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A semiorthogonal decomposition for Brauer–Severi schemes

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A semiorthogonal decomposition for the bounded derived category of coherent sheaves on a Brauer–Severi scheme is given. It relies on bounded derived categories of suitably twisted coherent sheaves on the base.

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1 Introduction

In this paper we give a semiorthogonal decomposition of the bounded derived category (the category of perfect complexes in the nonsmooth case) of coherent sheaves on a Brauer–Severi scheme $f : X \to S$. Brauer–Severi schemes can be seen as twisted projective bundles. This leads us to generalize the semiorthogonal decomposition given in [8] for projective bundles, by considering twisted sheaves on the base S instead of untwisted ones.

Let us recall what happens in the case of a projective bundle. Let S be a smooth projective variety, E a vector bundle of rank $r + 1$ over S. We consider its projectivization $p : X = \mathbb{P}(E) \to S$. We then have the following semiorthogonal decomposition for the bounded derived category $\mathbf{D}(X)$ of coherent sheaves on X.

Theorem 1.1 ([8, Theorem 2.6]) *Let $\mathbf{D}(S)_k$ be the full and faithful subcategory of $\mathbf{D}(X)$ whose objects are all objects of the form $p^*A \otimes \mathcal{O}_X(k)$ for an object A of $\mathbf{D}(S)$. Then the set of admissible subcategories*

$$(\mathbf{D}(S)_0, \ldots, \mathbf{D}(S)_r)$$

is a semiorthogonal decomposition for the bounded derived category $\mathbf{D}(X)$ of coherent sheaves on X.

The aim of the paper is to give the following generalization. Let $f : X \to S$ be a Brauer–Severi scheme of relative dimension r over a locally notherian scheme S. Let α be the corresponding class in $H^2(S, \mathbb{G}_m)$. Let us denote by $\mathbf{D}(X)$ the category of perfect complexes of coherent sheaves on X and by $\mathbf{D}(S, \alpha)$ the category of perfect complexes of α-twisted coherent sheaves on S. Notice that in the smooth case they actually correspond to the bounded derived categories.

Theorem 4.1 *There exist admissible full subcategories $\mathbf{D}(S, X)_k$ of $\mathbf{D}(X)$, such that $\mathbf{D}(S, X)_k$ is equivalent to the category $\mathbf{D}\big(S, \alpha^{-k}\big)$ for all $k \in \mathbb{Z}$. The set of admissible subcategories*

$$(\mathbf{D}(S, X)_0, \ldots, \mathbf{D}(S, X)_r)$$

is a semiorthogonal decomposition for the category $\mathbf{D}(X)$ of perfect complexes of coherent sheaves on X.

It will be clear in the proof of the theorem that the construction of the full admissible subcategories $\mathbf{D}(S, X)_k$ is closely related to the definition of the full admissible sucategories $\mathbf{D}(S)_k$ in Orlov's theorem.

The paper is organized as follows: in Section 2 we give the definition of twisted sheaves and we state the basic facts about their connection with Brauer–Severi schemes. We recall then basic facts about derived categories, categories of perfect complexes and derived functors in twisted case, following [4]. In Section 3 we recall the definition of admissible subcategories and semiorthogonal decomposition in a triangulated category and we state some basic results about it. The main theorem and its proof are given in Section 4, together with a simple example.

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Notation 1.2 All schemes considered are locally noetherian and we suppose that any pair of points has an affine open neighborhood. This technical property will be used in Section 2 to refine étale coverings as needed in the construction of the twisted sheaf associated to a Brauer–Severi variety.

S_{et} denotes the étale site of a scheme S. For the definition of a (étale) site, see [10] or [7].

Given an étale covering $U \to S$, we denote U'' the fibered product $U \times_S U$ and U''' the fibered product $U \times_S U \times_S U$. We call p_1 and p_2 the projections $U'' \to U$ and q_{ij} the projections $U''' \to U''$.

$f : X \to S$ is a Brauer–Severi scheme of relative dimension r, that means that f is flat and proper and each geometric fiber of f is isomorphic to $\mathbb{P}^r$.

X_U denotes $f^{-1}(U)$ for $U \to S$ an étale covering. We use notations X''_U, X'''_U, $p_{i,X}$ and $q_{ij,X}$ in the natural way. Notice that $X''_U = X_{U''}$.

