}(S, \mathcal{C}_0)) + V(S)$.
(ii) *When the discriminant divisor of q is smooth and n is even, we have* $V(Q) \Leftrightarrow V(\widetilde{S}) + V(S)$. *As a consequence, $V(Q)$ holds when* $\dim(S) \leqslant 2$, *and becomes equivalent to* $V(\widetilde{S})$ *when S is an Abelian 3-fold and k is of characteristic zero.*

(iii) *When the discriminant divisor of q is smooth and n is odd, we have $V(Q) \Leftrightarrow V_{NC}(\mathrm{D}^{\mathrm{b}}(\widehat{S}, \mathcal{B}_0)) + V(S)$. As a consequence, $V(Q)$ becomes equivalent to $V_{NC}(\mathrm{D}^{\mathrm{b}}(\widehat{S}, \mathcal{B}_0))$ when* $\dim(S) \leqslant 2$. *This latter conjecture holds when S is a curve or a rational surface and k is algebraically closed.*

We notice that the difference between the even and the odd dimensional case is due to the fact that in the even dimensional case $\widetilde{S}$ is a smooth and proper scheme, so that $U(\mathrm{D}^{\mathrm{b}}(\widetilde{S}, \mathcal{B}_0))_F = NC(S)_F$ for any field F of characteristic zero, as a consequence of Tabuada and Van den Bergh's result [**TVdB15**]. On the other hand, if S is a curve or a rational surface, then an explicit description of $(\widehat{S}, \mathcal{B}_0)$ allows to conclude (see [**BMT14**] for more details).

Intersection of quadrics. Let X be a smooth complete intersection of r quadric hypersurfaces in $\mathbb{P}^m$. The linear span of these r quadrics gives rise to a hypersurface $Q \subset \mathbb{P}^{r-1} \times \mathbb{P}^m$, and the projection into the first factor to a flat quadric fibration $q : Q \to \mathbb{P}^{r-1}$ of relative dimension $m-1$. The decompositions described in Theorem 2.21 give the following result.

Theorem 3.19. *The following holds:*

(i) *We have $V(X) \Leftrightarrow V_{NC}(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^{r-1}, \mathcal{C}_0))$.*

(ii) *When the discriminant divisor of q is smooth and m is odd, we have $V(X) \Leftrightarrow V(\widetilde{\mathbb{P}^{r-1}})$. As a consequence, $V(X)$ holds when $r \leqslant 3$.*

(iii) *When the discriminant divisor of q is smooth and m is even, we have $V(X) \Leftrightarrow V_{NC}(\mathrm{D}^{\mathrm{b}}(\widehat{\mathbb{P}^{r-1}}, \mathcal{B}_0))$. This latter conjecture holds when $r \leqslant 3$ and k is algebraically closed.*

Remark 3.20. (Relative version) As Theorem 2.21 is stated for relative complete intersections, Theorem 3.19 has a relative analogue with X replaced by a generic relative complete intersection $X \to S$ of r quadric fibrations $Q_i \to S$ of relative dimension $m-1$. Items (i), (ii), and (iii), hold similarly with $\mathbb{P}^{r-1}$ replaced by a $\mathbb{P}^{r-1}$-bundle $T \to S$, with $V(\widetilde{\mathbb{P}^{r-1}})$ replaced by $V(\widetilde{T}) + V(S)$, and with $V_{NC}(\mathrm{D}^{\mathrm{b}}(\widehat{\mathbb{P}^{r-1}}, \mathscr{B}_0))$ replaced by $V_{NC}(\mathrm{D}^{\mathrm{b}}(\widehat{T}, \mathscr{B}_0)) + V(S)$, respectively. Note that thanks to the relative item (ii), conjecture $V(X)$ holds when $r = 2$ and S is a curve.

Linear sections of Grassmannians, determinantal varieties. Following Kuznetsov [**Kuz06**], consider the following two classes of schemes:

(i) Let X be a generic linear section of codimension r of the Grassmannian $\mathrm{Gr}(2, W)$ (with $W = k^{\oplus 6}$) under the Plücker embedding, and Y the corresponding dual linear section of the cubic Pfaffian $\mathrm{Pf}(4, W^*)$ in $\mathbb{P}(\Lambda^2 W^*)$.

For example when $r = 3$, X is a Fano 5-fold; when $r = 4$, X is a Fano 4-fold; and when $r = 6$, X is a $K3$ surface of degree 14 and Y a Pfaffian cubic 4-fold. Moreover, X and Y are smooth whenever $r \leqslant 6$.

(ii) Let X be a generic linear section of codimension r of the Grassmannian $\mathrm{Gr}(2, W)$ (with $W = k^{\oplus 7}$) under the Plücker embedding, and Y the corresponding dual linear section of the cubic Pfaffian $\mathrm{Pf}(4, W^*)$ in $\mathbb{P}(\Lambda^2 W^*)$.

For example when $r = 5$, X is a Fano 5-fold; when $r = 4$, X is a Fano 4-fold; when $r = 8$, Y is a Fano 4-fold; and when $r = 9$, Y is a Fano 5-fold. Moreover, X and Y are smooth whenever $r \leqslant 10$.

Homological Projective Duality between $\mathrm{Gr}(2, W)$ and $\mathrm{Pf}(4, W^*)$ was proved by Kuznetsov and gives in these cases a noncommutative k-scheme $\mathcal{A}$ as a component both of $\mathrm{D^b}(X)$ and $\mathrm{D^b}(Y)$ and in such a way that the orthogonal complements are both generated by exceptional objects, see [**Kuz06**]. Theorem 3.17 gives then another family of examples of varieties satisfying V.

Theorem 3.21. *Let X and Y be as in the above classes (i)-(ii). Under the assumption that X and Y are smooth, we have $V(X) \Leftrightarrow V(Y)$. This conjecture holds when $r \leqslant 6$ (class (i)), and when $r \leqslant 6$ and $8 \leqslant r \leqslant 10$ (class (ii)).*

Other similar examples can be obtained via Homological Projective Duality, let us just cite another example. Let $Z^r_{m,n}$ be as in Section IV. Set X and Y to be (geometric) resolutions of singularities of dual linear sections of $Z^r_{m,n}$ and $Z^{m-r}_{m,n}$ respectively. As above, Theorem 3.17 can be applied to have a family of examples of varieties satisfying V. Indeed, there is a noncommutative scheme $\mathcal{C}$, component of both $\mathrm{D^b}(X)$ and $\mathrm{D^b}(Y)$ whose orthogonal complements, if not empty, are generated by exceptional objects.

Theorem 3.22. *Let X and Y be (geometric) resolutions of singularities of dual linear sections of dual determinantal varieties. Then $V(X) \Leftrightarrow V(Y)$. In particular $V(X)$ and $V(Y)$ both hold whenever $\dim(Y) \leqslant 2$ or $\dim(X) \leqslant 2$.*

Moishezon manifolds. A *Moishezon manifold* X is a compact complex manifold such that the field of meromorphic functions on each component of X has transcendence degree equal to the dimension of the component. As proved by Moishezon [**Moi**], X is a smooth projective $\mathbb{C}$-scheme if and only if it admits a Kähler metric. In the remaining cases, Artin [**Art70**] showed that X is a proper algebraic space over $\mathbb{C}$.

Let $Y \to \mathbb{P}^2$ be one of the non-rational conic bundles described by Artin and Mumford in [**AM72**], and $X \to Y$ a small resolution. In this case, X is a smooth (non necessarily projective) Moishezon manifold, and we consider the noncommutative scheme $\mathrm{perf}_{\mathsf{dg}}(X)$. Ingalls and Kuznetsov construct in [**IK15**] a semiorthogonal decomposition:

$$\mathrm{perf}_{\mathsf{dg}}(X) = \langle \mathcal{A}, E_1, E_2 \rangle,$$

where E_i are exceptional and $\mathcal{A}$ is a noncommutative scheme, component of $\mathrm{D^b}(S)$ for S the Enriques surface associated to Y (see [**IK15**] for details). Since $V(S)$ holds, $V_{NC}(S)$ holds and so does $V_{NC}(\mathcal{A})$. We obtain then the following result.

Theorem 3.23. *Conjecture $V_{NC}(\mathrm{perf}_{\mathsf{dg}}(X))$ holds for the above resolutions.*

CHAPTER 4

Semiorthogonal decompositions in birational geometry

This Chapter is devoted to study how semiorthogonal decompositions and noncommutative schemes can detect birational properties of a given projective k-variety X. The idea of studying birational geometry, especially for Fano varieties, using semiorthogonal decompositions goes back to the Moscow school, and in particular to Bondal and Orlov, as they point out in their 2002 ICM address [**BO02**].

Later, Kuznetsov sketched in [**Kuz10**] the definition of a *Griffiths component* of the noncommutative scheme $\mathrm{D^b}(X)$ (or some of its resolution of singularities). Roughly speaking, it would be the noncommutative scheme that one gets as the orthogonal complement to all the noncommutative subschemes of $\mathrm{D^b}(X)$ which are representable in dimension $\dim(X) - 2$. Having such a component well defined would give a birational invariant, as explained in [**Kuz15**], but requires for example the Jordan-Hölder property for semiorthogonal decompositions, which does not hold in general (see Example 1.52).

It is then a very difficult task to know whether such an invariant exists, but it does in the case of Del Pezzo surfaces over a general field [**AB15**], as we will see in Section II. However, let us trace the motivating inspiration behind the idea of Kuznetsov, by recalling Clemens-Griffiths' construction of a birational invariant for complex threefolds with principally polarized intermediate Jacobian [**CG72**].

If X is a smooth complex threefold with $H^{1,0}(X) = H^{3,0}(X) = 0$, then $J(X) := J^3(X)$ is a principally polarized Abelian variety (indeed, this is a case of a verepresentable threefold, as defined in Def. 3.10). Suppose that X' is also smooth and projective and that $\rho : X \dashrightarrow X'$ is a birational map. In particular $H^{1,0}(X') = H^{3,0}(X') = 0$ as well. Hironaka's resolution of singularities provides a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\epsilon"'] \arrow[dr, "\sigma"] & \\ X \arrow[r, dashed, "\rho"] & X', \end{tikzcd} \tag{39}$$

where $\epsilon : \widetilde{X} \to X$ is a composition of a finite number of smooth blow-ups, and $\sigma : \widetilde{X} \to X'$ is a birational morphism. Clemens and Griffths show that then $J(X') \subset J(\widetilde{X})$ and that $J(\widetilde{X}) = J(X) \oplus J(C_1) \oplus \ldots \oplus J(C_r)$ for some smooth curve C_i (the centers of the blow-ups composing ϵ) as principally polarized Abelian varieties. They key of Clemens and Griffiths' idea is that the category of principally polarized Abelian varieties is semisimple, so that the maximal component $A_X \subset J(X)$ which is not split by Jacobian of curves is well defined and a birational invariant, see [**CG72**] for more details.

