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Mapping class group of a plane curve germ ☆

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ABSTRACT

We prove that every topological conjugacy between two germs of singular holomorphic curves in the complex plane is homotopic to another conjugacy which extends homeomorphically to the exceptional divisors of their minimal desingularisations. As an application we give an explicit presentation of a finite index subgroup of the mapping class group of the germ of such a singularity.

0. Notations

– If A is a subset of a topological space, we denote by $\mathring{A}$ its interior and by $\overline{A}$ its closure. If A is a manifold, its boundary is denoted by ∂A.

– $\mathbb{B}_r$ will denote the closed ball in $\mathbb{C}^2$ of radius $r > 0$ centred at the origin, $\mathbb{S}^3_r = \partial\mathbb{B}_r$ and $\mathbb{D}_r = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$. All the balls and disks considered in the paper will be closed unless if the contrary is explicitly stated.

– For an analytic curve X, $\mathrm{Sing}(X)$ denotes the set of its singular points and $\mathrm{Comp}(X)$ is the collection of its irreducible components. Two irreducible components are called *adjacent* if they are distinct with non-empty intersection. The number $v(Y)$ of components adjacent to $Y \in \mathrm{Comp}(X)$ is called *valence* of Y. A (geometric) chain of X is a singular point of X belonging to two different irreducible components of valence $\geqslant 3$ or[1] a maximal connected union of irreducible components of X of valence exactly 2 having two adjacent components of valence $\geqslant 3$. A (geometric) dead branch is a maximal connected union of irreducible components of X having valence $\leqslant 2$ which is not a (geometric) chain.

1. Introduction

Let $(S, 0)$ and $(S', 0)$ be two holomorphic germs of singular curves at $0 = (0, 0) \in \mathbb{C}^2$. A *topological conjugacy between* $(S, 0)$ *and* $(S', 0)$ is a germ of homeomorphism $h : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $(h(S), 0) = (S', 0)$. Not every conjugacy h can be lifted to the resolution of singularities of S and S'. Here we are interested in such conjugacies that satisfy in addition other regularity conditions. The precise notion of an *excellent conjugacy* will be stated in Definition 2.5 but roughly speaking a conjugacy h is excellent if

- h can be lifted to a homeomorphism H between some neighbourhoods of the exceptional divisors $\mathcal{E}$ and $\mathcal{E}'$ of the resolutions of the singularities of S and S' (see the beginning of Section 2.1),

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[1] In formulae (13) and (14) of Section 3.2 we will give an alternative combinatorial and unified definition of chain of components by adding the valence $\geqslant 3$ adjacent components. This is the reason for the adjective (geometric) here.

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- H is a topological conjugacy between $\mathcal{E}$ and $\mathcal{E}'$,
- H is compatible with the Hopf's fibrations (see Definition 2.1) of each irreducible component of $\mathcal{E}$ and $\mathcal{E}'$, outside some neighbourhoods of the singular sets $\mathrm{Sing}(\mathcal{D})$ and $\mathrm{Sing}(\mathcal{D}')$ of the total transforms $\mathcal{D}$ and $\mathcal{D}'$ of S and S',
- the restriction of H to a neighbourhood of $\mathrm{Sing}(\mathcal{D})$ is holomorphic.

The existence of excellent conjugacies is established by the classical results of W. Burau, O. Zariski [25] and M. Lejeune [7]. The plumbing calculus technique introduced by D. Mumford [13] and developed by W.D. Neumann [14,5,15] helps to clarify this problem and allows to compute some topological invariants as the *fundamental group of the complement of S inside a Milnor ball* [12]:

$$\varGamma_S := \pi_1(\mathbb{B}_\varepsilon \setminus S, \cdot), \quad \mathbb{B}_\varepsilon := \{|z_1|^2 + |z_2|^2 \leqslant \varepsilon\},\ 0 < \varepsilon \ll 1. \tag{1}$$

The objective of this work is to describe the "homotopy classes" of topological conjugacies between two germs of curves and to prove that each class contains an excellent conjugacy. This problem has naturally emerged in the study of the topological classification of germs of singular foliations. It is a merely topological result but it plays a key role in solving a dynamical conjecture of D. Cerveau and P. Sad, cf. [3,9,10]. The structure of the proof and the techniques that we use are familiar in dimension three topology and close for instance to the ones exposed and developed by P. Popescu-Pampu in [17], in particular in Theorem 9.1. However, our result cannot be deduced from this or other statement of that paper. In fact, one of our goals was to be the most self-contained as possible, and to give a complete and proper proof of our main result using well-known tools for researchers working in the field of dynamical systems that are not necessarily familiar with all the techniques developed by topologists.

More precisely we say that two of topological conjugacies germs f and g between $(S, 0)$ and $(S', 0)$ are *fundamentally equivalent* (denoted by $f \asymp g$) if the restrictions of f and g to $\mathbb{B}_\varepsilon \setminus S$ are homotopic[2] as maps taking values in $\mathbb{B}_{\varepsilon'} \setminus S'$, for a suitable choice of $0 < \varepsilon \ll \varepsilon' \ll 1$. Clearly $\asymp$ is an equivalence relation on the set consisting of all topological conjugacies between S and S'. Note that the conic structure over $\partial\mathbb{B}_\varepsilon \setminus S$ of the complement $\mathbb{B}_\varepsilon \setminus S$ and the homotopy exact sequence associated with its fibration structure over the circle, show that $\mathbb{B}_\varepsilon \setminus S$ is a $K(\varGamma_S, 1)$ Eilenberg–MacLane space. Then the classical homotopy theory implies that $f \asymp g$ if and only if the morphisms induced by f and g from $\pi_1(\mathbb{B}_\varepsilon \setminus S, \cdot)$ to $\pi_1(\mathbb{B}_{\varepsilon'} \setminus S', \cdot)$, are equal modulo left or right compositions by inner automorphisms.

We define a *marking of S' by S* as a fundamental equivalence class (for $\asymp$) of a conjugacy between S and S'. The main result of this work is:

Theorem A. *Any marking admits an excellent representative.*

It is worthwhile to note that unicity is not claimed. In fact, there is no natural choice for an excellent representative of any marking. Our construction is based on the results of Waldhausen [21,22], Jaco and Shalen [6] and Johannson [8] about the decompositions of 3-manifolds. It cannot be deduced from the Lejeune–Zariski theorems: for evidence, in the case $S = S'$ the Zariski–Lejeune results are without object, while Theorem A provides non-trivial results on the automorphisms-group of curves germs.

The set $\mathcal{G}_S$ consisting of markings of a curve germ S by itself, is equipped (by the composition law) with a group structure. It is an analogous of the mapping class group for Riemann surfaces. The classical homotopy theory for $K(\pi, 1)$-spaces proves that the group $\mathcal{G}_S$ is embedded in the outer automorphism group of the fundamental group $\varGamma_S$, defined in (1). The image of this embedding $\mathrm{Out}_g(\varGamma_S) \subset \mathrm{Out}(\varGamma_S)$ is characterised by the preservation of some algebraic data on $\varGamma_S$, of geometric nature: the *peripheric structure endowed by its meridians*, cf. Definition 3.17, Theorem 3.16 and Corollary 3.20.

The subgroup $\mathcal{G}_S^0$ of $\mathcal{G}_S$ consisting of those homeomorphism germs fixing each irreducible component of S, is normal and has finite index; it is equal to the kernel of the natural morphism from $\mathcal{G}_S$ in $\mathfrak{S}_S$, the permutation group of the irreducible components of S. The previous theorem allows us to make explicit a system of generators of $\mathcal{G}_S^0$. Denote by $E : \mathcal{B} \to \mathbb{C}^2$ the resolution map of S and by $\mathcal{D} = E^{-1}(S)$ the total divisor. Recall that the valence of an irreducible component D of $\mathcal{D}$ is the cardinal of the finite set $S(D) := \mathrm{Sing}(\mathcal{D}) \cap D$. We denote by $\mathfrak{R}$ the set of irreducible components of $\mathcal{D}$ of valence $\geqslant 3$ and by $\mathfrak{C}$ the set of chains of $\mathcal{D}$, see the section of notations at the beginning of the paper. With these notations we can state the following result.

Theorem B. *There is an epimorphism*

$$\bigoplus_{D \in \mathfrak{R}} \mathrm{A}(D^\bullet) \oplus \bigoplus_{\mathcal{C} \in \mathfrak{C}} \mathbb{Z}^2_{\mathcal{C}} \twoheadrightarrow \mathcal{G}_S^0,$$

where $\mathrm{A}(D^\bullet)$ is the pure mapping class group[3] *of $D \cong \mathbb{S}^2$ pointed by $S(D)$ and $\mathbb{Z}^2_{\mathcal{C}} := \mathbb{Z}^2$.*

[2] In Definition 2.6 we present a different and more precise statement of the fundamental equivalence relation $\asymp$ which is equivalent to the one introduced here after Proposition 2.8.

[3] I.e. the homotopy classes of self-homeomorphisms of D fixing pointwise $\mathrm{Sing}(\mathcal{D}) \cap D$. This group is isomorphic to the quotient of the pure braid group of the sphere on $v(D)$ strands, by its centre, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. It is also isomorphic to the quotient of the pure braid group of the plane on $v(D) - 1$ strands, by its centre, which is isomorphic to $\mathbb{Z}$, cf. [2].

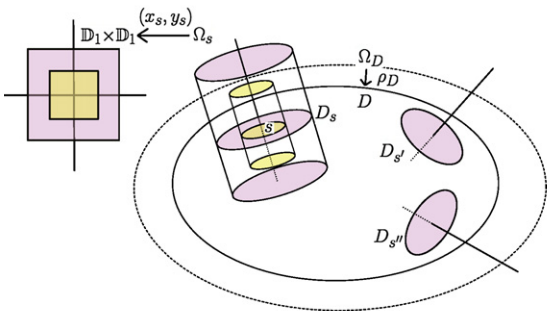

Fig. 1. The holomorphic coordinate system $(x_s, y_s) : \Omega_s \to \mathbb{D}_1 \times \mathbb{D}_1$ and the Hopf fibration $\rho_D : \Omega_D \to D$ of a local datum $\mathcal{L}$, associated to an irreducible component D of $\mathcal{D}$ and to a singularity $s \in S(D)$.

Note that the quotient group $\mathcal{G}_S/\mathcal{G}_S^0$ consists of "large symmetries of S". Note also that the graph of the topological JSJ decomposition of the 3-manifold obtained by removing to the sphere $\mathbb{S}^3_\varepsilon := \partial\mathbb{B}_\varepsilon$ a tubular neighbourhood of the link $S \cap \mathbb{S}_\varepsilon$ has $\mathfrak{R}$ as vertex set and $\mathfrak{C}$ as set of edges. Thus, $\mathcal{G}_S^0$ is a group of graph, in the sense of [19]. We will see with an explicit example, that in general the above epimorphism is not an isomorphism.

The structure of this work can be described as follows:

– In Section 2 we introduce some concepts on the (minimal) desingularisation of a germ of singular curve, as well as Milnor's tubes (of dimension three and four); they allow us to clarify the statement of main theorem and the key concept of marking.
– In Section 3 we establish the topological properties of Milnor's tubes, that will be used later. This chapter is divided into three sections. In the first one, we give an overview of the fundamental group of the complement of a singular curve. In the second one, we specify the Jaco–Johannson–Shalen decomposition of Milnor's 3-tube, which will play a key role in the proof of the main theorem. Finally in the third section, we study the algebraic properties of the action of a topological conjugacy between germs of curves, on some outstanding subgroups of the fundamental group, associated to the boundary components.
– In Section 4 we give the proof of Theorem A, structured into four sections: the reduction to three dimensions, the construction of a homeomorphism between Milnor's 3-tubes compatible with JSJ decompositions introduced in Section 3.2, the conjugation between the dual trees of the exceptional divisors and finally the extension to Milnor's 4-tubes.
– Finally, in Section 5 we study some algebraic properties of the group $\mathcal{G}_S$ and we prove Theorem B, using Theorem A already established.

2. Conjugacies and marked curves germs

2.1. Desingularisation and local data

In all the text, S denotes the intersection of an analytical curve in $\mathbb{C}^2$ with a closed ball $\mathbb{B} := \mathbb{B}_{r_0}$ of fixed centre $0 = (0, 0)$ and radius $r_0 > 0$. We assume that $\mathbb{B}$ is a *Milnor's ball for S*, i.e. $0 \in S$ and $S \setminus \{0\}$ is regular and transversal to the spheres $\partial\mathbb{B}_r$, $0 < r \leqslant r_0$. Let $E : \mathcal{B} \to \mathbb{B}$ be the (minimal) desingularisation map of S. We denote by $\mathcal{D} := E^{-1}(S)$ the *total divisor*, by $\mathcal{E} := E^{-1}(0)$ the *exceptional divisor* and by $\mathcal{S} := \overline{\mathcal{D} \setminus \mathcal{E}}$ the *strict transform* of S. We also denote by $S(D) := D \cap \mathrm{Sing}(\mathcal{D})$ the set of singular points of $\mathcal{D}$ belonging to $D \in \mathrm{Comp}(\mathcal{D})$. Two components $D, D' \in \mathrm{Comp}(\mathcal{D})$ are called *adjacent* if $D \neq D'$ and $D \cap D' \neq \emptyset$, in which case $D \cap D' = \{s\} \subset \mathrm{Sing}(\mathcal{D})$. We also consider a second analytic curve $S' \ni 0$ in a closed Milnor's ball $\mathbb{B}' := \mathbb{B}_{r'_0}$ for S'; $E' : \mathcal{B}' \to \mathbb{B}'$, $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{S}'$ denote respectively the resolution map, the total divisor, the exceptional divisor and the strict transform of S'. Throughout the paper, we adopt the following notations:

$$A^* := (A \setminus S), \qquad \mathcal{A}^* := (\mathcal{A} \setminus \mathcal{D}), \quad \text{for } A \subset \mathbb{B} \text{ and } \mathcal{A} \subset \mathcal{B}.$$

Similarly, for $A' \subset \mathbb{B}'$ and $\mathcal{A}' \subset \mathcal{B}'$, we denote $A'^* := (A' \setminus S')$ and $\mathcal{A}'^* := (\mathcal{A}' \setminus \mathcal{D}')$.

For each singular point $s \in \mathrm{Sing}(\mathcal{D})$, we fix a local holomorphic coordinate system $(x_s, y_s) : \Omega_s \xrightarrow{\sim} \mathbb{D}_1 \times \mathbb{D}_1$, defined on a closed neighbourhood Ω_s of s in $\mathcal{B}$ taking values on the closed polydisk $\mathbb{D}_1 \times \mathbb{D}_1$, such that $\mathcal{D} \cap \Omega_s = \{x_s y_s = 0\}$ and $\Omega_s \cap \Omega_{s'} = \emptyset$ if $s \neq s'$, where $\mathbb{D}_\varepsilon := \{|z| \leqslant \varepsilon\} \subset \mathbb{C}$. For each irreducible component $D \in \mathrm{Comp}(\mathcal{D})$, we fix a locally trivial fibration by closed disks, given by a differentiable submersion $\rho_D : \Omega_D \to D$, defined on a closed neighbourhood Ω_D of D in $\mathcal{B}$ (see Fig. 1). In Definition 2.1 we will precise the requirement of the compatibility of the fibration ρ_D with the polydisk structure on Ω_s for each $s \in S(D)$. We adopt the following notations, for $D, D' \in \mathrm{Comp}(\mathcal{D})$ we put

$$D_s := D \cap \Omega_s, \quad \text{for } s \in S(D) := \mathrm{Sing}(\mathcal{D}) \cap D \tag{2}$$

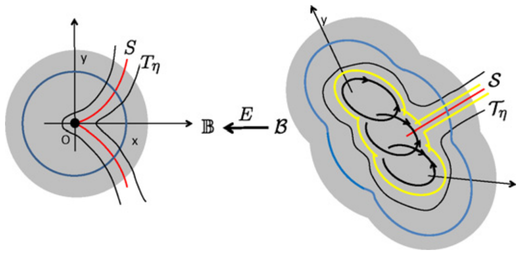

Fig. 2. The usual Milnor picture and the resolution map of the singular curve $S = f^{-1}(0)$ given by $f(x, y) = y^2 - x^3$.

and

$$K_D := \Big(D \setminus \bigcup_{s\in S(D)} \mathring{D}_s\Big). \tag{3}$$

For each subsets $X \subset \mathcal{B}$, $K \subset D$, not reduced to a single singular point, and each $s \in \mathrm{Sing}(\mathcal{D})$, we also denote

$$X(K) := X \cap \rho_D^{-1}(K) \quad \text{and} \quad X_s := X \cap \Omega_s. \tag{4}$$

Definition 2.1. We say that the collection $\mathcal{L} := ((x_s, y_s), \rho_D)_{s,D}$ is a *local datum* for S on $\mathcal{B}$, if it satisfies the following properties, for all $D \in \mathrm{Comp}(\mathcal{D})$ and $s \in \mathrm{Sing}(\mathcal{D})$:

(i) the restriction of ρ_D to D is the identity map;
(ii) if $D \subset \mathcal{E}$, then ρ_D is holomorphic on $\rho_D^{-1}(K_D)$;
(iii) if $D \subset \mathcal{S}$ and $m \in D \cap \partial\mathcal{B}$, then $\rho_D^{-1}(m) \subset \partial\mathcal{B}$;
(iv) if $z := x_s$ or y_s denotes the local coordinate which is $\not\equiv 0$ on D, then $z \circ \rho_D(m) = z(m)$ for $|z(m)| \leqslant 1/2$, $m \in \Omega_D \cap \Omega_s$.

The fibration ρ_D will be called the *Hopf fibration of base D*.

Note that ρ_D is holomorphic on a neighbourhood of each singular point of $\mathcal{D}$ and the local branches of $\mathcal{D}$ at these points are fibres of these fibrations. We leave the reader to prove that, if we fix local coordinates (x_s, y_s), there exists a fibration ρ_D, such that $\mathcal{L}$ is a local datum for S on $\mathcal{B}$.

2.2. Milnor's tubes and excellent homeomorphisms

Fix now a reduced holomorphic equation f of S, defined on an open neighbourhood of $\mathbb{B}$, with values in a closed disk $\mathbb{D}_\sigma := \{|z| \leqslant \sigma\} \subset \mathbb{C}$. For $\eta > 0$ small enough, denote

$$T_\eta := f^{-1}(\mathbb{D}_\eta) \cap \mathbb{B} \quad \text{and} \quad \mathcal{T}_\eta := E^{-1}(T_\eta) \subset \mathcal{B}.$$

When $\eta > 0$ is small enough, the restriction of f to T_η^* is a locally trivial $\mathcal{C}^\infty$-fibration with base $\mathbb{D}_\eta \setminus \{0\}$; we say then that T_η and $\mathcal{T}_\eta$ are *Milnor's* 4-*tubes of* S. Fix also a reduced equation of S' defined on an open neighbourhood of $\mathbb{B}'$. We define in the same way, the notion of Milnor's 4-tubes of S', denoted by $T'_{\eta'} \subset \mathbb{B}'$, $\mathcal{T}'_{\eta'} \subset \mathcal{B}'$ (see Fig. 2).

Remark 2.2. If $T_\eta \subset \mathbb{B}$ is a Milnor's 4-tube and $\mathbb{B}_\varepsilon$ is a closed ball contained in $\mathring{T}_\eta$, then the inclusions $\mathbb{B}_\varepsilon^* \subset T_\eta^* \subset \mathbb{B}^*$ induce isomorphisms at the fundamental group level.

Once the local datum $\mathcal{L}$ is fixed, we can precise the topology of the Milnor's 4-tubes. Classically for $\eta_0 > 0$ small enough, we construct[4] a smooth vector field $\mathcal{X}$ on $\mathcal{T}_{\eta_0}$ vanishing on $\mathcal{D}$ which is tangent to the fibres of ρ_D at each point of $\mathcal{T}_{\eta_0}(K_D)$, for each irreducible component D of $\mathcal{D}$, and fulfilling the equality $\mathcal{X} \cdot (f \circ E) = f \circ E$. This vector field blow down by E to a Lipschitz vector field X on T_{η_0} tangent to S and vanishing at the origin. Its flow is defined for all negative times.

We consider the following 3-manifolds with boundary, that we call *Milnor's* 3-*tubes*:

$$M_\eta := f^{-1}(\partial\mathbb{D}_\eta) \cap \mathbb{B} \subset \partial T_\eta \quad \text{and} \quad \mathcal{M}_\eta := E^{-1}(M_\eta) \subset \partial\mathcal{T}_\eta. \tag{5}$$

Using the flows of $\mathcal{X}$ and X we easily construct a retraction by deformation of $T_{\eta_0}^*$ on M_{η_0}—and then also a retraction by deformation of $\mathcal{T}_{\eta_0}^*$ on $\mathcal{M}_{\eta_0}$. The tangency properties of these flows allow us to be more specific.