2 Twisted sheaves

In this section, we recall the definition of twisted sheaves and we state the relationship between them and Brauer–Severi schemes. We are working in étale topology, but all can be defined and stated in the analytic topology as well (see [4, I, 1]).

Definition 2.1 Let S be a scheme with étale topology, let $U \to S$ be an étale covering, let $\alpha \in \Gamma(U''', \mathbb{G}_m)$ be a 2-cocycle.

A quasi-coherent *α-twisted sheaf* on S is given by a quasi-coherent sheaf E over U and an isomorphism $\phi : p_1^* E \to p_2^* E$, such that

$$(q_{23}^*\phi) \circ (q_{12}^*\phi) = \alpha(q_{13}^*\phi).$$

We say that such a sheaf is *coherent* if E is a coherent sheaf on U, and we denote $\mathrm{Coh}(S, \alpha)$ the category of coherent α-twisted sheaves on S and $\mathbf{D}(S, \alpha)$ the category of perfect complexes of such sheaves.

The category $\mathrm{Coh}(S, \alpha)$ does not change up to equivalence neither by refining the open cover $U \to X$, nor by changing α by a cochain.

Lemma 2.2 *If α and α' represent the same element of $H^2(S, \mathbb{G}_m)$, the categories $\mathrm{Coh}(S, \alpha)$ and $\mathrm{Coh}(S, \alpha')$ are equivalent.*

Proof. This is [4, Lemma 1.2.8]. Indeed, if α and α' are in the same cohomology class they differ by a 1-cochain: $\alpha = \alpha' + \delta\gamma$. But then sending any α'-twisted sheaf (E, ϕ) to the α-twisted sheaf $(E, \gamma\phi)$ gives the required equivalence. □

Remark 2.3 Notice that in general the choice of the cochain γ matters: different choices give different equivalences. Since we are just interested in the existence of such equivalences and not in a special one, in what follows this choice will not matter.

Now we can see how twisted sheaves arise naturally when we consider Brauer–Severi schemes. Let $f : X \to S$ be a flat and proper morphism between schemes such that each geometric fibre is isomorphic to $\mathbb{P}^r$. Then we call X a Brauer–Severi scheme of relative dimension r over S.

We can find a étale covering $U \to S$, so that $X_U = f^{-1}(U)$ is a projective bundle over U and $X_U \to X$ is an étale covering. Then we have a local picture $\mathbb{P}(E_U) \to U$, where E_U is a locally free sheaf of rank $r + 1$ on U and we have an isomorphism $\rho : \mathbb{P}(E_U) \tilde{\to} X_U$. This fact is an application of descent theory ([5, I, 8]).

Consider the diagram

$$X'''_U \overrightarrow{\overrightarrow{\rightarrow}} X''_U \rightrightarrows X_U$$

and call the projections $p_{i,X}$ and $q_{ij,X}$. We have an isomorphism

$$\psi := p_{1,X}^* \rho^{-1} \circ p_{2,X}^* \rho : \mathbb{P}(p_1^* E_U) \xrightarrow{\sim} \mathbb{P}(p_2^* E_U).$$

We would like to lift it to an isomorphism $\phi : p_1^* E_U \tilde{\to} p_2^* E_U$.

Consider U such that $p_1^* E_U$ and $p_2^* E_U$ can be trivialized. This implies that ψ is an automorphism of $U'' \times \mathbb{P}^r$ and then it gives a section of $PGL(r + 1, U'')$. We can again refine U in order to obtain from it a section of

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$GL(r+1, U'')$, which will give us the required isomorphism $\phi : p_1^* E_U \tilde{\to} p_2^* E_U$. Notice that this is not canonical since it can be done up to a choice of an element of $\Gamma(U'', \mathbb{G}_m)$. This can be done since any pair of point on S has an affine open neighborhood (see [9]).

For this reason, we have $(q_{12}^* \phi) \circ (q_{23}^* \phi) = \alpha_U (q_{13}^* \phi)$, where $\alpha_U \in \Gamma(U''', \mathbb{G}_m)$. We can see that α_U gives a cocycle and then (E_U, ϕ) is an α-twisted sheaf.