On the other hand, for k any field with resolution of singularities, X of any dimension, and $\rho : X \dashrightarrow X'$ birational with X' smooth, one can consider the diagram (39). We have that $\sigma^*\mathrm{D}^{\mathrm{b}}(X') \subset \mathrm{D}^{\mathrm{b}}(\widetilde{X})$ is admissible, and $\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle \epsilon^*\mathrm{D}^{\mathrm{b}}(X), \mathrm{D}^{\mathrm{b}}(Z_1), \ldots, \mathrm{D}^{\mathrm{b}}(Z_s)\rangle$ where Z_i are the centers of the blow-ups composing ϵ (repetitions must be allowed). Based on these arguments, on motivic constructions in the case where weak factorization holds, and inspired by Kuznetsov's original idea, we define *categorical representability* and ask about it providing criteria of nonrationality. Moreover, we can construct a *noncommutative motivic rational defect*, a quite weak invariant, which anyway gives some evidence to categorical representability to be an interesting notion to explore. This, together with some more detailed analysis of the Mori fiber space case, is the content of the first Section.

In the rest of the Chapter, we describe how this approach is fruitful in the case of geometrically rational surfaces, complex threefolds such as conic bundles, and some complex fourfolds.

I. Categorical representability and motivic measures

I.1. Categorical representability for schemes. In this section, we provide some general argument to motivate the idea that representability, as defined in 1.44, should play a role in studying birational geometry. Let X be a projective k-scheme. Recall from 1.38 the notion of noncommutative resolution of singularities for X. First of all, we define the notion of categorical representability of schemes.

Definition 4.1 ([**BB12**]). Let X be a projective k-variety. We say that X is *categorically representable* in dimension m (or equivalently in codimension $\dim(X) - m$) if there is a noncommutative resolution of singularities $\mathcal{B}$ of X which is representable in dimension m.

We will use the following notations:

$$\mathrm{rdim}_{\mathrm{cat}}(X) := \min\{\mathrm{rdim}\,\mathcal{B} \mid \mathcal{B} \text{ is a nc resolution of singularities of } X\}$$

$$\mathrm{rcodim}_{\mathrm{cat}}(X) := \dim(X) - \mathrm{rdim}_{\mathrm{cat}}(X),$$

whenever at least one such a representable $\mathcal{B}$ exists[1].

Example 4.2. If $X = \mathbb{P}^n$ is a projective space, then $\mathrm{rdim}_{\mathrm{cat}}(X) = 0$. Indeed, $\mathrm{D}^{\mathrm{b}}(X)$ is generated by the full exceptional sequence $\langle \mathcal{O}_X, \ldots, \mathcal{O}_X(n)\rangle$, see [**Bei78**].

If X is smooth and such that $\mathrm{D}^{\mathrm{b}}(X)$ has no nontrivial semiorthogonal decomposition, then $\mathrm{rdim}_{\mathrm{cat}}(X) = \dim(X)$ if we assume that there is no fully faithful functor $\mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ for any Y with $\dim(Y) < \dim(X)$. These two conditions hold for example if X is a curve of positive genus, see [**Oka11**] for indecomposability, and use Theorem 3.12 to prove the second property. Other examples of semiorthogonally indecomposable derived categories include some complex surfaces [**KO15**], and connected varieties with trivial canonical bundle [**Bri99**, Ex. 3.2]. The second property is more subtle in this case, but can still be shown in some cases using deeper geometrical cycle-theoretic arguments, as for example the infinite dimension of the Griffths group of a Calabi–Yau threefold.

[1]Notice that, thanks to Kuznetsov-Lunts' result [**KL12**, Thm. 1.4], such a $\mathcal{B}$ exists whenever X is separated of finite type and k is of characteristic zero.

Let us restrict for some paragraph to the smooth and projective case. In this case categorical representability has a geometric interpretation.

Proposition 4.3. *Let X be a smooth projective k-variety. We have that* $\mathrm{rdim}_{\mathrm{cat}}(X) \leqslant m$ *if and only if there exist smooth projective varieties $Y_1, \ldots, Y_r$ and objects $\mathcal{E}_i$ in* $\mathrm{D^b}(Y_i \times X)$*, such that, for any $i = 1, \ldots, r$, we have* $\dim(Y_i) \leqslant m$*, and $\Phi_{\mathcal{E}_i}$ are splitting functors whose images form a semiorthogonal set and generate* $\mathrm{D^b}(X)$*.*

Proof. First of all, suppose that such Y_i and $\mathcal{E}_i$ exist. Let us denote by $\mathcal{A}_i := \mathrm{Im}\Phi_{\mathcal{E}_i}$ the admissible subcategories of $\mathrm{D^b}(X)$. The noncommutative schemes $\mathcal{A}_i$ are components of $\mathrm{D^b}(Y_i)$, so that $\mathrm{rep}\mathcal{A}_i \leqslant m$ for any $i = 1, \ldots, r$. Since $\mathrm{D^b}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_r \rangle$, we get that $\mathrm{rdim}_{\mathrm{cat}}(X) \leqslant m$.

On the other hand, suppose that $\mathrm{rdim}_{\mathrm{cat}}(X) \leqslant m$. Then there is a semiorthogonal decomposition $\mathrm{D^b}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_r \rangle$ with $\mathrm{rdim}\, \mathcal{A}_i \leqslant m$ for all $i = 1, \ldots, r$. Up to refining the semiorthogonal decomposition, we have, by Definition 1.44 of representability of a noncommutative scheme, that $\mathcal{A}_i$ is admissible in some $\mathrm{D^b}(Y_i)$ with $\dim(Y_i) \leqslant m$. We obtain then splitting functors $\Phi_i : \mathrm{D^b}(Y_i) \to \mathrm{D^b}(X)$ of dg enhanced triangulated categories, which are Fourier-Mukai by Lemma 2.2. $\square$

Remark 4.4. Suppose that X is a projective, nonsmooth k-scheme. We would like to extend Proposition 4.3 to this case, where $\mathrm{D^b}(X)$ is replaced by some resolution of singularities $\mathcal{B}$, which satisfies the minimality defining $\mathrm{rdim}_{\mathrm{cat}}(X)$. The resolution that most fit this aim is probably the one given in Definition 1.42, in which case $\mathcal{B}$ is, locally over X, described as an $\mathcal{O}_X$-algebra B. In this case we can extend Proposition 4.3 by asking that the objects $\mathcal{E}_i$ are, locally over X, sheaves of B-algebras.

Proposition 4.3 also explains the choice of the terminology, since the categorical representability of a variety X gives in particular that the noncommutative motive $NC(X)$ is a submotive of $NC(Y_1) \oplus \ldots \oplus NC(Y_r)$ which is in turn equal to $NC(Y_1 \amalg \ldots \amalg Y_r)$, the noncommutative motive of a (disconnected) variety of dimension bounded by m.

We start investigating the motivations that led us to define categorical representability as a useful tool in birational geometry with the following simple observation.

Lemma 4.5. *Let X and Y be smooth and projective k-schemes and $\sigma : Y \to X$ a morphism such that $\sigma_* \mathcal{O}_Y = \mathcal{O}_X$ and $R^i \sigma_* \mathcal{O}_Y = 0$ for $i \neq 0$. For example, σ is a divisorial contraction, or a Mori fiber space. Then $L\sigma^* : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ is fully faithful.*

Proof. For any A and B objects in $\mathrm{D^b}(X)$, we have

$$\mathrm{Hom}_Y(L\sigma^* A, L\sigma^* B) = \mathrm{Hom}_X(A, R\sigma_* L\sigma^* B) = \mathrm{Hom}_X(A, B \otimes R\sigma_* \mathcal{O}_Y) = \mathrm{Hom}_X(A, B),$$

by adjunction, projection formula and by our assumption respectively. $\square$

Thanks to the work of Orlov [**Orl93**], we have a description of the derived category of a blow-up. This implies that this operation preserves representability in codimension 2.

Lemma 4.6. *Let $Y \to X$ be a blow-up of smooth projective varieties along a smooth center. If we assume* $\mathrm{rcodim}_{\mathrm{cat}}(X) \geqslant 2$*, then* $\mathrm{rcodim}_{\mathrm{cat}}(Y) \geqslant 2$*.*

Proof. If $Z \subset X$ is the center of the blow-up and has codimension c in X, then $\mathrm{D^b}(Y) = \langle \mathrm{D^b}(X), \mathrm{D^b}(Z)_1, \ldots, \mathrm{D^b}(Z)_{c-1} \rangle$, where $\mathrm{D^b}(Z)_i \simeq \mathrm{D^b}(Z)$ for any $i = 1, \ldots, c-1$, see [**Orl93**]. The proof follows by definition of categorical representability. $\square$

Corollary 4.7. *Suppose that $\epsilon : Y \to \mathbb{P}^n$ is a composition of blow-ups along smooth centers, and $n \geqslant 2$. Then* $\mathrm{rcodim_{cat}}(Y) \geqslant 2$.

Proof. This follows straightforward from Lemma 4.6 and $\mathrm{rdim_{cat}}(\mathbb{P}^n) = 0$. $\square$

Now, let $\rho : \mathbb{P}^n \dashrightarrow X$ be a birational map. Resolving the singularities of ρ give a diagram like (39), with a smooth and projective Y obtained by iterated smooth blow-ups of $\mathbb{P}^n$ and a birational morphism $\sigma : Y \to X$. Using Lemma 4.5 and Corollary 4.7, we get that $\mathrm{D^b}(X)$ is admissible inside $\mathrm{D^b}(Y)$ and that $\mathrm{rcodim_{cat}}(Y) \geqslant 2$.

Notice anyway that the converse implication of Lemma 4.6 is highly nontrivial and not known in general. Even more general, as remarked in §III.3, if a noncommutative scheme $\mathcal{A}$ is such that $\mathrm{rdim}\,\mathcal{A} = n$, it is not known whether any noncommutative scheme $\mathcal{B}$, admissible in $\mathcal{A}$, satisfies $\mathrm{rdim}\,\mathcal{B} \leqslant n$. Hence, the fact that $\mathrm{rcodim_{cat}}(Y) \geqslant 2$ does not give any information neither on $\mathrm{rcodim_{cat}}(X)$ nor on any other admissible subcategory of $\mathrm{D^b}(Y)$.

I.2. Motivic measures and a rational defect. We consider here the Grothendieck ring of varieties $K_0(\mathrm{Var}(k))$, which is defined by taking the free $\mathbb{Z}$-module generated by varieties over k, and taking the quotient by the so-called *scissor relation*, that is: $[X] = [Z] + [U]$ whenever $Z \subset X$ is a closed subvariety with open complement U, and $[-]$ denotes the class of a given variety in $K_0(\mathrm{Var}(k))$. The sum of classes corresponds to the disjoint union, $[X] + [Y] = [X \amalg Y]$ and the product to the product of varieties $[X] \times [Y] = [X \times Y]$. The unit $\mathbf{1}$ of $K_0(\mathrm{Var}(k))$ is the class of $\mathrm{Spec}(k)$, while we denote by $\mathbb{L} = [\mathbb{A}^1_k]$ the class of the affine line. Notice that $[\mathbb{P}^1_k] = \mathbf{1} + \mathbb{L}$.

