

On the existence, regularity and decay of solitary waves to a generalized Benjamin-Ono equation

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Abstract

We consider a two-dimensional generalization of the Benjamin-Ono equation and prove that it admits solitary-wave solutions that are analytic functions. We find the optimal decay rate at infinity of these solitary waves.

1 Introduction

We study the solitary waves of the following generalization of the Benjamin-Ono (BO) equation

$$(1) \quad A_t + \alpha A A_x - \beta (-\Delta)^{\frac{1}{2}} A_x = 0$$

in \mathbf{R}^2 , where $\alpha, \beta > 0$ and $(-\Delta)^{\frac{1}{2}}$ is the operator defined by $\mathcal{F}((-\Delta)^{\frac{1}{2}} u)(\xi) = |\xi| \widehat{u}(\xi)$. \mathcal{F} or $\widehat{\cdot}$ represent the Fourier transform.

Equation (1) describes the dynamics of three-dimensional, slightly nonlinear disturbances in boundary-layer shear flows (without the assumption of a difference in their scales along and across the flow), see [2], [6].

The solitary waves of (1) are solutions of the form $A(x, y, t) = v(x - ct, y)$ where c is the speed of the solitary wave. It seems that solitary waves play an important role in the evolution of (1). Such a solution must satisfy the equation

$$(2) \quad -cv_x + \frac{\alpha}{2} (v^2)_x - \beta (-\Delta)^{\frac{1}{2}} v_x = 0.$$

Numerical experiments ([6]) show the existence of solitary waves (solitons). It has also been observed that the solitons decay at infinity like some power of $r = \sqrt{x^2 + y^2}$.

Our aim is to give rigorous proofs of these facts.

We suppose throughout that the wave speed c is positive.

In the next section, we show that solitary waves exist and are smooth (analytic) functions. Since the techniques we use are classical, we only sketch the proofs. In the last section we prove that the solutions of some generalization of equation (2) in \mathbf{R}^n decay at infinity as $\frac{1}{|x|^{n+1}}$ and this algebraic rate is nearly optimal. We hope that our results about the decay of solutions of a quite general equation should be useful elsewhere (see also Remark 8 below).

Our method to study analyticity and decay of solutions was inspired by the ideas developed by BONA and LI in [3], [4] for one-dimensional problems.

2 Existence and regularity

In order to simplify equation (2), we integrate it once in x and make the scale change $v(x, y) = au(bx, by)$, where $a = \frac{2c}{\alpha}$ and $b = \frac{c}{\beta}$. Then (2) reduces to

$$(3) \quad u + (-\Delta)^{\frac{1}{2}}u = u^2$$

or, using the Fourier transform,

$$(4) \quad (1 + |\xi|)\widehat{u} = \widehat{u^2}.$$

Let us introduce the functionals

$$V(u) = \frac{1}{2} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}}u|^2 + |u|^2 dx = \frac{1}{2(2\pi)^2} \int_{\mathbf{R}^2} (1 + |\xi|)|\widehat{u}|^2 d\xi$$

and

$$I(u) = \frac{1}{3} \int_{\mathbf{R}^2} u^3 dx.$$

Clearly V and I are well defined and of class C^2 on the Sobolev space $H^{\frac{1}{2}}(\mathbf{R}^2)$.

For $\mu \neq 0$, we consider the minimization problem

$$(\mathcal{P}) \quad \text{minimize } V(u) \text{ under the constraint } I(u) = \mu.$$

A minimizer of (\mathcal{P}) is called a ground state. If u is such a minimizer, there exists a Lagrange multiplier λ such that

$$(1 + (-\Delta)^{\frac{1}{2}})u = \lambda u^2.$$

It is easy to see that $\lambda\mu$ is positive (because the above equation gives $2V(u) = 3\lambda I(u)$). Then λu is a non-trivial solution of (3). Clearly λu minimizes $V(v)$ subject to the constraint $I(v) = I(\lambda u)$.

Theorem 1. *There exists minimizers of problem (\mathcal{P}) . Consequently, equation (3) admits non-trivial solutions.*

Proof. One may prove Theorem 1 by using the concentration-compactness principle, as it was done in [1] to show the existence of solitary waves for the ILW equation. The main difficulty is to eliminate dichotomy. To do this, one needs to estimate the L^2 -norm of the commutator $(L\chi - \chi L)u$ in terms of the derivatives of order ≥ 1 of χ and the $H^{\frac{1}{2}}$ -norm of u , where $Lu = \mathcal{F}^{-1}((1 + |\xi|)^{\frac{1}{2}}\widehat{u})$ and $\chi \in C_0^\infty(\mathbf{R}^2)$. But this can be done and we obtain the existence of ground states.