From now on, given a Brauer–Severi scheme $f : X \to S$, we will consider the α-twisted sheaf (E_U, ϕ) described above and the category $\mathbf{D}(S, \alpha)$. Notice that the choice of α_U could be modified by a 1-cochain, but, by Lemma 2.2, this would give an equivalent category. Whence everything depends just on the cohomology class of α.

The class α represents the obstruction to $f : X \to S$ to be the projectivization of a locally free sheaf. To express this via cohomology, recall the exact sequence of sheaves over S:

$$1 \longrightarrow \mathbb{G}_m \longrightarrow GL(r+1) \longrightarrow PGL(r+1) \longrightarrow 1.$$

It gives a long cohomology sequence

$$\ldots \longrightarrow H^1(S, GL(r+1)) \longrightarrow H^1(S, PGL(r+1)) \xrightarrow{\delta} H^2(S, \mathbb{G}_m)$$

and especially a connecting homomorphism δ.

Let $[X]$ be the cohomology class of X in $H^1(S, PGL(r+1))$ and $\alpha' := \delta([X])$ in $H^2(S, \mathbb{G}_m)$. If $\alpha' = 0$, the class $[X]$ would lift to an element of $H^1(S, GL(r+1))$, that is a rank $r+1$ vector bundle on S. Since X is not a projective bundle, α' is a nonzero element of the cohomological Brauer group $\mathrm{Br}'(S) := H^2(S, \mathbb{G}_m)$ and it is exactly the cohomology class α of the α_U described above, see also [4].

As a projective bundle $\mathbb{P}(E_U)$ over U, on X_U there exists a tautological line bundle $\mathcal{O}_{X_U}(1)$. We will also write $\mathcal{O}_{X_U}(k)$ for $k \in \mathbb{Z}$.

Notice that the choice of the bundle $\mathcal{O}_{X_U}(1)$ over X_U depends on the choice of E_U, moreover $\mathcal{O}_{X_U}(1)$ does not glue as a global untwisted sheaf $\mathcal{O}_X(1)$ on X. However, the existence of a section for the morphism f ensures the existence of a global $\mathcal{O}_X(1)$.

Lemma 2.4 *Let $f : X \to S$ be a Brauer–Severi scheme. If $s : S \to X$ is a section of f, then there exists a vector bundle G on S such that $\mathbb{P}(G) \cong X \to S$.*

P r o o f. The result is known, but since it is hard to find a reference, we give a proof.

Consider the diagram

$$\begin{tikzcd}
X'''_U \arrow[r, "q_{ij,X}", shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left, "f"] & X''_U \arrow[r, "p_{1,X}", shift left] \arrow[r, "p_{2,X}"', shift right] \arrow[d, shift left, "f"] & X_U \arrow[r] \arrow[d, shift left, "f"] & X \arrow[d, shift left, "f"] \\
U''' \arrow[r, shift left=2] \arrow[r] \arrow[r, "q_{ij}"', shift right=2] \arrow[u, shift left, "s"] & U'' \arrow[r, "p_1", shift left] \arrow[r, "p_2"', shift right] \arrow[u, shift left, "s"] & U \arrow[r] \arrow[u, shift left, "s"] & S \arrow[u, shift left, "s"]
\end{tikzcd}$$

Here s and f are improperly used to mean their pull-backs to U, U'' and U''' in order to keep a clearer notation.

We can choose $\mathcal{O}_{X_U}(1)$ such that $s^* \mathcal{O}_{X_U}(1) = \mathcal{O}_U$.

Consider now $p_{1,X}^* \mathcal{O}_{X_U}(1)$ and $p_{2,X}^* \mathcal{O}_{X_U}(1)$, the two pull-backs of $\mathcal{O}_{X_U}(1)$ to $X_{U''}$. There exists an invertible sheaf L on S such that $p_{1,X}^* \mathcal{O}_{X_U}(1) \cong p_{2,X}^* \mathcal{O}_{X_U}(1) \otimes f^* L$. Since

$$s^* p_{i,X}^* \mathcal{O}_{X_U}(1) = \mathcal{O}_{U''}$$

we have L trivial. We choose an isomorphism

$$\phi : p_{1,X}^* \mathcal{O}_{X_U}(1) \longrightarrow p_{2,X}^* \mathcal{O}_{X_U}(1)$$

such that $s^* \phi = \mathrm{Id}_{\mathcal{O}_{U''}}$.