A *motivic measure* is a ring homomorphism $\mu : K_0(\mathrm{Var}(k)) \to R$ to some commutative ring R. There are many natural examples of motivic measures, but we are here mainly interested in the one defined by Larsen and Lunts in [**LL03**] in the case where weak factorization holds, for example if k has characteristic zero [**AKMW02**]. We assume working over such a k in this subsection.

Using weak factorization, Bittner [**Bit04**] has provided another presentation of $K_0(\mathrm{Var}(k))$ as the $\mathbb{Z}$-module generated by isomorphism classes of smooth proper varieties with the relations $[X] - [Z] = [Y] - [E]$ whenever $Y \to X$ is the blow-up along the smooth center Z with exceptional divisor E, see [**Bit04**]. Larsen and Lunts [**LL03**] have then shown that there is a motivic measure $\mu_{LL} : K_0(\mathrm{Var}(k)) \to \mathbb{Z}[SB]$ to the free $\mathbb{Z}$-module generated by stable birationality classes. Moreover, $\ker \mu_{LL} = \langle \mathbb{L} \rangle$ is the ideal generated by the class $\mathbb{L}$ of the affine line. It follows that (as remarked in [**GS14**]), if X is rational of dimension n, then:

$$[X] = [\mathbb{P}^n] + \mathbb{L} M_X \tag{40}$$

in $K_0(\mathrm{Var}(k))$, where M_X is a $\mathbb{Z}$-linear combination of classes of varieties of dimension bounded above by $n-2$. Galkin and Shinder define then $([X] - [\mathbb{P}^n])/\mathbb{L} \in K_0(\mathrm{Var}(k))[\mathbb{L}^{-1}]$ as the *rational defect* of X [**GS14**].

The second motivic measure we want to consider was defined by Bondal, Larsen, and Lunts [**BLL04**], still in the case where weak factorization holds. Recall the definition of the Grothendieck ring of noncommutative k-schemes from §III.4. Then there is a measure defined by

$$\begin{aligned} \mu_{BLL} : K_0(\mathrm{Var}(k)) &\longrightarrow PT(k) \\ [X] &\longmapsto x := I(\mathrm{D^b}(X)). \end{aligned}$$

We notice that we have a natural filtration by the dimension on the ring $K_0(\mathrm{Var}(k))$ and that we have $\mu_{BLL}(K_0(\mathrm{Var}(k))_d) \subset PT_d(k)$. However, for any d, we have $\mu_{BLL}(\mathbb{P}^d) = (d+1)\mathbf{e}$, which is in $PT_0(k)$, while $[\mathbb{P}^d]$ is not in $K_0(\mathrm{Var}(k))_i$ for any $i < d$.

Recall again that $I(\mathrm{D^b}(\mathbb{P}^1)) = 2\mathbf{e}$ and that $[\mathbb{P}^1] = 1 + \mathbb{L}$ in $K_0(\mathrm{Var}(k))$. It follows that $\mu_{BLL}(\mathbb{L}) = \mathbf{e}$ in $PT(k)$. Applying μ_{BLL} to (40), we obtain that if X is rational of dimension n, then x is in $PT_{n-2}(k)$. This give a proof to the following Proposition.

Proposition 4.8. *If X is a smooth and projective variety of dimension n such that x is not in $PT_{n-2}(k)$, then X is not rational.*

Definition 4.9. If X is a smooth and projective variety of dimension n, the class of the element x in the group $PT(k)/PT_{n-2}(k)$ is called the *noncommutative motivic rational defect* of X.

We end by commenting the fact that Proposition 4.8 is a very weak result. Indeed, as remarked above, we have an implication $\mathrm{rdim}_{\mathrm{cat}}(X) \leqslant i \Rightarrow x \in PT_i(k)$, but the converse implication is not known, even for $i = 0$. For example, it is not known that $PT_n(k) \neq PT_{n-2}(k)$ in full generality (though, this is true for $n \leqslant 4$). Anyway, this let us conclude the speculations of these two subsections by the following question.

Question 4.10. Is categorical representability in codimension 2 a necessary condition for rationality? That is, if X is rational, do we have $\mathrm{rcodim}_{\mathrm{cat}}(X) \geqslant 2$?

Before getting into more details about Question 4.10 for Mori fiber spaces, let us briefly remark how noncommutative motives could play a role in this context as well. Recall the definition of the category $\mathrm{NChow}(k)$ from IV, and the filtration by thick subcategories $\mathrm{NChow}_d(k) \subset \mathrm{NChow}(k)$. As remarked previously, we have that if $\mathcal{A}$ has $\mathrm{rdim}\,\mathcal{A} = d$, then $U(\mathcal{A})$ is in $\mathrm{NChow}_d(k)$.

Moreover, one can also consider the Grothendieck ring $K_0(\mathrm{NChow}(k))$ of the additive category $\mathrm{NChow}(k)$, and the motivic measure μ_{nc} defined by Tabuada (see [**Tab15**, 4.2.1]):

$$\begin{aligned} \mu_{nc} : K_0(\mathrm{Var}(k)) &\longrightarrow K_0(\mathrm{NChow}(k)) \\ [X] &\longmapsto [NC(X)], \end{aligned}$$

where $[-]$ also denotes the class of a noncommutative Chow motive in the Grothendieck ring. As noticed by Tabuada, the map μ_{nc} factors through the map μ_{BLL}, via the natural map $PT(k) \to K_0(\mathrm{NChow}(k))$ [**Tab15**, Prop. 4.8]. This would let us give an analog (and, in general, weaker) statement of Proposition 4.8, and also to ask whether X being rational of dimension n implies $NC(X)$ being in $\mathrm{NChow}_{n-2}(k)$.

I.3. The case of Mori fiber spaces. Question 4.10 is certainly intriguing but probably too challenging, at least in whole generality. In this last subsection, we explain a possible approach via Mori fiber spaces. Again, we start by assuming X to be smooth and projective, even if we could consider noncommutative resolution of singularities. However, this first assumption makes the explanations much easier and we will come back to more general case at the end of this subsection.

By a *Mori fiber space* we mean a flat morphism $\pi : X \to Y$ between projective varieties of relative dimension $m > 0$ and such that the relative Picard group $\mathrm{Pic}(X/Y) \simeq \mathbb{Z}$ is free of rank one, and the relative canonical bundle $\omega_{X/Y}$ is antiample. In particular, if $\mathcal{O}_{X/Y}(1)$ is an ample generator of $\mathrm{Pic}(X/Y)$, then $\omega_{X/Y} = \mathcal{O}_{X/Y}(-i)$ and we call i the *index* of X over Y. Notice that if $Y = \mathrm{Spec}(k)$, a Mori fiber space X is a Fano variety of Picard rank one and index i. Lemma 4.5 gives that $\pi^*\mathrm{D^b}(Y)$ is an admissible subcategory of $\mathrm{D^b}(X)$. Moreover, let us denote by $\mathrm{D^b}(Y)(j) := \pi^*\mathrm{D^b}(Y) \otimes \mathcal{O}_{X/Y}(j)$. Then $\mathrm{D^b}(Y)(j)$ is also admissible in $\mathrm{D^b}(X)$ and equivalent to $\mathrm{D^b}(Y)$. A relative Kawamata-Viehweg vanishing gives the following result.

Proposition 4.11. *Let $\pi : X \to Y$ be a Mori fiber space of relative index i. Then there is a semiorthogonal decomposition*

$$\mathrm{D^b}(X) = \langle \mathcal{A}_{X/Y}, \mathrm{D^b}(Y), \dots, \mathrm{D^b}(Y)(i-1) \rangle,$$

for a noncommutative scheme $\mathcal{A}_{X/Y}$.

Notice that for any integer r, the sequence $\langle \mathrm{D^b}(Y)(r), \dots, \mathrm{D^b}(Y)(i+r-1) \rangle$ is semiorthogonal, and its complement is a noncommutative scheme, dg-equivalent to $\mathcal{A}_{X/Y}$. A simple application Proposition 4.8 indicates that in the case of Mori fiber spaces the noncommutative scheme $\mathcal{A}_{X/Y}$ is the one which could obstruct nonrationality.

Corollary 4.12. *Let $X \to Y$ be a Mori fiber space of relative dimension m, and let $n =$ $\dim(X)$. Assuming that either Y is rational or $m > 1$, we have that x is in $PT_{n-2}(k)$ if and only if $I(\mathcal{A}_{X/Y})$ is in $PT_{n-2}(k)$.*

Proof. By the assumptions on Y, there exists $d \leqslant n-2$ such that the class y is in $PT_d(k) \subset PT_{n-2}(k)$. Using the semiorthogonal decomposition from Proposition 4.11, we get that $x = I(\mathcal{A}_{X/Y})$ modulo elements in $PT_{n-2}(k)$, and the proof follows. □

Notice that we can extend our analysis to any rational map $\rho : X \dashrightarrow Y$, which can be resolved into a Mori fiber space $\widetilde{X} \to Y$. For example, if X is a cubic threefold, and $X \dashrightarrow \mathbb{P}^2$ is a projection off a line contained in X, then $\widetilde{X} \to \mathbb{P}^2$ is a conic bundle. We denote $\mathcal{A}_{X/Y,\rho} := \mathcal{A}_{\widetilde{X}/Y}$

Corollary 4.13. *Suppose that there is a rational map $\rho : X \dashrightarrow Y$ and a commutative diagram:*

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "\epsilon"'] \arrow[dr, "\pi"] & \\
X \arrow[r, dashed, "\rho"] & Y,
\end{tikzcd}$$

where $\pi : \widetilde{X} \to Y$ is a Mori fiber space of relative dimension m and $\epsilon : \widetilde{X} \to X$ is a blow up along a smooth center. Assuming that either Y is rational or $m > 1$, we have that x is in $PT_{n-2}(k)$ if and only if $I(\mathcal{A}_{X/Y,\rho})$ is in $PT_{n-2}(k)$.

Proof. We notice that $\mathrm{D^b}(\widetilde{X})$ has two decompositions, one given by the Mori fiber space map $\widetilde{X} \to Y$ as in Proposition 4.11 and the other given by the blow-up of X, hence containing a copy of $\mathrm{D^b}(X)$ and a finite number of copies of the blown-up loci. A proof similar to the one of Corollary 4.12 applies then again: once we write the two decompositions of $\widetilde{x}$, we deduce that $x = I(\mathcal{A}_{X/Y,\rho})$ modulo $PT_{n-2}(k)$. □

Remark 4.14. The previous arguments are explained in the cases where all the involved varieties are smooth. However, this is just a comfortable but not necessary assumption, since we can replace $\mathrm{D^b}(X)$ by a noncommutative resolution of singularities whenever it exists.