[4] By transversality, there exists a vector field $\mathcal{X}_D$ fulfilling these properties on an open neighbourhood W_D of K_D. On Ω_s, $s \in \mathrm{Sing}(\mathcal{D})$, the existence of such vector fields can be deduced from the quasi-homogeneity of the function $f \circ E$ coming from the fact that it is locally a monomial. All these vector fields can be glued together using a partition of unity consisting of functions $u_D : W_D \to \mathbb{R}$ which are identically 1 on $\mathcal{T}_{\eta_0}(K_D)$ and $u_s : \Omega_s \to \mathbb{R}$ which are identically zero on $\Omega_s \cap (\bigcup_D \mathcal{T}_{\eta_0}(K_D))$, cf. [23].

Proposition 2.3. *There exists a diffeomorphism $\Theta : \mathcal{M}_{\eta_0} \times]0, \eta_0] \xrightarrow{\sim} \mathcal{T}^*_{\eta_0}$ such that*

$$\Theta\big(\mathcal{M}_{\eta_0} \times \{\eta\}\big) = \mathcal{M}_\eta, \qquad \Theta\big(\partial\mathcal{M}_{\eta_0} \times]0, 1]\big) = \mathcal{T}'^*_{\eta_0} \cap \partial\mathcal{B}', \qquad \Theta(m, \eta_0) = m,$$

for all $m \in \mathcal{M}_{\eta_0}$ and $0 < \eta \leqslant \eta_0$. Furthermore, with notations (4)*,*

$$\Theta\big(\mathcal{M}_{\eta_0}(K_D) \times]0, \eta_0]\big) = \mathcal{T}^*_{\eta_0}(K_D), \quad D \in \mathrm{Comp}(\mathcal{D}),$$

and the restriction of Θ to $\mathcal{M}_{\eta_0}(K_D) \times]0, \eta_0]$ extends to a differentiable map $\Theta_D : \mathcal{M}_{\eta_0}(K_D) \times [0, \eta_0] \to \mathcal{T}_{\eta_0}(K_D)$ fulfilling the relations

$$\rho_D \circ \Theta_D(m, s) = \rho_D(m), \qquad \Theta_D(m, 0) = \rho_D(m) \in K_D.$$

This diffeomorphism blow down to a diffeomorphism

$$\Theta^\flat : M_{\eta_0} \times]0, \eta_0] \xrightarrow{\sim} T^*_{\eta_0} \tag{6}$$

which induce a retraction by deformation of $(T^*_{\eta_0}, T^*_{\eta_0} \cap \partial\mathbb{B})$ on $(M_{\eta_0}, \partial M_{\eta_0})$. $T^*_{\eta_0}$ being a retract by deformation of $\mathbb{B}^*$, cf. [12], M_{η_0} is also a retract by deformation of $\mathbb{B}^*$. Then for $\eta > 0$ small enough, the restriction of ρ_D to $\mathcal{T}_\eta(K_D)$ is a fibration by disks.

Remark 2.4. When they occur, the inclusions $\mathbb{B}^*_\varepsilon \subset T^*_\eta \subset \mathbb{B}^*$ are homotopy equivalences and consequently induce[5] isomorphisms at the fundamental group level. Since M_η fibres by f over the circle $\partial\mathbb{D}_\eta$, the associated exact homotopy sequence shows that M_η is a $K(\pi, 1)$ Eilenberg–MacLane space. It is the same for T^*_η and $\mathbb{B}^*$ which retracts to M_η, and for $\mathcal{T}^*_\eta$, $\mathcal{B}^*$ and $\mathcal{M}_\eta$ which are homeomorphic to them.

The local datum $\mathcal{L}$ for S on $\mathcal{B}$ being always fixed, consider also a local datum for S' on $\mathcal{B}'$, denoted by

$$\mathcal{L}' := \big(\big(x'_{s'}, y'_{s'}\big) : \Omega'_{s'} \to \mathbb{D}_1 \times \mathbb{D}_1, \rho'_{D'} : \Omega'_{D'} \to D'\big)_{s', D'}.$$

We use for $\mathcal{L}'$, the same notations (2), (3) and (4) introduced for $\mathcal{L}$.

Definition 2.5. A homeomorphism $\Phi : T_\eta \to T'_{\eta'}$ between two Milnor's 4-tubes for S and S', such that $\Phi(S) = S'$, is called *excellent for $\mathcal{L}$ and $\mathcal{L}'$*, if it lifts to a homeomorphism $\phi : \mathcal{T}_\eta \to \mathcal{T}'_{\eta'}$, $E' \circ \phi = \Phi \circ E$, fulfilling the following properties:

(a) ϕ is holomorphic on a neighbourhood of each singular point of $\mathcal{D}$;
(b) for each irreducible component D of $\mathcal{D}$, we have the equality

$$\phi\big(\mathcal{T}_\eta(K_D)\big) = \mathcal{T}'_{\eta'}\big(K'_{\phi(D)}\big);$$

moreover the fibrations ρ_D and $\rho'_{\phi(D)}$ are conjugated by ϕ on these sets, i.e. $\rho'_D(\phi(m)) = \phi(\rho_D(m))$, $m \in \mathcal{T}_\eta(K_D)$.

2.3. Marking between germs of curves

Classically the conic structure of $\mathbb{B}^*$, which will be specified in Section 4.1, induces a retraction by deformation of $\mathbb{B}^*$ on each pointed closed subball $\mathbb{B}^*_\varepsilon \subset \mathbb{B}^*$. Thus, if the Milnor's 4-tube $T_\eta \subset \mathbb{B}$ contains $\mathbb{B}_\varepsilon$, the inclusion $\mathbb{B}^*_\varepsilon \subset T^*_\eta \subset \mathbb{B}^*$ induces isomorphisms at the fundamental group level. Hence each continuous map from one of these sets into $\mathbb{B}'^*$ defines a morphism from the fundamental group of $\mathbb{B}^*$ into the fundamental group of $\mathbb{B}'^*$. More precisely, consider the set $\mathfrak{C}(\mathbb{B}^*, \mathbb{B}'^*)$ consisting of all the continuous maps $F : U \to \mathbb{B}'^*$, from any arc-wise connected subset U of $\mathbb{B}^*$, such that the inclusion map $i_U : U \hookrightarrow \mathbb{B}^*$ induces an isomorphism $i_{U*} : \pi_1(U, p) \xrightarrow{\sim} \pi_1(\mathbb{B}^*, p)$. Then we denote

$$\underline{F}_* := F_* \circ (i_U)^{-1} : \pi_1\big(\mathbb{B}^*, p\big) \to \pi_1\big(\mathbb{B}'^*, F(p)\big).$$

Definition 2.6. We say that two elements $F : U \to \mathbb{B}'^*$ and $G : V \to \mathbb{B}'^*$ of $\mathfrak{C}(\mathbb{B}^*, \mathbb{B}'^*)$ are *fundamentally equivalent* (denoted by $F \asymp G$), if for any path α in $\mathbb{B}^*$, from a point $p \in U$ to a point $q \in V$, there exists a path α' in $\mathbb{B}'^*$ from $F(p)$ to $G(q)$ such that

$$\alpha'_* \circ \underline{F}_* = \underline{G}_* \circ \alpha_*, \tag{7}$$

where $\alpha_* : \pi_1(\mathbb{B}^*, p) \to \pi_1(\mathbb{B}^*, q)$ and $\alpha'_* : \pi_1(\mathbb{B}'^*, F(p)) \to \pi_1(\mathbb{B}'^*, G(q))$ are the natural isomorphisms induced by α and α'.

[5] This is easy to see using the conic structure [12] of the pair $(\mathbb{B}, S)$.

It is easy to see that $\asymp$ defines an equivalence relation on $\mathfrak{C}(\mathbb{B}^*, \mathbb{B}'^*)$ and that $F \asymp G$ as soon as there exists a pair of paths (α, α') satisfying (7).

Definition 2.7. An equivalence class $\mathfrak{f}$ by $\asymp$ will be called a *marking of S' by S*, if there exists an open neighbourhood U of the origin in $\mathbb{B}$ and an element $\breve{F} : U^* \to \mathbb{B}'^*$ of $\mathfrak{f}$, which extends to a homeomorphism $F : U \xrightarrow{\sim} F(U) \subset \mathbb{B}'$ preserving the orientations,[6] such that $F(S \cap U) = S' \cap F(U)$.

From now on all the homeomorphism conjugating two germs of curves which we consider, are supposed to preserve the orientations of the ambient space and those of the holomorphic curves.

Obviously two homeomorphisms conjugating S to S' on neighbourhoods of the origin, define the same marking of S' by S as soon as their germs are equal. We therefore speak about *homeomorphism germs which represent a marking*. Since from Remark 2.4 we have that $\mathbb{B}^*_\epsilon$ is a $K(\pi, 1)$-space, a classical theorem of algebraic topology[7] give us the following characterisation:

Proposition 2.8. *Two germs of homeomorphisms conjugating the germs of curves S and S', represent the same marking if and only if they induce homotopic maps from $\mathbb{B}^*_\varepsilon$ into $\mathbb{B}'^*$, with $\varepsilon > 0$ small enough.*

This leads us to ask the following question:

Question. It is true that two germs of homeomorphisms $h_0, h_1 : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $h_i(S, 0) = (S', 0)$, $i = 0, 1$, define the same marking if and only if there exist a germ of homeomorphism $H : (\mathbb{C}^3, I) \to (\mathbb{C}^3, I)$ along the compact set $I := 0 \times 0 \times [0, 1]$, such that $H(x, y, t) = (H_t(x, y), t)$, $H_0 = h_0$, $H_1 = h_1$ and such that the set germs along I, $H(S \times [0, 1])$ and $S' \times [0, 1]$, are equal?

The main result of this work is the following theorem.

Theorem 2.9. *Let $\mathcal{L} = ((x_s, y_s), \rho_D)_{s,D}$, respectively $\mathcal{L}' = ((x'_{s'}, y'_{s'}), \rho'_{D'})_{s',D'}$, be a local datum for S, respectively S', on $\mathcal{B}$, respectively $\mathcal{B}'$ and let $h : \mathbb{B}_\varepsilon \xrightarrow{\sim} h(\mathbb{B}_\varepsilon) \subset \mathbb{B}'$ be a homeomorphism such that $h(S \cap \mathbb{B}_\varepsilon) = S' \cap h(\mathbb{B}_\varepsilon)$. Then there exists a homeomorphism $\Phi : T_\eta \xrightarrow{\sim} T'_{\eta'}$, $\Phi(S) = S'$ which is excellent for the local data $\mathcal{L}$ and $\mathcal{L}'$, such that the restrictions $h_{|\mathbb{B}^*_\varepsilon}$ and $\Phi_{|T^*_\eta} : T^*_\eta \to T'^*_{\eta'}$ are fundamentally equivalent.*

In other terms, we obtain Theorem A stated in the introduction which can be reformulated as follows:

Each marking of S' by S can be represented by an excellent homeomorphism between two Milnor's 4-tubes.

3. Topology of Milnor's tubes

Before starting the proof of Theorem 2.9, we bring to light in this section the topological properties of the Milnor's tubes that we shall use later.

3.1. Fundamental group and homology

We shall give an explicit presentation of the fundamental group Γ of T^*_η. For this, recall that the dual tree[8] $\mathbb{A}$ of the desingularisation of S has one vertex for each element $D \in \mathrm{Comp}(\mathcal{D})$ and, for each singular point $s \in \mathrm{Sing}(\mathcal{D})$, $\{s\} = D \cap D'$, it has an edge joining the vertices corresponding to D and D'.

Fix a local datum $\mathcal{L}$ for S and a topological embedding j of a geometrical realisation $|\mathbb{A}|$ of $\mathbb{A}$ in $\mathcal{T}^*_\eta$, such that:

- for each $D \in \mathrm{Comp}(\mathcal{D})$, $j^{-1}(\mathcal{T}^*_\eta(K_D))$ is connected and it contains a unique vertex $\mathbf{s}_D$, which is the one associated to D; furthermore, $\rho_D \circ j$ restricts to a neighbourhood of $\mathbf{s}_D$ as an embedding;
- for each $s \in \mathrm{Sing}(\mathcal{D})$, $j^{-1}(\mathcal{T}^*_\eta(K_s))$ is connected and it is contained in a single edge, which is the one associated to s. Without lost of generality we also assume that the point having coordinates $(x_s, y_s) = (\epsilon, \epsilon)$ belongs to $j(\mathbb{A})$, $0 < \epsilon \ll \eta$.

In the sequel we will assume that the base point of the fundamental group Γ belongs to $j(\mathbb{A})$. Since $j(\mathbb{A})$ is contractible, we can identify the groups $\pi_1(\mathcal{T}^*_\eta, j(\mathbb{A}))$ and Γ, by an isomorphism that we shall not make explicit.

[6] If $S = S'$ is given by an equation with real coefficients, then $F(x, y) = (\bar{x}, \bar{y})$ preserves the ambient space orientation, but reverse the orientation of S.

[7] Confer for example [24, Corollary 4.4, p. 226].

[8] The dual graphs are usually weighted by either Euler numbers (self-intersections) or multiplicities. Both are relevant in this paper, see Corollary 3.4.

Definition 3.1. We call *meridian* associated to a component $D \in \mathrm{Comp}(\mathcal{D})$, the conjugacy class of the element $\mathfrak{c}_D \in \Gamma$ which corresponds to the loop $\rho_D^{-1}(\rho_D(j(\mathbf{s}_D))) \cap \mathcal{M}_\eta$, oriented as the boundary of a holomorphic curve.

Remark 3.2. Let $s \in \mathrm{Sing}(\mathcal{D})$ be the intersection of D and $D' \in \mathrm{Comp}(\mathcal{D})$. Assume that $D \cap \Omega_s = \{x_s = 0\}$ and $D' \cap \Omega_s = \{y_s = 0\}$. Then $\mathfrak{c}_D$, respectively $\mathfrak{c}_{D'}$, are the homotopy classes of the loops $(x_s, y_s) = (e^{2i\pi t}\epsilon, \epsilon)$, respectively $(x_s, y_s) = (\epsilon, e^{2i\pi t}\epsilon)$.

Let us denote by $(\cdot,\cdot) : \mathrm{Comp}(\mathcal{D}) \times \mathrm{Comp}(\mathcal{D}) \to \mathbb{Z}$ the intersection pairing on the components of the total divisor $\mathcal{D}$. When $D \neq D'$ then $(D, D') = 1$ if $D \cap D' \neq \emptyset$ and $(D, D') = 0$ otherwise. When $E \in \mathrm{Comp}(\mathcal{E})$ the self-intersection number (E, E) coincides with the integral over the fundamental class of the Chern class of the normal bundle of E inside $\mathcal{B}$. It also coincides with the Euler number of the unit normal bundle of E. When C is an irreducible component of the strict transform of S we simply put $(C, C) = 0$.

Proposition 3.3. *The fundamental group Γ is defined by the generators system $\{\mathfrak{c}_D\}_{D \in \mathrm{Comp}(\mathcal{D})}$, whose relations are given by the families*

$$\prod_{D' \in \mathrm{Comp}(\mathcal{D})} \mathfrak{c}_{D'}^{(D',E)} = 1, \qquad [\mathfrak{c}_D, \mathfrak{c}_E]^{(D,E)} = 1 \tag{8}$$

indexed by $E \in \mathrm{Comp}(\mathcal{E})$ and $D \in \mathrm{Comp}(\mathcal{D})$.

Here we consider the product $\prod_{D' \in \mathrm{Comp}(\mathcal{D})} \mathfrak{c}_{D'}^{(D',E)}$ with the order induced by the cyclic order of the wedges of $\rho_E \circ j(\mathrm{star}(\mathbf{s}_E))$, obtained by projection of the $\mathbf{s}_E$-star, in the component E. The proof is done by induction, by applying the classical Seifert–Van Kampen's theorem; see for example [13,4,9].

We will use a multiplicative notation for writing the elements of Γ and an additive notation for their classes in $\Gamma/[\Gamma, \Gamma] \cong H_1(\mathcal{T}_\eta^*; \mathbb{Z})$; but we will keep the same names.

Corollary 3.4. *The homology group $H_1(\mathcal{T}_\eta^*; \mathbb{Z})$ is a rank $r := \#\mathrm{Comp}(S)$ free-abelian group, generated by the classes $\mathfrak{c}_{\mathcal{S}_j}$ associated to the irreducible components $\mathcal{S}_1, \ldots, \mathcal{S}_r$ of $\mathcal{S}$. Furthermore, denoting by $\{E_1, \ldots, E_n\}$, the components of $\mathcal{E}$ and by $\mathfrak{c}_{\mathcal{E}}$ and $\mathfrak{c}_{\mathcal{S}}$, the column-matrix obtained by transposing $(\mathfrak{c}_{E_1}, \ldots, \mathfrak{c}_{E_n})$ and $(\mathfrak{c}_{\mathcal{S}_1}, \ldots, \mathfrak{c}_{\mathcal{S}_r})$, we have that*

$$\mathfrak{c}_{\mathcal{E}} = -(\mathcal{E}, \mathcal{E})^{-1}(\mathcal{E}, \mathcal{S}) \cdot \mathfrak{c}_{\mathcal{S}}, \tag{9}$$

being $(\mathcal{E}, \mathcal{E})$, respectively $(\mathcal{E}, \mathcal{S})$, the matrices whose entries are the intersection numbers (E_i, E_j), respectively $(E_i, \mathcal{S}_k)$. Finally, the (i,k)-entry of the matrix $-(\mathcal{E}, \mathcal{E})^{-1} \cdot (\mathcal{E}, \mathcal{S})$, is equal to the multiplicity $\nu_{E_i}(f_k \circ E)$ of $f_k \circ E$ along E_i, being f_k a reduced equation of S_k.

Proof. From (8), we deduce the relations:

$$0 = \sum_{D \in \mathrm{Comp}(\mathcal{D})} (E_i, D)\mathfrak{c}_D = \sum_{j=1}^{n} (E_i, E_j)\mathfrak{c}_{E_j} + \sum_{k=1}^{r} (E_i, \mathcal{S}_k)\mathfrak{c}_{\mathcal{S}_k}. \tag{10}$$

Then it is enough to write them in matrix form and to express $\mathfrak{c}_{E_i}$ depending of $\mathfrak{c}_{\mathcal{S}_k}$, using the well-know fact that $\det(\mathcal{E}, \mathcal{E})$ is equal to ± 1. Finally,

$$\begin{aligned}\nu_{E_i}(f_k \circ E) &= \frac{1}{2i\pi} \int_{\mathfrak{c}_{E_i}} E^*\left(\frac{df_k}{f_k}\right) = \frac{1}{2i\pi} \int_{-\sum_{\ell=1}^{r}((\mathcal{E},\mathcal{E})^{-1}(\mathcal{E},\mathcal{S}))_{i\ell} E(\mathfrak{c}_{\mathcal{S}_\ell})} \frac{df_k}{f_k} \\ &= -\left((\mathcal{E}, \mathcal{E})^{-1} \cdot (\mathcal{E}, \mathcal{S})\right)_{ik},\end{aligned} \tag{11}$$

because $\frac{1}{2i\pi} \int_{E(\mathfrak{c}_{\mathcal{S}_\ell})} \frac{df_k}{f_k} = \delta_{\ell k}$. □

3.2. The JSJ decomposition

The following is well known by the specialists in topology of 3-manifolds. These are applications to singularities of curves, of the classification's results of 3-manifolds, due to Waldhausen [21,22], Jaco and Shalen [6] and Johannson [8]. This study was done by Michel and Weber [11] and by Neumann [14,15] via plumbing calculus. In this section we specify these technics in order to highlight the properties that we will need in the next section. For precise statements of the used theorems, we refer to [20] from which we adopt the vocabulary. The reader may also refers to the C.T.C. Wall's monography [23] and to the Neumann and Swarup's article [16].

By using still the notations (3) and (4), we define for each singular point s and each component D of $\mathcal{D}$, the following sub-manifolds (with boundary) of $\mathcal{M}_\eta$:

$$\mathcal{M}_s := \mathcal{M}_\eta \cap \Omega_s \quad \text{and} \quad \mathcal{M}_D := \mathcal{M}_\eta(K_D). \tag{12}$$

We call them *elementary blocks* of $\mathcal{M}_\eta$. The Jaco–Shalen–Johannson decomposition (JSJ for short) of $\mathcal{M}_\eta$ in Seifert blocks and thick tori that we will now define, will be obtained by aggregating such elementary blocks.