We may also observe that problem (\mathcal{P}) is exactly of the type discussed by O. LOPES in a recent paper ([5]). The functionals V and I satisfy the assumptions **HH₁** – **HH₆** of Lopes and using the Theorems 3.1 and 3.15 in [5], we infer that any minimizing sequence (u_n) of (\mathcal{P}) possesses a subsequence that converges strongly in $H^{\frac{1}{2}}(\mathbf{R}^2)$ (modulo translation in \mathbf{R}^2) to an element u which is a ground state. \square

We give another variational characterization of the ground states. We consider the functionals

$$E(u) = \frac{1}{2} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}}u|^2 dx - \frac{1}{3} \int_{\mathbf{R}^2} u^3 dx$$

and

$$Q(u) = \frac{1}{2} \int_{\mathbf{R}^2} |u|^2 dx.$$

Proposition 2. *Let u_\star be a minimizer of V under the constraint $I(u) = \mu$. Suppose that u_\star satisfies the equation (3). Then $E(u_\star) = 0$ and u_\star is a solution of the problem*

$$(\mathcal{P}') \quad \text{minimize } E(v) \text{ under the constraint } Q(v) = Q(u_\star).$$

Proof. Multiplying (3) by u and integrating we obtain the identity

$$(5) \quad \int_{\mathbf{R}^2} |u|^2 dx + \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u|^2 dx = \int_{\mathbf{R}^2} u^3 dx.$$

For $u \in H^{\frac{1}{2}}(\mathbf{R}^2)$ we denote $u_{a,b}(x) = bu(ax)$. Then $I(u_{a,b}) = b^3 a^{-2} I(u)$, $\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u_{a,b}|^2 dx = b^2 a^{-1} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u|^2 dx$ and $Q(u_{a,b}) = b^2 a^{-2} Q(u)$.

We have $I(u_{\star, a, a^{\frac{2}{3}}}) = I(u_\star)$ and since u_\star is a minimizer of the problem (\mathcal{P}) , the function

$$f(a) = V(u_{\star, a, a^{\frac{2}{3}}}) = \frac{1}{2} a^{\frac{1}{3}} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u_\star|^2 dx + \frac{1}{2} a^{-\frac{2}{3}} \int_{\mathbf{R}^2} |u_\star|^2 dx$$

has a minimum at $a = 1$. Hence $f'(1) = 0$, that is

$$(6) \quad \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u_\star|^2 dx = 2 \int_{\mathbf{R}^2} |u_\star|^2 dx.$$

Combining (5) and (6) we obtain $E(u_\star) = 0$.

Let $v \in H^{\frac{1}{2}}(\mathbf{R}^2)$ such that $Q(v) = Q(u_\star)$. We want to show that $E(v) \geq E(u_\star) = 0$. This clearly holds if $I(v) \leq 0$. Suppose that $I(v) > 0$. For $a > 0$, let $b(a) = a^{\frac{2}{3}} \left(\frac{\int u_\star^3 dx}{\int v^3 dx} \right)^{\frac{1}{3}}$. Then $I(v_{a,b(a)}) = I(u_\star)$, hence $V(v_{a,b(a)}) \geq V(u_\star)$ and this gives

$$(7) \quad \left(\frac{\int u_\star^3 dx}{\int v^3 dx} \right)^{\frac{2}{3}} \cdot \left[a^{\frac{1}{3}} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} v|^2 dx + a^{-\frac{2}{3}} \int_{\mathbf{R}^2} |v|^2 dx \right] \geq \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u_\star|^2 dx + \int_{\mathbf{R}^2} |u_\star|^2 dx.$$

The minimum of the left side of (7) for $a \in (0, \infty)$ is

$$3 \left(\frac{\int u_\star^3 dx}{\int v^3 dx} \right)^{\frac{2}{3}} \cdot \left(\frac{1}{2} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} v|^2 dx \right)^{\frac{2}{3}} \left(\int_{\mathbf{R}^2} |v|^2 dx \right)^{\frac{1}{3}}.$$

Hence

$$\begin{aligned} & 3 \left(\frac{\frac{1}{2} \int |(-\Delta)^{\frac{1}{4}} v|^2 dx}{\int v^3 dx} \right)^{\frac{2}{3}} \left(\int_{\mathbf{R}^2} u_\star^3 dx \right)^{\frac{2}{3}} \left(\int_{\mathbf{R}^2} |v|^2 dx \right)^{\frac{1}{3}} \\ & \geq \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u_\star|^2 dx + \int_{\mathbf{R}^2} |u_\star|^2 dx. \end{aligned}$$