The isomorphism ϕ satisfies an untwisted cocycle condition. Indeed,

$$s^* \left((q_{12,X}^* \phi) \circ (q_{23,X}^* \phi) \circ (q_{13,X}^* \phi)^{-1} \right) = \mathrm{Id}_{\mathcal{O}_{U'''}}.$$

This shows that $\mathcal{O}_{X_U}(1)$ gives a global untwisted sheaf $\mathcal{O}_X(1)$ and that means X is a projective bundle over S. $\square$

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Let us briefly recall what happens to most common derived functors when we consider the category of perfect complexes of twisted sheaves on a scheme. A more satisfying description can be found in [4]. It is in fact an adaptation to twisted case of the results of [6].

Remark 2.5 In order to extend the ideas to a more general context, we will deal with categories of perfect complexes of coherent (α-twisted) sheaves. A perfect complex of coherent sheaves is a complex whose cohomology sheaves are coherent and which has finite global Tor-dimension. Equivalentely, it is quasi isomorphic, over any affine open set, to a bounded complex of locally free sheaves of finite rank in any degree. A complete treatement of perfect complexes on a site is given in [11]. Everything is defined in the very general context of fibered categories, hence all definitions fit for twisted sheaves. In the smooth case, the category of perfect complexes of coherent (α-twisted) sheaves turn out to be the same as the bounded derived category of coherent (α-twisted) sheaves, but keep in mind that in the non smooth case what we call here $\mathbf{D}(S)$ (resp. $\mathbf{D}(S,\alpha)$) is not the bounded derived category of (α-twisted) coherent sheaves on S but just a full triangulated subcategory.

Theorem 2.6 ([4, Theorem 2.2.6]) *Let $f : X \to S$ be a morphism between schemes, let α, α' be in $H^2(S, \mathbb{G}_m)$, and $\mathcal{AB}$ be the category of abelian groups. Then the following derived functors are defined:*

$$\begin{aligned} \underline{R\mathrm{Hom}} &: \mathbf{D}(S,\alpha)^\circ \times \mathbf{D}(S,\alpha') \longrightarrow \mathbf{D}(S,\alpha^{-1}\alpha'), \\ R\mathrm{Hom} &: \mathbf{D}(S,\alpha)^\circ \times \mathbf{D}(S,\alpha) \longrightarrow \mathbf{D}^b(\mathcal{AB}), \\ \bigotimes_S &: \mathbf{D}(S,\alpha) \times \mathbf{D}(S,\alpha') \longrightarrow \mathbf{D}(S,\alpha\alpha'), \\ Lf^* &: \mathbf{D}(S,\alpha) \longrightarrow \mathbf{D}(X, f^*\alpha). \end{aligned}$$

If $f : X \to S$ is a projective lci (*locally complete intersection*) *morphism, then we can define*

$$Rf_* : \mathbf{D}(X, f^*\alpha) \longrightarrow \mathbf{D}(S,\alpha).$$

Let us recall without explicit statements that Projection Formula, Adjoint Property of Rf_* and Lf^* and Flat Base Change are still valid in the α-twisted context. The only thing to care of is the choice of the right twist. All this and much more is detailed in [4] and can easily be generalized to categories of perfect complexes in a nonsmooth case.

3 Semiorthogonal decompositions

Let k be a field and $\mathbf{D}$ a k-linear triangulated category.

Definition 3.1 A full triangulated subcategory $\mathbf{D}' \subset \mathbf{D}$ is *admissible* if the inclusion functor $i : \mathbf{D}' \to \mathbf{D}$ admits a right adjoint.

Definition 3.2 The *orthogonal complement* $\mathbf{D}'^{\perp}$ of $\mathbf{D}'$ in $\mathbf{D}$ is the full subcategory of all objects $A \in \mathbf{D}$ such that $\mathrm{Hom}(B, A) = 0$ for all $B \in \mathbf{D}'$.

We remark firstly that the orthogonal complement of an admissible subcategory is a triangulated subcategory.

It can be shown that a full triangulated subcategory $\mathbf{D}' \subset \mathbf{D}$ is admissible if and only if for all object A of $\mathbf{D}$, there exists a distinguished triangle $B \to A \to C$ where $B \in \mathbf{D}'$ and $C \in \mathbf{D}'^{\perp}$, see [1]. We also have the following theorem.