Denote by $\mathfrak{R}$ the set of all irreducible components of $\mathcal{D}$ having valence $\geqslant 3$. Now, we precise the notion of *chain of components* given at the end of the notations section: it is a finite collection of irreducible components of $\mathcal{E}$,

$$\mathcal{C} := \{D_0, \dots, D_{l_\mathcal{C}+1}\}, \quad l_\mathcal{C} \geqslant 0,\ D_0, D_{l_\mathcal{C}+1} \in \mathfrak{R}, \tag{13}$$

such that

$$v(D_1) = \dots = v(D_{l_\mathcal{C}}) = 2 \quad \text{and} \quad D_j \cap D_{j+1} \neq \emptyset, \quad j = 0, \dots, l_\mathcal{C}. \tag{14}$$

Denote by $\mathfrak{C}$ the set of all the chains of components of $\mathcal{E}$. The number $l_\mathcal{C}$ is called the length of the chain $\mathcal{C} \in \mathfrak{C}$. Notice that chains of length zero, which are explicit allowed, consist in two irreducible components of $\mathfrak{R}$ meeting at a single singular point. A *dead branch of $\mathcal{E}$ adjacent to $D \in \mathfrak{R}$* is a finite sequence $\mathcal{C} := \{D_0, \dots, D_{l_\mathcal{C}}\}$, $l_\mathcal{C} \geqslant 1$, of components of $\mathcal{E}$, such that

$$D_0 = D, \qquad v(D_j) = 2, \qquad v(D_{l_\mathcal{C}}) = 1, \qquad D_k \cap D_{k+1} \neq \emptyset, \tag{15}$$

with $1 \leqslant j \leqslant l_\mathcal{C} - 1$ and $0 \leqslant k \leqslant l_\mathcal{C} - 1$. The component $D_{l_\mathcal{C}}$ is called the *end component of* $\mathcal{C}$ and the intersection point of D_0 with D_1, the *attaching point of* $\mathcal{C}$. We denote by $\mathfrak{M}$ the set of all dead branches of $\mathcal{E}$. Notice that a chain of components is not oriented unlike the case of a dead branch in which we take $D_0 \in \mathfrak{R}$ and $v(D_{l_\mathcal{C}}) = 1$. Chains of components of length > 0 and dead branches are also called *bambous* in the literature.

Fix $\mathcal{C} := \{D_0, \dots, D_{\ell_\mathcal{C}+1}\} \in \mathfrak{C}$ and denote by s_j the intersection point of D_{j-1} and D_j, for $j = 0, \dots, l_\mathcal{C}$. If $\eta > 0$ is small enough, as we will assume, then $\mathcal{M}_{s_j}$ is a *thick torus*, i.e. $\mathcal{M}_{s_j}$ is homeomorphic to the product of the standard torus $\mathbb{T} := \partial\mathbb{D}_1 \times \partial\mathbb{D}_1$ with a compact interval. Each $\mathcal{M}_{D_j}$, $j = 1, \dots, l_\mathcal{C}$, is also a thick torus and by gluing them, we obtain a 3-manifold with boundary $\mathcal{M}_\mathcal{C}$, endowed with a homeomorphism:

$$\breve{\sigma}_\mathcal{C} : \mathcal{M}_\mathcal{C} := \bigcup_{j=1}^{l_\mathcal{C}} \mathcal{M}_{D_j} \cup \bigcup_{j=0}^{l_\mathcal{C}} \mathcal{M}_{s_j} \xrightarrow{\sim} \mathbb{T} \times [-1, 1].$$

This product structure extends to a neighbourhood of the boundary of $\mathcal{M}_\mathcal{C}$ over a 3-manifold with boundary $\widetilde{\mathcal{M}}_\mathcal{C}$, endowed with a homeomorphism

$$\sigma_\mathcal{C} : \widetilde{\mathcal{M}}_\mathcal{C} \xrightarrow{\sim} \mathbb{T} \times [-1-\epsilon, 1+\epsilon], \qquad \sigma_\mathcal{C}^{-1}\big(\mathbb{T} \times [-1, 1]\big) = \mathcal{M}_\mathcal{C}, \qquad \sigma_{\mathcal{C}|\mathcal{M}_\mathcal{C}} = \breve{\sigma}_\mathcal{C}. \tag{16}$$

Consider the 2-torus $\mathbb{T}_\mathcal{C} := \sigma_\mathcal{C}^{-1}(\mathbb{T} \times \{0\})$. The adherence B of each connected component of $\mathcal{M}_\eta \setminus (\bigcup_{\mathcal{C}\in\mathfrak{C}} \mathbb{T}_\mathcal{C})$ contains a unique elementary block $\mathcal{M}_D$, $D \in \mathfrak{R}$. We say that B is the *JSJ block of $\mathcal{M}_\eta$ associated to D* and we will denote it by B_D. We will also denote by $B_D^\flat$ the connected component of the adherence of $\mathcal{M}_\eta \setminus \bigcup_{\mathcal{C}\in\mathfrak{C}} \mathcal{M}_\mathcal{C}$ inside B_D.

For each dead branch $\mathcal{C} := \{D_0, \dots, D_{l_\mathcal{C}}\} \in \mathfrak{M}$ of $\mathcal{E}$ we still denote

$$\mathcal{M}_\mathcal{C} := \bigcup_{j=1}^{l_\mathcal{C}} \mathcal{M}_{D_j} \cup \bigcup_{j=0}^{l_\mathcal{C}-1} \mathcal{M}_{s_j}, \quad \text{where } \{s_j\} := D_j \cap D_{j+1}. \tag{17}$$

Then, if $D \in \mathfrak{R}$, $B_D^\flat$ is the union of $\mathcal{M}_D$ and the manifolds $\mathcal{M}_\mathcal{C}$, where $\mathcal{C}$ describes the set of dead branches whose attaching points belong to D. Notice that if $\mathcal{C} \in \mathfrak{M}$ then $\mathcal{M}_\mathcal{C}$ is homeomorphic to a solid torus $\mathbb{D} \times \mathbb{S}^1$; remark also that the complement $\mathcal{M}_\mathcal{C}^\circ$, inside $\mathcal{M}_\mathcal{C}$, of a Hopf fibre (not contained in $D_{l_\mathcal{C}-1}$) of the divisor $D_{l_\mathcal{C}}$ having valence 1, has the homotopy type of a torus $\mathbb{S}^1 \times \mathbb{S}^1$.

Definition 3.5. For each $\mathcal{C} \in \mathfrak{C} \cup \mathfrak{M}$, we put $H_1^\mathcal{C} = H_1(\mathcal{M}_\mathcal{C}, \mathbb{Z})$, if $\mathcal{C} \in \mathfrak{C}$ and $H_1^\mathcal{C} = H_1(\mathcal{M}_\mathcal{C}^\circ, \mathbb{Z})$, if $\mathcal{C} \in \mathfrak{M}$. For each $D_j \in \mathcal{C}$, the class $\mathfrak{c}_j$ of a fibre of ρ_{D_j} restricted to $\mathcal{M}_{D_j}$ and oriented as the boundary of a holomorphic curve of $\mathcal{T}_\eta$, will be called meridian associated to D_j. If $\mathcal{C} \in \mathfrak{M}$, we define the *exceptional meridian* $\mathfrak{c}_{l_\mathcal{C}+1}$ *of the dead branch* $\mathcal{C}$ as the generator of the kernel of the morphism $H_1(\mathcal{M}_\mathcal{C}^\circ, \mathbb{Z}) \to H_1(\mathcal{M}_\mathcal{C}, \mathbb{Z})$, oriented as the boundary of a holomorphic curve in $\mathcal{T}_\eta$.

Proposition 3.6. *If $\mathcal{C} \in \mathfrak{C} \cup \mathfrak{M}$ then*:

(i) *$H_1^\mathcal{C}$ is the free-abelian group of rank 2 generated by $\mathfrak{c}_0, \dots, \mathfrak{c}_{l_\mathcal{C}+1}$ having the following relations*:

$$\mathfrak{c}_{j-1} + e_j \mathfrak{c}_j + \mathfrak{c}_{j+1} = 0, \qquad e_j = (D_j, D_j), \quad j = 1, \dots, l_\mathcal{C}; \tag{18}$$

(ii) *for each* $j = 0, \dots, l_\mathcal{C}$, *the elements* $\mathfrak{c}_j$, $\mathfrak{c}_{j+1}$ *define a basis of* $H_1^\mathcal{C}$; *all theses bases define the same orientation*; *the canonical* $\mathbb{Z}$*-linear* 2*-form* $\det(\cdot,\cdot)$ *over* $H_1^\mathcal{C}$ *such that* $\det(\mathfrak{c}_j, \mathfrak{c}_{j+1}) = 1$ *and* $\det(\mathfrak{c}_j, \mathfrak{c}_j) = 0$, *corresponds to the intersection form of each connected component of* $\partial\mathcal{M}_\mathcal{C}$ *with the induced orientation*;

(iii) *we have that* $\mathfrak{c}_0 = a\mathfrak{c}_{l_\mathcal{C}} + b\mathfrak{c}_{l_\mathcal{C}+1}$, *with* $a = \pm\det(A) \neq 0$, *where A denotes the matrix of the restriction to the divisor* $\bigcup_{j=1}^{l_\mathcal{C}} D_j$, *of the intersection form of* $\mathcal{E}$;

(iv) *the elements* $\mathfrak{c}_0 \otimes 1$, $\mathfrak{c}_{l_\mathcal{C}+1} \otimes 1$ *define a* $\mathbb{Q}$*-basis of* $H_1^\mathcal{C} \otimes \mathbb{Q}$.

Proof. Assertion (i) follows directly from relations (10) and the fact that $H_1^\mathcal{C}$ is canonically identified with the integer homology of a torus. Assertion (ii) follows easily from the relations (18), see also [4,5], which can be written in matrix form as follows

$$\underbrace{\begin{bmatrix} e_1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & e_2 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & e_{l_\mathcal{C}-1} & 1 \\ 0 & \cdots & \cdots & 0 & 1 & e_{l_\mathcal{C}} \end{bmatrix}}_{A} \begin{bmatrix} \mathfrak{c}_1 \\ \mathfrak{c}_2 \\ \vdots \\ \vdots \\ 0 \\ \mathfrak{c}_{l_\mathcal{C}} \end{bmatrix} = - \begin{bmatrix} \mathfrak{c}_0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \mathfrak{c}_{l_\mathcal{C}+1} \end{bmatrix}.$$

By applying Cramer's formula, it is easy to see that the coefficient a in the expression $\mathfrak{c}_0 = a\mathfrak{c}_{l_\mathcal{C}} + b\mathfrak{c}_{l_\mathcal{C}+1}$ is $a = \pm\det A$, where A is the matrix of the restriction to the divisor $\bigcup_{j=1}^{l_\mathcal{C}} D_j \subset \mathcal{E}$ of the intersection form of $\mathcal{E}$, which is negative definite. This gives us (iii), because $\det A \neq 0$. Finally, Assertion (iv) follows directly from (iii). □

Denote by $S_\mathfrak{M}(D) \subset S(D)$ the set of attaching points of the dead branches over D and put

$$\widehat{S}(D) := S(D) \setminus S_\mathfrak{M}(D), \qquad \widehat{K}_D := D \setminus \bigcup_{s \in \widehat{S}(D)} \mathring{D}_s = K_D \cup \bigcup_{s \in S_\mathfrak{M}(D)} D_s. \tag{19}$$

Corollary 3.7. *For each irreducible component D of* $\mathcal{E}$ *having valence* $\geqslant 3$, *the restriction to* $\mathcal{M}_D$ *of the fibration* ρ_D *of the local datum* $\mathcal{L}$, *extends to a Seifert fibration* $\widehat{\rho}_D : B_D^\flat \to \widehat{K}_D$, *having exceptional fibres* $\widehat{\rho}_D^{-1}(s)$, $s \in S_\mathfrak{M}(D)$. *Moreover*, $\widehat{\rho}_D^{-1}(s)$ *is the intersection of* $B_D^\flat$ *with a fibre of the Hopf fibration corresponding to the end component of the dead brach whose attaching point is s.*

Proof. Consider a meridian $\mathfrak{m} := [\partial\mathbb{D} \times \{1\}]$ and a parallel $\mathfrak{p} := [\{1\} \times \mathbb{S}^1]$ inside $H_1(\mathbb{D}^* \times \mathbb{S}^1, \mathbb{Z})$. It is well known that a curve of $\mathbb{D}^* \times \mathbb{S}^1$ having integer homology class $a\mathfrak{p} + b\mathfrak{m}$ is the fibre of a Seifert fibration of $\mathbb{D} \times \mathbb{S}^1$, if and only if $a \neq 0$. We conclude by applying Assertion (iii) in the previous proposition because in this case $\mathfrak{m} = \mathfrak{c}_{l_\mathcal{C}+1}$. □

Remark 3.8. The product structure of the thick tori $\mathcal{M}_\mathcal{C}$, $\mathcal{C} \in \mathfrak{C}$, allows us to extend $\widehat{\rho}_D$ into a Seifert fibration

$$\widehat{\rho}_D^{\text{ext}} : B_D \to \widehat{K}_D^{\text{ext}}, \qquad \widehat{K}_D^{\text{ext}} := D \setminus \bigcup_{s \in \widehat{S}(D)} \mathring{\underline{D}}_s, \qquad \widehat{\rho}^{\text{ext}}_{D|B_D^\flat} = \widehat{\rho}_D, \tag{20}$$

whose fibres are contained in the fibres of $\sigma_\mathcal{C}$, where $\underline{D}_s$ denotes a conformal closed disk centred at s contained in $\mathring{D}_s$.

Thus, each torus $\mathbb{T}_\mathcal{C}$, $\mathcal{C} \in \mathfrak{C}$, which is the intersection of two JSJ blocks is endowed with two circle fibrations obtained by restricting the Seifert fibrations of each adjacent block. The homology classes of these two fibrations are precisely $\mathfrak{c}_0$ and $\mathfrak{c}_{l_\mathcal{C}+1}$. They can be considered as elements of $H_1(\mathbb{T}_\mathcal{C}, \mathbb{Z})$ because the inclusion $\mathbb{T}_\mathcal{C} \subset \mathcal{M}_\mathcal{C}$ induces an isomorphism in homology.

Remark 3.9. If $\mathcal{C} \in \mathfrak{C}$ then it is easy to see that $\mathbb{T}_\mathcal{C}$ is incompressible in B_D by using that $\nu(D_j) \geqslant 3$ for $j \in \{0, l_\mathcal{C} + 1\}$, cf. [9]. This gives the monomorphism $H_1(\mathbb{T}_\mathcal{C}, \mathbb{Z}) \hookrightarrow H_1(B_{D_j}, \mathbb{Z})$. Thus, $\mathfrak{c}_0$ and $\mathfrak{c}_{l_\mathcal{C}+1}$ are also independent in $H_1(B_{D_j}, \mathbb{Z})$ and therefore the Seifert fibrations of B_{D_0} and $B_{D_{l_\mathcal{C}+1}}$ are incompatible. Moreover, by using the relations (9), it is easy to see that the image of $H_1(\mathbb{T}_\mathcal{C}, \mathbb{Z})$ inside $H_1(\mathcal{M}_\eta, \mathbb{Z})$ is different from the images of the tori contained in the boundary of $\mathcal{M}_\eta$.

Since the hypotheses of Theorem 1.2.3 of [20] are verified we obtain:

Corollary 3.10. *The collection $(\mathbb{T}_{\mathcal{C}})_{\mathcal{C}\in\mathfrak{C}}$ is a characteristic family*[9] *of essential tori*[10] *of the 3-manifold $\mathcal{M}_\eta$ and determines its JSJ decomposition, which is entirely constituted by Seifert blocks.*

Remark 3.11. The vertices of the tree of the JSJ decomposition of $\mathcal{M}_\eta$ (corresponding to the Seifert blocks B_D) are in one-to-one correspondence with the irreducible components $D \in \mathfrak{R}$, and their edges (joining two vertices corresponding to two adjacent Seifert blocks) are in one-to-one correspondence with the chains $\mathcal{C} \in \mathfrak{C}$.

3.3. Peripheral structures and geometric isomorphisms

For each irreducible component S_k of S we consider a tubular neighbourhood W_k of $S_k \cap (\mathbb{B}_r \setminus \mathring{\mathbb{B}}_s)$ with $0 < s < r \ll 1$, such that the restrictions of the fibrations $\rho_{\mathcal{S}_k}$ and ρ_{D_k} to $\mathcal{W}_k := E^{-1}(W_k)$ are topologically trivial, where $D_k \in \mathrm{Comp}(\mathcal{D})$ denotes the irreducible component of $\mathcal{E}$ adjacent to $\mathcal{S}_k$. The fundamental group $\mathcal{P}_k := \pi_1(W_k^*)$ is isomorphic to $\mathbb{Z}\mathfrak{m}_k \oplus \mathbb{Z}\mathfrak{p}_k$, where $\mathfrak{m}_k$ and $\mathfrak{p}_k$ are the oriented boundaries of a fibre of the restriction to $\mathcal{W}_k^*$ of $\rho_{\mathcal{S}_k}$ and ρ_{D_k} respectively. The commutativity of $\mathcal{P}_k$ allows us do not make explicit the choice of a base point in W_k^*. Notice that $\mathfrak{m}_k$ generates the kernel of the morphism $\pi_1(W_k^*) \to \pi_1(W_k)$ induced by the inclusion. Let $s = \mathcal{S}_k \cap D_k \in \mathrm{Sing}(\mathcal{D})$ be the attaching point of $\mathcal{S}_k$. Up to permutation of the coordinates (x_s, y_s) we will assume that $x_s = 0$ is a reduced equation of $\mathcal{S}_k$. We suitably choose $\varepsilon_1, \varepsilon_2 > 0$ such that $\mathcal{W}_k^*$ retracts over the 2-torus $\{|x_s| = \varepsilon_1, |y_s| = \varepsilon_2\}$. The loops m and p of $\mathcal{W}_k^*$ defined by $(x_s, y_s) \circ m(t) = (\varepsilon_1 e^{2i\pi t}, \varepsilon_2)$ and $(x_s, y_s) \circ p(t) = (\varepsilon_1, \varepsilon_2 e^{2i\pi t})$ are representatives of $\mathfrak{m}_k$ and $\mathfrak{p}_k$ respectively.

Definition 3.12. We will call $\mathfrak{m}_k$ and $\mathfrak{p}_k$ the canonical *meridian* and *parallel* of S_k respectively.

Remark 3.13. It should be noticed that there are several choices for the parallel in the literature. The most widely used is the Seifert one, denoted here $\mathfrak{p}_k^S$, which verifies $\mathrm{lk}(\mathfrak{p}_k, S_k) = 0$. This linking number coincides with the intersection number of $\mathfrak{p}_k$ with a Seifert surface of S_k that we can take as a Milnor fibre $f_k = c \neq 0$ of a reduced equation f_k of S_k. We can write locally $f_k(x_s, y_s) = x_s y_s^{\nu_k} u(x_s, y_s)$, where $u(0,0) \neq 0$ and $\nu_k = \nu_{D_k}(f_k \circ E)$ is the multiplicity considered in (11). Consequently we have $\mathfrak{p}_k^S = \mathfrak{p}_k - \nu_k \mathfrak{m}_k$. The relationship with other choices can be deduced from this formula, see for instance [11].

Proposition 3.14. *The subset W_k^* is incompressible in T_η^*, i.e. the morphism $i_k : \mathcal{P}_k \to \Gamma$ induced by the inclusion $W_k^* \subset T_\eta^*$, which is given explicitly by $i_k(\mathfrak{m}_k) = \mathfrak{c}_{S_k}$, $i_k(\mathfrak{p}_k) = \mathfrak{c}_{D_k}$, is injective.*

Proof. This can be proved[11] by using iteratively Van Kampen's theorem, as we have already done in the construction of an adapted neighbourhood of $\mathcal{D}$ by boundary assembly in [9]. We shall present here other proof based on the incompressibility inside T_η^* of the Milnor fibre[12] F of a reduced equation f of S. Let us denote by i_{CW}, i_{WT}, i_{CF}, i_{FT} the morphisms induced at the fundamental groups by the inclusions $F \cap W_k^* \subset W_k^*$, $W_k^* \subset T_\eta^*$, $F \cap W_k^* \subset F$, $F \subset T_{\eta^*}$ respectively. Note that $\pi_1(F)$ is the kernel of the morphism $f_* : \Gamma \to \mathbb{Z}$ sending $\mathfrak{c}_D$ into the multiplicity $\nu_D(f \circ E)$ of $f \circ E$ along D. Let us denote by $\nu_k := \nu_{D_k}(f \circ E)$ the vanishing order of $E^* f$ along D_k. Since $f \circ E = x_s y_s^{\nu_k}$, we have the isomorphism $\pi_1(F \cap W_k^*) \cong \mathbb{Z}\mathfrak{b}_k$, where $\mathfrak{b}_k$ denotes the oriented connected component of the boundary of F contained in W_k^* and $i_{CW}(\mathfrak{b}_k) = \mathfrak{p}_k - \nu_k \mathfrak{m}_k$. On the other hand, if $\mathfrak{k} = \alpha \mathfrak{p}_k + \beta \mathfrak{m}_k \in \pi_1(W_k^*)$ belongs to the kernel of i_{WT}, then $f_*(i_{WT}(\alpha \mathfrak{p}_k + \beta \mathfrak{m}_k)) = \alpha \nu_k + \beta = 0$; hence $\mathfrak{k} = i_{CW}(\alpha \mathfrak{b}_k)$. Since $i_{WT} \circ i_{CW} = i_{FT} \circ i_{CF}$, i_{CF} and i_{FT} are injective, $\alpha = 0$ and i_{WT} is also injective. □

In the sequel we will identify $\mathcal{P}_k$ with its image inside the fundamental group Γ, taking the base point in W_k^* lying also on $j(\mathbb{A})$. If we need to consider more than one subgroup $\mathcal{P}_k$ at the same time, it will be necessary to consider the family of all the conjugate subgroups of $\mathcal{P}_k$ inside Γ. The following result make precise this situation.

