Solutions to $x^3 + y^3 + z^3 + w^3 = (x + y + z + w)^3$: cubic surfaces, 27 lines, and the icosahedron.

Marcello Bernardara

Institut de Mathématiques de Toulouse

Perspective and infinite

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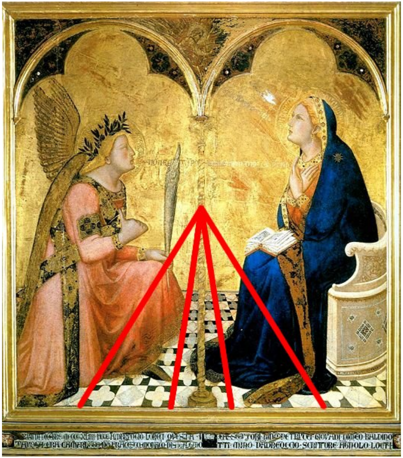

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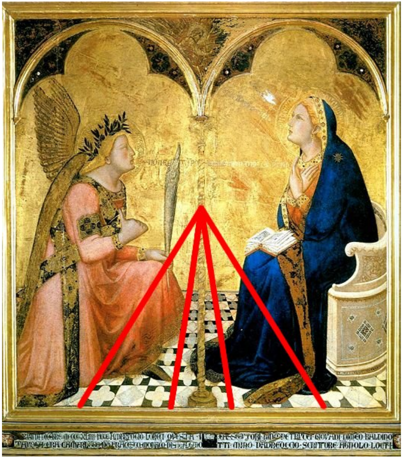

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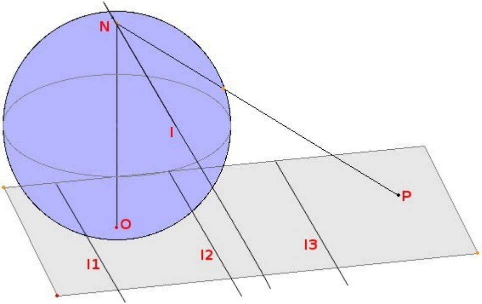

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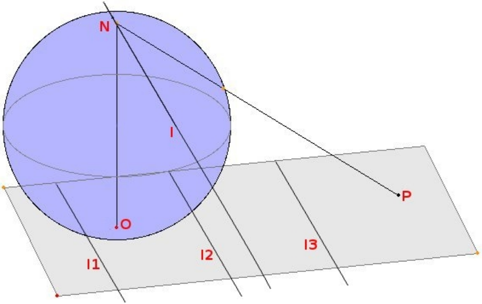

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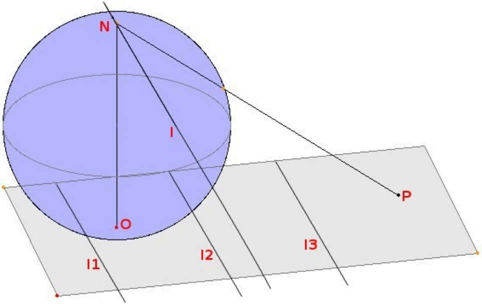

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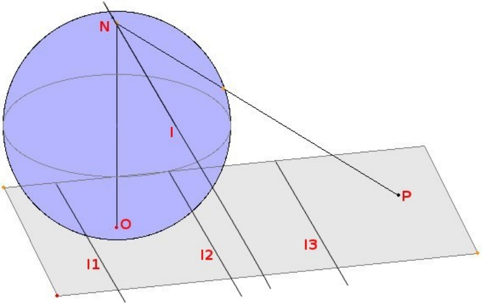

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Equation of such a line l: $\lambda(x_0, x_1, x_2)$, where $P = (x_0, x_1, x_2)$ is a point on l, and $(x_0, x_1, x_2) \neq (0, 0, 0)$.