II. Birational geometry of geometrically rational surfaces

Let k be a field, k^s a separable closure, and $\bar{k}$ an algebraic closure. A smooth projective geometrically integral surface X over k such that $\overline{X} = X \times_k \bar{k}$ is $\bar{k}$-rational is called a *geometrically rational surface*. Recall that X is a *del Pezzo* surface if $\omega_X^\vee$ is ample. The *degree* of a geometrically rational surface is the self-intersection number $d = \omega_X \cdot \omega_X$.

When the base field k is not algebraically closed, the existence of k-rational points on a variety X (being a necessary condition for rationality) is a major open question in arithmetic geometry. We consider, in the case where X is a del Pezzo surface, as a natural extension of Question 4.10, the following question formulated by H. Esnault.

Question 4.15. Let X be a smooth projective variety over a field k. Can the bounded derived category $\mathrm{D^b}(X)$ of coherent sheaves detect the existence of a k-rational point on X?

In this section, we describe how to get a positive answer to Question 4.10 for del Pezzo surfaces over any field, and we moreover construct a noncommutative scheme $\mathcal{GK}_X$ (the *Griffiths-Kuznetsov component*), which is a birational invariant for such surfaces. Moreover, we show how semiorthogonal decompositions naturally provide vector bundles whose second Chern classes are related to the existence of closed point of given degree, giving a positive answer to Question 4.15. All the material is taken from [**AB15**].

II.1. Generalities on geometrically rational surfaces. We say that a field extension l of k is a *splitting field* for X if $X \times_k l$ is birational to $\mathbb{P}^2_l$ via a sequence of monoidal transformations centered at closed l-points. An important fact is that geometrically rational surfaces are separably split, see [**Coo88**, Thm. 1], [**VA13**, Thm. 1.6].

A surface X is *minimal* over k, or *k-minimal*, if any birational morphism $f : X \to Y$, defined over k, is an isomorphism. Over a separably closed field, the only minimal rational surfaces are the projective plane and projective bundles over the projective line. Over a general field, this is no longer true. Minimal geometrically rational surfaces have been completely classified, and we have the following list (see [**Man74**], and the recent survey [**Has09**]):

(1) $X = \mathbb{P}^2_k$ is a projective plane, so $\mathrm{Pic}(X) = \mathbb{Z}$, generated by the hyperplane $\mathcal{O}(1)$;

(2) $X \subset \mathbb{P}^3_k$ is a smooth quadric and $\mathrm{Pic}(X) = \mathbb{Z}$, generated by the hyperplane section $\mathcal{O}(1)$;
(3) X is a del Pezzo surface with $\mathrm{Pic}(X) = \mathbb{Z}$, generated by the canonical class ω_X;
(4) X is a conic bundle $f : X \to C$ over a geometrically rational curve, with $\mathrm{Pic}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Examples of del Pezzo surfaces with $\mathrm{Pic}(X) = \mathbb{Z}$ generated by ω_X include non-k-rational *Severi-Brauer surfaces*, that is non-k-rational surfaces X over k such that $\overline{X} \simeq \mathbb{P}^2_{\bar{k}}$. The set of isomorphism classes of Severi-Brauer surface is in bijection with the set of k-isomorphism classes of central simple algebras A of degree 3 over k, and we will write $X = \mathrm{SB}(A)$ accordingly. By a theorem of Châtelet, a Severi-Brauer surface X is k-rational if and only if X is k-isomorphic to projective space if and only if $X(k) \neq \varnothing$, cf. [**GS06**, Thm. 5.1.3]. In this case, we say that X *splits* and remark that X always splits after a finite separable field extension. As intersection numbers do not change under scalar extension, X has degree 9.

We finally notice that $\mathrm{SB}(A) \simeq \mathrm{SB}(B)$ if and only if $A \simeq B$ are Brauer-equivalent, while $\mathrm{SB}(A)$ and $\mathrm{SB}(B)$ are birationally equivalent if and only if A and B generate the same subgroup of the Brauer group $\mathrm{Br}(k)$. This latter fact is conjectured by Amitsur (cf. [**Ami55**]) to hold in any dimension. In particular, $\mathrm{SB}(A)$ and $\mathrm{SB}(A^2)$ are birational but not isomorphic k-varieties.

Denote by $\rho(X)$ the Picard rank of X. Minimal surfaces in cases (1), (2) and (3) are Fano varieties of Picard rank one and index[2] 3, 2 and 1 respectively. Minimal surfaces in cases (4) have a structure of Mori fiber space $X \to C$. Notice that a necessary condition for the rationality of X is the rationality of C, that is $C \simeq \mathbb{P}^1$. This isomorphism is equivalent $\mathrm{rdim}_{\mathrm{cat}}(C) = 0$.

Manin has proved [**Man74**, Thm. 29.4] that, given a (non necessarily minimal) del Pezzo surface of degree $d \geqslant 2$, the existence of a k-rational point (not lying on any exceptional curve if $d \leqslant 4$) implies the existence of a unirational parametrization, i.e., a map $\mathbb{P}^2_k \dashrightarrow X$ of finite degree. In particular, if $\deg(X) \geqslant 5$, this map has degree one. It follows that if X is a del Pezzo surface of degree $d \geqslant 5$, $X(k)$ is nonempty if and only if X is k-rational.

II.2. Categorical representability and the Griffiths-Kuznetsov component for del Pezzo surfaces. Let X be a minimal del Pezzo surface, so that the structure map $X \to \mathrm{Spec}(k)$ is a Mori fiber space of index $i \leqslant 3$. We consider the noncommutative scheme $\mathcal{A}_X = \mathcal{A}_{X/k}$, and notice that $\mathrm{rk} K_0(\mathcal{A}_X) = 3 - i$ and

- $\mathcal{A}_X = 0$ if $i = 3$, that is if $X = \mathbb{P}^2_k$ and
$$\mathrm{D^b}(X) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$$
- $\mathcal{A}_X = \mathrm{D^b}(k, \mathcal{C}_0)$ if $i = 2$, that is if $X \subset \mathbb{P}^3_k$ is a quadric, where $\mathcal{C}_0$ is the even Clifford algebra associated to X (see [**ABB14**, **Kuz08**]), and
$$\mathrm{D^b}(X) = \langle \mathrm{D^b}(k, \mathcal{C}_0), \mathcal{O}, \mathcal{O}(1) \rangle$$

[2]By index, we mean here the index of X as a Fano variety. Another notion of index for such surfaces is given by the greatest common divisor of degrees of closed points. Later, we will consider this notion and name it *point-index* just to mark the difference.

- $\mathcal{A}_X \simeq \langle \omega_X \rangle^\perp$ if $i = 1$, and

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O} \rangle.$$

In order to understand the structure of $\mathcal{A}_X$ in the third case, we rely on the description of all possible full exceptional collections for del Pezzo surfaces over algebraically closed fields which was obtained by Rudakov, Gorodentsev-Rudakov, Kuleshov-Orlov and Karpov-Nogin (see, e.g., [**GR87**], [**Rud89**], [**KO94**], [**KN98**]). While these authors restrict to working over algebraically closed fields of characteristic zero, their proofs are based on properties of vector bundles and on the description of a del Pezzo surface as a blow-up of $\mathbb{P}^2$ (or of a quadric surface as $\mathbb{P}^1 \times \mathbb{P}^1$), hence they actually hold for any totally split del Pezzo surface, in particular, they hold over any separably closed field.

Let us describe what is known for $k = k^s$. Kuleshov-Orlov show that any exceptional object in $\mathrm{D^b}(X)$ is (a shift of) a vector bundle [**KO94**]. Recall the definition of exceptional block from §I.4. A *3-block exceptional collection* is a full exceptional collection on X whose exceptional objects form three exceptional blocks. Thanks to Rudakov [**Rud89**], Gorodentsev-Rudakov [**GR87**] and Karpov-Nogin [**KN98**], there is a finite set (actually $\max\{5-d, 1\}$ if $d = \deg(X)$) of 3-block exceptional collections from which any other 3-block exceptional collection can be obtained by mutations and tensoring the whole category by line bundles; this holds unless X is the blow-up of one or two points in $\mathbb{P}^2$, in which case there is no 3-block exceptional collection. We notice however that in the latter cases X is never minimal. Moreover, these 3-block collections are completely classified, that is, we know the minimal ones (where a collection is minimal if the bundles have minimal possible rank in the set of all the collections that one obtain via mutations).

If k is general, and X is minimal, we have $K_0(X) = \mathbb{Z}^3$. In the cases where $X = \mathbb{P}^2$, or X is a quadric, we already have either $\mathcal{A}_X = 0$ or an algebraic description of $\mathcal{A}_X$. We notice moreover that the decomposition $\mathrm{D^b}(\mathbb{P}^2) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$ is already the unique minimal 3-block decomposition. Moreover, if X is a quadric, the three components of the decomposition $\mathrm{D^b}(X) = \langle \mathrm{D^b}(k, \mathcal{C}_0), \mathcal{O}, \mathcal{O}(1) \rangle$ base-change to the three blocks of the unique minimal 3-block decomposition of X_{k^s}.

We then restrict to the cases of index 1, and X minimal. In order to produce semiorthogonal decompositions of $\mathrm{D^b}(X)$, we consider the minimal 3-block decompositions of X_{k^s} and recall Proposition 1.57: if such a decomposition descend to the base field k, then we would have a semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathrm{D^b}(l_1, A_1), \mathrm{D^b}(l_2, A_2), \mathrm{D^b}(l_3, A_3) \rangle, \tag{41}$$

for l_i/k étale algebras and $A_i \in \mathrm{Br}(l_i)$ central simple algebras. We also notice that X minimal is equivalent to $\rho(X) = 1$ which is in turn equivalent to $K_0(X) = \mathbb{Z}^3$. The latter is then equivalent to have that l_i/k is a field extension. An analysis of all the possible minimal 3-block decompositions gives the following result [**AB15**].

Theorem 4.16. *Let X be a del Pezzo surface over a general field k of degree d.*

If $d \leqslant 4$, and X is minimal then there is no decomposition of the form (41)*, and the decomposition*

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O} \rangle$$

does not base-change to any 3-block decomposition of $\mathrm{D^b}(X_{k^s})$*.*

If $d \geqslant 5$, then there is always a decomposition of the form (41)*, base changing to the unique 3-block decomposition of* $\mathrm{D^b}(X_{k^s})$*. Moreover,* $\mathrm{D^b}(l_1, A_1) = \langle \mathcal{O} \rangle$*, so that $l_1 = k$ and A_1 is trivial. The k-algebras A_2 and A_3 arise as endomorphism algebras of vector bundles V_2 and V_3 respectively. If X is minimal, then* $\mathcal{A}_X = \langle \mathrm{D^b}(l_2, A_2), \mathrm{D^b}(l_3, A_2) \rangle$ *and l_i are field extensions of k.*

The statement of Theorem 4.16 gives a clear separation between high degree and low degree surfaces. This separation should reflect the fact that low degree surfaces have a much more complicated arithmetic than the high degree ones. Moreover, surfaces of high degree are always either toric (for $d \geqslant 6$) or rational (any degree 5 del Pezzo is rational, [**SD72**]). Moreover, any minimal rational surface has degree $d \geqslant 5$.