Proposition 3.15. *The normaliser of $\mathcal{P}_k$ in Γ equals $\mathcal{P}_k$, i.e. if $\zeta \in \Gamma$ and $\zeta \mathcal{P}_k \zeta^{-1} \subset \mathcal{P}_k$ then $\zeta \in \mathcal{P}_k$. In particular, the decomposition $\mathcal{P}_k = \mathbb{Z}\mathfrak{m}_k \oplus \mathbb{Z}\mathfrak{p}_k$ is intrinsic*[13] *in Γ.*

Proof. The proof of the previous proposition shows that $\pi_1(F) \cap \mathcal{P}_k = \pi_1(F \cap W_k^*) = \mathbb{Z}\mathfrak{b}_k$. Consider the element $\zeta' := \zeta \mathfrak{m}_k^{-\ell}$ with $\ell := f_*(\zeta) = \frac{1}{2i\pi} \int_\zeta E^*(\frac{df}{f})$. Since $f_*(\mathfrak{m}_k) = 1$, it follows that $f_*(\zeta') = 0$ and then $\zeta' \in \pi_1(F)$. Thus, $\zeta' \mathfrak{b}_k \zeta'^{-1} \in \pi_1(F) \cap$

[9] I.e. a minimal family of tori such that the adherence of each connected component of its complement is either a Seifert or atoroidal manifold, cf. [20, p. 144].

[10] I.e. incompressible in $\mathcal{M}_\eta$ and non-isotopic to any connected component of $\partial\mathcal{M}_\eta$.

[11] When S is not irreducible, i.e. $r > 1$, we can argue directly in homology because in this case $\pi_1(W_k^*) \cong H_1(W_k^*; \mathbb{Z}) \cong \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^r \cong H_1(T_\eta^*; \mathbb{Z})$. This last inclusion follows from (11) because $f_\ell \circ E$ vanish on D_k, for $\ell = 1, \ldots, r$.

[12] This follows trivially from the exact long sequence associated to the Milnor fibration.

[13] I.e. the decomposition $P = \mathbb{Z}\mathfrak{m}_P \oplus \mathbb{Z}\mathfrak{p}_P$ of each conjugated subgroup $P := \zeta \mathcal{P}_k \zeta^{-1}$ given by $\mathfrak{m}_P := \zeta \mathfrak{m}_k \zeta^{-1}$ and $\mathfrak{p}_P := \zeta \mathfrak{p}_k \zeta^{-1}$ does not depend on ζ.

$\zeta'\mathcal{P}_k\zeta'^{-1} = \pi_1(F) \cap \mathcal{P}_k = \mathbb{Z}\mathfrak{b}_k$. Hence $\zeta'\mathfrak{b}_k\zeta'^{-1} = \mathfrak{b}_k^n$ for some $n \in \mathbb{Z}$. By passing this last equality to the homology we obtain that $n = 1$ and consequently $[\zeta', \mathfrak{b}_k] = 1$. This equality can be thought as a relation inside the free group $\pi_1(F)$. Since the subgroup $\langle \zeta', \mathfrak{b}_k \rangle$ of $\pi_1(F)$ is also free by Schreier's classical result, we deduce that it is monogeneous, i.e. $\langle \zeta', \mathfrak{b}_k \rangle = \langle \theta \rangle$ for some $\theta \in \pi_1(F) = \langle u_1, v_1, \ldots, u_g, v_g, b_1, \ldots, b_r | \prod_{i=1}^g [u_i, v_i] \prod_{j=1}^r b_j = 1 \rangle$, where $b_j \subset \partial F$. We can assume that $\mathfrak{b}_k = b_1$. It suffices to prove that b_1 cannot be a non-trivial power in $\pi_1(F)$ because in that case $\zeta' \in \langle \theta \rangle = \langle \mathfrak{b}_k \rangle \subset \mathcal{P}_k$. If $r > 1$ then b_1 belongs to a free system of generators $u_1, v_1, \ldots, u_g, v_g, b_1, \ldots, b_{r-1}$ of $\pi_1(F)$; it cannot be then a non-trivial power of another element. If $r = 1$, then $b_1^{-1} = \prod_{i=1}^g [u_i, v_i]$ is a cyclicly reduced word in the free group $\pi_1(F) = \langle u_1, v_1, \ldots, u_g, v_g | \rangle$; it is easy to see in that case that it cannot be a non-trivial power. □

Theorem 3.16. *Let U be an open neighbourhood of 0 in $\mathbb{B}$ and let h be a homeomorphism*[14] *from U onto a neighbourhood U' of 0 in $\mathbb{B}'$, such that $h(S \cap U) = S' \cap U'$. Assume that the inclusion $U^* \subset \mathbb{B}^*$ induces an isomorphism $\pi_1(U^*) \cong \Gamma$. Then, for each irreducible component S_k of S the isomorphism $h_* : \Gamma \to \Gamma'$ induced by h transforms $\mathcal{P}_k$ onto the subgroup $\mathcal{P}'_k$ associated to the irreducible component $S'_k = h(S_k \cap U)$ of $S' \cap U'$ and sends meridian into meridian, i.e. $h_*(\mathfrak{m}_k) = \mathfrak{m}'_k$.*

Proof. Consider tubular neighbourhoods W_k of $S_k \cap (\mathbb{B}_r \setminus \mathring{\mathbb{B}}_s)$ and $W''_k \subset W'_k$ of $S'_k \cap (\mathbb{B}'_{r'} \setminus \mathring{\mathbb{B}}'_{s'})$ contained in U and U' respectively such that $W''_k \subset h(W_k) \subset W'_k$, $\mathcal{P}_k = \pi_1(W_k^*)$ and $\pi_1(W''^*_k) = \pi_1(W'^*_k) = \mathcal{P}'_k$ via the inclusion $W''^*_k \subset W'^*_k$. Thus, we have that $h_*(\mathcal{P}_k) \subset \mathcal{P}'_k$ and we deduce that the composition $\mathcal{P}'_k \to h_*(\mathcal{P}_k) \to \mathcal{P}'_k$ is an isomorphism. Therefore $h_*(\mathcal{P}_k) = \mathcal{P}'_k$ and the restriction of h_* to $\mathcal{P}_k \cong \mathbb{Z}^2$ is onto over $\mathcal{P}'_k \cong \mathbb{Z}^2$. Since every epimorphism of $\mathbb{Z}^2$ onto itself is also one-to-one, we deduce that $h_* : \mathcal{P}_k \to \mathcal{P}'_k$ is an isomorphism. In the same way, we have that $h_* : \pi_1(W_k) \to \pi_1(W_{k'})$ is also an isomorphism. Thus, h_* conjugate the kernels of the morphisms induced by the inclusions $W_k^* \subset W_k$ and $W'^*_k \subset W'_k$, which are generated by $\mathfrak{m}_k$ and $\mathfrak{m}'_k$ respectively. We obtain that $h_*(\mathfrak{m}_k) = \mathfrak{m}'^{\pm 1}_k$; but the exponent must be $+1$ because h preserves the orientations. □

In the statement of the previous theorem, once k is given, we have arbitrarily chosen the base points of Γ and Γ' in W_k and W''_k respectively. We would like to have a more intrinsic notion of the morphism h_* independent on that choices. To this end, we go back to the notion of fundamental equivalence introduced in Section 2.6. Notice that the ambiguity of the action of $h_* : \Gamma \to \Gamma'$ is controlled by the left and/or right composition of h_* by inner automorphisms. This leads to the well-known notion of *exterior isomorphism*, as an equivalence class of an isomorphism $\Gamma \to \Gamma'$ modulo composition by inner automorphisms. Now, using Proposition 3.15, we can define the following notion.

Definition 3.17. We will say that an exterior isomorphism $\varphi : \Gamma \to \Gamma'$ *preserves the peripheral structures* if it sends all the subgroups conjugated to $\mathcal{P}_k$ onto subgroups conjugated to $\mathcal{P}'_{k'}$. Furthermore, the isomorphism φ is called *geometric* if moreover it sends all the conjugates of the meridians $\mathfrak{m}_k$ into conjugates of the meridians $\mathfrak{m}'_{k'}$.

Remark 3.18. Theorem 3.16 asserts that if $h : (U, S) \to (U', S')$ is a germ of homeomorphism then $h_* : \Gamma \to \Gamma'$ is a geometric isomorphism. The first half of the proof of Theorem 3.16 implies that if $h : U^* \to U'^*$ is a homeomorphism, then $h_* : \Gamma \to \Gamma'$ preserves the peripheral structure; however, it could be not geometric as the following example shows: $U = U' = \mathbb{C}^2$, $S = S' = \{xy = 0\}$ and $h : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*$ defined by $h(x, y) = (xy, y)$.

We recall here a very important result of F. Waldhausen [22, Corollary 6.5]:

Theorem 3.19. *Let M and M' be 3-manifolds which are irreducible and boundary-irreducible. Suppose M is sufficiently large. Let $\varphi : \pi_1(M) \to \pi_1(M')$ be an isomorphism which respects the peripheral structure, i.e. for each connected component F of ∂M, there exists a connected component F' of $\partial M'$, such that $\varphi(\pi_1(F))$ is conjugated to $\pi_1(F')$. Then there exists a homeomorphism $\phi : M \to M'$ inducing φ in homotopy, i.e. $\varphi = \phi_*$.*

Corollary 3.20. *If $\varphi : \Gamma \to \Gamma'$ is an isomorphism which respects the peripheral structure then there exists a homeomorphism $h : T_\eta^* \to T'^*_\eta$, such that $h_* = \varphi : \pi_1(T_\eta^*) \to \pi_1(T'^*_\eta)$. If in addition φ is geometric, then h extends to a homeomorphism from T_η onto T'_η, such that $h(S) = S'$. Thus, every geometric isomorphism is induced by a (unique) marking.*

Proof. We can apply Waldhausen's theorem to the isomorphism $\varphi : \Gamma \cong \pi_1(M_\eta) \to \pi_1(M'_\eta) \cong \Gamma'$, because M_η and $M_{\eta'}$ are irreducible after Remark 2.4 with non-empty incompressible boundary (so that they are sufficiently large) thanks to Proposition 3.14. Hence, there exists a homeomorphism $\phi : M_\eta \to M_{\eta'}$, which extends trivially to $h : T_\eta^* \to T^*_{\eta'}$, via the product structures $T_\eta^* \cong M_\eta \times]0, \eta]$ and $T'^*_\eta \cong M'_\eta \times]0, \eta]$ given by (6). On the other hand, if φ conjugate the meridians of the boundary tori of M_η and M'_η, then ϕ extends to a homeomorphism from $T_\eta \cap \partial\mathbb{B}$ onto $T'_\eta \cap \partial\mathbb{B}'$. By using the conical structure of S and S', it is easy to extend ϕ to a homeomorphism of pairs $h : (T_\eta, S) \to (T'_\eta, S')$. □

[14] Preserving the orientations, as usual.

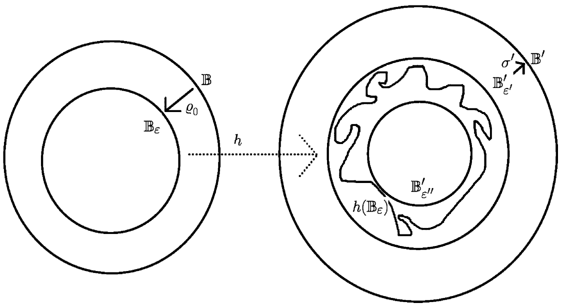

Fig. 3. The continuous map h_1 defined from h and the retractions ϱ_0 and σ'.

4. Proof of the main theorem

Given a homeomorphism $h : \mathbb{B}_\varepsilon \xrightarrow{\sim} h(\mathbb{B}_\varepsilon) \subset \mathbb{B}'$ such that $h(S \cap \mathbb{B}_\varepsilon) = S' \cap h(\mathbb{B}_\varepsilon)$, in the first section of this chapter we construct a map $\breve{h}_1$ from M_η into $M'_{\eta'}$, for $0 < \eta \ll \eta' \ll 1$, which is fundamentally equivalent to h. Thanks to the results of Waldhausen, we shall modify this map by a homotopy, in order to obtain a homeomorphism h_2 between Milnor 3-tubes of S and S'.

In the next section, we will use the classical result of Jaco–Shalen–Johannson to give an isotopy between h_2 and a new homeomorphism h_3 which preserves some precise realisations of the JSJ decomposition of the Milnor 3-tubes. Nevertheless, it is worthwhile to recall that Waldhausen's theory is enough for the 3-manifolds appearing in singularity theory.

Next, in Section 4.3, we shall construct an explicit isomorphism between the dual trees of the minimal desingularisations of S and S'.

This allows us to extend h_3 to the Milnor 4-tubes in the next section. This extension to four dimensions will be done in four steps: In the first one, we only deal with the blocks $\mathcal{T}_\eta(D)$ associated to the components D of valence $\geqslant 3$. Secondly, we treat the case of chains $\mathcal{C}$ of components of valence 2, by using the product structure of the blocks $\mathcal{T}_\eta(\mathcal{C})$. Next, we consider the case of dead branches and that of the strict transforms of the original curves. Finally, in the last step, we will modify the constructed homeomorphism by suitable isotopies in order to assure that it is fundamentally equivalent to the initial homeomorphism h.

4.1. Reduction to dimension three

Without lost of generality we can assume that the radii of the Milnor balls $\mathbb{B}$ and $\mathbb{B}'$ for S and S' are both equal to 1. We fix $\varepsilon > 0$ and $0 < \varepsilon'' < \varepsilon' < 1$ small enough such that

$$\mathbb{B}'_{\varepsilon''} \subset h(\mathbb{B}_\varepsilon) \subset \mathbb{B}'_{\varepsilon'}.$$

Endow the pair $(\mathbb{B}, S)$ with a *conic structure*, i.e. a diffeomorphism $\varphi : \partial\mathbb{B} \times [0,1] \to \mathbb{B}$ outside the origin, satisfying $\varphi(\partial\mathbb{B} \times \{r\}) = \partial\mathbb{B}_r$, for all $r \in]0,1]$, $\varphi((S \cap \partial\mathbb{B}) \times [0,1]) = S$ and $\varphi(m,0) = 0$, $\varphi(m,1) = m$, for all $m \in \partial\mathbb{B}$. We also have a conic structure $\varphi' : \partial\mathbb{B}' \times [0,1] \to \mathbb{B}'$ for the pair $(\mathbb{B}', S')$. Denote by $\varrho_0 : \mathbb{B} \to \mathbb{B}_\varepsilon$ the retraction by deformation which corresponds to the usual radial retraction replacing the standard radii by the ones given by the conic structure. In other words, ϱ_0 conjugated by φ writes as $(m,t) \mapsto (m,\varepsilon)$, for $\varepsilon \leqslant t \leqslant 1$ and $(m,t) \mapsto (m,t)$ for $0 \leqslant t \leqslant \varepsilon$. Denote also by $\sigma'_0 : (\mathbb{B}' \setminus \{0\}) \to \partial\mathbb{B}'$ the retraction by deformation corresponding, via φ', to the map $(m,s) \mapsto (m,1)$. Finally, we denote by $\sigma' : \mathbb{B}' \to \mathbb{B}'$, the continuous map corresponding, via φ', to the map $(m,t) \mapsto (m, \varsigma(t))$, where $\varsigma(t)$ is affine for $\varepsilon'' \leqslant t \leqslant \varepsilon'$ and satisfies $\varsigma(t) = t$, for $t \leqslant \varepsilon''$ and $\varsigma(t) = 1$, for $t \geqslant \varepsilon'$. Clearly we have that

$$\varrho_0^{-1}(S) = S, \qquad \sigma'\bigl(S'\bigr) = S', \qquad \sigma'^{-1}\bigl(S'\bigr) = S'.$$

Notice that ϱ_0 and σ' are the identity in a neighbourhood of the origin and that σ' coincides with σ'_0 outside of $\mathbb{B}'_{\varepsilon'}$. We define

$$h_1 := \sigma' \circ h \circ \varrho_0 : \mathbb{B} \to \mathbb{B}'.$$

This map is continuous, not necessarily surjective neither injective, and satisfies $h_1(\partial\mathbb{B}) = \partial\mathbb{B}'$, $h_1(S) = S'$ and $h_1^{-1}(S') = S$. Hence, it defines a map from $\mathbb{B}^*$ into $\mathbb{B}'^*$. On the other hand, h_1 coincides with h in a neighbourhood of the origin, and consequently the restrictions of h and h_1 to $\mathbb{B}^*_\varepsilon$ are fundamentally equivalent, i.e. $h \asymp h_1$ (see Fig. 3).

We fix now Milnor 4-tubes $T_\eta \subset \mathbb{B}$ for S and $T'_{\eta'} \subset \mathbb{B}'$ for S', such that $h_1(T_\eta) \subset T'_{\eta'}$. We denote by $r : T^*_\eta \to M_\eta$ the retraction by deformation over the Milnor 3-tube (5), given by the product structure described in Proposition 2.3 and we denote by $r' : T'^*_{\eta'} \to M'_{\eta'} := f'^{-1}(\partial\mathbb{D}_{\eta'}) \cap \mathbb{B}'$ the similar retraction corresponding to S'. We put

$$\breve{h}_1 := r' \circ h_1 \circ \iota_{M_\eta} : M_\eta \to M'_{\eta'},$$

where $\iota_{M_\eta} : M_\eta \hookrightarrow T^*_\eta$ denotes the inclusion map.

Remark 4.1. Clearly we have $h \asymp h_1 \asymp \breve{h}_1$.

By multiplying the equation f' by $\frac{\eta}{\eta'}$, we shall assume that $\eta = \eta'$. From now on we will identify T_η to $\mathcal{T}_\eta$, T'_η to $\mathcal{T}'_\eta$ and we continue to denote by $\breve{h}_1$ the map $E'^{-1} \circ \breve{h}_1 \circ E$ defined on $\mathcal{M}_\eta := E^{-1}(M_\eta)$ taking values on $\mathcal{M}'_{\eta'} := E'^{-1}(M'_{\eta'})$. This map satisfies the hypotheses of Theorem 6.1 of Waldhausen [22], because $\breve{h}_1(\partial\mathcal{M}_\eta) \subset \partial\mathcal{M}'_{\eta'}$. The thesis of that theorem is a dichotomy: either $\mathcal{M}_\eta$ is the total space of a line bundle over a closed Riemann surface, or $\breve{h}_1$ is homotopic to a covering map. Since the first situation do not occur in our case and the morphism induced by $\breve{h}_1$ on the fundamental groups is surjective, there exists a homotopy $F : \mathcal{M}_\eta \times [0,1] \to \mathcal{M}'_{\eta'}$ satisfying $F(\partial\mathcal{M}_\eta \times [0,1]) \subset \partial\mathcal{M}'_{\eta'}$, $F(\cdot,0) = \breve{h}_1$ and such that $F(\cdot,1)$ is a homeomorphism. We put

$$h_2 := F(\cdot,1) : \mathcal{M}_\eta \xrightarrow{\sim} \mathcal{M}'_\eta.$$

Remark 4.2. The relation $h_2 \asymp \breve{h}_1$ is verified.

Recall that one of our goals is to construct a homeomorphism $\mathcal{T}_\eta \to \mathcal{T}'_\eta$ fundamentally equivalent to the original homotopy equivalence.

4.2. Construction of a JSJ compatible homeomorphism

Consider the analogous JSJ decomposition of $\mathcal{M}'_{\eta'}$ to that we have already described for $\mathcal{M}_\eta$. We keep the notations (12) for the elementary blocks of $\mathcal{M}'_{\eta'}$; we denote by $\mathfrak{R}'$ the collection of irreducible components of $\mathcal{D}'$ having $\geqslant 3$ and by $\mathfrak{C}'$ the collection of the chains of components of $\mathcal{D}'$ joining two elements of $\mathfrak{R}'$; for each $\mathcal{C}' \in \mathfrak{C}'$, we consider the thick tori $\mathcal{M}'_{\mathcal{C}'}$ and $\widetilde{\mathcal{M}}'_{\mathcal{C}'}$, as well as its product structures $\sigma'_{\mathcal{C}'} : \widetilde{\mathcal{M}}'_{\mathcal{C}'} \xrightarrow{\sim} \mathbb{T} \times [-1-\epsilon, +1+\epsilon]$ constructed as in (16); the 2-torus $\mathbb{T}'_{\mathcal{C}'} = \sigma'^{-1}_{\mathcal{C}'}(\mathbb{T} \times \{0\})$ is properly embedded in $\mathcal{M}'_{\eta'}$ and the adherences of the connected components of $\mathcal{M}'_{\eta'} \setminus \bigcup_{\mathcal{C}' \in \mathfrak{C}'} \mathbb{T}'_{\mathcal{C}'}$ constitute the *JSJ blocks* of $\mathcal{M}'_{\eta'}$; each of them is denoted by $B'_{D'}$, because it contains a unique elementary block $\mathcal{M}'_{D'}$ with $v(D') \geqslant 3$; we consider an extended Seifert fibration $\widehat{\rho}'^{\,\mathrm{ext}} : B'_{D'} \to \widehat{K}'^{\,\mathrm{ext}}_{D'}$, defined as in (20), which prolongs the Hopf fibration $\rho'_D : \mathcal{M}'_{D'} \to K'_{D'}$; finally the collection $(\mathbb{T}'_{\mathcal{C}'})_{\mathcal{C}' \in \mathfrak{C}'}$ is, by the same reasons, a characteristic family of essential tori properly embedded in $\mathcal{M}'_{\eta'}$.