Since $\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u_\star|^2 dx = 2 \int_{\mathbf{R}^2} |u_\star|^2 dx$ and $\int_{\mathbf{R}^2} u_\star^3 dx = 3 \int_{\mathbf{R}^2} |u_\star|^2 dx$ by (5) and (6) and $\int_{\mathbf{R}^2} |v|^2 dx = \int_{\mathbf{R}^2} |u_\star|^2 dx$ by assumption, we obtain

$$\frac{\frac{1}{2} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} v|^2 dx}{\int_{\mathbf{R}^2} v^3 dx} \geq \frac{1}{3},$$

that is $E(v) \geq 0$. □

Remark 3. The converse of Proposition 2 is valid modulo a scale change. More precisely, let u_\star be as above and let v be a solution of problem (\mathcal{P}') . Then there exists $a > 0$ such that $v_{a,a}$ is a solution of problem (\mathcal{P}) .

Indeed, $Q(v_{a,a}) = Q(v)$ and $E(v_{a,a}) = aE(v)$. Since v is a minimizer of (\mathcal{P}') , necessarily $E(v) = 0$. Moreover, there exists a Lagrange multiplier λ such that

$$(-\Delta)^{\frac{1}{2}}v - v^2 = -\lambda v.$$

Then $\lambda \int_{\mathbf{R}^2} |v|^2 dx = - \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}}v|^2 dx + \int_{\mathbf{R}^2} v^3 dx = \frac{1}{2} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}}v|^2 dx$, hence $\lambda > 0$. Denote $v_\star = v_{\frac{1}{\lambda}, \frac{1}{\lambda}}$. Then v_\star satisfies (3). Multiplying this by v_\star and integrating, we find that v_\star also satisfies (5). But $E(v_\star) = 0$ and so we deduce that $\frac{1}{2} \int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}}v_\star|^2 dx = \frac{1}{3} \int_{\mathbf{R}^2} v_\star^3 dx = \int_{\mathbf{R}^2} |v_\star|^2 dx = \int_{\mathbf{R}^2} |u_\star|^2 dx$. Now it is clear that $I(v_\star) = I(u_\star)$ and $V(v_\star) = V(u_\star)$, i.e. $V(v_\star)$ achieves the minimum of $V(w)$ for all $w \in H^{\frac{1}{2}}(\mathbf{R}^2)$ such that $I(w) = I(u_\star)$.

Now we turn our attention to the regularity of solitary waves.

Theorem 4. *Let $u \in H^{\frac{1}{2}}(\mathbf{R}^2)$ be a solution of (3). Then $u \in W^{k,p}(\mathbf{R}^2)$ for all $k \in \mathbf{N}$ and all $p \in [1, \infty]$. In particular, u is a C^∞ function and tends to zero at infinity.*

Proof. By the Sobolev imbedding theorem, $H^{\frac{1}{2}}(\mathbf{R}^2) \subset L^4(\mathbf{R}^2)$, so that $u^2 \in L^2(\mathbf{R}^2)$. From (4) we deduce that $|\xi|\widehat{u} \in L^2(\mathbf{R}^2)$, hence $u \in H^1(\mathbf{R}^2)$. Again by Sobolev's imbedding we have $u \in L^p(\mathbf{R}^2)$ for $2 \leq p < \infty$.

It is easy to check that the functions $m(\xi) = \frac{1}{1+|\xi|}$ and $m_i(\xi) = \frac{\xi_i}{1+|\xi|}$ satisfy $|\partial^\alpha m(\xi)| \leq C|\xi|^{-|\alpha|}$ and $|\partial^\alpha m_i(\xi)| \leq C|\xi|^{-|\alpha|}$ for $|\alpha| = 0, 1, 2$ and a classical theorem of Mikhlin implies that $m, m_i \in M_q(\mathbf{R}^2)$ for $1 < q < \infty$, i.e. m, m_i are Fourier multipliers for $L^q(\mathbf{R}^2)$, $1 < q < \infty$. Equation (4) gives

$$\widehat{u}(\xi) = m(\xi)\widehat{u^2}(\xi) \text{ and } \widehat{u_{x_j}}(\xi) = im_j(\xi)\widehat{u^2}(\xi)$$

and Mikhlin's theorem implies that $u, u_{x_j} \in L^p(\mathbf{R}^2)$ for all $p \in]1, \infty[$. Hence $u \in W^{1,p}(\mathbf{R}^2)$, $\forall p \in]1, \infty[$. In particular, u is continuous and tends to zero at infinity.