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Projective varieties and Homogeneous equations

Homogeneous polynomials: polynomials $F(x_0, \ldots, x_n)$ whose monomials with nonzero coefficients all have the same total degree.
If $F(a_0, \ldots, a_n) = 0$, then $F(\lambda a_0, \ldots, \lambda a_n) = 0$ for all λ, so the Zero Locus $Z(F) = \{\text{points of } \mathbb{P}^n \text{ where } F \text{ vanishes}\}$ is well defined.

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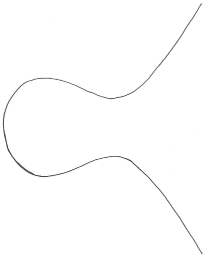

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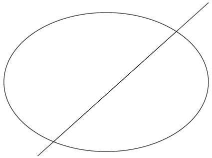

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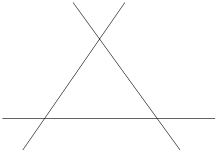

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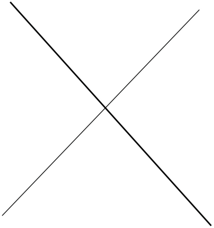

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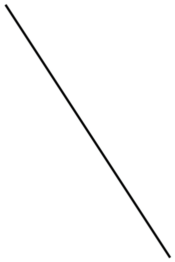

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Lines on cubic surfaces

A Cubic Surface in $\mathbb{P}^3$ is the zero locus $S = Z(F)$ of a degree 3 homogeneous polynomial. We suppose that S is smooth.

Theorem (Cayley-Salmon 1849)

There are exactly 27 lines on S.

Arthur Cayley (1821 - 1895)
George Salmon (1819 - 1904)

Proof of the Theorem

First step

There are at most 3 lines in S through any point P of S. If there are 2 or 3, then they are coplanar.
Every plane intersects S along either an irreducible cubic or a conic plus a line or three distinct lines.

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Proof: wlog, let $w = 0$ be the equation of the plane Π, then $\Pi \cap S$ is a degree three curve on $\mathbb{P}^2$ The line $l : (z = w = 0)$ in Π is double if

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Second Step

There is at least a line on S.

Proof of the Theorem

Third Step

l a line on S. There are exactly 5 pairs (l_i, l'_i) of lines on S meeting l s.t.

- for $i = 1, \ldots 5$, l, l_i, and l'_i are coplanar
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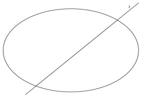
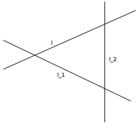
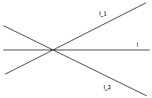
PROOF: take l and a plane $\Pi \supset l$, Then $\Pi \cap S$ is one of the following:







We want exactly 5 planes giving a) or b). If $l : (z = w = 0)$ then we can write $F = Ax^2 + Bxy + Cy^2 + Dx + Ey + G$ which is the equation of a conic in the plane (x, y) depending on z, w. Exactly 5 of them are reducible.

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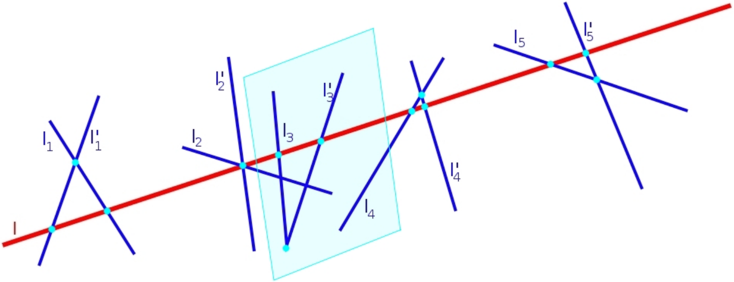

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LINES	l		m		l_i		l'_i		m'_i		l_{ijk}		
	1	+	1	+	5	+	5	+	5	+	10	=	27