Clearly $(\mathbb{T}_\mathcal{C})_{\mathcal{C} \in \mathfrak{C}}$ and $(h_2^{-1}(\mathbb{T}'_{\mathcal{C}'}))_{\mathcal{C}' \in \mathfrak{C}'}$ are two characteristic families of essential tori of $\mathcal{M}_\eta$. After the unicity theorem of characteristic families, cf. [20, (1.2.6)], there exists a bijection

$$\kappa_2 : \mathfrak{C} \xrightarrow{\sim} \mathfrak{C}'$$

and a homeomorphism ψ from $\mathcal{M}_\eta$ onto itself, isotopic to the identity, such that $h_2(\psi(\mathbb{T}_\mathcal{C})) = \mathbb{T}'_{\kappa_2(\mathcal{C})}$ for each $\mathcal{C} \in \mathfrak{C}$. Putting $\widetilde{h}_2 := h_2 \circ \psi$ we have that

$$\widetilde{h}_2 \asymp h_2 \asymp h \quad \text{and} \quad \widetilde{h}_2(\mathbb{T}_\mathcal{C}) = \mathbb{T}'_{\kappa_2(\mathcal{C})}, \quad \text{for each } \mathcal{C} \in \mathfrak{C}.$$

Remark 4.3. Clearly $\widetilde{h}_2$ transforms every JSJ block of $\mathcal{M}_\eta$ into a JSJ block $\mathcal{M}'_{\eta'}$, defining thus a (unique) bijection $\kappa_3 : \mathfrak{R} \xrightarrow{\sim} \mathfrak{R}'$ such that $\widetilde{h}_2(B_D) = B'_{\kappa_3(D)}$.

Lemma 4.4. *There exists a homeomorphism $\breve{h}_2$ isotopic to $\widetilde{h}_2$, conjugating the family of thick tori and preserving their product structures. More precisely, $\breve{h}_2(\mathcal{M}_\mathcal{C}) = \mathcal{M}'_{\kappa_2(\mathcal{C})}$ and $\sigma'_{\kappa_2(\mathcal{C})} \circ \breve{h}_2 = \sigma_\mathcal{C}$.*

Proof. Consider $\mathcal{C} \in \mathfrak{C}$ and a JSJ block B_D of $\mathcal{M}_\eta$ such that $\mathbb{T}_\mathcal{C} \subset \partial B_D$. The torus $\mathbb{T}'_{\mathcal{C}'}$, $\mathcal{C}' := \kappa_2(\mathcal{C})$, is a connected component of $\partial B'_{\kappa_3(D)}$. We can assume that $\partial B^\flat_D \supset \sigma_\mathcal{C}^{-1}(\mathbb{T} \times \{1\})$ and $\partial B'^\flat_{\kappa_3(D)} \supset \sigma'^{-1}_{\mathcal{C}'}(\mathbb{T} \times \{1\})$. The homeomorphism r_s from $\mathbb{T} \times [0, 1+\epsilon]$ onto $\mathbb{T} \times [s, 1+\epsilon]$, defined by

$$r_s(p,t) := \left(p, s + t\frac{1+\epsilon-s}{1+\epsilon}\right), \quad s \in [0,1],$$

lifts to a homeomorphism from $\sigma_\mathcal{C}^{-1}(\mathbb{T} \times [0, 1+\epsilon])$ onto $\sigma_\mathcal{C}^{-1}(\mathbb{T} \times [s, 1+\epsilon])$, which extends as the identity to a homeomorphism

$$R_s : B_D \xrightarrow{\sim} B_D(s) := B_D \setminus \sigma_\mathcal{C}^{-1}\bigl(\mathbb{T} \times [0, s[\bigr), \quad R_{s|B^\flat_D} = \mathrm{id}_{B^\flat_D}.$$

A similar homeomorphism R'_s from $B'_{\kappa_3(D)}$ onto

$$B'_{\kappa_3(D)}(s) := B'_{\kappa_3(D)} \setminus \sigma'^{-1}_{\mathcal{C}'}\bigl(\mathbb{T} \times [0, s[\bigr)$$

can be constructed. For all $s \in [0,1]$, we denote by $F_s : \mathcal{M}_\eta \to \mathcal{M}'_{\eta'}$ the map defined by

$$\begin{cases} F_s(m) = \breve{h}_2(m), & \text{if } m \notin B_D, \\ F_s(m) := R'_s \circ \breve{h}_2 \circ R_s^{-1}(m), & \text{if } m \in B_D(s), \\ F_s(m) := \sigma'^{-1}_{\mathcal{C}'} \circ (H_2 \times \mathrm{id}_{[0,s]}) \circ \sigma_{\mathcal{C}}(m), & \text{if } m \in \sigma_{\mathcal{C}}^{-1}\big(\mathbb{T} \times [0,s]\big), \end{cases}$$

where $H_2(p) := \sigma'_{\mathcal{C}'}(\breve{h}_2(\sigma_{\mathcal{C}}^{-1}(p,0)))$. Clearly F_s is an isotopy between $F_0 = \breve{h}_2$ and $F_1(\mathcal{M}_{\mathcal{C}} \cap B_D) = \mathcal{M}'_{\mathcal{C}'} \cap B'_{\kappa_3(D)}$. We conclude the proof by making a successive composition of such isotopies, one for each JSJ block of $\mathcal{M}_\eta$. □

Lemma 4.5. *There exists a homeomorphism h_3 isotopic to $\breve{h}_2$, satisfying the same properties as $\breve{h}_2$ in Lemma 4.4 and conjugating the Seifert fibrations of the complements of the thick tori, i.e. there are homeomorphisms $\varsigma_D : \widehat{K}_D \xrightarrow{\sim} \widehat{K}'_{\kappa_3(D)}$, $D \in \mathfrak{R}$, such that $\widehat{\rho}'_{\kappa_3(D)} \circ h_{3|B_D^\flat} = \varsigma_D \circ \widehat{\rho}_D$.*

Proof. Clearly $B_D^\flat$ is endowed with two Seifert fibrations [18]: $\widehat{\rho}_D$ over $\widehat{K}_D$ and $\widehat{\rho}'_{\kappa_3(D)} \circ \breve{h}_{2|B_D^\flat}$ over $\widehat{K}'_{\kappa_3(D)}$. Since $B_D^\flat$ is not a solid torus neither a thick torus, the unicity for Seifert fibrations of [20, Theorem 1.2.5] (see also [21, Satz 10.1]), gives us an isotopy $\psi_{D,s} : B_D^\flat \to B_D^\flat$, $s \in [0,1]$, such that $\psi_{D,0}$ is the identity and $\psi_{D,1}$ conjugates the foliations defined by these two fibrations. More precisely, if $\varsigma_D : \widehat{K}_D \to \widehat{K}'_{\kappa_3(D)}$ is the homeomorphism induced by $\psi_{D,1}$ over the leaf spaces, we have that $\widehat{\rho}'_{\kappa_3(D)} \circ \breve{h}_2 \circ \psi_{D,1} = \varsigma_D \circ \widehat{\rho}_D$. Thanks to Assertion (a) of Lemma 4.6 below, these isotopies can be glued into a global isotopy $\psi_s : \mathcal{M}_\eta \to \mathcal{M}_\eta$, satisfying $\psi_0 = \mathrm{id}_{\mathcal{M}_\eta}$, $\psi_{s|B_D^\flat} = \psi_{D,s}$, $D \in \mathfrak{R}$ and $\psi_{s|\mathbb{T}_{\mathcal{C}}} = \mathrm{id}_{\mathbb{T}_{\mathcal{C}}}$, $\mathcal{C} \in \mathfrak{C}$. We conclude by defining $h_3 = \breve{h}_2 \circ \psi_1$. □

Although the following result about extension of isotopies is classical, we include here the precise statement that we need with a short proof.

Lemma 4.6. *Let B be a manifold with boundary and let $B^\flat \subset B$ be a sub-manifold with boundary having the same dimension, such that there exists a homeomorphism σ from $\overline{B \setminus B^\flat}$ onto $\partial B \times [0,1]$, satisfying $\sigma(\partial B) = \partial B \times \{1\}$ and $\sigma(\partial B^\flat) = \partial B \times \{0\}$. Then*

(a) *if $F_s : B^\flat \to B^\flat$, $s \in [0,1]$, is an isotopy such that $F_0 = \mathrm{id}_{B^\flat}$, then there is an isotopy $F'_s : B \to B$ such that $F'_{s|B^\flat} = F_s$ and $F'_{s|\partial B} = \mathrm{id}_{\partial B}$, $s \in [0,1]$;*
(b) *if $G_s : \partial B \to \partial B$, $s \in [0,1]$, is an isotopy such that $G_0 = \mathrm{id}_{\partial B}$, then there exists an isotopy $G'_s : B \to B$ such that $G'_{s|B^\flat} = \mathrm{id}_{B^\flat}$ and $G'_{s|\partial B} = G_s$, $s \in [0,1]$.*

Proof. (a) Let us denote $\widetilde{F}_s := \sigma \circ F_s \circ \sigma^{-1}_{|\partial B^\flat}$. For $m \in \overline{B \setminus B^\flat}$, we define $F'_s(m) := \sigma^{-1} \circ \widetilde{F}'_s \circ \sigma(m)$, with $\widetilde{F}'_s(p,t) := \widetilde{F}_{s-t}(p,t)$, if $0 \leqslant t \leqslant s$ and $\widetilde{F}'_s(p,t) := (p,t)$, if $s \leqslant t \leqslant 1$. The proof of (b) is analogous. □

For each $D \in \mathfrak{R}$, the homeomorphism ς_D given by Lemma 4.5, induces a bijection ϖ_D between the singular points of $\mathcal{D}$ lying on D and those of $\mathcal{D}'$ lying on $D' := \kappa_3(D)$. With the notations (19), it verifies

$$\varpi_D\big(S_{\mathfrak{M}}(D)\big) = S_{\mathfrak{M}}\big(D'\big) \quad \text{and consequently} \quad \varpi_D\big(\widehat{S}(D)\big) = \widehat{S}\big(D'\big),$$

because the attaching points of the dead branches correspond to the exceptional fibres of the Seifert fibrations and the elements of $\widehat{S}(D)$, respectively $\widehat{S}(D')$, correspond to the connected components of $\partial\widehat{K}_D$, respectively $\partial\widehat{K}'_{D'}$. It is easy to see that, up to modifying h_3 by an isotopy, ς_D sends the disk D_s, $s \in S_{\mathfrak{M}}(D)$, onto the disk $D'_{\varpi_D(s)}$. Hence h_3 sends the elementary block $\mathcal{M}_D$ onto the elementary block $\mathcal{M}'_{D'}$, and it conjugates the corresponding Hopf fibrations.

4.3. Conjugation of the dual trees of the divisors

We summarise the results that we are obtained so far. We have constructed a homeomorphism $h_3 : \mathcal{M}_\eta \to \mathcal{M}'_{\eta'}$ such that $h_3 \asymp h$, as well as bijections

$$\kappa_3 : \mathfrak{R} \to \mathfrak{R}, \qquad \kappa_2 : \mathfrak{C} \to \mathfrak{C}', \qquad \kappa_1 : \mathfrak{M} \to \mathfrak{M}', \tag{21}$$

satisfying for all $\mathcal{C} \in \mathfrak{C}$ and $\widetilde{\mathcal{C}} \in \mathfrak{M}$ the following properties:

(a) the image by κ_3 of the extremal components of $\mathcal{C}$, are the extremal components of $\kappa_2(\mathcal{C})$;
(b) if $D_0 \in \mathfrak{R}$ is the attaching component of $\widetilde{\mathcal{C}}$, then $\kappa_3(D_0)$ is the attaching component of $\kappa_1(\mathcal{C})$;
(c) the homeomorphism h_3 sends $\mathcal{M}_{\mathcal{C}}$ onto $\mathcal{M}'_{\kappa_2(\mathcal{C})}$, $\mathcal{M}_{\widetilde{\mathcal{C}}}$ onto $\mathcal{M}'_{\kappa_1(\widetilde{\mathcal{C}})}$ and $\mathbb{T}_{\mathcal{C}}$ onto $\mathbb{T}'_{\kappa_2(\mathcal{C})}$; for each $D \in \mathfrak{R}$, it also transforms B_D into $B'_{\kappa_3(D)}$, $B_D^{\prime\flat}$ into $B'^{\flat}_{\kappa_3(D)}$ and $\mathcal{M}_D$ into $\mathcal{M}'_{\kappa_3(D)}$, conjugating, in addition, the corresponding Seifert fibrations.

The following proposition extends the one-to-one correspondences (21) to all the components of the divisors. It makes precise the classical results of Zariski–Lejeune, giving, by means of the property (c), the relationship between the homeomorphism h conjugating S to S' and the one-to-one correspondence between the dual trees of $\mathcal{D}$ and $\mathcal{D}'$. It should be also mentioned that using plumbing calculus, Neumann [14] proves unicity of the graph under negativity conditions on the intersection form.

Proposition 4.7. *There exists a one-to-one correspondence $\kappa : \mathrm{Comp}(\mathcal{D}) \to \mathrm{Comp}(\mathcal{D}')$ between the sets of irreducible components of $\mathcal{D}$ and $\mathcal{D}'$, such that*:

(1) *it is compatible with the intersection numbers, i.e. $(\kappa(D), \kappa(D')) = (D, D')$, for each $D, D' \in \mathrm{Comp}(\mathcal{D})$;*
(2) *for all $\mathcal{C} \in \mathfrak{C}$, respectively $\widetilde{\mathcal{C}} \in \mathfrak{M}$, we have the equivalence: $(D \in \mathcal{C}) \Leftrightarrow (\kappa(D) \in \kappa_2(\mathcal{C}))$, respectively $(D \in \widetilde{\mathcal{C}}) \Leftrightarrow (\kappa(D) \in \kappa_1(\widetilde{\mathcal{C}}))$; in particular, $\mathcal{C}$ and $\kappa_2(\mathcal{C})$ have the same length, as well as $\widetilde{\mathcal{C}}$ and $\kappa_1(\widetilde{\mathcal{C}})$;*
(3) *the restriction of κ to $\mathfrak{R} \subset \mathrm{Comp}(\mathcal{D})$ coincides with κ_3.*

In particular, the properties (a), (b) *and* (c) *above are verified by κ.*

Before proving Proposition 4.7 we need an auxiliary result. Fix $\mathcal{C} \in \mathfrak{C} \cup \mathfrak{M}$.

- If $\mathcal{C} = \{D_0, \ldots, D_{l_\mathcal{C}+1}\} \in \mathfrak{C}$, we consider the chain
$$\mathcal{C}' = \{D'_0, \ldots, D'_{l_{\mathcal{C}'}+1}\} = \kappa_2(\mathcal{C}) \in \mathfrak{C}',$$
which we will numerate so that $D'_0 = \kappa_3(D_0)$ and $D'_{l_{\mathcal{C}'}+1} = \kappa_3(D_{l_\mathcal{C}+1})$.
- If $\mathcal{C} = \{D_0, \ldots, D_{l_\mathcal{C}}\} \in \mathfrak{M}$, we consider $\mathcal{C}' = \{D'_0, \ldots, D'_{l_{\mathcal{C}'}}\} = \kappa_2(\mathcal{C}) \in \mathfrak{M}'$.

We denote by $\mathfrak{c}_j \in H_1(\mathcal{M}_\mathcal{C}, \mathbb{Z})$ the meridian associated to D_j and by $\mathfrak{c}'_k \in H_1(\mathcal{M}'_{\mathcal{C}'}, \mathbb{Z})$ that one associated to D'_k, cf. Definition 3.5. The equality $h_3(\mathcal{M}_\mathcal{C}) = \mathcal{M}'_{\mathcal{C}'}$ induces an isomorphism
$$h_{3*} : H_1(\mathcal{M}_\mathcal{C}, \mathbb{Z}) \to H_1(\mathcal{M}'_{\mathcal{C}'}, \mathbb{Z}).$$
In the case that $\mathcal{C}$ and $\mathcal{C}'$ are dead branches, we denote by $\mathfrak{c}_{l_\mathcal{C}+1}$ and $\mathfrak{c}'_{l_{\mathcal{C}'}+1}$ the corresponding exceptional meridians. Since h_3 conjugates the Seifert fibrations of B_{D_0} and $B'_{D'_0}$, it conjugates also the exceptional fibres. Hence h_3 transforms $\mathcal{M}^\circ_\mathcal{C}$ into $\mathcal{M}'^\circ_{\mathcal{C}'}$ and it induces an isomorphism
$$h_{3*} : H_1(\mathcal{M}^\circ_\mathcal{C}, \mathbb{Z}) \to H_1(\mathcal{M}'^\circ_{\mathcal{C}'}, \mathbb{Z}).$$

Lemma 4.8. *For each $j = 0, \ldots, l+1$, we have the equalities $l_\mathcal{C} = l_{\mathcal{C}'} =: l$ and $h_{3*}(\mathfrak{c}_j) = \mathfrak{c}'_j \in H_1^{\mathcal{C}'}$.*

First, we will prove the following technical result[15]:

Sub-Lemma 4.9. *Fix $a = (\alpha_1, \alpha_2)$ and $b = (\beta_1, \beta_2) \in \mathbb{Z}^2$, $\gcd(\alpha_1, \alpha_2) = 1$, $\gcd(\beta_1, \beta_2) = 1$ such that $\det(a, b) > 0$. Then there is a unique $n \in \mathbb{N}$ and a unique ordered collection $\mathbf{c} := (c_0, \ldots, c_{n+1})$ of elements of $\mathbb{Z}^2 \cap (\mathbb{Q}a + \mathbb{Q}b) \subset \mathbb{Q}^2$ such that*
$$\begin{cases} \det(c_j, c_{j+1}) = 1, & j = 0, \ldots, n, \\ \det(c_{k-1}, c_{k+1}) > 1, & k = 1, \ldots, n, \\ c_0 = a, & c_{n+1} = b. \end{cases} \tag{22}$$
In particular, if $\det(a, b) = 1$, the unique collection $\mathbf{c}$ satisfying (22) *is given by $n = 0$, $c_0 = a$ and $c_1 = b$.*

Proof. The existence of n and c_j follows easily by a standard argument concerning continuous fractions. To prove the unicity, we will use the following straightforward assertion:

($\diamond$) *For each $u := (\nu_1, \nu_2)$, $v := (\upsilon_1, \upsilon_2) \in \mathbb{Z}^2$ and $\lambda, \mu \in \mathbb{Q}_{>0}$, such that $\det(u, v) > 0$, $\nu_1 + \nu_2 > 1$, $\upsilon_1 + \upsilon_2 > 1$ and $(1, 1) = \lambda u + \mu v$, we have that $\det(u, v) > 1$.*

Indeed, clearly $0 < \lambda, \mu < 1$; hence $(1, 1)$ belongs to the interior of the parallelogram having vertices 0, u, $u + v$, v; but this is impossible when u and v span $\mathbb{Z}^2$ over $\mathbb{Z}$. To prove the unicity we will make a double recurrence with the following induction hypothesis $\mathcal{H}_{N,N'}$:

[15] We thank to Mark Spivakovski for his help and his suggestions concerning the proof of this lemma.

“if $\mathbf{c} := (c_j)_{j=0}^{n+1}$ and $\mathbf{c}' := (c'_k)_{k=0}^{n'+1}$ are two finite sequences of elements of $\mathbb{Z}^2 \cap (\mathbb{Q}a + \mathbb{Q}b)$ satisfying (22), and if $0 \leqslant n \leqslant N$ and $0 \leqslant n' \leqslant N'$, then $n = n'$ and $\mathbf{c} = \mathbf{c}'$”.

First, we shall see that $\mathcal{H}_{0,N} \Rightarrow \mathcal{H}_{0,N+1}$; by symmetry we will also have $\mathcal{H}_{N,0} \Rightarrow \mathcal{H}_{N+1,0}$; since $\mathcal{H}_{0,0}$ is obvious, it will suffice then to prove the implication $\mathcal{H}_{N-1,N'-1} \Rightarrow \mathcal{H}_{N,N'}$.

$\mathcal{H}_{0,N'} \Rightarrow \mathcal{H}_{0,N'+1}$: Up to an automorphism of $\mathbb{Z}^2$ we can assume that $c_0 = a = (1,0)$ and $c_1 = b = (0,1)$. The property ($\diamond$) provides the existence of an index $\widetilde{k} \in \{1,\dots,N'\}$, such that $c'_{\widetilde{k}} = (1,1)$, $1 \leqslant i_0 \leqslant N$. We conclude by applying the induction hypothesis to the two sequences $((1,0),(1,1))$ and $(c'_0,\dots,c'_{\widetilde{k}})$, as well as to the two sequences $((1,1),(0,1))$ and $\widetilde{\mathbf{c}}' := (c'_{\widetilde{k}},\dots,c'_{N'+1})$.