Theorem 3.3 ([2, Proposition 1.5], or [1, Lemma 3.1]) *Let $\mathbf{D}'$ be a full triangulated subcategory of a triangulated category $\mathbf{D}$. Then $\mathbf{D}'$ is admissible if and only if $\mathbf{D}$ is generated by $\mathbf{D}'$ and $\mathbf{D}'^{\perp}$.*

Admissible subcategories occur when we have a fully faithful exact functor $F : \mathbf{D}' \to \mathbf{D}$ that admits a right adjoint. To be precise, this functor defines an equivalence between $\mathbf{D}'$ and an admissible subcategory of $\mathbf{D}$.

Definition 3.4 A sequence of admissible triangulated subcategories $\sigma = (\mathbf{D}_1, \ldots, \mathbf{D}_n)$ is *semiorthogonal* if, for all $i > j$, one has $\mathbf{D}_j \subset \mathbf{D}_i^{\perp}$. If σ generates the category $\mathbf{D}$, we call it a *semiorthogonal decomposition* of $\mathbf{D}$.

Lemma 3.5 *Let $\sigma = (\mathbf{D}_1, \ldots, \mathbf{D}_n)$ be a sequence of full subcategories of $\mathbf{D}$ sucht that $\mathbf{D}_j \subset \mathbf{D}_i^{\perp}$ for all $i > j$ and σ generates $\mathbf{D}$. Then $\mathbf{D}_i$ is admissible for $i = 1, \ldots, n$, and then σ is a semiorthogonal decomposition of $\mathbf{D}$.*

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Proof. Consider $\mathbf{D}_n$ and $\mathbf{D}_n^{\perp}$: they generate the category $\mathbf{D}$ and then they are admissible. In general, consider $\mathbf{D}_i$ and $\mathbf{D}_i^{\perp}$ for $1 \leq i < n$: they generate the category $\mathbf{D}_{i+1}^{\perp}$ and then they are admissible. □

For further information about admissible subcategories and semiorthogonal decomposition, see [1, 2, 3].

4 The main theorem

Let now $f : X \to S$ be a Brauer–Severi scheme of relative dimension r and α in $\mathrm{Br}(S)$ the element associated to it as explained in Section 2. This section is dedicated to the proof of the following theorem.

Theorem 4.1 *There exist admissible full subcategories $\mathbf{D}(S,X)_k$ of $\mathbf{D}(X)$, such that $\mathbf{D}(S,X)_k$ is equivalent to the category $\mathbf{D}\big(S,\alpha^{-k}\big)$ for all $k \in \mathbb{Z}$. The set of admissible subcategories*

$$\sigma = (\mathbf{D}(S,X)_0, \dots, \mathbf{D}(S,X)_r)$$

is a semiorthogonal decomposition for the category $\mathbf{D}(X)$ of perfect complexes of coherent sheaves on X.

Recall that there exist a rank $r+1$ locally free sheaf E_U on U, such that $X_U = \mathbb{P}(E_U)$ and that E_U gives an α-twisted sheaf on S. Moreover on X_U we have a tautological line bundle $\mathcal{O}_{X_U}(1)$. In this case, we consider α as a cocycle chosen once for all in the cohomology class $[\alpha]$. By Lemma 2.2 this choice does not affect the category $\mathbf{D}(S,\alpha)$ up to equivalence.

We split this section in three parts: in the first one we define the full subcategories $\mathbf{D}(S,X)_k$ of $\mathbf{D}(X)$ and we show the equivalence between $\mathbf{D}(S,X)_k$ and $\mathbf{D}\big(S,\alpha^{-k}\big)$; all this is inspired by a construction by Yoshioka [12]. In the second one we show that the sequence σ is indeed a semiorthogonal decomposition. In the third one we give a simple example.

4.1 Construction of $\mathbf{D}(S,X)_k$

Definition 4.2 We define $\mathbf{D}(S,X)_k$, for $k \in \mathbb{Z}$, to be the full subcategory of $\mathbf{D}(X)$ generated by objects A such that

$$A_{|X_U} \simeq_{\mathrm{q.iso}} f^* A_U \otimes \mathcal{O}_{X_U}(k) \tag{4.1}$$

where A_U is an object in $\mathbf{D}(U)$.