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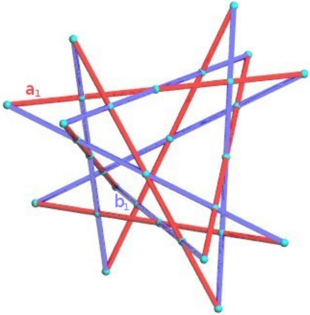

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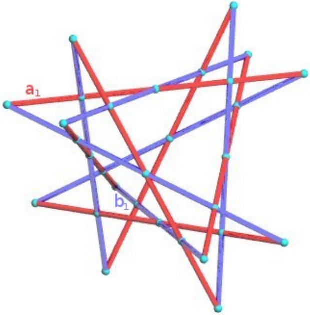

Schläfli's Theorem (1858)

The 27 lines are completely determined by a double six. There are 36 double sixes for a given cubic surface.

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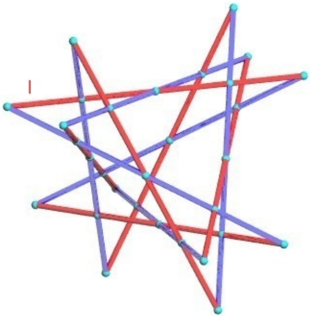

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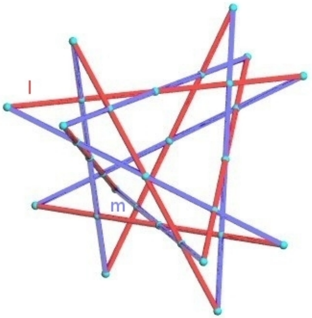

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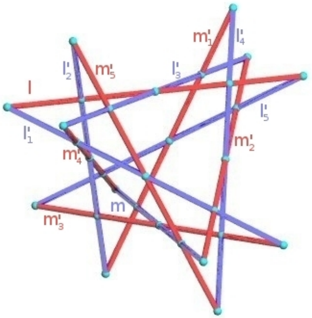

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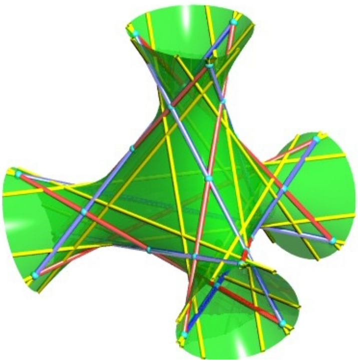
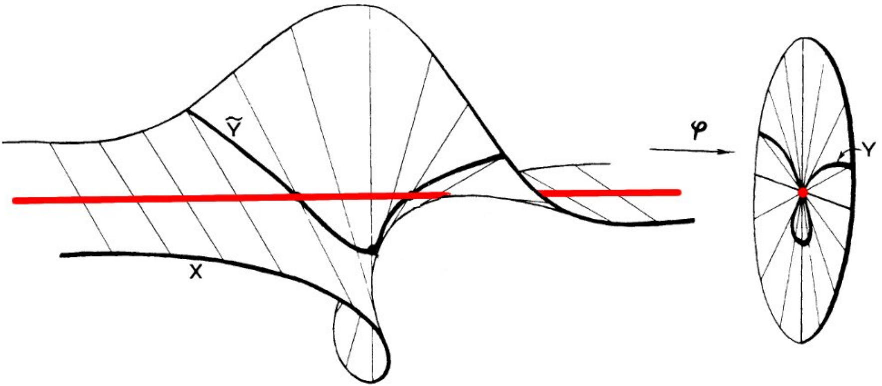


The blow-up of a point

Consider a point on a surface, and "replace it with a line"



The red line (the exceptional divisor) parameterizes all the lines through the red point (the center of the blow-up).