$\mathcal{H}_{N-1,N'-1} \Rightarrow \mathcal{H}_{N,N'}$: Always up to an automorphism of $\mathbb{Z}^2$, we can assume that $a = (1,0)$ and $b = (\beta_1,\beta_2)$, with $\beta_1 < \beta_2$. Thanks to Assertion ($\diamond$) we obtain two indices $\widetilde{j} \in \{1,\dots,N\}$ and $\widetilde{k} \in \{1,\dots,N'\}$ such that $c_{\widetilde{j}} = c'_{\widetilde{k}} = (1,1)$. The inductive hypothesis applies to the two sequences $(c_j)_{j=0,\dots,\widetilde{j}}$ and $(c_k)_{k=0,\dots,\widetilde{k}}$, as well as to the two sequences $(c_j)_{j=\widetilde{j},\dots,N}$ and $(c_k)_{k=\widetilde{k},\dots,N'}$, showing that $N = N'$ and $c = c'$.

This achieves the proof of the sub-lemma. □

Proof of Lemma 4.8. After Lemma 4.5, h_3 sends every connected component of $\partial\mathcal{M}_{\mathcal{C}}$ over a connected component of $\partial\mathcal{M}'_{\mathcal{C}'}$, by conjugating the corresponding Seifert fibrations. Hence the isomorphism h_{3*} induced in homology satisfies the following equalities:

$$h_{3*}(\mathfrak{c}_0) = \mathfrak{c}'_0 \quad \text{and} \quad h_{3*}(\mathfrak{c}_{l+1}) = \mathfrak{c}'_{l'+1}, \tag{23}$$

where we have put $l := l_{\mathcal{C}}$ and $l' = l_{\mathcal{C}'}$. Thanks to Assertion (ii) of Proposition 3.6 we obtain that

$$\det(\mathfrak{a},\mathfrak{b}) = \det{}'\big(h_{3*}(\mathfrak{a}), h_{3*}(\mathfrak{b})\big), \tag{24}$$

for all $\mathfrak{a},\mathfrak{b} \in H_1^{\mathcal{C}}$. From relations (18) we deduce that

$$\det(\mathfrak{c}_{j-1},\mathfrak{c}_{j+1}) = -(D_j,D_j) \geqslant 2, \quad j = 1,\dots,l,$$

because the resolution map E of S is minimal.

Set $\mathfrak{c}''_j := h_{3*}^{-1}(\mathfrak{c}'_j)$. The two finite sequences $\mathbf{c} := (\mathfrak{c}_j)_{j=0,\dots,l+1}$ and $\mathbf{c}'' := (\mathfrak{c}''_j)_{j=0,\dots,l'+1}$ of elements of $H_1(\mathcal{M}'_{\mathcal{C}'},\mathbb{Z}) \simeq \mathbb{Z}^2$, have the same first and last terms (23); they satisfy the relations (22) of Sub-Lemma 4.9. The conclusion follows from the unicity of these families. □

Proof of Proposition 4.7. Since the lengths of the chains and the dead branches corresponding by κ_2 and by κ_1 are the same, there exists a unique bijection $\kappa : \mathrm{Comp}(\mathcal{D}) \to \mathrm{Comp}(\mathcal{D}')$ extending κ_3 and satisfying Assertions (2) and (3), as well as the equivalence $(D \cap D' \neq \emptyset \Leftrightarrow \kappa(D) \cap \kappa(D') \neq \emptyset)$. Since the self-intersections of all the compact irreducible components being $\leqslant -1$ (even $\leqslant -2$ if $v(D) \leqslant 2$), to prove (1) it suffices to show the relations

$$(D,D) = \big(\kappa(D),\kappa(D)\big), \quad \text{for all } D \in \mathrm{Comp}(\mathcal{E}). \tag{25}$$

When D_j is contained in a chain or a dead branch of $\mathcal{E}$, the relations (18) follow directly from the equalities $(D_j,D_j) = -\det(\mathfrak{c}_{j-1},\mathfrak{c}_{j+1})$. Since h_{3*} preserves the determinant forms (24), the relations $h_{3*}(\mathfrak{c}_j) = \mathfrak{c}'_j$ of Lemma 4.8 give us $(D_j,D_j) = (\kappa(D_j),\kappa(D_j))$, for each $j = 1,\dots,l_{\mathcal{C}}+1$.

It remains to prove (25) when D has valence $\geqslant 3$. Notice that then $\mathcal{M}_D$ is a retract by deformation of

$$\mathcal{M}_D^{\sharp} := \mathcal{M}_D \bigcup_{j=1}^{v(D)} \mathcal{M}_{s_j}, \qquad D \cap D_j =: \{s_j\},$$

where $D_1,\dots,D_{v(D)}$ are the irreducible components of $\mathcal{D}$ adjacent to D. The singular point s_j is the attaching point of a chain or a dead branch $\mathcal{C}_j$, or even of a strict transform. Consider the meridian $\mathfrak{c}_j \in H_1(M_j,\mathbb{Z}) \simeq H_1(\mathcal{M}_{s_j} \cap \mathcal{M}_D,\mathbb{Z})$ associated to D_j, where M_j is the thick torus $\mathcal{M}_{\mathcal{C}_j}$, or $\mathcal{M}^{\circ}_{\mathcal{C}_j}$ in the two first cases, or $\mathcal{M}_{s_j} \cup \mathcal{M}_{D_j}$ in the last one. Denote by $\widetilde{\mathfrak{c}}_j \in H_1(\mathcal{M}_D,\mathbb{Z})$ the image of $\mathfrak{c}_j$ by the monomorphism induced in homology by the natural inclusion $\mathcal{M}_D \cap \mathcal{M}_{s_j} \subset \mathcal{M}_D$. We can rewrite the *index formula along D* given by (10) in the following way, cf. [4,5]:

$$(D,D)\mathfrak{c} + \sum_{j=1}^{v(D)} \widetilde{\mathfrak{c}}_j = 0 \quad \text{in } H_1(\mathcal{M}_D,\mathbb{Z}),$$

where $\mathfrak{c}$ is the homology class of a fibre $\rho_D^{-1}(p)$, $p \in K_D$, that we call *the meridian associated to D*. Therefore $\{\kappa(D_j)\}_{j=1,\dots,v(D)}$ is the collection of the irreducible components of $\mathcal{D}$ adjacent to $\kappa(D)$. Thanks to Lemma 4.8, their corresponding meridians are $h_{3*}(\mathfrak{c}_j) \in H_1(\mathcal{M}'_{\kappa(D)} \cap \mathcal{M}'_{s'_j}, \mathbb{Z})$, where $\{s'_j\} := \kappa(D) \cap \kappa(D_j)$. In the same way $h_{3*}(\mathfrak{c})$ is the meridian associated to $\kappa(D)$, because h_3 conjugates the Seifert fibrations of B_D and $B'_{\kappa(D)}$. The index formula along $\kappa_3(D)$ gives us the equality (25). □

4.4. Extension to dimension four

We continue to use the notations (3) and (4) and we define now the collection of *elementary blocks of the Milnor* 4-*tube* $\mathcal{T}_\eta$ by means of

$$\mathcal{T}_s := \mathcal{T}_\eta \cap \Omega_s \quad \text{and} \quad \mathcal{T}_D := \mathcal{T}_\eta(K_D), \quad s \in \mathrm{Sing}(\mathcal{D}),\ D \in \mathrm{Comp}(\mathcal{D}). \tag{26}$$

A 4-*tube associated to a chain* $\mathcal{C} \in \mathfrak{C}$, respectively *to a dead branch* $\widetilde{\mathcal{C}} \in \mathfrak{M}$ is, with the notations (13) and (14), respectively (15) and (17):

$$\mathcal{T}_\mathcal{C} := \bigcup_{j=1}^{l_\mathcal{C}} \mathcal{T}_{D_j} \cup \bigcup_{j=0}^{l_\mathcal{C}} \mathcal{T}_{s_j}, \quad \text{respectively} \quad \mathcal{T}_{\widetilde{\mathcal{C}}} := \bigcup_{j=1}^{l_{\widetilde{\mathcal{C}}}} \mathcal{T}_{D_j} \cup \bigcup_{j=0}^{l_{\widetilde{\mathcal{C}}}-1} \mathcal{T}_{s_j}. \tag{27}$$

We define in a similar way the elementary blocks of $\mathcal{T}'_{\eta'}$, that we denote by $\mathcal{T}'_{s'}$, $s' \in \mathrm{Sing}(\mathcal{D}')$ and $\mathcal{T}'_{D'}$, $D' \in \mathrm{Comp}(\mathcal{D}')$, as well as the 4-tubes $\mathcal{T}'_{\mathcal{C}'}$, $\mathcal{C}' \in \mathfrak{C}'$ and $\mathcal{T}'_{\widetilde{\mathcal{C}}'}$, $\widetilde{\mathcal{C}}' \in \mathfrak{M}'$.

Now, for $\star \in \mathfrak{R}$, we shall construct homeomorphisms $G_\star : \mathcal{T}_\star \to \mathcal{T}'_{\kappa(\star)}$ satisfying the properties (a), (b) of Definition 2.5 and coinciding with h_3 on $\mathcal{T}_\star \cap \mathcal{M}_\eta = \mathcal{M}_\star$. After that we will construct $G_\star$ when $\star$ is a chain, then when it is a dead branch or a strict transform, satisfying always the properties (a), (b) of Definition 2.5. It will be able to be glued with the homeomorphisms G_D, $D \in \mathfrak{R}$, already constructed, but it will not necessarily coincide with h_3 over $\mathcal{M}_\star$. Finally, by using suitable Dehn twists, we will modify the global homeomorphism

$$G : \mathcal{T}_\eta \to \mathcal{T}'_{\eta'}, \qquad G_{|\mathcal{M}_\star} = G_\star, \qquad \star \in \mathfrak{R} \cup \mathfrak{C} \cup \mathfrak{M}, \tag{28}$$

in order to that its restriction to $\mathcal{M}_\eta$ becomes isotopic to h_3. At that moment we will have a homeomorphism Φ satisfying the properties of Theorem 2.9.

4.4.1. Construction of G_D, for $D \in \mathfrak{R}$

The restrictions of the Hopf fibrations to the elementary blocks $\mathcal{T}_D$ and $\mathcal{T}'_{\kappa(D)}$, $D \in \mathrm{Comp}(\mathcal{D})$, are globally trivial disk fibrations.

There exist differentiable vector fields Z and Z' on these blocks which are tangent to the Hopf fibres and whose restriction to each of them correspond to the real radial vector field $u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$, in the trivialising coordinates $(u + iv, \rho_D) : \mathcal{T}_D \to \mathbb{D}_1 \times K_D$. We define a homeomorphism extending $h_{3|\mathcal{M}_D}$ and conjugating the Hopf fibrations, by means of

$$\begin{cases} G_D : \mathcal{T}_D \to \mathcal{T}'_{\kappa(D)}, \qquad G_{D|\mathcal{M}_D} = h_3, \qquad \rho'_{\kappa(D)} \circ G_D = \rho_{D|\mathcal{T}_D}, \\ G_D\big(\phi_t^Z(m)\big) := \phi_t^{Z'}\big(h_3(m)\big), \quad \text{if } m \in \mathcal{M}_D,\ t < 0, \\ G_D(m) := \varsigma_D(m)\big(h_3(m)\big), \quad \text{if } m \in K_D, \end{cases}$$

where ϕ_t^Z and $\phi_t^{Z'}$ denote the flows of Z and Z' respectively.

4.4.2. Construction of $G_\mathcal{C}$ when $\mathcal{C}$ is a chain

Let $\mathcal{C} \in \mathfrak{C}$ be a chain of $\mathcal{D}$ and let $\mathcal{C}' := \kappa_2(\mathcal{C})$ be the associated chain of $\mathcal{D}'$,

$$\mathcal{C} = \{D_j\}_{j=0,\dots,l+1} \in \mathfrak{C}, \qquad \mathcal{C}' = \big\{D'_j\big\}_{j=0,\dots,l+1}, \qquad D'_j := \kappa(D_j),$$

the components D_0, D_{l+1} having valence $\geqslant 3$. We will assume that $l \geqslant 1$; the case of a chain of length $l = 0$, i.e. without components of valence 2 and having a unique singular point $\{s\} = D_0 \cap D_1$ can be treated in a similar way by putting $\mathcal{M}_\mathcal{C} = \mathcal{M}_s$ and $\mathcal{M}'_{\mathcal{C}'} = \mathcal{M}'_{s'}$, $\{s'\} := D'_0 \cap D'_1$.

First, we shall construct homeomorphisms g_{s_j} which are holomorphic over some neighbourhoods W_{s_j} of the singularities $\{s_j\} := D_{j-1} \cap D_j$. Then we shall construct homeomorphisms g_{D_j} defined over the elementary blocks $\mathcal{T}_{D_j}$, conjugating the Hopf fibrations. Finally, we will glue these homeomorphisms in order to obtain a global homeomorphism

$$G_\mathcal{C} : \mathcal{T}_\mathcal{C} \to \mathcal{T}'_{\mathcal{C}'}$$

satisfying the properties (a) and (b) in Definition 2.5 of excellent homeomorphisms.

Step 1. The map $f \circ E$, composition of the reduced equation of S fixed in Section 2.2 and the resolution of singularities map, is a global equation for $\mathcal{D}$. Corollary 3.4 gives us universal formulae (see also [5, Theorem 18.2]) expressing the multiplicities $\nu_D(f \circ E)$ along each irreducible component D of $\mathcal{D}$, from the intersection matrix of $\mathcal{D}$. The intersection matrices (D', D'') and $(\kappa(D'), \kappa(D''))$, D', $D'' \in \mathrm{Comp}(\mathcal{D})$, coincide thanks to Assertion (1) of Proposition 4.7. Using always the notations of Section 2.2, we have then

$$\nu_{D'_j}\big(f' \circ E'\big) = \nu_{D_j}(f \circ E) =: m_j, \quad j = 0, \ldots, l+1.$$

Let s_j be the intersection point of D_j and D_{j+1} and let s'_j be that of D'_j and D'_{j+1}. There exist local holomorphic coordinates at these points

$$(u_j, v_j) : W_{s_j} \xrightarrow{\sim} \mathbb{D}_1 \times \mathbb{D}_1, \qquad \big(u'_j, v'_j\big) : W'_{s'_j} \xrightarrow{\sim} \mathbb{D}_1 \times \mathbb{D}_1, \tag{29}$$

with $W_{s_j} \subset \mathring{\Omega}_{s_j}$ and $W'_{s'_j} \subset \mathring{\Omega}'_{s'_j}$, such that $v_j = 0$, respectively $v'_j = 0$, is a local equation of D_j, respectively D'_j, making monomial the functions $f \circ E$ and $f' \circ E'$, i.e.

$$f \circ E_{|W_{s_j}} = u_j^{m_{j+1}} v_j^{m_j} \quad \text{and} \quad f' \circ E'_{|W'_{s'_j}} = u_j^{\prime m_{j+1}} v_j^{\prime m_j}.$$

We obtain thus a local biholomorphism g_{s_j}, between $W_{s_j} \cap \mathcal{T}_{s_j} = W_{s_j} \cap \mathcal{T}_\eta$ and $W'_{s'_j} \cap \mathcal{T}'_{s'_j} = W'_{s'_j} \cap \mathcal{T}'_{\eta'}$, by putting

$$g_{s_j} := \big(u'_j, v'_j\big)^{-1} \circ (u_j, v_j) : W_{s_j} \cap \mathcal{T}_\eta \to W'_{s'_j} \cap \mathcal{T}'_{\eta'}. \tag{30}$$

By taking $\eta > 0$ small enough, the 3-manifold $W_{s_j} \cap \mathcal{M}_\eta$, as well as the connected components $\mathfrak{T}_j$ and $\mathfrak{T}_{j+1}$ of $\overline{\mathcal{M}_{s_j} \setminus W_{s_j}}$, with $\mathfrak{T}_j \cap \mathcal{M}_{D_j} \neq \emptyset$, are thick tori. Their inclusions in $\mathcal{M}_\mathcal{C}$ induce isomorphisms in homology. Assume that $W'_{s'_j} \cap \mathcal{M}'_{\eta'}$ satisfies the same properties. Up to decreasing $\eta' > 0$ if necessary, we can assume that the restriction of g_{s_j} to $W_{s_j} \cap \mathcal{M}_\eta$, taking values in $W'_{s'_j} \cap \mathcal{M}'_{\eta'}$, induces an isomorphism

$$g_{s_j *} : H_1(\mathcal{M}_\mathcal{C}, \mathbb{Z}) \to H_1\big(\mathcal{M}'_{\mathcal{C}'}, \mathbb{Z}\big). \tag{31}$$

Lemma 4.10. *Let $\mathfrak{c}_j$ and $\mathfrak{c}'_j$ be the meridians associated to the components D_j and D'_j respectively, cf. Definition* 3.5. *Then $g_{s_j*}(\mathfrak{c}_k) = \mathfrak{c}'_k$, $k := j, j+1$, for each $j = 0, \ldots, l$.*

Proof. Assume that $k = j$, the case $k = j+1$ is completely analogous. Up to permuting the coordinates of the local datum if necessary, we can also assume that $y_{s_j} = 0$ is an equation for D_j. If $\eta > 0$ is small enough, the fibres of x_{s_j} and u_j are transversal to the *Milnor fibres* (i.e. that of $f \circ E$). Since the simple curves $x_{s_j}^{-1}(p) \cap \mathcal{M}_\eta$ and $u_j^{-1}(p) \cap \mathcal{M}_\eta$ turn once around D_j and do not turn around D_{j+1} they are homologous in $\mathcal{M}_{s_j} \cong \mathcal{M}_\mathcal{C}$. We left the reader to complete the details. Now, it suffices to note that, by construction, g_{s_j} transforms fibres of u_j (resp. v_j) into fibres u'_j (resp. v'_j). □

Step 2. Consider now homeomorphisms

$$g_{D_j} : \mathcal{T}_{D_j} \to \mathcal{T}'_{D'_j}, \quad j = 1, \ldots, l \tag{32}$$

fulfilling the following properties:

(a) $g_{D_j}(\mathcal{T}_{D_j} \cap \mathcal{T}_{s_j}) = \mathcal{T}'_{D'_j} \cap \mathcal{T}'_{s'_j}$,
(b) g_{D_j} conjugate the Hopf fibrations, i.e. there is a homeomorphism $\varsigma_{D_j} : K_{D_j} \to K'_{D'_j}$ such that $\varsigma_{D_j} \circ \rho_{D_j}(m) = \rho'_{D'_j} \circ g_{D_j}(m)$, $m \in \mathcal{T}_{D_j}$,
(c) the morphism $g_{D_j *} : H_1(\mathcal{M}_\mathcal{C}, \mathbb{Z}) \to H_1(\mathcal{M}'_{\mathcal{C}'}, \mathbb{Z})$ induced[16] by the restriction of g_{D_j} to $\mathcal{M}_{D_j}$, taking values in $\mathcal{M}'_{D'_j}$, verifies $g_{D_j *}(\mathfrak{c}_k) = \mathfrak{c}'_k$, for $k = j \pm 1$.

Notice that the equality (c) for $k = j$ follows from (b) and that the case $k = j-1$ is equivalent to the case $k = j+1$, by applying the index formulae (18) and Assertion (1) of Proposition 4.7. Hence, the construction of the homeomorphisms g_{D_j} is straightforward once one has trivialised the Hopf fibrations, using that the Euler numbers of D_j and D'_j coincide after Proposition 4.7.

Step 3. It remains to construct a homeomorphism from each connected component $\mathfrak{T}$ of $\mathcal{T}_{s_j} \setminus W_{s_j}$, $j = 0, \ldots, l$, onto a connected component $\mathfrak{T}'$ of $\mathcal{T}'_{s'_j} \setminus W'_{s'_j}$, which could be glued with g_{s_j} and $g_{D_{j'}}$, $j' = j$ or $j+1$. To do this we fix a homeomorphism Λ from $\mathfrak{T}$ onto $[0,1] \times \mathbb{S}^1 \times \mathbb{D}_1$ and a disk fibration $\rho_\mathfrak{T} : \mathfrak{T} \to D_{j'} \cap \mathfrak{T}$, coinciding with $\rho_{D_{j'}}$ on a connected

[16] Via the identifications $H_1(\mathcal{M}_{D_j}, \mathbb{Z}) \simeq H_1(\mathcal{M}_\mathcal{C}, \mathbb{Z})$ and $H_1(\mathcal{M}'_{D'_j}, \mathbb{Z}) \simeq H_1(\mathcal{M}'_{\mathcal{C}'}, \mathbb{Z})$ given by the natural inclusions.

component of $\Lambda^{-1}(\{0,1\}\times\mathbb{S}^1\times\mathbb{D}_1)$ and with a coordinate of (29) on the other connected component. We proceed in the same way for $\mathfrak{T}'$. Notice that the restrictions of g_{s_j} and $g_{D_{j'}}$ to $\partial\mathfrak{T}$ conjugate the constructed fibrations. To conclude it suffices to apply the following lemma by using also Lemma 4.10.