Let us first comment the low degree case. The proof of the statement is a verification that the descent of any 3-block decomposition would contradict the minimality of X, because it would imply that some exceptional divisor is defined on X. A minimal surface X with $d \leqslant 4$ is never rational. Moreover, unless $d = 4$ and $X(k) \neq \varnothing$, such an X is birationally rigid, that is, if $X' \dashrightarrow X$, then $X' \to X$ is a birational morphism. It follows that $\mathcal{A}_X$ is a birational invariant. If $d = 4$ and x is a k-rational point, then the blow-up $\widetilde{X} \to X$ of x has a structure of conic bundle $X \to \mathbb{P}^1$, and $\mathcal{A}_X = \mathcal{A}_{X/\mathbb{P}^1} \simeq \mathrm{D^b}(\mathbb{P}^1, \mathcal{B}_0)$, where $\mathcal{B}_0$ is the sheaf of even parts of the Clifford algebra associated to the conic bundle.

If $d \geqslant 5$, then Theorem 4.16 recovers some decompositions which were already known in the case of Brauer-Severi surfaces [**Ber09**], involution surfaces [**Blu12**] and degree 6 del Pezzo surfaces [**BSS11**]. Besides, Theorem 4.16 gives a general criterion to construct semiorthogonal decompositions of the form (41): given an exceptional block $\mathbf{E} = \{E_1, \ldots, E_r\}$ in $\mathrm{D^b}(X_{k^s})$, generated by vector bundles, consider the vector bundle $W := \bigoplus_{i=1}^r E_i$. Then W is a generator for $\mathbf{E}$. If there is a vector bundle V on X such that $V_{k^s} = W^{\oplus s}$ for some s, then the category $\langle V \rangle$ base changes to $\mathbf{E}$, and we are in the setup of Proposition 1.57. This is the way we produce the vector bundles V_2 and V_3 in Theorem 4.16, by considering the explicit description of the minimal 3-block decompositions.

Definition 4.17. Let X be a minimal del Pezzo surface over a general field k of degree d. We define the *Griffiths-Kuznetsov component* $\mathcal{GK}_X$ as the following noncommutative scheme:

- if $d \leqslant 4$, set $\mathcal{GK}_X := \mathcal{A}_X$,
- if $d \geqslant 5$, set $\mathcal{GK}_X := \coprod_{A_i \text{ nontrivial}} \mathrm{D^b}(l_i, A_i)$,

where A_i are the algebras appearing in Theorem 4.16, and A_i nontrivial means that the class of A_i is nontrivial in the Brauer group $\mathrm{Br}(l_i)$.

Notice that A_i nontrivial is equivalent to $\mathrm{rep}\mathrm{D^b}(l_i, A_i) \neq 0$. Indeed, one can show (see, e.g., [**AB15**, Thm. 1.4.6], or the more general [**Ant14**]), that, given two fields K_1 and K_2 and two Azumaya algebras A_1 and A_2 over K_1 and K_2 respectively, then $\mathrm{D^b}(K_1, A_1) \simeq$

$\mathrm{D}^{\mathrm{b}}(K_2, A_2)$ if and only if $K_1 \simeq K_2$ and A_1 is Brauer equivalent to A_2. The results from Theorem 4.16 tell then that $\mathcal{GK}_X$ encodes the information on the noncommutative schemes, admissible in $\mathrm{D}^{\mathrm{b}}(X)$, which are not representable in dimension 0.

Theorem 4.18. *Let X be a del Pezzo surface of degree d over an arbitrary field k, and assume that $X(k) \neq \varnothing$ if $d = 4$, and that there is either a rational point or no point of degree 2 if X has degree 8. Then $\mathcal{GK}_X$ is well defined and is a birational invariant.*

In particular, X is rational if and only if $\mathcal{GK}_X = 0$, and the latter is equivalent to $\mathrm{rdim}_{\mathrm{cat}}(X) = 0$.

In the case where $d \leqslant 4$, the proof relies on the fact that such surfaces are birationally rigid. Notice moreover that we are excluding exactly those two cases where blowing a point of minimal degree would give a conic bundle $\widetilde{X} \to C$, where C is a conic, and $\mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}(C, \mathcal{B}_0)$ for $\mathcal{B}_0$ the Clifford algebra of the conic bundle. Hence these two del Pezzo surfaces can be considered, both from a birational and a categorical point of view, as conic bundles.

II.3. Del Pezzo surfaces of degree $d \geqslant 5$. Let X be a del Pezzo surface of degree $d \geqslant 5$. In this case, the semiorthogonal decomposition from Theorem 4.16 not only gives the birational invariant $\mathcal{GK}_X$, but also a way to obtain information on the degree of closed points on X from three vector bundles, and an answer to Question 4.15. Let us give some more detail on the algebras A_i in all the cases.

Severi-Brauer surfaces. As recalled, if $d = 9$, then $X_{k^s} \simeq \mathbb{P}^2_{k^s}$, and there is an Azumaya algebra A identifying the isomorphism class of X. The Brauer class of A has order 3, so that the subgroup $\langle A \rangle \subset \mathrm{Br}(k)$ is just $\langle A \rangle = \{1, A, A^2\}$. In this case, we have $A_2 = A$ and $A_3 = A^2$, so that the statements of Theorem 4.16 are a consequence of Amitsur's results. However, one can explicitly show that $\mathcal{GK}_X$ is a birational invariant by checking all the possible Sarkisov links of such a surface (see [**AB15**, App. A]).

Involution surfaces. If $d = 8$, either X is the blow-up of $\mathbb{P}^2_k$ in one point, and there is nothing to show, or X is an involution variety, that is X_{k^s} is a quadric in $\mathbb{P}^3_{k^s}$. In this case, there is an Azumaya algebra A with an involution, such that $\mathrm{SB}(A)$ is a Severi-Brauer threefold containing X. The involution gives a semisimple algebra B which is central over a degree 2 extension l/k. In this case, $A_2 = A$ and $A_3 = B$. First of all X is rational if and only if A and B are trivial in $\mathrm{Br}(k)$ and $\mathrm{Br}(l)$ respectively. The fact that $\mathcal{GK}_X$ is a birational invariant can be proved by checking all the possible Sarkisov links of such a surface.

Degree 7. If $d = 7$, then X is the blow-up of $\mathbb{P}^2_k$ in either one point of degree 2 or in two rational points. Anyway, there is nothing to show here.

Degree 6. If $d = 6$, then Colliot-Thélène, Karpenko and Merkurjev have constructed two algebras B and Q associated to X [**CTKM07**]. We have that B is a degree 3 Azumaya algebra over a quadratic extension of k, and Q is a degree 2 Azumaya algebra over a cubic extension of k. We refer to [**CTKM07**] for more details, we just mention that these algebras are related to toric presentations of X (see also [**Blu10**]). In this case, $A_2 = B$ and $A_3 = Q$. The fact that rationality is equivalent to Brauer triviality of these two algebras goes back to

Colliot-Thélène, Karpenko and Merkurjev [**CTKM07**]. The fact that $\mathcal{GK}_X$ is a birational invariant can be proved by checking all the possible Sarkisov links of such a surface.

Degree 5. If $d = 5$, we have that $X(k) \neq \varnothing$ and X is rational (see [**SD72**]). In this case, we resolve the birational transformation $X \dashrightarrow \mathbb{P}^2_k$ by blowing-up a rational point. Explicit calculations on the blow-up allow to show that $A_2 = k$ is the endomorphism algebra of a simple rank 2 vector bundle, and $A_3 = l$ is a degree 5 extension arising as the endomorphism algebra of a rank 5 vector bundle.

We turn now to Question 4.15. If $d \geqslant 5$, we have that $X(k) \neq \varnothing$ if and only if X is k-rational (see [**Man74**, Thm. 29.4]). We have then the following simple Corollary of Theorem 4.18.

Corollary 4.19. *Let X be a del Pezzo surface of degree $d \geqslant 5$. Then X has a k-rational point if and only if* $\mathrm{rdim}_{\mathrm{cat}}(X) = 0$.

Let us recall that if X is a smooth projective surface, Hassett and Tschinkel [**HT14**, Lemma 8] prove that the point-index of X can be recovered from $\mathrm{D^b}(X)$ as the greatest common divisor of the second Chern classes of objects. Here, the *point-index*[3] $\mathrm{ind}_0(X)$ of a variety X over k is the greatest common divisor of the degrees of closed points of X.

Let X be a del Pezzo surface of degree $d \geqslant 5$. The semiorthogonal decompositions from Theorem 4.16 provide, via the vector bundles generating each component, a way to calculate the point-index $\mathrm{ind}_0(X)$. Indeed, any such X has a semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathrm{D^b}(l_1, A_1), \mathrm{D^b}(l_2, A_2), \mathrm{D^b}(l_3, A_3)\rangle, \tag{42}$$

where each $\mathrm{D^b}(l_i, A_i)$ can be generated by a vector bundle (notice that $\mathrm{D^b}(l_1, A_1) = \langle \mathcal{O}\rangle \simeq \langle \omega_X \rangle$, but we could pick-up higher rank vector bundles, such as $\omega_X^{\oplus 2}$ to have nontrivial second Chern classes).

Theorem 4.20. *Let X be a del Pezzo surface of degree $d \geqslant 5$. Then there are vector bundles V_1, V_2, V_3 generating the components of* (42)*, such that* $\mathrm{ind}_0(X) = \gcd\{c_2(V_i)\}$.

III. Rationality criteria for complex threefolds

Let $k = \mathbb{C}$ (or an algebraically closed field of characteristic zero). In this Section we consider the approach outlined in Section I to threefolds. We denote hence by $\pi : X \to Y$ a Mori fiber space with $\dim(X) = 3$ and Y rational. In particular, since $\dim(Y) \leqslant 2$ and k is algebraically closed, we have that Y is rational and $\mathrm{rdim}_{\mathrm{cat}}(Y) = 0$.

III.1. Intermediate Jacobians vs representability. Suppose that X is verepresentable[4]. Using the theory of noncommutative motives, we have shown then that $\mathbf{J}(X) \simeq$

[3]this number is generally called index in the literature, but we don't want to confuse it with the index of X as a Fano variety, and we apologize for this unusual terminology. The $_0$ in the notation is meant to recall that we consider degree of 0-dimensional subschemes.