It follows easily by induction that $u \in W^{k,p}(\mathbf{R}^2)$ for all $k \in \mathbf{N}$ and $p \in]0, \infty[$. Indeed, suppose that $u \in W^{n,p}(\mathbf{R}^2)$ for all $p \in]1, \infty[$. If $\alpha_1, \alpha_2 \in \mathbf{N}$, $\alpha_1 + \alpha_2 = n$, we have for example

$$\mathcal{F}(\partial_{x_1}^{\alpha_1+1} \partial_{x_2}^{\alpha_2} u) = \frac{i\xi_1}{1+|\xi|} \mathcal{F}(\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} (u^2)).$$

The induction hypothesis implies that $\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} (u^2) \in L^p(\mathbf{R}^2)$, $\forall p \in]1, \infty[$. Again by Mikhlin's theorem we obtain $\partial_{x_1}^{\alpha_1+1} \partial_{x_2}^{\alpha_2} u \in L^p(\mathbf{R}^2)$, $1 < p < \infty$ and so $u \in W^{n+1,p}(\mathbf{R}^2)$ for all $p \in]1, \infty[$.

The fact that $u \in W^{k,1}(\mathbf{R}^2)$ for all $k \in \mathbf{N}$ can be easily proved by writing (3) as a convolution equation and using Lemma 7 below (see also the proof of Theorem 11 and Remark 12). \square

Theorem 5. *Let $u \in H^{\frac{1}{2}}(\mathbf{R}^2)$ be a solution of (3). Then there exists $\sigma > 0$ and an holomorphic function U of two complex variables z_1, z_2 defined in the domain*

$$\Omega_\sigma = \{(z_1, z_2) \in \mathbf{C}^2 \mid |Im(z_1)| < \sigma, |Im(z_2)| < \sigma\}$$

such that $U(x, y) = u(x, y)$ for all $(x, y) \in \mathbf{R}^2$.

Proof. By Theorem 4 we have $(1 + |\xi|^2)^{\frac{m}{2}} \widehat{u}(\xi) \in L^2(\mathbf{R}^2)$ for all $m \geq 0$. We take $m > 1$ and apply Cauchy-Schwarz' inequality to get

$$\int_{\mathbf{R}^2} |\widehat{u}|(\xi) d\xi \leq \left(\int_{\mathbf{R}^2} (1 + |\xi|^2)^m |\widehat{u}|^2(\xi) d\xi \right)^{\frac{1}{2}} \cdot \left(\int_{\mathbf{R}^2} (1 + |\xi|^2)^{-m} d\xi \right)^{\frac{1}{2}} < \infty.$$

Hence $\widehat{u} \in L^1(\mathbf{R}^2)$. Equation (4) implies that

$$|\widehat{u}|(\xi) \leq |\widehat{u}| \star |\widehat{u}|(\xi) \text{ and } |\xi| |\widehat{u}|(\xi) \leq |\widehat{u}| \star |\widehat{u}|(\xi).$$

We note $\mathcal{C}_1 |\widehat{u}| = |\widehat{u}|$ and for $n \geq 1$, $\mathcal{C}_{n+1} |\widehat{u}| = (\mathcal{C}_n |\widehat{u}|) \star |\widehat{u}|$.

Lemma 6. *We have for all $k \in \mathbf{N}$*

$$|\xi|^k |\widehat{u}|(\xi) \leq (k+1)^{k-1} \mathcal{C}_{2(k+1)} |\widehat{u}|(\xi).$$

The lemma follows easily by induction, using the identity

$$\sum_{j=0}^k \mathcal{C}_k^j (1+j)^{j-1} (1+k-j)^{k-j+1} = 2(2+k)^{k-1}$$

(which is a specialization of Abel's identity).

Using Lemma 6, we have

$$\begin{aligned} |\xi|^k |\widehat{u}|(\xi) &\leq (k+1)^{k-1} \|\mathcal{C}_{2(k+1)} |\widehat{u}|\|_{L^\infty} \leq (k+1)^{k-1} \|\mathcal{C}_{2k+1} |\widehat{u}|\|_{L^2} \cdot \|\widehat{u}\|_{L^2} \\ &\leq (k+1)^{k-1} \|\widehat{u}\|_{L^1}^{2k} \cdot \|\widehat{u}\|_{L^2}^2. \end{aligned}$$

Let $a_k = \frac{(k+1)^{k-1} \|\widehat{u}\|_{L^1}^{2k} \cdot \|\widehat{u}\|_{L^2}^2}{k!}$. Clearly $\frac{a_{k+1}}{a_k} = \|\widehat{u}\|_{L^1}^2 \cdot \left(\frac{k+2}{k+1}\right)^k \rightarrow e \|\widehat{u}\|_{L^1}^2$ as $k \rightarrow \infty$. Let $\sigma = \frac{1}{e \|\widehat{u}\|_{L^1}^2}$.