Lemma 4.11. *Consider $\tilde{\phi}_0$, $\tilde{\phi}_1$ two homeomorphisms of $\mathbb{S}^1\times\mathbb{D}_1$ onto itself, commuting to the first projection, i.e. $\tilde{\phi}_k(\theta,z)=(\theta,\phi_{k,\theta}(z))$, satisfying $\phi_{k,\theta}(0)=0$, for $k=0,1$, and whose restrictions to $\mathbb{S}^1\times\partial\mathbb{D}_1$ induce the identity in homology. Then there exists a homeomorphism $\tilde{\Phi}$ from $[0,1]\times\mathbb{S}^1\times\mathbb{D}_1$ onto itself, commuting to the two first projections, i.e. $\tilde{\Phi}(t,\theta,z)=(t,\theta,\Phi_{t,\theta}(z))$ such that*

(1) $\Phi_{k,\theta}=\phi_{k,\theta}$ *for* $k=0,1$,
(2) $\Phi_{t,\theta}(z)=z$ *for all* $t\in[\frac{1}{3},\frac{2}{3}]$.

Proof. The existence of $\tilde{\Phi}=\tilde{\Phi}(\tilde{\phi}_0,\tilde{\phi}_1)$ fulfilling conditions (1) and (2) follows from that of $\tilde{\Phi}(\tilde{\phi}_0,\mathrm{id})$ satisfying only condition (1) by considering

$$\tilde{\Phi}(\tilde{\phi}_0,\tilde{\phi}_1)(t,\theta,z)=\begin{cases}\tilde{\Phi}(\tilde{\phi}_0,\mathrm{id})(3t,\theta,z) & \text{if } t\in[0,\frac{1}{3}],\\ (t,\theta,z) & \text{if } t\in[\frac{1}{3},\frac{2}{3}],\\ \tilde{\Phi}(\tilde{\phi}_1,\mathrm{id})(3(1-t),\theta,z) & \text{if } t\in[\frac{2}{3},1].\end{cases}$$

Now we proceed to construct $\tilde{\Phi}$ fulfilling condition (1) in the case $\tilde{\phi}_1=\mathrm{id}$. The homotopy class of the map

$$\begin{aligned}\mathbb{S}^1&\to\mathrm{Aut}(S^1),\\ \theta&\mapsto\phi_{0,\theta}|_{\partial\mathbb{D}_1}\end{aligned}$$

defines an element in the fundamental group $\pi_1(\mathrm{Aut}(\mathbb{S}^1))$ which can be identified to $\mathbb{Z}$ via the isomorphism

$$\big[(\psi_t)_{t\in[0,1]}\big]\mapsto\frac{1}{2i\pi}\int_{t\mapsto\psi_t(1)}\frac{dz}{z}.$$

Since the restriction of $\tilde{\phi}_0$ to $\mathbb{S}^1\times\partial\mathbb{D}_1$ induces the identity in homology we have that

$$\int_{t\mapsto\phi_{0,e^{2i\pi t}}(1)}\frac{dz}{z}=0,$$

and consequently there exists a homotopy

$$\begin{aligned}[0,1]\times\mathbb{S}^1&\to\mathrm{Aut}(\mathbb{S}^1),\\ (t,\theta)&\mapsto\phi^{\partial}_{t,\theta}\end{aligned}$$

such that $\phi^{\partial}_{0,\theta}=\phi_{0,\theta}|_{\partial\mathbb{D}_1}$ and $\phi^{\partial}_{1,\theta}=\mathrm{id}_{\partial\mathbb{D}_1}$. Now it suffices to define $\Phi_{t,\theta}(z)$ as follows:

– $\Phi_{t,\theta}(0)=0$,
– if $t\in[0,\frac{1}{2}]$ then

$$\Phi_{t,\theta}(z)=\begin{cases}|z|\phi_{0,\theta}(\frac{z}{|z|}) & \text{if } |z|\geqslant 1-2t,\ z\neq 0,\\ (1-2t)\phi_{0,\theta}(\frac{z}{1-2t}) & \text{if } 0<|z|<1-2t,\end{cases}$$

– if $t\in]\frac{1}{2},1]$ and $z\neq 0$ then $\Phi_{t,\theta}(z)=|z|\phi^{\partial}_{2t-1,\theta}(\frac{z}{|z|})$,

based on a combination of Lemma 4.6 and Alexander's trick. Checking the continuity at the points on $z=0$ others than $(t,z)=(0,0)$ is straightforward. The continuity at the points $(t,z)=(0,0)$ follows easily from the assumption $\phi_{0,\theta}(0)=0$. The continuity at the points on $t=\frac{1}{2}$ follows from the fact $\phi_{0,\theta}|_{\partial\mathbb{D}_1}=\phi^{\partial}_{0,\theta}$. □

4.4.3. Construction of $G_{\mathcal{C}}$ when $\mathcal{C}$ is a dead branch or a strict transform

We consider first the case of a dead branch of $\mathcal{D}$, denoted by $\mathcal{C}=\{D_j\}_{j=0,\dots,l}$, $\nu(D_0)\geqslant 3$, and we denote by $\mathcal{C}':=\kappa_1(\mathcal{C})=\{D'_j\}_{j=0,\dots,l}$, $D'_j:=\kappa(D_j)$, the corresponding dead branch of $\mathcal{D}'$. We can do, in this context, all the precedent construction unless for the extremal component, i.e. for $\{s_j\}:=D_j\cap D_{j+1}$, $j=0,\dots,l-1$, with the same notations that in (30), we construct a homeomorphism g_{s_j} and, for each component of valence two, a homeomorphism g_{D_j} as in (32). In $H_1(\mathcal{M}'^{\circ}_{\mathcal{C}'}, \mathbb{Z})$, we have yet the equalities

$$g_{s_j *}(\mathfrak{c}_k) = \mathfrak{c}'_k, \quad k = j, j+1,\ j = 0, \ldots, l-1,$$

for the same reasons as in Lemma 4.10 and thanks to Lemma 4.8. Hence the homeomorphisms g_{s_j} and g_{D_j} can be glued as in the step 3 above. It only remains to extend $g_{s_{l-1}}$ along D_l. To do this we will assume as before that $\eta, \eta' > 0$ are small enough so that the connected components of $\overline{\mathcal{T}_{s_{l-1}} \setminus W_{s_{l-1}}}$ and $\overline{\mathcal{T}'_{s'_{l-1}} \setminus W'_{s'_{l-1}}}$ are the thick tori. It suffices then to construct a homeomorphism g from the connected component $\mathfrak{T}$ of $(\mathcal{T}_{s_{l-1}} \setminus W_{s_{l-1}}) \cup \mathcal{T}_{D_l}$ containing D_l, onto the connected component $\mathfrak{T}'$ of $(\mathcal{T}'_{s_{l-1}} \setminus W'_{s_{l-1}}) \cup \mathcal{T}'_{D'_l}$ containing D'_l, which coincides with $g_{s_{l-1}}$ over the solid torus $\mathfrak{T} \cap W_{s_{l-1}}$. Fix again fibrations $\rho_{\mathfrak{T}} : \mathfrak{T} \to \mathfrak{T} \cap D_l$ and $\rho'_{\mathfrak{T}'} : \mathfrak{T}' \to \mathfrak{T}' \cap D'_l$, coinciding with the Hopf fibrations on $\mathcal{T}_{D_l}$, respectively $\mathcal{T}'_{D_l}$, and with a coordinate of (29) on $\mathfrak{T} \cap W_{s_{l-1}}$, respectively $\mathfrak{T}' \cap W'_{s_{l-1}}$. Clearly, $\mathfrak{T}$ and $\mathfrak{T}'$ are homeomorphic to $\mathbb{D}_1 \times \mathbb{D}_1$, and the fibrations $\rho_{\mathfrak{T}}$ and $\rho'_{\mathfrak{T}}$ correspond to the first projection. To achieve the construction of $G_{\mathcal{C}}$, it suffices to use the following lemma whose proof is similar to that of Lemma 4.11.

Lemma 4.12. *Let ϕ be a homeomorphism from $\partial\mathbb{D}_1 \times \mathbb{D}_1$ onto itself, commuting to the first projection, i.e. $\phi(\theta, p) = (\theta, \underline{\phi}(\theta, p))$, and such that restricted to $\partial\mathbb{D}_1 \times \partial\mathbb{D}_1$, induces the identity map in homology. Then ϕ extends to a homeomorphism Φ from $\mathbb{D}_1 \times \mathbb{D}_1$ onto itself, commuting also to the first projection.*

Proof. As for Lemma 4.11, there exists a continuous map

$$t \in [0,1] \mapsto \tilde{\Phi}_t \in C^0\big(\mathbb{S}^1, \operatorname{Aut}(\mathbb{S}^1)\big),$$

such that $\tilde{\Phi}_0(\theta)(\vartheta) = \vartheta$ and $\tilde{\Phi}_1(\theta)(\vartheta) = \phi(\theta, \vartheta)$. We put $\Phi(z', z'') := (z', \underline{\Phi}(z', z''))$, with

$$\underline{\Phi}(z', z'') := \begin{cases} |z''| \cdot \tilde{\Phi}_{|z'|}(\frac{z'}{|z'|})(\frac{z''}{|z''|}), & \text{if } |z'| \leqslant |z''| \leqslant 1, \\ |z''| \cdot \tilde{\Phi}_{1+|z'|-\frac{|z''|}{|z'|}}(\frac{z'}{|z'|})(\frac{z''}{|z''|}), & \text{if } |z'|^2 \leqslant |z''| \leqslant |z'|, \\ |z'|^2 \cdot \underline{\phi}(\frac{z'}{|z'|}, \frac{z''}{|z'|^2}), & \text{if } |z''| \leqslant |z'|^2 \leqslant 1. \end{cases} \qquad \square$$

Consider now the case that D_1 and $D'_1 := \kappa(D_1)$ are the strict transforms of irreducible components of S and S' respectively. The adjacent components $D_0 \in \operatorname{Comp}(\mathcal{D})$, respectively $D'_0 := \kappa(D_0) \in \operatorname{Comp}(\mathcal{D}')$, have valence $\geqslant 3$. Denote $\{s\} := D_0 \cap D_1$ and $\{s'\} := D'_0 \cap D'_1$, $\mathcal{C} := \{D_0, D_1\}$, $\mathcal{C}' := \{D'_0, D'_1\}$ and put $\mathcal{M}_{\mathcal{C}} := \mathcal{M}_s \cup \mathcal{M}_{D_1}$, $\mathcal{T}_{\mathcal{C}} := \mathcal{T}_s \cup \mathcal{T}_{D_1}$, $\mathcal{M}'_{\mathcal{C}'} := \mathcal{M}'_{s'} \cup \mathcal{M}'_{D'_1}$ and $\mathcal{T}'_{\mathcal{C}'} := \mathcal{T}'_{s'} \cup \mathcal{T}'_{D'_1}$. With the same notations we construct as in (31) a biholomorphism $g_s : W_s \cap \mathcal{T}_\eta \to W'_{s'} \cap \mathcal{T}'_{\eta'}$. For the same reasons as in Lemma 4.10, it verifies the equalities $g_{s*}(\mathfrak{c}_k) = \mathfrak{c}'_k$, $k = 0, 1$, where $\mathfrak{c}_k$, respectively $\mathfrak{c}'_k$, are the homology classes in $H_1(\mathcal{M}_{\mathcal{C}}, \mathbb{Z})$, respectively $H_1(\mathcal{M}'_{\mathcal{C}'}, \mathbb{Z})$, of an arbitrary fibre of the Hopf fibration ρ_{D_k} restricted to $\mathcal{M}_s \cap \mathcal{M}_{D_k}$, respectively $\rho'_{D'_k}$ restricted to $\mathcal{M}'_{s'} \cap \mathcal{M}'_{D'_k}$. Notice that the restriction of h_3 to $\mathcal{M}_{\mathcal{C}} \cap \partial\mathcal{B}$ (which is a connected component of the boundary of $\mathcal{M}_\eta$), taking values in $\mathcal{M}'_{\mathcal{C}'} \cap \partial\mathcal{B}'$, verifies also the equality[17]

$$h_{3*}(\mathfrak{c}_k) = \mathfrak{c}'_k \quad \text{in } H_1\big(\mathcal{M}'_{\mathcal{C}'}, \mathbb{Z}\big),\ k = 0, 1.$$

Indeed, by construction h_3 and h are fundamentally equivalent, so that their actions on the fundamental group Γ differ by an inner automorphism. By passing to the homology we have then $h_* = h_{3*}$. Theorem 3.16 claims that the image by h_* of the meridian $\mathfrak{m}_{D_1}$ of the peripheral subgroup $\mathcal{P} \subset \Gamma$ associated to $\mathcal{C}$ is just the meridian $\mathfrak{m}_{D'_1} \in \mathcal{P}' \subset \Gamma'$. Since the isomorphisms $\mathcal{P} \cong H_1(\mathcal{M}_{\mathcal{C}}, \mathbb{Z})$ and $\mathcal{P}' \cong H_1(\mathcal{M}'_{\mathcal{C}'}, \mathbb{Z})$ identify $\mathfrak{m}_{D_1}$ to $\mathfrak{c}_1$ and $\mathfrak{m}'_{D'_1}$ to $\mathfrak{c}'_1$ we obtain the equality $h_{3*}(\mathfrak{c}_1) = \mathfrak{c}'_1$. On the other hand, from Remark 3.9 follows that the natural inclusion $H_1(\mathcal{M}_{\mathcal{C}}, \mathbb{Z}) \hookrightarrow H_1(B_{D_0})$ sends $\mathfrak{c}_0$ into the homology class of $\mathfrak{c}_{D_0} \in \pi_1(B_{D_0}) \subset \Gamma$ represented[18] by a fibre of the Seifert fibration of D_0. We have an analogous description for $\mathcal{M}'_{\mathcal{C}}$. Since h_3 conjugates the Seifert fibrations of B_{D_0} and $B'_{D'_0}$, it follows that $h_{3*}(\mathfrak{c}_0) = \mathfrak{c}'_0$.

Let $H_{D_1} : \mathcal{T}_{D_1} \to \mathcal{T}'_{D'_1}$ be a homeomorphism whose restriction to $\mathcal{M}_{\mathcal{C}} \cap \mathcal{B}$ coincides with h_3 and commutes to the Hopf fibrations, i.e. $H_{D_1}(K_{D_1}) = K'_{D'_1}$ and $H_{D_1} \circ \rho_{D_1} = \rho'_{D'_1} \circ H_{D_1}$. As in the previous step 3, we construct a homeomorphism $G_{\mathcal{C}} : \mathcal{T}_{\mathcal{C}} \to \mathcal{T}'_{\mathcal{C}'}$ extending g_s, which coincides with H_{D_0} when restricted to $\mathcal{M}_{\mathcal{C}} \cap \mathcal{M}_{D_0}$ and with H_{D_1} when restricted to $\mathcal{M}_{\mathcal{C}} \cap \partial\mathcal{B}$.

[17] With the identifications given by the natural inclusions, i.e. $H_1(\mathcal{M}_{\mathcal{C}} \cap \partial\mathcal{B}, \mathbb{Z}) \simeq H_1(\mathcal{M}_{\mathcal{C}}, \mathbb{Z})$ and $H_1(\mathcal{M}'_{\mathcal{C}'} \cap \partial\mathcal{B}', \mathbb{Z}) \simeq H_1(\mathcal{M}'_{\mathcal{C}'}, \mathbb{Z})$.

[18] Here we use that the fixed desingularisations of S and S' are minimal and therefore $v(D_0) = v(D'_0) \geqslant 3$.

4.4.4. Modification by Dehn twists

We will now modify the homeomorphism G obtained in (28), by composing it at the right with a homeomorphism $\Psi : \mathcal{T}_\eta \to \mathcal{T}_\eta$ which is the identity over each block $\mathcal{T}_D$, $D \in \mathfrak{R}$, in such a way that $G \circ \Psi$ satisfies Theorem 2.9. Denoting by $\mathcal{C} =: \{D_j\}_{j=0}^{l}$ a chain of $\mathfrak{C}$, a dead branch or a pair of components associated to a strict transform, and putting $\Psi_{\mathcal{C}} := \Psi_{|\mathcal{T}_{\mathcal{C}}}$, it suffices to prove the following assertion:

($\star\star$) *There exists a homeomorphism $\Psi_{\mathcal{C}} : \mathcal{T}_{\mathcal{C}} \to \mathcal{T}_{\mathcal{C}}$, $\Psi_{\mathcal{C}}(\mathcal{T}_{\mathcal{C}} \cap \mathcal{D}) = \mathcal{T}_{\mathcal{C}} \cap \mathcal{D}$, whose support is contained in the interior of $(\Omega_{s_0} \setminus \{s_0\})$, $\{s_0\} := D_0 \cap D_1$, such that $\Psi_{\mathcal{C}|\mathcal{M}_{\mathcal{C}}}$ and $G^{-1} \circ h_3 : \mathcal{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}$ are homotopic relatively to the boundary of $\mathcal{M}_{\mathcal{C}}$, i.e. there is a homotopy $F_t : \mathcal{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}$, $t \in [0,1]$, such that $F_0 = G^{-1} \circ h_3$, $F_1 = \Psi_{|\mathcal{M}_{\mathcal{C}}}$ and $F_t(m) = m$, for all $t \in [0,1]$ and $m \in \partial\mathcal{M}_{\mathcal{C}}$.*

Recall that to every continuous map K from a manifold with boundary bord X into itself, which is the identity when restricted to a subset $A \subset X$, we can associate a *variation morphism relative to A*, cf. [1], by means of

$$\mathrm{var}_K : H_1(X, A; \mathbb{Z}) \to H_1(X, \mathbb{Z}), \quad [\delta] \mapsto \big[K(\delta) - \delta\big].$$

This morphism is an invariant of the relative to A homotopy class of K. Notice that if $K_* : H_1(X,\mathbb{Z}) \to H_1(X,\mathbb{Z})$ denotes the morphism induced by K and $i_* : H_1(X,\mathbb{Z}) \to H_1(X, A; \mathbb{Z})$ that of the inclusion $(X, \emptyset) \subset (X, A)$, we have the equality $K_* = \mathrm{id}_{H_1(X,\mathbb{Z})} + \mathrm{var}_K \circ i_*$. We will use the following result.

Proposition 4.13. *Two homeomorphisms χ_0 and $\chi_1 : \mathcal{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}$ whose restrictions to $\partial\mathcal{M}_{\mathcal{C}}$ are the identity, are homotopic relatively to $\partial\mathcal{M}_{\mathcal{C}}$, if and only if their variation morphisms are equal*:

$$\mathrm{var}_{\chi_0} = \mathrm{var}_{\chi_1} : H_1(\mathcal{M}_{\mathcal{C}}, \partial\mathcal{M}_{\mathcal{C}}; \mathbb{Z}) \to H_1(\mathcal{M}_{\mathcal{C}}, \mathbb{Z}).$$

Notice that if $\mathcal{C}$ is a dead branch then $(\mathcal{M}_{\mathcal{C}}, \partial\mathcal{M}_{\mathcal{C}})$ is homeomorphic to $(\mathbb{S}^1 \times \mathbb{D}_1, \mathbb{S}^1 \times \mathbb{S}^1)$ and $H_1(\mathcal{M}_{\mathcal{C}}, \partial\mathcal{M}_{\mathcal{C}}; \mathbb{Z}) = 0$. To obtain ($\star\star$), we then define $\Psi_{\mathcal{C}} = \mathrm{id}_{\mathcal{T}_{\mathcal{C}}}$. If $\mathcal{C}$ is not a dead branch then Assertion ($\star\star$) follows directly from the following realisation lemma.

Lemma 4.14. *Assume that $\mathcal{C}$ is a chain or it is associated to a strict transform. Then for each morphism $L : H_1(\mathcal{M}_{\mathcal{C}}, \partial\mathcal{M}_{\mathcal{C}}; \mathbb{Z}) \to H_1(\mathcal{M}_{\mathcal{C}}, \mathbb{Z})$, there exists a homeomorphism $\Psi : \mathcal{T}_{\mathcal{C}} \to \mathcal{T}_{\mathcal{C}}$ with support contained in $\mathring{\Omega}_{s_0} \setminus \{s_0\}$, satisfying $\Psi_{\mathcal{C}}(\mathcal{T}_{\mathcal{C}} \cap \mathcal{D}) = \mathcal{T}_{\mathcal{C}} \cap \mathcal{D}$ and such that L is the variation morphism of the restriction of Ψ to $\mathcal{M}_{\mathcal{C}}$, i.e. $L = \mathrm{var}_{\Psi_{|\mathcal{M}_{\mathcal{C}}}}$.*

Proof. Clearly $H_1(\mathcal{M}_{\mathcal{C}}, \partial\mathcal{M}_{\mathcal{C}}, \mathbb{Z}) = \mathbb{Z}\mathfrak{d}$ is generated by the homotopy class of an arbitrary path δ joining the two connected components of $\partial\mathcal{M}_{\mathcal{C}}$. Thanks to the formula[19] $\mathrm{var}_{\chi_1 \circ \chi_2} = \mathrm{var}_{\chi_1} + \mathrm{var}_{\chi_2}$, it suffices to construct Ψ for $L = L_k : [\delta] \mapsto \mathfrak{c}_k$, $k = 0, 1$, where $\mathfrak{c}_0$ and $\mathfrak{c}_1$ are the meridians associated to D_0 and D_1. Indeed, they are a $\mathbb{Z}$-basis of $H_1(\mathcal{M}_{\mathcal{C}}, \mathbb{Z})$, by Proposition 3.6. For $k = 0$ or 1, we fix as in (29) local coordinates (u, v) at the point s_0 such that the map $f \circ E$ is monomial and $v = 0$ is a reduced local equation of D_k. The homeomorphism (Dehn twist) $\Psi : \mathcal{T}_{\mathcal{C}} \to \mathcal{T}_{\mathcal{C}}$ defined by

$$u \circ \Psi = u, \qquad v \circ \Psi = \begin{cases} e^{2i\pi(3|u|-1)} \cdot v, & \text{if } \frac{1}{3} \leqslant |u| \leqslant \frac{2}{3}, \\ v, & \text{otherwise}, \end{cases}$$

fulfils the desired properties. □

Proof of Proposition 4.13. The proof consists to suitably apply Eilenberg's classification theorem, cf. [24, Theorem V.6.7], which we recall here:

Theorem 4.15. *Let Y be an $(n-1)$-connected topological space whose group $\pi = \pi_n(Y)$ is abelian, let (X, A) be a relative CW-complex and let $f_0 : X \to Y$ be a continuous map. Assume that*

(1) *Y is q-simple for $n + 1 \leqslant q \leqslant \dim(X, A)$,*
(2) *$H^q(X, A; \pi_q(Y)) = 0$ for $n + 1 \leqslant q \leqslant \dim(X, A)$,*
(3) *$H^{q+1}(X, A; \pi_q(Y)) = 0$ for $n + 1 \leqslant q \leqslant \dim(X, A) - 1$.*

Then the correspondence $f \mapsto (f_0, f)^* \iota^n(Y)$ induces a bijection between the set of relative to A homotopy classes of extensions of $f_{0|A}$ and the cohomology group $H^n(X, A; \pi)$.