[4]this is the case for almost all X as above: they all have a single principally polarized intermediate Jacobian $J(X)$, but there are cases where the polarization is not known to be an incidence polarization.

$\mathbf{J}(\mathcal{A}_{X/Y})$ are isogenous Abelian varieties, see Theorem 3.12. If we moreover assume that $\mathrm{rdim}\,\mathcal{A}_{X/Y} \leqslant 1$, that is

$$\mathcal{A}_{X/Y} = \langle \mathrm{D^b}(C_1), \ldots, \mathrm{D^b}(C_r), E_1, \ldots, E_s \rangle, \tag{43}$$

for smooth and projective curves C_i and exceptional objects E_j, Theorem 3.13 gives us that $J(X) = J(C_1) \oplus \ldots \oplus J(C_r)$ as a principally polarized Abelian variety, so that the Clemens-Griffiths component is trivial. This latter particular case of Theorem 3.12 was shown without using noncommutative motives (but Fourier-Mukai functors instead, which actually amounts to consider the same case in a different language) in [**BB13**, **BB12**].

Proposition 4.21. *Let $X \to Y$ be a verepresentable threefold Mori fiber space, with Y rational. If* $\mathrm{rep}\mathcal{A}_{X/Y} \leqslant 1$*, then* $\mathrm{rcodim_{cat}}(X) \geqslant 2$ *and the Griffiths component $A_X \subset J(X)$ is trivial.*

More precisely, if $\mathcal{A}_{X/Y}$ is decomposed as in (43)*, then $J(X) = J(C_1) \oplus \ldots \oplus J(C_r)$ as principally polarized Abelian varieties.*

Corollary 4.22. *Let X be a threefold with nontrivial Griffiths component $A_X \subset J(X)$. Then* $\mathrm{rcodim_{cat}}(X) \leqslant 1$*, for any Mori fiber space $X \to Y$, we have* $\mathrm{rdim}\,\mathcal{A}_{X/Y} \geqslant 2$ *and X is not rational.*

Proposition 4.21 can be considered as an evidence to Question 4.10: if $\mathrm{rcodim_{cat}}(X) \geqslant 2$, then one of the "classical" obstructions to rationality, that is, the Griffths component, vanishes. The converse implication is not true in general, as we will see at the end of this section. However, for many examples of rational Fano threefolds of Picard number one, for which $J(X)$ is split by curves, the corresponding semiorthogonal decomposition of $\mathcal{A}_X$ can be obtained by explicit constructions.

Trivial Jacobian. If $X = \mathbb{P}^3$, or a quadric in $\mathbb{P}^4$, or a Fano of index 2 and degree 5, or a Fano index 1 and degree 22, then $J(X) = 0$ and X is rational. In all these cases $\mathrm{rep}\mathcal{A}_X = 0$, so that $\mathrm{rcodim_{cat}}(X) = 3$, see [**Bei78**], [**Kap88**], [**Orl91**] and [**Kuz96**] respectively.

Jacobian of a curve. If X is the intersection of two quadrics and C a genus 2 curve, if X has index one and degree 18 and C a genus 2 curve, if X has index 1 and degree 16 and C is a plane quartic, if X has index 1 and degree 12 and C a genus 7 curve, then $J(X) = J(C)$ and X is rational. In all these cases, $\mathrm{rep}\mathcal{A}_X = 1$ and $\mathrm{D^b}(C) \subset \mathcal{A}_X$ is admissible and its orthogonal complement is generated by a finite number (possibly, none) of exceptional vector bundles; whence $\mathrm{rcodim_{cat}}(X) = 2$, see [**BO95**], [**Kuz05b**], [**Kuz05b**], and [**Kuz05a**] respectively.

By a classification argument we are not far from saying that there are very few examples which could contradict that a smooth Fano threefold X of Picard rank one being rational implies $\mathrm{rcodim_{cat}}(X) \geqslant 2$.

On the other hand, Corollary 4.22 is certainly very nice, but we notice that there non-rational threefolds with trivial Griffiths invariant, as, for example the Artin and Mumford example [**AM72**]. In this case, X is singular but can be resolved by blowing-up its ten double points $\widetilde{X} \to X$.

Proposition 4.23. *Let X be the Artin-Mumford quartic double solid and $\widetilde{X} \to X$ be the blow-up of its ten double points. In particular $J(\widetilde{X}) = 0$ and $\widetilde{X}$ is not rational. Then* $\mathrm{D^b}(\widetilde{X})$ *is a noncommutative resolution of singularities of X and* $\mathrm{rcodim_{cat}}(\widetilde{X}) = 1$.

PROOF. The fact that $\widetilde{X}$ is nonrational and has trivial Jacobian goes back to Artin and Mumford original paper [**AM72**]: indeed, X has these properties, so does $\widetilde{X}$.

Hosono and Takagi [**HT15**] consider the Enriques surface S associated to X (the so-called Reye congruence), and show that there is a semiorthogonal decomposition

$$\mathrm{D^b}(\widetilde{X}) = \langle \mathrm{D^b}(S), E_1, \dots, E_{12} \rangle,$$

where E_i are exceptional objects. This is implies first that $\mathrm{rcodim_{cat}}(\widetilde{X}) \geqslant 1$.

To prove the converse inequality, notice that we have $K_0(\widetilde{X}) = \mathbb{Z}^{12} \oplus K_0(S)$. Moreover, the 2-torsion subgroup $K_0(S)_2$ of $K_0(S)$ is nontrivial: we have $K_0(S)_2 = \mathbb{Z}/2\mathbb{Z}$. Indeed, if S is an Enriques surface, the Chern character is integral and gives an isomorphism between $K_0(S)$ and the singular cohomology of S (similarly, one can argue by using the Bloch conjecture, which is true for S, and the topological filtration of the Grothendieck group of S). In particular, $K_0(S) = \mathbb{Z} \oplus \mathrm{Pic}(S) \oplus \mathbb{Z}$ and $\mathrm{Pic}(S) \simeq \mathbb{Z}^{10} \oplus \mathbb{Z}/2\mathbb{Z}$ (see, e.g., [**BHPVdV04**, VIII Prop. 15.2]). We conclude that $K_0(\widetilde{X})_2 \simeq K_0(S)_2 \simeq \mathbb{Z}/2\mathbb{Z}$.

Now, since $J(\widetilde{X}) = 0$, if $\mathrm{rdim_{cat}}(\widetilde{X}) \leqslant 1$, then $\widetilde{X}$ has a full exceptional collection. But this latter fact would imply that $K_0(\widetilde{X})$ is a free $\mathbb{Z}$-module of finite rank. $\square$

Proposition 4.23, could be seen as an evidence to the fact that noncommutative schemes and their representability should provide a finer invariant than the intermediate Jacobian. We finally notice that there could be other noncommutative resolution of singularities $\mathcal{B}$ of X, such as the Moishezon manifold $X^+ \to X$ giving a small resolution, which we already considered in 3.23. Anyway, similar arguments show that $\mathrm{D^b}(X^+)$ is not representable in dimension one either.

III.2. Conic bundles and del Pezzo fibrations of degree 4. We now turn our attention to the case where Y is a rational (minimal) surface, so that $\pi : X \to Y$ is a conic bundle. Moreover, we denote by $\Delta \subset Y$ the discriminant divisor, that is the curve parameterizing singular fibers. We assume that Δ has at most double points and that the smooth points of Δ correspond to simply degenerate fibers, while double points correspond to double lines. We say that X is standard if, moreover, it is relatively minimal. Finally notice that Δ comes with a double cover $\widetilde{\Delta} \to \Delta$ which is ramified along the double points.

In the case of standard conic bundles, we have a full understanding of the relationship between intermediate Jacobians, semiorthogonal decompositions of $\mathcal{A}_{X/Y}$ and of $\mathrm{D^b}(X)$, categorical representability and rationality. Notice moreover that the case of cubic threefolds is covered by these conic bundles: if Z is a cubic threefold, the projection $Z \dashrightarrow \mathbb{P}^2$ along any line in Z can be resolved by blowing up Z along the line to get a standard conic bundle $X \to \mathbb{P}^2$. Moreover, it can be shown that $\mathcal{A}_{X/\mathbb{P}^2} = \langle \mathcal{A}_Z, E \rangle$ for an exceptional object E (see [**BMMS12**]).

Finally, let us notice that, for a conic bundle $\pi : X \to Y$, we have an equivalence $\mathrm{D^b}(Y, \mathcal{B}_0) \simeq \mathcal{A}_{X/Y}$, where $\mathcal{B}_0$ is the sheaf of the even parts of the Clifford algebra associated to

X. The following result from [**BB13**] completely resolves the question on relating rationality questions and intermediate Jacobians to $\mathcal{A}_{X/Y}$ for conic bundles over minimal surfaces.

Theorem 4.24. *Let $\pi : X \to Y$ be a standard conic bundle over a minimal rational surface. Then X is rational if and only if $\mathrm{rcodim}_{\mathrm{cat}}(X) \geqslant 2$ if and only if $\mathrm{rdim}\,\mathcal{A}_{X/Y} \leqslant 1$. In particular, this is the case if and only if there are smooth projective curves C_i and exceptional objects E_j such that*

$$\mathcal{A}_{X/Y} = \langle \mathrm{D^b}(C_1), \ldots, \mathrm{D^b}(C_r), E_1, \ldots, E_r \rangle,$$

which is in turn equivalent to $J(X) = J(C_1) \oplus \ldots \oplus J(C_r)$ as principally polarized Abelian varieties.

The proof of Theorem 4.24 relies on the classification of rational conic bundles $\pi : X \to Y$ over minimal surfaces achieved by Beauville [**Bea77**] and Shokurov [**Sho84**]. This classification is based on the fact that $J(X)$ is identified, as a principally polarized Abelian variety, to the Prym variety $\mathrm{Prym}(\widetilde{\Delta}/\Delta)$ of the discriminant double cover of Δ. There are only five cases where such a Prym variety is split by curves, and they all correspond to a rational X. In all of these five cases, explicit constructions and calculations of mutations allow to construct the required semiorthogonal decomposition of $\mathcal{A}_{X/Y}$.

Suppose on the other hand that $\mathcal{A}_{X/Y}$ has the required decomposition. The reconstruction Theorem 3.13 for the intermediate Jacobian was proved in [**BB13**] in this case, hence showing that the decomposition of $\mathcal{A}_{X/Y}$ gives the splitting of $J(X)$. Then Beauville–Shokurov classification applies.

If $\pi : X \to Y$ is a Mori fiber space of relative dimension 2, then we consider $Y = \mathbb{P}^1$, and $\pi : X \to \mathbb{P}^1$ is a del Pezzo fibration, i.e. the fibers are del Pezzo surfaces. In this case, less is known, there is no particular algebraic description of $\mathcal{A}_{X/\mathbb{P}^1}$ in general[5].