If $0 < \tau < \sigma$, the series $\sum_{k=0}^{\infty} \frac{(\tau |\xi|)^k}{k!} |\widehat{u}|(\xi)$ converges uniformly in L^∞ -norm (because each term is dominated by $\tau^k a_k$ and the series $\sum_{k=0}^{\infty} \tau^k a_k$ converges absolutely).

Hence $e^{\tau |\xi|} \widehat{u}(\xi) \in L^\infty(\mathbf{R}^2)$ for $\tau < \sigma$.

We define the function

$$U(z_1, z_2) = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} e^{i(z_1 \xi_1 + z_2 \xi_2)} \widehat{u}(\xi_1, \xi_2) d\xi_1 d\xi_2.$$

By the Paley-Wiener Theorem, U is well defined and analytic in Ω_σ and by Plancherel's Theorem we have $U(x, y) = u(x, y)$ for all $(x, y) \in \mathbf{R}^2$. \square

3 Decay properties

We consider a generalization of equation (3) in \mathbf{R}^n , namely

$$(8) \quad (1 + (-\Delta)^{\frac{1}{2}})u = g(u)$$

with the following assumptions on g :

- i) $g : \mathbf{C} \rightarrow \mathbf{C}$ is continuous and

ii) there exists $\gamma > 1$ and $C > 0$ such that $|g(z)| \leq C|z|^\gamma$, $\forall z \in \mathbf{C}$.

The aim of this paragraph is to prove that the solutions of (8) that tend to zero at infinity must decay (at least) as $\frac{1}{|x|^{n+1}}$.

Equation (8) may be written in the equivalent forms

$$(9) \quad \widehat{u} = \frac{1}{1 + |\xi|} \widehat{g(u)}$$

or

$$(10) \quad u = k \star g(u),$$

where $k = \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|}\right)$. We begin with some estimates on the kernel k .

Lemma 7.

i) We have

$$k(x) = c_n \int_0^\infty e^{-s} \cdot \frac{s}{(|x|^2 + s^2)^{\frac{n+1}{2}}} ds, \quad \text{where } c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}.$$

ii) $k \in C^\infty(\mathbf{R}^n \setminus \{0\})$ and there exist positive constants A_1^n, A_2^n such that

$$\begin{aligned} A_1^n |x|^{-n+1} &\leq k(x) < \frac{c_n}{n-1} |x|^{-n+1} && \text{if } 0 < |x| \leq 1, n \geq 2, \quad \text{respectively} \\ -c_1 e^{-1} \ln|x| &< k(x) < -c_1 \ln|x| + c_1 && \text{if } 0 < |x| \leq 1, n = 1 \quad \text{and} \\ A_2^n |x|^{-n-1} &\leq k(x) < c_n |x|^{-n-1} && \text{if } |x| \geq 1, n \geq 1. \end{aligned}$$

iii) $|x|^{n+1}k(x) \in L^\infty(\mathbf{R}^n)$ and for $1 \leq p < \infty$ we have $|x|^\alpha k(x) \in L^p(\mathbf{R}^n)$ if and only if

$$(11) \quad n - 1 - \frac{n}{p} < \alpha < n + 1 - \frac{n}{p}.$$

In particular, $k \in L^p(\mathbf{R}^n)$ if and only if $1 \leq p < \frac{n}{n-1}$.

Remark 8. From now on, we use only equation (10), the assumptions i) and ii) on g and the estimates on k given by Lemma 7, iii). Hence our result about the decay of solutions (Theorem 11 below) holds for any equation that can be written in the form (10) with a kernel k that satisfies the conclusion iii) of Lemma 7.

Proof of Lemma 7. i) For any $\phi \in \mathcal{S}$ (the Schwartz' space of rapidly decreasing functions) we have

$$\begin{aligned} \langle k, \phi \rangle_{\mathcal{S}', \mathcal{S}} &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{1}{1 + |\xi|} \int_{\mathbf{R}^n} e^{ix \cdot \xi} \phi(x) dx d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \int_0^\infty e^{-(1+|\xi|)s} ds \cdot \int_{\mathbf{R}^n} e^{ix \cdot \xi} \phi(x) dx d\xi \\ &= \int_0^\infty e^{-s} \int_{\mathbf{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix \cdot \xi} e^{-|\xi|s} d\xi \right) \phi(x) dx ds \\ &= \int_0^\infty e^{-s} \int_{\mathbf{R}^n} P_s(x) \phi(x) dx \end{aligned}$$

$$\text{where } P_s(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix \cdot \xi} e^{-|\xi|s} d\xi = \frac{c_n s}{(|x|^2 + s^2)^{\frac{n+1}{2}}}$$

is the Poisson kernel

$$= \int_{\mathbf{R}^n} c_n \int_0^\infty e^{-s} \cdot \frac{s}{(|x|^2 + s^2)^{\frac{n+1}{2}}} ds \phi(x) dx.$$