In this statement $\iota^n(Y) \in H^n(Y; \pi) \cong \mathrm{Hom}(H_n(Y), \pi)$ is identified to the inverse of the Hurewicz isomorphism $\pi_n(Y) \xrightarrow{\sim} H_n(Y)$. If Y is a CW-complex, then $\iota^n(Y)$ sends each n-cell of Y into the unique element of $\pi = \pi_n(Y)$ obtained by collapsing the $(n-1)$-skeleton of Y to the base point. On the other hand, we denote by $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ and

[19] Indeed, $\mathrm{var}_{\chi_1\chi_2}\mathfrak{d} = [\chi_1\chi_2\delta - \delta] = [\chi_1\chi_2\delta - \chi_2\delta] + [\chi_2\delta - \delta] = \mathrm{var}_{\chi_1}\mathfrak{d} + \mathrm{var}_{\chi_2}\mathfrak{d}$ because $[\chi_2\delta] = [\delta] = \mathfrak{d}$ in $H_1(\mathcal{M}_{\mathcal{C}}, \partial\mathcal{M}_{\mathcal{C}}; \mathbb{Z})$.

$\mathbb{I} = [0,1]$ and we consider the map $F_{(f_0,f_1)} : X \times \partial\mathbb{I} \cup A \times \mathbb{I} \to Y$ defined by $F_{(f_0,f_1)}(x,t) = f_t(x)$ if $x \in X$ and $t \in \partial\mathbb{I}$ and by $F_{(f_0,f_1)}(a,t) = f_0(a) = f_1(a)$ if $a \in A$ and $t \in \mathbb{I}$. Then we have $(f_0, f_1)^* = (\mathrm{i}^*\times)^{-1} \circ \partial^* \circ F^*_{(f_0,f_1)}$, where

$$\partial^* : H^n(X \times \partial\mathbb{I} \cup A \times \mathbb{I}; \pi) \to H^{n+1}(X \times \mathbb{I}, X \times \partial\mathbb{I} \cup A \times \mathbb{I}; \pi)$$

is the connecting morphism and $\mathrm{i}^*\times : H^n(X, A; \pi) \to H^{n+1}(X \times \mathbb{I}, X \times \partial\mathbb{I} \cup A \times \mathbb{I}; \pi)$ is the isomorphism induced by the product by the dual of the natural generator $\mathrm{i} \in H_1(\mathbb{I}, \partial\mathbb{I})$, using that $(X \times \mathbb{I}, X \times \partial\mathbb{I} \cup A \times \mathbb{I}) = (X, A) \times (\mathbb{I}, \partial\mathbb{I})$.

If $\mathcal{C}$ is a chain, respectively a dead branch, we apply the theorem with $X = Y := \mathcal{M}_{\mathcal{C}}$ which is homeomorphic to $\mathbb{T} \times \mathbb{I}$, respectively to $X = Y \cong \mathbb{D} \times \mathbb{S}^1$. Hence, it is an Eilenberg–MacLane space $K(\pi, 1)$, with $\pi = \pi_1(\mathbb{T} \times \mathbb{I}) = H_1(\mathbb{T} \times \mathbb{I}) \cong \mathbb{Z}^2$ (resp. $\pi = \mathbb{Z}$). The hypotheses of the previous theorem are trivially satisfied. We also put $A := \partial\mathcal{M}_{\mathcal{C}} \cong \mathbb{T} \times \partial\mathbb{I}$, respectively $A \cong \partial\mathbb{D} \times \mathbb{S}^1$ and $f_0 = \mathrm{id}$.

If $\mathcal{C}$ is a dead branch then

$$H^1(X, A; \pi) = H^1\big(\mathbb{D} \times \mathbb{S}^1, \partial\mathbb{D} \times \mathbb{S}^1, \mathbb{Z}\big) = H^1\big((\mathbb{D}, \partial\mathbb{D}) \times \big(\mathbb{S}^1, \emptyset\big)\big) = 0,$$

by the relative Künneth formula and by the fact that $H^i(\mathbb{D}, \partial\mathbb{D}) = 0$ for $i = 0, 1$. In this case, we obtain that all the extensions of the identity map on A are homotopic relatively to A.

In the case that $\mathcal{C}$ is a chain (of $\mathfrak{C}$ or a pair of components associated to a strict transform) we obtain that the set of relative to A homotopy classes of extensions of the identity are in one-to-one correspondence with $H^1(\mathbb{T} \times \mathbb{I}, \mathbb{T} \times \partial\mathbb{I}; \mathbb{Z}^2) \cong \mathbb{Z}^2$. It suffices to show that if $f : \mathbb{T} \times \mathbb{I} \to \mathbb{T} \times \mathbb{I}$ is an extension of the identity on $\mathbb{T} \times \partial\mathbb{I}$ such that $\mathrm{var}_f = 0$, then $(\mathrm{id}, f)^*\iota^1(\mathbb{T} \times \mathbb{I}) = (\mathrm{id}, \mathrm{id})^*\iota^1(\mathbb{T} \times \mathbb{I})$. In fact, to avoid working with the connecting morphism, it suffices to see that

$$F^*_{(\mathrm{id},f)}\iota^1(\mathbb{T} \times \mathbb{I}) = F^*_{(\mathrm{id},\mathrm{id})}\iota^1(\mathbb{T} \times \mathbb{I}) \in H^1\big(\mathbb{T} \times \mathbb{I} \times \partial\mathbb{I} \cup \mathbb{T} \times \partial\mathbb{I} \times \mathbb{I}; \mathbb{Z}^2\big).$$

Since $\mathbb{T} \times \mathbb{I} \times \partial\mathbb{I} \cup \mathbb{T} \times \partial\mathbb{I} \times \mathbb{I} = \mathbb{T} \times \partial(\mathbb{I} \times \mathbb{I})$, we see that

$$H_1(\mathbb{T} \times \mathbb{I} \times \partial\mathbb{I} \cup \mathbb{T} \times \partial\mathbb{I} \times \mathbb{I}) \cong H_1(\mathbb{T}) \oplus H_1\big(\partial(\mathbb{I} \times \mathbb{I})\big) \cong \mathbb{Z}^3,$$

hence $H^1(\mathbb{T} \times \mathbb{I} \times \partial\mathbb{I} \cup \mathbb{T} \times \partial I \times \mathbb{I}; \mathbb{Z}^2) \cong \mathrm{Hom}(H_1(\mathbb{T} \times \mathbb{I} \times \partial\mathbb{I} \cup \mathbb{T} \times \partial\mathbb{I} \times \mathbb{I}), \mathbb{Z}^2) \cong \mathbb{Z}^3 \otimes \mathbb{Z}^2$.

Recall that $\mathfrak{c}_0$, $\mathfrak{c}_1$ is a basis of $H_1(\mathbb{T} \times \mathbb{I}) = H_1(\mathbb{T})$ such that $\mathfrak{c}_0 \subset \mathbb{D}^* \times \{e^{i\theta}\}$ and $\mathfrak{c}_1 \subset \{z\} \times \mathbb{S}^1$. Let $\mathfrak{e}$ be a generator of $H_1(\partial(\mathbb{I} \times \mathbb{I})) \cong \mathbb{Z}$. It is easy to see that $\iota^1(\mathbb{T} \times \mathbb{I}) \in H^1(\mathbb{T} \times \mathbb{I}; \pi_1(\mathbb{T} \times \mathbb{I})) \cong \mathrm{Hom}(H_1(\mathbb{T} \times \mathbb{I}), H_1(\mathbb{T} \times \mathbb{I}))$ can be identified with the identity map and consequently $F^*_{(\mathrm{id},f)}\iota^1(\mathbb{T} \times \mathbb{I}) \cong F_{(\mathrm{id},f)*}$, where

$$F_{(\mathrm{id},f)*} : H_1\big(\mathbb{T} \times \partial(\mathbb{I} \times \mathbb{I})\big) \cong \mathbb{Z}\mathfrak{c}_0 \oplus \mathbb{Z}\mathfrak{c}_1 \oplus \mathbb{Z}\mathfrak{e} \to \mathbb{Z}\mathfrak{c}_0 \oplus \mathbb{Z}\mathfrak{c}_1 \cong H_1(\mathbb{T} \times \mathbb{I})$$

is represented by a matrix of the form

$$\begin{pmatrix} 1 & 0 & k \\ 0 & 1 & m \end{pmatrix},$$

where $(m, k) \in \mathbb{Z}^2$ verify that $\mathrm{var}_f(\mathfrak{d}) = k\mathfrak{c}_0 + m\mathfrak{c}_1$ and $\mathfrak{d}$ is the generator of $H_1(\mathbb{T} \times \mathbb{I}, \mathbb{T} \times \partial\mathbb{I}; \mathbb{Z}) \cong \mathbb{Z}$ joining the two connected components of $\mathbb{T} \times \partial\mathbb{I}$. The proof of the proposition is now complete. □

5. Mapping class group of a germ of curve

Given a germ of plane curve S we denote by

- $\mathcal{G}_S$ the set of markings of S by itself, which is a group with the composition;
- Γ_S the fundamental group of the pointed Milnor tube $T_\eta \setminus S$;
- $\mathrm{Out}(\Gamma_S) := \mathrm{Aut}(\Gamma_S)/\mathrm{Inn}(\Gamma_S)$ the group of exterior automorphisms of Γ_S;
- $\mathrm{Out}_g(\Gamma_S)$ the subgroup of $\mathrm{Out}(\Gamma_S)$ consisting of geometric exterior automorphisms, see Definition 3.17.

Theorem 5.1. *The map* $* : \mathcal{G}_S \to \mathrm{Out}(\Gamma_S)$ *sending each marking* $[h]$ *to its action* h_* *into the fundamental group* Γ_S, *is an isomorphism onto* $\mathrm{Out}_g(\Gamma_S)$.

Proof. The map $*$ is well defined precisely because when considering the outer automorphism group we are eliminating the ambiguity in the choice of h in its fundamental class $[h]$. The map is trivially a monomorphism of groups thanks to Proposition 2.8 because $T_\eta \setminus S$ is a $K(\Gamma_S, 1)$ space. Finally, that the image of $*$ is $\mathrm{Out}_g(\Gamma_S)$ follows from Corollary 3.20. □

Corollary 5.2. *Every element of* $\mathrm{Out}_g(\Gamma_S)$ *can be realised by an excellent homeomorphism of* (T_η, S) *onto itself.*

Let $\mathbb{A}_S$ be the weighted dual tree of the minimal resolution of singularities of S and let $\mathfrak{S}_S$ be the permutation group of the set of irreducible components of S. There exist two well-defined natural morphisms $\sigma : \mathcal{G}_S \to \mathfrak{S}_S$ and $\bar{\sigma} : \mathrm{Aut}(\mathbb{A}_S) \to \mathfrak{S}_S$.

The existence of an excellent homeomorphism in each homotopy class of $\mathcal{G}_S$ and the fact that $\bar\sigma$ is one-to-one, proved in the next lemma, allows us to consider a well-defined morphism

$$\alpha : \mathcal{G}_S \to \operatorname{Aut}(\mathbb{A}_S)$$

such that $\sigma = \bar\sigma \circ \alpha$.

Lemma 5.3. *With the precedent notations we have that*

(i) $\bar\sigma$ *is one-to-one, and consequently* $\ker\sigma = \ker\alpha$;
(ii) α *is onto and therefore* $\operatorname{Im}\sigma = \operatorname{Im}\bar\sigma$.

Proof. The first assertion can easily proved by induction on the number r of irreducible components of S. The case $r = 1$ is proved by induction on the number g of Puiseux pairs of S. When $g = 1$ a completely explicit description of the situation shows that $\bar\sigma$ is one-to-one in this case. The second assertion can also be proved by induction on the number of irreducible components of S. When S is irreducible then $\operatorname{Aut}(\mathbb{A}_S) = \{\mathrm{id}\}$ by Assertion (i). If S_i and S_j are two irreducible components of S exchanged by $g \in \operatorname{Aut}(\mathbb{A}_S)$ then the weighted subtrees corresponding to the resolutions of S_i and S_j are isomorphic. In this case, it is easy to see that there is a homeomorphism from $(\mathcal{T}_\eta, \mathcal{D})$ onto itself which induces g and which is the identity outside of a neighbourhood of the part of the divisor $\mathcal{D}$ not intersecting the subtrees corresponding to S_i and S_j. □

Always with the notations (3), (4), (26), (27), for each chain $\mathcal{C} \in \mathfrak{C}$ we put $K_\mathcal{C} := \mathcal{T}_\mathcal{C} \cap \mathcal{D}$, $\mathcal{T}_\eta(\partial K_\mathcal{C}) = \mathcal{T}_\eta(\partial(K_\mathcal{C} \cap D_0)) \cup \mathcal{T}_\eta(\partial(K_\mathcal{C} \cap D_{l_\mathcal{C}+1}))$.

Definition 5.4. For each element $B \in \mathfrak{B} := \mathfrak{R} \cup \mathfrak{C}$ we consider the group $\mathcal{G}_B$ of homotopy classes relatively to $K_B \cup \mathcal{T}_\eta(\partial K_B)$ of homeomorphisms from $\mathcal{T}_B$ onto itself, preserving K_B and which are the identity on $\mathcal{T}_\eta(\partial K_B)$.

Every element of $\mathcal{G}_B$ induces an excellent marking whose support is contained in $\mathcal{T}_\eta(K_B)$. Hence we have a well-defined morphism

$$\beta : \bigoplus_{B \in \mathfrak{B}} \mathcal{G}_B \to \mathcal{G}_S.$$

Proposition 5.5. *Fix $D \in \mathfrak{R}$ and $\mathcal{C} \in \mathfrak{C}$.*

(i) *The group $\mathcal{G}_D$ is isomorphic to the group $A(D^\bullet)$ of relative to $S(D)$ homotopy classes of homeomorphisms of D fixing pointwise $S(D)$.*
(ii) *Each element of $\mathcal{G}_\mathcal{C}$ is a Dehn twist along $\mathcal{C}$, cf. Section* 4.4.4. *In particular, $\mathcal{G}_\mathcal{C} \cong \mathbb{Z}^2$.*

Proof. To prove Assertion (i) we trivialise $\mathcal{T}_\eta(K_D) \cong K_D \times \mathbb{D}$ and we express an excellent representative of an arbitrary element $\mathfrak{f}$ of $\mathcal{G}_D$ under the form (f, g), where $f : K_D \to K_D$ is a homeomorphism which is the identity on ∂K_D and $g : K_D \to \operatorname{Homeo}(\mathbb{D}, 0) \simeq \mathbb{S}^1$. Since $g_{|\partial K_D}$ is constant equal to $\mathrm{id}_\mathbb{D}$ it follows that (f, g) is isotopic to $(f, \mathrm{id}_\mathbb{D})$. Thus, $\mathfrak{f} = [(f, g)]$ is completely determined by $[f] \in A(D^\bullet)$. Conversely, each element $[f] \in A(D^\bullet)$ determines a unique element $[(f, \mathrm{id}_\mathbb{D})] \in \mathcal{G}_D$. On the other hand, Assertion (ii) follows directly from Proposition 4.13. □

The pure mapping class group $A(D^\bullet)$ can be identified to the quotient of the Artin pure braid group on $\nu(D)$ strands over the 2-sphere by its centre (which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$), see for instance [2]. It can also be compared with the pure braid group on $\nu(D) - 1$ strands over the disk. We call the elements of $\mathcal{G}_D$ *Artin twist over D*.

After Proposition 5.5, Theorem B claims that the image of β is the kernel $\mathcal{G}_S^0$ of σ, in other words, the Artin twists and the Dehn twists generate the finite index subgroup $\mathcal{G}_S^0$ of the mapping class group $\mathcal{G}_S$.

Proof of Theorem B. By Theorem A every element of $\mathcal{G}_S^0$ can be represented by an excellent homeomorphism $f : \mathcal{T}_\eta \to \mathcal{T}_\eta$ fixing each irreducible component of $\mathcal{D}$ and which is the identity[20] over the boundary of each block $\mathcal{T}_\eta(K_D)$ and $\mathcal{T}_\eta(\mathcal{C})$. By Seifert–Van Kampen theorem, the fundamental group Γ_S is the amalgamated product of the fundamental groups $\Gamma_S(D) = \pi_1(B_D)$, $D \in \mathfrak{R}$, of the Seifert blocks in the JSJ decomposition of M_η over the fundamental groups of the essential tori $\Gamma_S(\mathcal{C}) = \pi_1(\mathbb{T}_\mathcal{C})$, $\mathcal{C} \in \mathfrak{C}$. Let $D \in \mathfrak{R}$ be a terminal vertex of the JSJ tree of M_η, cf. Remark 3.11, and let $\mathcal{C} \in \mathfrak{C}$ be its adjacent chain. By composing f by suitable elements of $\mathcal{G}_D$ and $\mathcal{G}_\mathcal{C}$ we can assume that $f_* : \Gamma_S \to \Gamma_S$ is the identity over $\Gamma_S(D) \supset \Gamma_S(\mathcal{C})$. We conclude by reasoning by induction on the number of Seifert blocks on which f_* is not the identity.

[20] This is possible thanks to the fact that f is holomorphic in a neighbourhood of each singularity of $\mathcal{D}$.

This proves that the subgroups $\mathcal{G}_D$ and $\mathcal{G}_{\mathcal{C}}$ generate $\mathcal{G}_S^0$. On the other hand, their direct product structure follows from the fact that they have disjoint supports. □

The following example shows that the epimorphism of Theorem B is not one-to-one in general. Thus, it could exist other relations between the generators of $\mathcal{G}_D$ and $\mathcal{G}_{\mathcal{C}}$ apart from those just we make explicit.

Example 5.6. The curve S defined by the equation

$$f(x, y) = \left(y^2 - x^3\right)^2 - \alpha x^5 y - \beta x^2 y^3 = 0$$

is irreducible for generic (α, β), it has two Puiseux pairs, the exceptional divisor of its minimal desingularisation consists of five lines, E_i, $i = 1, \dots, 5$, numbered in order of appearance, and having intersection matrix

$$\begin{pmatrix} -3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}.$$

In this case, there are two irreducible components E_3 and E_5 having valence three with two (resp. one) adjacent dead branches E_1, E_2 (resp. E_4). There is only one chain, $\mathcal{C}$, having length 0, which correspond to the point $E_3 \cap E_5$. The fundamental group Γ_S of a pointed Milnor tube of f admits as generators the homotopy classes a_1, b_1, c_1, b_2, c_2, d of loops contained in Hopf fibres of the components E_1, E_2, E_3, E_4, E_5 and S respectively. The relations of these generators are generated by

$$a_1^3 = c_1 = b_1^2, \qquad a_1 b_1 c_2 = c_1^3, \qquad c_2 = b_2^2, \qquad c_1 b_2 d = c_2$$

and

$$[c_1, a_1] = [c_1, b_1] = [c_1, c_2] = [c_2, b_2] = [c_2, d] = 1. \tag{33}$$

By taking suitably the base point, the action on Γ_S of a Dehn twist along $\mathcal{C}$ of type (p, q) has the form

$$a_1 \mapsto a_1, \qquad b_1 \mapsto b_1, \qquad c_1 \mapsto c_1, \qquad b_2 \mapsto c_1^p b_2 c_1^{-p}, \qquad c_2 \mapsto c_2, \qquad d \mapsto c_1^p d c_1^{-p}.$$

After relations (33), it coincides with the inner automorphism associated to the element $c_1^p c_2^q \in \Gamma_S$. Thus, in this case, $\beta(\mathcal{G}_{\mathcal{C}}) \subset \ker(*)$ which is trivial.

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