Lemma 4.3 *For all $k \in \mathbb{Z}$, there is a functor*

$$f_k^* : \mathbf{D}\big(S,\alpha^{-k}\big) \longrightarrow \mathbf{D}(S,X)_k$$

given by the association

$$A_{|U} \longmapsto f^* A_{|U} \otimes \mathcal{O}_{X_U}(k). \tag{4.2}$$

Proof. Firstly, X_U is the projective bundle $\mathbb{P}(E_U)$ over U. We then have on X_U the surjective morphism $f^*E_U \twoheadrightarrow \mathcal{O}_{X_U}(1)$. Given F an α^{-1}-twisted sheaf on S, we have the surjective morphism

$$f^*(F_U \otimes E_U) = f^*F_U \otimes f^*E_U \twoheadrightarrow f^*F_U \otimes \mathcal{O}_{X_U}(1).$$

Since F_U and E_U give respectively an α^{-1}-twisted and an α-twisted sheaf on S, their tensor product $F_U \otimes E_U$ gives an untwisted sheaf on S. We can naturally see $f^*F_U \otimes f^*E_U$ as an untwisted sheaf on X: the glueing isomorphism is obtained by pull-back with f and this makes naturally $f^*F_U \otimes \mathcal{O}_{X_U}(1)$ an untwisted sheaf as well. It is now clear that given an object A in $\mathbf{D}\big(S,\alpha^{-1}\big)$, the object given locally by (4.2) belongs to $\mathbf{D}(S,X)_1$.

The proof is similar for any $k \in \mathbb{Z}$. □

Theorem 4.4 *The functor f_k^* defined in Lemma 4.3 is an equivalence between the category $\mathbf{D}\big(S,\alpha^{-k}\big)$ and the category $\mathbf{D}(S,X)_k$.*

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Proof. Given A in $\mathbf{D}(S,X)_1$, consider the association over U

$$A_{|X_U} \longmapsto Rf_*\big(A_{|X_U} \otimes \mathcal{O}_{X_U}(-1)\big).$$

We show that it gives a functor Λ from $\mathbf{D}(S,X)_1$ to $\mathbf{D}\big(S,\alpha^{-1}\big)$ and that is the quasi-inverse functor of f_1^*.

Firstly, since A is in $\mathbf{D}(S,X)_1$, on X_U we have $A_{|X_U} = f^*A_U \otimes \mathcal{O}_{X_U}(1)$, with A_U in $\mathbf{D}(U)$. Evaluating Λ on $A_{|X_U}$ we get

$$Rf_*\big(A_{|X_U} \otimes \mathcal{O}_{X_U}(-1)\big) = Rf_*f^*A_U.$$

Now use projection formula:

$$Rf_*f^*A_U = Rf_*\mathcal{O}_X \otimes A_U.$$

We have $R^if_*\mathcal{O}_X = 0$ for $i > 0$ and $f_*\mathcal{O}_X = \mathcal{O}_S$, and then

$$Rf_*f^*A_U \simeq_{\mathrm{q.iso}} A_U. \tag{4.3}$$

It follows that Λ associates to $A_{|X_U}$ the object A_U in $\mathbf{D}(U)$.

At a level of coherent sheaves, by the same reasoning used in Lemma 4.3, we have the surjective morphism

$$f^*(F_U \otimes E_U) \twoheadrightarrow f^*F_U \otimes \mathcal{O}_{X_U}(1).$$

Since E_U is an α-twisted sheaf on S, we can give to F_U the structure of α^{-1}-twisted sheaf over S. This shows that Λ is actually a functor from the subcategory $\mathbf{D}(S,X)_1$ to the category $\mathbf{D}\big(S,\alpha^{-1}\big)$.

It is now an evidence by (4.3) that Λ and f_1^* are each other quasi-inverse.

The proof for $k \in \mathbb{Z}$ is similar. □

We then have constructed full subcategories $\mathbf{D}(S,X)_k$ of $\mathbf{D}(X)$, each one equivalent to a category of perfect complexes of suitably twisted sheaves on S.

Notice that we have $f_0^* = Lf^* = f^*$ since f is flat and the full subcategory of $\mathbf{D}(X)$ which is the image of $\mathbf{D}(S)$ under the functor f^* is in fact the category $\mathbf{D}(S,X)_0$ defined earlier.