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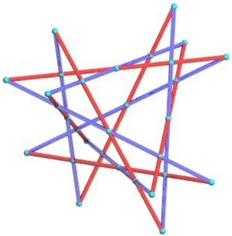

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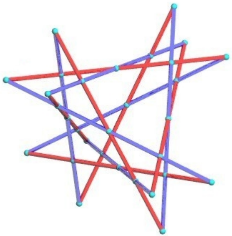

- There are 72 choices of 6 lines giving a map $S \to \mathbb{P}^2$.

The 27 lines from this point of view

- 6 skew lines come from the 6 points

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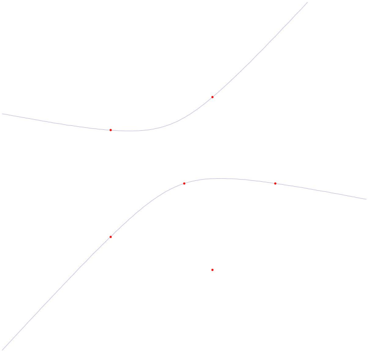
- There is a single conic for any choice of 5 points. Each of these conics give a line (6 skew lines)



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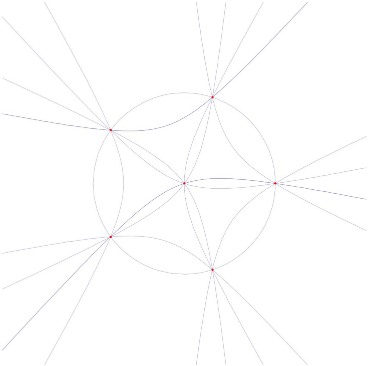
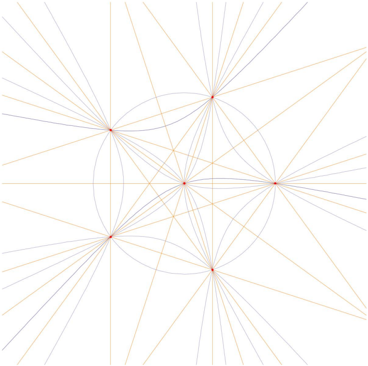


The 27 lines from this point of view

- Each line through two points gives a line on the surface (15 of them)



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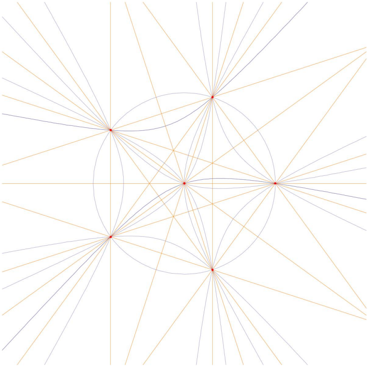

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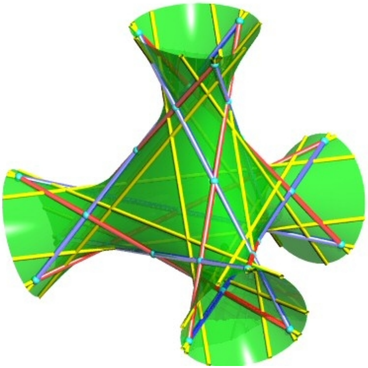

The Clebsch cubic

Consider the homogeneous polynomial

$$F(x, y, z, w) = x^3 + y^3 + z^3 + w^3 - (x + y + z + w)^3$$

the cubic surface in $\mathbb{P}^3$ defined as $Z(F)$ is the locus of points $(x : y : w : z)$ satisfying

$$x^3 + y^3 + z^3 + w^3 = (x + y + z + w)^3$$

This cubic is smooth and is called the Clebsch cubic

Theorem: Clebsch (1871), Klein (1873)

The six points in the plane corresponding to the Clebsch cubic are given by the lines in the euclidean space joining opposite vertices of a regular icosahedron, with assigned vertex coordinates.

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- The 6 points in $\mathbb{P}^2$ have to be defined over $\mathbb{Q}[\sqrt{-5}]$.
- There are 72 such icosahedra.

The Icosahedron