This proves i).

ii) It is obvious that $k \in C^\infty(\mathbf{R}^n \setminus \{0\})$. Using i), for $n \geq 2$ and $0 < |x| \leq 1$ we clearly have $k(x) > c_n \int_0^{|x|} \frac{se^{-1}}{(|x|^2+s^2)^{\frac{n+1}{2}}} ds = c_n \frac{e^{-1}}{n-1} \left(1 - \frac{1}{2^{\frac{n-1}{2}}}\right) \frac{1}{|x|^{\frac{1}{n-1}}}$ and

$$k(x) < c_n \int_0^\infty \frac{s}{(|x|^2+s^2)^{\frac{n+1}{2}}} ds = \frac{c_n}{n-1} \frac{1}{|x|^{\frac{1}{n-1}}}.$$

For $n = 1$ and $0 < |x| \leq 1$, integrating by parts and using the elementary inequality $\ln(x^2 + s^2) \leq \ln(x^2 + 1) < s^2$ for $s \neq 0$ we obtain $k(x) = -\frac{1}{2}c_1 \ln x^2 + \frac{1}{2}c_1 \int_0^\infty e^{-s} \ln(x^2 + s^2) ds < -c_1 \ln |x| + \frac{1}{2}c_1 \int_0^\infty e^{-s} s^2 ds = -c_1 \ln |x| + c_1$ and obviously $k(x) > c_1 \int_0^1 \frac{se^{-1}}{x^2+s^2} ds = \frac{1}{2}c_1 e^{-1} (\ln(x^2 + 1) - \ln x^2) > -c_1 e^{-1} \ln |x|$.

For $|x| \geq 1$ we get $k(x) > c_n \int_0^1 \frac{se^{-s}}{(2|x|^2)^{\frac{n+1}{2}}} ds = c_n \left(1 - \frac{2}{e}\right) 2^{-\frac{n+1}{2}} \frac{1}{|x|^{\frac{n+1}{2}}}$ and $k(x) < c_n \int_0^\infty \frac{se^{-s}}{|x|^{\frac{n+1}{2}}} ds = \frac{c_n}{|x|^{\frac{n+1}{2}}}$.

iii) is a direct consequence of ii). \square

Lemma 9. *Let l and m be two constants satisfying $0 < l < m - n$. Then there exists $B > 0$ depending only on l, m and n such that for all $\varepsilon > 0$ we have*

$$\begin{aligned} a) \int_{\mathbf{R}^n} \frac{|y|^l}{(1 + \varepsilon|y|)^m (1 + |x - y|)^m} dy &\leq \frac{B|x|^l}{(1 + \varepsilon|x|)^m} \text{ for all } x \in \mathbf{R}^n, |x| \geq 1 \text{ and} \\ b) \int_{\mathbf{R}^n} \frac{1}{(1 + \varepsilon|y|)^m (1 + |x - y|)^m} dy &\leq \frac{B}{(1 + \varepsilon|x|)^m} \text{ for all } x \in \mathbf{R}^n. \end{aligned}$$

The proof of Lemma 9 is elementary and is essentially the same as the proof of Lemma 3.1.1 in [4], p. 383.

After this preparation, we may prove an integral estimate of the solutions of the convolution equation (10). This is given in the next lemma.

Lemma 10. *Suppose that $f \in L^\infty(\mathbf{R}^n)$ satisfies (10), i.e. $f = k \star g(f)$ and $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$.*

Then $|x|^\beta f(x) \in L^q(\mathbf{R}^n)$ for all $q \in]n, \infty[$ and for all $\beta \in [0, 1[$.

Proof. We remark first that $k \in L^1(\mathbf{R}^n)$ and $g(f) \in L^\infty(\mathbf{R}^n)$, so f is continuous. Choose $p \in]1, \frac{n}{n-1}[$. Then choose α such that

$$(12) \quad n - \frac{n}{p} < \alpha < n + 1 - \frac{n}{p}.$$

By Lemma 7 we have $k \in L^p(\mathbf{R}^n)$ and $|\cdot|^\alpha k \in L^p(\mathbf{R}^n)$. Let $K_{\alpha,p} = \|(1 + |x|)^\alpha k(x)\|_{L^p}$.