4.2 σ is a semiorthogonal decomposition

Lemma 4.5 *For any A in $\mathbf{D}(S,X)_k$ and B in $\mathbf{D}(S,X)_n$ we have $R\mathrm{Hom}(A,B) = 0$ for $r \geq k-n > 0$.*

Proof. We have locally $A_{|X_U} = f^*A_U \otimes \mathcal{O}_{X_U}(k)$ and $B_{|X_U} = f^*B_U \otimes \mathcal{O}_{X_U}(n)$.

We have

$$\begin{aligned} \underline{R\mathrm{Hom}}(A_{|X_U}, B_{|X_U}) &= \underline{R\mathrm{Hom}}(f^*A_U \otimes \mathcal{O}_{X_U}(k), f^*B_U \otimes \mathcal{O}_{X_U}(n)) \\ &= \underline{R\mathrm{Hom}}(f^*A_U, f^*B_U \otimes \mathcal{O}_{X_U}(n-k)). \end{aligned}$$

We now use the adjoint property of f^* and Rf_*:

$$\underline{R\mathrm{Hom}}(f^*A_U, f^*B_U \otimes \mathcal{O}_{X_U}(n-k)) = \underline{R\mathrm{Hom}}(A_U, Rf_*(f^*B_U \otimes \mathcal{O}_{X_U}(n-k)).$$

Now by projection formula

$$Rf_*(f^*B_U \otimes \mathcal{O}_{X_U}(n-k)) = B_U \otimes Rf_*(\mathcal{O}_{X_U}(n-k)).$$

We have $Rf_*(\mathcal{O}_{X_U}(n-k)) = 0$ for $-r \leq n-k < 0$ and hence the sheaves $\underline{R\mathrm{Hom}}(A,B)$ are zero.

Using the local to global Ext spectral sequence, we get the proof. □

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We thus have an ordered set $\sigma = (\mathbf{D}(S,X)_0, \ldots, \mathbf{D}(S,X)_r)$ of orthogonal subcategories of $\mathbf{D}(X)$. Last step in proving Theorem 4.1 is to show that it generates the whole category.

Consider the fiber square over S

$$\begin{tikzcd} P := X \times_S X \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

We have $g = f : X \to S$. We call P the product $X \times_S X$.

Consider the diagonal embedding $\Delta : X \to P$. It is a section for the projection morphism $p : P \to X$. By Lemma 2.4, there exists a vector bundle G on X such that $P \cong \mathbb{P}(G) \to X$.

Consider now on P the surjective morphism: $p^*G \twoheadrightarrow \mathcal{O}_P \to 0$. We also have the Euler short exact sequence on P:

$$0 \longrightarrow \Omega_{P/X}(1) \longrightarrow p^*G \longrightarrow \mathcal{O}_P(1) \longrightarrow 0.$$

Combining the exact sequence and the surjective morphism, we get a section of $\mathrm{Hom}(\Omega_{P/X}(1), \mathcal{O}_P)$ whose zero locus is the diagonal Δ of P. Remark that $\Omega_{P/X}(1) = p^*\Omega_{X/S} \otimes \mathcal{O}_P(1)$ and

$$\Lambda^k(p^*\Omega_{X/S} \otimes \mathcal{O}_P(1)) = p^*\Omega^k_{X/S} \otimes \mathcal{O}_P(k).$$

We get a Koszul resolution

$$0 \longrightarrow p^*\Omega^r_{X/S} \otimes \mathcal{O}_P(r) \longrightarrow \ldots \longrightarrow p^*\Omega_{X/S} \otimes \mathcal{O}_P(1) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_\Delta \longrightarrow 0.$$

By this complex we deduce that $\mathcal{O}_\Delta$ belongs, as an element of the category $\mathbf{D}(P)$, to the subcategory generated from

$$\left\{p^*\Omega^r_{X/S} \otimes \mathcal{O}_P(r), \ldots, p^*\Omega_{X/S} \otimes \mathcal{O}_P(1), \mathcal{O}_X \boxtimes \mathcal{O}_X\right\} \tag{4.4}$$

by exact triangles and shifting.