An algebraic description is known in the case where $X \to \mathbb{P}^1$ is a del Pezzo fibration of degree 4. Indeed, in this case, X is the relative complete intersection of two quadric fibrations $Q_i \to \mathbb{P}^1$ of relative dimension 3. In this case we have a Hirzebruch surface $S \to \mathbb{P}^1$ and a Clifford algebra $\mathcal{C}_0$ on S corresponding to the linear span $\mathcal{Q} \to \mathbb{P}^1$ of the two quadric fibrations, see Definition 2.18. Homological Projective Duality, see Theorem 2.21 gives then $\mathcal{A}_{X/\mathbb{P}^1} \simeq \mathrm{D^b}(S, \mathcal{C}_0)$.

To study the Clifford algebra $\mathcal{C}_0$, notice that the linear span $\mathcal{Q} \to \mathbb{P}^1$ has a section, thanks to the existence of a Section of $X \to \mathbb{P}^1$ (guaranteed by Campana-Peternell-Pukhlikov [**CPP02**] and Graber-Harris-Starr [**GHS03**], or simply constructed explicitly by Alexeev [**Ale87**]) and the Amer-Brumer Theorem (see, e.g., [**EKM08**, Thm. 17.14] or [**ABB14**, Thm. 1.9.1]). Along this section, we can perform reduction by hyperbolic splitting to get a conic bundle $Q \to S$, as recalled in §III.3.

On the other hand, Alexeev [**Ale87**] shows that X is birational to a conic bundle $Q' \to S'$, for a Hirzebruch surface S'. The following result was established in [**ABB14**] and is an analog of Theorem 4.24 in the case of del Pezzo fibrations of degree 4.

[5] Notice however that the results of Section II apply to the generic fiber of X as a variety over the function field of $\mathbb{P}^1$.

Theorem 4.25. *Let $\pi : X \to \mathbb{P}^1$ be a del Pezzo fibration of degree 4 and $Q \to S$ and $Q' \to S'$ the two conic bundles described above. There are equivalences $\mathcal{A}_{X/\mathbb{P}^1} \simeq \mathcal{A}_{Q/S} \simeq \mathcal{A}_{Q'/S'}$.*

Moreover, X is rational if and only if $\mathrm{rcodim}_{\mathrm{cat}}(X) \geqslant 2$, which is equivalent to $\mathrm{rdim}\, \mathcal{A}_{X/\mathbb{P}^1} \leqslant 1$. These facts are also equivalent to $J(X) = J(C_1) \oplus \ldots \oplus J(C_r)$ for smooth projective curves C_i. Finally, this hold if and only if there is a semiorthogonal decomposition

$$\mathcal{A}_{X/\mathbb{P}^1} = \langle \mathrm{D^b}(C_1), \ldots, \mathrm{D^b}(C_r), E_1, \ldots, E_s \rangle,$$

with E_i exceptional objects.

The proof of the above Theorem uses the fact that Clifford algebras are Morita invariant under hyperbolic splitting, recalled in Theorem 2.22. It follows then that $\mathcal{A}_{X/\mathbb{P}^1} \simeq \mathrm{D^b}(S, \mathcal{C}_0) \simeq \mathcal{A}_{Q/S}$ via Homological Projective Duality. The equivalence $\mathcal{A}_{Q/S} \simeq \mathcal{A}_{Q'/S'}$ is obtained via an explicit geometric comparison between Alexeev's construction and the hyperbolic splitting, which actually gives $S = S'$. The second part of the statement is based on Theorem 4.24 and on the isomorphism $J(X) \simeq J(Z)$ as principally polarized Abelian varieties given by Alexeev [**Ale87**].

IV. Some complex fourfolds

Let $k = \mathbb{C}$ (or an algebraically closed field of characteristic zero). In this Section we consider the approach outlined in Section I to fourfolds. In this case, since the dimension is even, there is no intermediate Jacobian carrying a principal polarization. However, the noncommutative and motivic consideration carried on in Section I are still valid. We will explain in detail two examples of fourfolds where one can conjecture the existence of a noncommutative criterion of rationality, based on categorical representability.

IV.1. Cubic Fourfolds. Cubic fourfolds are certainly the most famous character in birational geometry in the last 30 years. Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. Such X is a Fano variety of Picard rank 1 and index 3. The most interesting part of the cohomology of X is the integral primitive cohomology lattice $H^4_{pr}(X, \mathbb{Z})$, that is, the sublattice of $H^4(X, \mathbb{Z})$, orthogonal to the double hyperplane section h^2. The quadratic form considered here is the intersection pairing. The deepest feature of the lattice $H^4_{pr}(X, \mathbb{Z})$, which can be described by Hodge theory and by the study of the scheme of lines $F(X)$ on X, is that it "looks almost like" a K3 lattice $H^2(S, \mathbb{Z})(-1)$, up to a Tate twist; indeed, $H^4_{pr}(X, \mathbb{Z})$ and $H^2(S, \mathbb{Z})(-1)$ could share a corank 1 primitive sublattice.

Based on the seminal work of Beauville and Donagi [**BD85**] on Pfaffian cubic fourfolds, Hassett the defined a *special cubic fourfold* to be a cubic fourfold X containing an algebraic 2-cycle T which is not homologous to h^2. The generic cubic is not special, since the only algebraic 2-cycle is h^2 (see [**Has00**, Thm. 3.1.2]). A special cubic fourfold is identified by a positive definite rank 2 saturated sublattice K of $H^4(X, \mathbb{Z})$ containing h^2. Indeed, given an abstract rank 2 positive definite lattice K with a distinguished element h^2 of self-intersection 3, a *labeling* of a special cubic fourfold is the choice of a primitive embedding $K \hookrightarrow H^4(X, \mathbb{Z})$ identifying the distinguished element with the double hyperplane section h^2.

The *discriminant* of a labeled special cubic fourfold (X, K) is defined to be the determinant of the intersection matrix of K.

Let $\mathcal{C}$ be the moduli space of cubic fourfolds. It is a 20-dimensional moduli space. Hassett shows that the moduli space $\mathcal{C}_d$ of special cubic fourfolds of discriminant d is a (possibly empty) irreducible algebraic divisor of $\mathcal{C}$. Other numerical constraints are described in [**Has00**, §4]. Low discriminant examples are cubic fourfolds containing a plane ($d = 8$), cubic fourfolds containing a cubic scroll ($d = 12$) and cubic fourfolds containing a quartic scroll ($d = 14$). Pfaffian cubics (see Example 4.27 for their definition) always contain a quartic scroll, and hence belong to $\mathcal{C}_{14}$ (actually, they are dense in $\mathcal{C}_{14}$). Nonsmooth cases can be also considered: cubic fourfolds with a double point have $d = 6$, while determinantal cubic fourfolds have $d = 2$.

The expectation is that Hodge theoretical information contained in the labeling should witness the vanishing of some obstruction to rationality. Indeed, it is expected that the general (hence, nonspecial) cubic fourfold is not rational. On the other hand, there are two classes of smooth cubics which are known to be rational: the first one is the one of Pfaffian cubics [**BD85**] and the second one is given by a particular case of cubic fourfolds containing a plane [**Has99**], as explained in Example 4.26. Finally, both nodal and determinantal cubics are rational.

Example 4.26 ([**Has00**], [**Has99**]). Let $X \subset \mathbb{P}^5$ be a smooth cubic containing a plane $P \subset X$, and consider the projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ along the plane P. Restricting this projection to X give rise to a rational map $X \dashrightarrow \mathbb{P}^2$ which can be resolved, by blowing up P, into a quadric surface bundle $\pi : \widetilde{X} \to \mathbb{P}^2$, ramified along a sextic curve (generically smooth) $C \subset \mathbb{P}^2$. If π has an odd section, then $\widetilde{X}$ and, a fortiori, X are rational. Cubic fourfolds with a plane and an odd section form a countable union of divisors in the moduli space $\mathcal{C}_8$.

Notice moreover that the discriminant double cover $S \to \mathbb{P}^2$, ramified along the sextic curve C, is a (generically smooth) K3 surface of degree 2.

Example 4.27 ([**BD85**]). Let V be a six-dimensional complex vector space and consider $\mathrm{Gr}(2, V) \subset \mathbb{P}(\wedge^2 V)$ via the Plücker embedding. The variety $\mathrm{Pf}(4, \wedge^2 V^\vee) \subset \mathbb{P}(\wedge^2 V^\vee)$ is defined as (the projectivization of) the set of skew symmetric six by six matrices with rank bounded above by 4. It is a (nonsmooth) cubic hypersurface of $\mathbb{P}(\wedge^2 V^\vee)$. Let $L \subset \mathbb{P}(\wedge^2 V)$ be a linear subspace of dimension 8, and denote by $L^\perp \subset \mathbb{P}(\wedge^2 V)$ its orthogonal subspace, which has dimension 5. If we take L general enough, then $X := L^\perp \cap \mathrm{Pf}(4, \bigwedge^2 V^\vee)$ and $S := L \cap \mathrm{Gr}(2, V)$ are a smooth cubic in $\mathbb{P}^5$ and a smooth degree 14 K3 surface in $\mathbb{P}^8$ respectively. Cubic fourfolds arising from this construction are called *Pfaffian* cubics with associated K3 surface S.

The interplay between K3 surfaces and special cubic fourfolds goes beyond the Examples 4.26 and 4.27 and abstract lattices. A given labeled special cubic fourfold (X, K_d) of discriminant d, is defined to have an *associated K3 surface* if there exists a polarized K3 surface (S, l) and a lattice isomorphism $H^2_0(S, \mathbb{Z})(-1) \simeq K_d^\perp \subset H^4_{pr}(X, \mathbb{Z})$, where $H^2_0(S, \mathbb{Z}) := l^\perp$ is the primitive cohomology lattice of (S, l). It is expected that having an associated K3 surface should be a necessary condition to rationality.

Let now semiorthogonal decompositions come into play. The interplay between cubic fourfolds and K3 surfaces extend to the noncommutative context. Indeed, if X is a cubic fourfold, Kuznetsov shows that the Serre functor of $\mathcal{A}_X$ is $S_{\mathcal{A}_X} = [2]$. As defined in Definition 1.20, we say that $\mathcal{A}_X$ is a Calabi-Yau noncommutative scheme of dimension 2. Even better, we say that $\mathcal{A}_X$ is a noncommutative K3 surface. Notice that Question 4.10 asks whether $\mathrm{rdim}\,\mathcal{A}_X \geqslant 3$ is an obstruction to rationality. Kuznetsov actually conjectures a stronger fact, which could be interpreted as a noncommutative analog of the Hodge theoretical expectation.