Now choose $l \in [0, \alpha - \frac{n(p-1)}{p}[$. For $0 < \varepsilon < 1$ we denote

$$h_\varepsilon(x) = \frac{|x|^l}{(1 + \varepsilon|x|)^\alpha} f(x).$$

Let q be the conjugate of p , i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then $h_\varepsilon \in L^q(\mathbf{R}^n)$ by the choice of l . Since

$$f(x) = (k \star g(f))(x) = \int_{\mathbf{R}^n} k(x - y)(1 + |x - y|)^\alpha \cdot \frac{g(f(y))}{(1 + |x - y|)^\alpha} dy,$$

using Hlder's inequality we obtain

$$(13) \quad |f(x)| \leq K_{\alpha,p} \left(\int_{\mathbf{R}^n} \frac{|g(f(y))|^q}{(1 + |x - y|)^{\alpha q}} dy \right)^{\frac{1}{q}}.$$

The assumption ii) on the function g and the fact that $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$ imply that for every $\delta > 0$ there exists $R_\delta > 1$ such that if $|x| \geq R_\delta$ we have

$$|g(f(x))| \leq \delta |f(x)|.$$

If $0 < r < q$, by (13) and Hlder's inequality we obtain

$$\begin{aligned} \int_{\mathbf{R}^n \setminus B(0, R_\delta)} |h_\varepsilon(x)|^q dx &= \int_{\mathbf{R}^n \setminus B(0, R_\delta)} |h_\varepsilon(x)|^{q-r} \left(\frac{|x|^l}{(1 + \varepsilon|x|)^\alpha} \right)^r |f(x)|^r dx \\ &\leq \int_{\mathbf{R}^n \setminus B(0, R_\delta)} |h_\varepsilon(x)|^{q-r} \left(\frac{|x|^l}{(1 + \varepsilon|x|)^\alpha} \right)^r \cdot K_{\alpha,p}^r \left(\int_{\mathbf{R}^n} \frac{|g(f(y))|^q}{(1 + |x-y|)^{\alpha q}} dy \right)^{\frac{r}{q}} dx \\ &\leq K_{\alpha,p}^r \left(\int_{\mathbf{R}^n \setminus B(0, R_\delta)} |h_\varepsilon(x)|^q dx \right)^{\frac{q-r}{q}} \\ &\quad \times \left[\int_{\mathbf{R}^n \setminus B(0, R_\delta)} \left(\frac{|x|^l}{(1 + \varepsilon|x|)^\alpha} \right)^q \cdot \int_{\mathbf{R}^n} \frac{|g(f(y))|^q}{(1 + |x-y|)^{\alpha q}} dy dx \right]^{\frac{r}{q}}. \end{aligned}$$

The last sequence of inequalities gives

$$(15) \quad \int_{\mathbf{R}^n \setminus B(0, R_\delta)} |h_\varepsilon(x)|^q dx \leq K_{\alpha,p}^q \int_{\mathbf{R}^n \setminus B(0, R_\delta)} \left(\frac{|x|^l}{(1 + \varepsilon|x|)^\alpha} \right)^q \cdot \int_{\mathbf{R}^n} \frac{|g(f(y))|^q}{(1 + |x-y|)^{\alpha q}} dy dx.$$

(since $h_\varepsilon \in L^q(\mathbf{R}^n)$, we may divide by $\int_{\mathbf{R}^n \setminus B(0, R_\delta)} |h_\varepsilon(x)|^q dx$). Observe that $lq < \alpha q - n$ by the choice of l . Using Fubini's Theorem and Lemma 9 we obtain

$$(16) \quad \begin{aligned} &\int_{\mathbf{R}^n \setminus B(0, R_\delta)} \left[\left(\frac{|x|^l}{(1 + \varepsilon|x|)^\alpha} \right)^q \cdot \int_{\mathbf{R}^n} \frac{|g(f(y))|^q}{(1 + |x-y|)^{\alpha q}} dy \right] dx \\ &= \int_{\mathbf{R}^n} |g(f(y))|^q \left[\int_{\mathbf{R}^n \setminus B(0, R_\delta)} \frac{|x|^{lq}}{(1 + \varepsilon|x|)^{\alpha q}} \cdot \frac{1}{(1 + |x-y|)^{\alpha q}} dx \right] dy \\ &\leq \int_{\mathbf{R}^n \setminus B(0, R_\delta)} |g(f(y))|^q \frac{B|y|^{lq}}{(1 + \varepsilon|y|)^{\alpha q}} dy \\ &\quad + \int_{B(0, R_\delta)} |g(f(y))|^q \cdot \int_{\mathbf{R}^n \setminus B(0, R_\delta)} \frac{|x|^{lq}}{(1 + \varepsilon|x|)^{\alpha q}} \cdot \frac{1}{(1 + |x-y|)^{\alpha q}} dx dy \end{aligned}$$

where B depends on n, l, q and α , but not on ε . The last integral is majorized by a constant C depending on f and R_δ (but not on ε).