Given A an element of $\mathbf{D}(X)$, we remark that $A = Rq_*(p^*A \otimes \mathcal{O}_\Delta)$. Since all involved functors (pull-back, direct image and tensor product) are exact functors, A belongs to the subcategory of $\mathbf{D}(X)$ generated by

$$\left\{Rq_*\left(p^*\left(A \otimes \Omega^r_{X/S}\right) \otimes \mathcal{O}_P(r)\right), \ldots, Rq_*\left(p^*\left(A \otimes \Omega_{X/S}\right) \otimes \mathcal{O}_P(1)\right), Rq_*p^*A\right\}.$$

Lemma 4.6 *The object $Rq_*\left(p^*\left(A \otimes \Omega^k_{X/S}\right) \otimes \mathcal{O}_P(k)\right)$ in $\mathbf{D}(X)$ belongs to the subcategory $\mathbf{D}(S,X)_k$.*

Proof. We look at it in a local situation. In this case X_U is a projective bundle over U, and we have

$$q^*\mathcal{O}_{X_U}(k) = \mathcal{O}_P(k)_{|X_{U''}}.$$

This leads us to write locally

$$\begin{aligned} Rq_*\left(p^*\left(A \otimes \Omega^k_{X/S}\right) \otimes \mathcal{O}_P(k)\right)_{|X_{U''}} &= Rq_*\left(\left(p^*\left(A \otimes \Omega^k_{X/S}\right)\right)_{|X_U} \otimes q^*\mathcal{O}_{X_U}(k)\right) \\ &= Rq_*\left(p^*\left(A \otimes \Omega^k_{X/S}\right)\right)_{|X_U} \otimes \mathcal{O}_{X_U}(k) \\ &= f^*Rg_*\left(\left(A \otimes \Omega^k_{X/S}\right)_{|X_U}\right) \otimes \mathcal{O}_{X_U}(k), \end{aligned} \tag{4.5}$$

where we used projection formula and flat base change in the last two equalities. Then we have an object locally of the form finally given in (4.5), and then it is an object in $\mathbf{D}(S,X)_k$. $\square$

We have shown that all objects A in $\mathbf{D}(X)$ belong to the subcategory generated by the orthogonal sequence σ. This implies, by Lemma 3.5, that the subcategories $\mathbf{D}(S,X)_k$ are admissible and then σ is in fact a semiorthogonal decomposition of $\mathbf{D}(X)$. This completes the proof of Theorem 4.1. $\square$

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4.3 An example

We finally treat the simplest example of a Brauer–Severi scheme. Let K be a field and X a Brauer–Severi variety over the scheme $Spec(K)$. In this case Theorem 4.1 gives a very explicit semiorthogonal decomposition of the bounded derived category $\mathbf{D}(X)$ of coherent sheaves on X in terms of central simple algebras over K.

The cohomological Brauer group of $Spec(K)$ is indeed the Brauer group $\mathrm{Br}(K)$ of the field K. The elements of $\mathrm{Br}(K)$ are equivalence classes of central simple algebras over K and its composition law is tensor product. To each α in $\mathrm{Br}(K)$ corresponds the choice of a central simple algebra over K.

Given the α corresponding to the Brauer–Severi variety X, an α^{-1}-twisted sheaf is then a module over a properly chosen central simple algebra A, and it is coherent if it is finitely generated. The category $\mathbf{D}(Spec(K), \alpha^{-1})$ is the bounded derived category of finitely generated modules over the algebra A. Concerning the element α^{-k} in $\mathrm{Br}(K)$, just remind that the composition law is tensor product, to see that we can choose $A^{\otimes k}$ to represent it. The construction of $\mathbf{D}\big(Spec(K), \alpha^{-k}\big)$ is then straightforward. We can state the following corollary of the Theorem 4.1.

Corollary 4.7 *Let K be a field, X a Brauer–Severi variety over $Spec(K)$ of dimension r. Let α be the class of X in $\mathrm{Br}(K)$ and A a central simple algebra over K representing α^{-1}. The bounded derived category $\mathbf{D}(X)$ of coherent sheaves on X has a semiorthogonal decomposition $\sigma = (\mathbf{D}(K,X)_0, \dots, \mathbf{D}(K,X)_r)$, where $\mathbf{D}(K,X)_i$ is equivalent to the bounded derived category of finitely generated $A^{\otimes i}$-modules.*

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