Conjecture 4.28 (Kuznetsov, [**Kuz10**]). *Let $X \subset \mathbb{P}^5$ be a cubic fourfold. Then X is rational if and only if there is a K3 surface S such that $\mathcal{A}_X \simeq \mathrm{D^b}(S)$ (if X is not smooth, replace $\mathcal{A}_X$ by a crepant noncommutative resolution of singularities).*

Recently, Addington and Thomas have shown that Kuznetsov's conjecture is equivalent to the Hodge-theoretical expectation, that is that $\mathcal{A}_X \simeq \mathrm{D^b}(S)$ for some K3 surface S if and only if X has an associated K3 surface [**AT14**], confirming that noncommutative methods match commutative ones in this case as well.

However, the proof of Addington and Thomas relies, to show the existence of S, on a deformation argument. In all the known examples of rational cubics, the description of S can be made very explicit, and could give hints and evidences to Kuznetsov's conjecture.

Pfaffian cubics ($d = 14$). If X is a Pfaffian and S the associated K3 as in example 4.27, then $\mathcal{A}_X \simeq \mathrm{D^b}(S)$ is a consequence of Homological Projective Duality between $\mathrm{Gr}(2, V)$ and $\mathrm{Pf}(4, \bigwedge^2 V^*)$, which was established by Kuznetsov [**Kuz10**].

Nodal cubics ($d = 6$). If X is nodal, then there is birational map $X \dashrightarrow \mathbb{P}^4$ induced by the projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ along the singular point of X. The resolution of this map is given by the blow up of the singular point on X on one side and the blow-up of a complete intersection of type $(2, 3)$ in $\mathbb{P}^4$, i.e. a degree 6 K3 surface S, on the other side. Then there is noncommutative resolution of singularities $\mathcal{B}$ of $\mathcal{A}_X$ such that $\mathcal{B} \simeq \mathrm{D^b}(S)$, [**Kuz10**].

Determinantal cubics ($d = 2$). If X is determinantal, Homological Projective Duality established in IV shows that there is a noncommutative resolution of singularities $\mathcal{B}$ of $\mathcal{A}_X$ generated by 6 exceptional objects [**BBF16**]. One should expect $\mathcal{B}$ to be the resolution of singularities of a degenerate K3 surface S of degree 2 into two cubic scrolls joining along an elliptic sextic (see [**Laz10**] for the geometric construction).

Cubics with a plane ($d = 8$). This is probably the most intriguing case. As recalled in Example 4.26, if $P \subset X$ is a plane, the blow-up of X along P is a quadric surface bundle $\widetilde{X} \to \mathbb{P}^2$ with discriminant curve $C \subset \mathbb{P}^2$ of degree 6. We suppose that C is smooth (this is the general case). Let $S \to \mathbb{P}^2$ be the degree 2 K3 surface given by the discriminant double cover. Notice that, since X is in $\mathcal{C}_8$, the surface S is not a K3 surface associated to X[6], and such X has in general no associated K3 surface.

If $\mathcal{C}_0$ is the sheaf of even parts of the Clifford algebra of the quadric bundle, then $\mathcal{C}_0$ lifts to an Azumaya algebra A on S with Brauer class $\alpha \in \mathrm{Br}(S)$. The description of the derived category of a quadric fibration [**Kuz08**] gives then $\mathcal{A}_X = \mathrm{D^b}(S, \alpha)$. We call such α

[6]Voisin [**Voi86**] shows that one can identify $K_d^\perp$ with an index 2 sublattice of $H^2_0(S, \mathbb{Z})$.

the *Clifford invariant* associated to the quadric fibration, or, equivalently to X. Moreover, $\alpha = 0$ if and only if the quadric bundle has an odd section [**Kuz10**].

Kuznetsov shows moreover that, in the general case, that is if S has Picard number one, then the Clifford invariant is not trivial and there exists no smooth K3 surface S' such that $\mathrm{D^b}(S, \alpha) \simeq \mathrm{D^b}(S')$, see [**Kuz10**]. It follows that one should expect such cubic fourfolds to be nonrational, or to provide a counterexample to Kuznetsov's Conjecture 4.28.

On the other hand, suppose that X contains a plane and has associated nontrivial Clifford invariant. This is equivalent to the quadric fibration $\widetilde{X} \to \mathbb{P}^2$ not having a section. However, this is condition is not sufficient to have X not rational. A natural question is to wonder whether there exist such rational fourfolds, and find the K3 surface S' realizing $\mathrm{D^b}(S') \simeq \mathrm{D^b}(S, \alpha) \simeq \mathcal{A}_X$. As recalled, we should have S of Picard rank at least 2.

Examples of such fourfolds are achieved by completely describing the locus of those cubics containing a plane and a rational quartic scroll, that is, by describing the intersection $\mathcal{C}_8 \cap \mathcal{C}_{14}$. Notice that such fourfolds contain at least three non homologous algebraic cycles, so that the K3 surface S has Picard rank at least 2. The following description of cubics in $\mathcal{C}_8 \cap \mathcal{C}_{14}$ was provided in [**ABBVA14**].

Theorem 4.29. *There are five irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{14}$, indexed by the discriminant $d_X \in \{21, 29, 32, 36, 37\}$ of the intersection form on the algebraic cohomology lattice $A(X) \subset H^4(X, \mathbb{Z})$. The Clifford invariant of a general cubic fourfold X in $\mathcal{C}_8 \cap \mathcal{C}_{14}$ is trivial if and only if d_X is odd. The Pfaffian locus is dense in the $d_X = 32$ component.*

Notice that if d_X is odd, then X is rational and Kuznetsov's Conjecture 4.28 holds since the Clifford invariant is trivial. On the other hand, if d_X is even, since $A(X)$ has rank at least 3, we have that the degree 2 K3 surface S has Picard rank at least 2. Hence, it is not known whether there could be a K3 surface S' with $\mathrm{D^b}(S') \simeq \mathrm{D^b}(S, \alpha) \simeq \mathcal{A}_X$. As a consequence of Theorem 4.29, we get that for $d_X = 32$, there in general such a surface S', which is the degree 14 K3 surface associated to X as a Pfaffian cubic. In particular, (it is known that) X is rational and Kuznetsov's Conjecture 4.28 holds.

We finally notice that in [**ABBVA14**] is provided an explicit (i.e. via Pfaffian equations) example of a cubic fourfold X which is Pfaffian, contains a plane, has rank 3 algebraic cohomology and nontrivial Clifford invariant.

IV.2. Fibrations in intersections of quadrics. Another example of a Mori fiber space $\pi : X \to Y$ with X of dimension 4 and such that $\mathcal{A}_{X/Y}$ can be explicitly described via a surface S and a Brauer class $\alpha \in \mathrm{Br}(S)$ was considered in [**ABB14**].

Recall from Definition 2.18 the notion of generic relative intersection of quadrics. We consider $\pi : X \to \mathbb{P}^1$ to be such intersection of two quadric fibrations $Q_i \to \mathbb{P}^1$ of relative dimension 4, defined by line-bundle valued quadratic forms (q_1, L_1, E) and (q_2, L_2, E), where E has rank 6, $X \subset \mathbb{P}(E)$, and L_i are line bundles on $\mathbb{P}^1$. Then $\pi : X \to \mathbb{P}^1$ is a Mori fiber space of relative dimension 3 and index 2. Moreover, we have a Hirzebruch surface $F := \mathbb{P}(L_1 \oplus L_2)$, and 4-dimensional quadric fibration $Q \to F$ with smooth degeneration divisor $\Delta \subset S$.

The first direct application of Homological Projective Duality, see Theorem 2.20 is that, denoting by $\mathcal{C}_0$ the sheaf of even parts of the Clifford algebra of $Q \to F$, we have that $\mathcal{A}_{X/\mathbb{P}^1} \simeq \mathrm{D}^{\mathrm{b}}(F, \mathcal{C}_0)$. Moreover, since the relative dimension of Q is even, we can consider the double cover $S \to F$, ramified along Δ, and $\mathcal{C}_0$ lifts to an Azumaya algebra A on S with Brauer class $\alpha \in \mathrm{Br}(S)$. Notice that the composition map $S \to F \to \mathbb{P}^1$ endow S with a hyperelliptic fibration over $\mathbb{P}^1$, the fibers being double covers of the fibers of $F \to \mathbb{P}^1$, ramified along the 6 points where Δ meets a fiber. The following result from [**ABB14**] gives a positive answer to Question 4.10, at least in the case where $\alpha = 0$.

Theorem 4.30. *Let $X \to \mathbb{P}^1$ be a generic relative intersection of two 4-dimensional quadrics, and S and α the surface and the class in* $\mathrm{Br}(S)$ *as above, respectively. Then* $\mathcal{A}_{X/\mathbb{P}^1} \simeq \mathrm{D}^{\mathrm{b}}(S, \alpha)$. *If $\alpha = 0$, then X is rational and* $\mathrm{rcodim}_{\mathrm{cat}}(X) \geqslant 2$. *In particular, this is the case if X contains a surface which is generically ruled over $\mathbb{P}^1$ by the restriction of π.*

The proof of the previous Theorem is based on the existence of a section for the quadric fibration $Q \to F$. This section comes, via Amer-Brumer Theorem (see, e.g., [**EKM08**, Thm. 17.14] or [**ABB14**, Thm. 1.9.1]), from a smooth section of $\pi : X \to \mathbb{P}^1$, which exists thanks to Campana-Peternell-Pukhlikov [**CPP02**] and Graber-Harris-Starr [**GHS03**] results on fibrations over curves with rationally connected fibers. It follows that we can perform hyperbolic splitting along this section to get a quadric surface bundle $Q' \to F$ whose sheaf of even parts of the Clifford algebra is Morita equivalent to the one of $Q \to S$, see Theorem 2.22.

Now we have that X and Q' are birational to each other, see [**ABB14**, §5] - this can be seen as a higher dimensional analog of Alexeev's (see [**Ale87**]) birational map between a del Pezzo fibration of degree 4 and a conic bundle over a Hirzebruch surface we considered in Theorem 4.25. Having a regular section of $Q' \to F$ is now equivalent to the vanishing of the class α in $\mathrm{Br}(S)$, and is a sufficient condition for rationality. If X contains a surface generically ruled over $\mathbb{P}^1$ by the restriction of π, then the section of $Q' \to F$ is constructed explicitly. In [**ABB14**] we can then state a Conjecture which is inspired both by Question 4.10 and by Kuznetsov's Conjecture 4.28.

Conjecture 4.31. *Let $X \to \mathbb{P}^1$ be a fibration in complete intersections of two four-dimensional quadrics over an algebraically closed field of characteristic zero.*

- **Weak version.** *The fourfold X is rational if and only if it* $\mathrm{rcodim}_{\mathrm{cat}}(X) \geqslant 2$.
- **Strong version.** *The fourfold X is rational if and only if* $\mathrm{rdim}\, \mathcal{A}_{X/\mathbb{P}^1} \leqslant 2$.

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