Combining (15) and (16) and taking into account the fact that $|g(f(y))| < \delta |f(y)|$ on $\mathbf{R}^n \setminus B(0, R_\delta)$, we get

$$(17) \quad \int_{\mathbf{R}^n \setminus B(0, R_\delta)} |h_\varepsilon(x)|^q dx \leq K_{\alpha,p}^q \left[B\delta^q \int_{\mathbf{R}^n \setminus B(0, R_\delta)} |h_\varepsilon(x)|^q dx + C \right].$$

Choosing δ such that $K_{\alpha,p}B^{\frac{1}{q}}\delta < 1$, from (17) we deduce that

$$(18) \quad \int_{\mathbf{R}^n \setminus B(0, R_\delta)} |h_\varepsilon(x)|^q dx \leq C'$$

where C' is a constant that does not depend on ε . We let $\varepsilon \rightarrow 0$ in (18) and apply Fatou's Lemma to obtain

$$\int_{\mathbf{R}^n \setminus B(0, R_\delta)} |x|^{lq} |f(x)|^q dx \leq C'.$$

Hence $|x|^l f(x) \in L^q(\mathbf{R}^n)$ for $q = \frac{p}{p-1}$.

To summarize, we proved that for any $p \in]1, \frac{n}{n-1}[$, for any $\alpha \in]n - \frac{n}{p}, n+1 - \frac{n}{p}[$ and for any $l \in [0, \alpha - \frac{n(p-1)}{p}[$ we have $|x|^l f(x) \in L^{\frac{p}{p-1}}$.

We choose sequences (p_k) , (α_k) and (l_k) such that

$$\begin{aligned} p_k &\in]1, \frac{n}{n-1}[, & p_k &\longrightarrow \frac{n}{n-1} \text{ as } k \longrightarrow \infty \\ \alpha_k &\in]n - \frac{n}{p_k}, n+1 - \frac{n}{p_k}[, & \alpha_k &\longrightarrow 2 \text{ as } k \longrightarrow \infty \text{ and} \\ l_k &\in [0, \alpha_k - \frac{n(p_k-1)}{p_k}[, & l_k &\longrightarrow 1 \text{ as } k \longrightarrow \infty. \end{aligned}$$

Then $q_k = \frac{p_k}{p_k-1} \rightarrow n$ as $k \rightarrow \infty$ and $|x|^{l_k} f(x) \in L^{q_k}$ for all k . This proves the lemma. \square

We may now state our main result.

Theorem 11. *Suppose that f satisfies equation (10) and*

-either $f \in L^p(\mathbf{R}^n)$ for a $p \in](\gamma-1)n, \infty[$, $p \geq \gamma$,

-or $f \in L^\infty(\mathbf{R}^n)$ and $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

Then $|x|^{n+1} f(x) \in L^\infty(\mathbf{R}^n)$.

Proof. First we show that we always have $f \in L^\infty(\mathbf{R}^n)$ and $f(x) \rightarrow 0$ at infinity.

Suppose that $f \in L^p(\mathbf{R}^n)$ and $p > \gamma n$. Then $g(f) \in L^{\frac{p}{\gamma}}(\mathbf{R}^n)$. Since $\frac{p}{\gamma} > n$ and $k \in L^q(\mathbf{R}^n)$ for all $q \in [1, \frac{n}{n-1}[$, it clearly follows from equation (10) that $f \in L^\infty(\mathbf{R}^n)$, f is continuous and tends to zero at infinity.

If $f \in L^{\gamma n}(\mathbf{R}^n)$, then $g(f) \in L^n(\mathbf{R}^n)$. Equation (10) and Young's theorem imply that $f \in L^q(\mathbf{R}^n)$ for all $q \in [\gamma n, \infty[$. Then the preceding argument shows that $f \in L^\infty(\mathbf{R}^n)$ and $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

Now suppose that $p \in](\gamma-1)n, \gamma n[$ and $p \geq \gamma$. Then $g(f) \in L^{\frac{p}{\gamma}}(\mathbf{R}^n)$ and by (10) and Young's theorem we obtain $f \in L^q(\mathbf{R}^n)$ for all $q \in [\frac{p}{\gamma}, \frac{pn}{\gamma n-p}[$. Iterating this argument, after a finite number of steps we get $f \in L^p(\mathbf{R}^n)$ for a $p \geq \gamma n$. As above we obtain $f \in L^\infty(\mathbf{R}^n)$ and $f(x) \rightarrow 0$ at infinity.

The rest of the proof is a standard bootstrap argument. We make use of the inequality

$$(19) \quad \||x|^\delta f| \leq C \left((|x|^\delta k) \star |g(f)| + k \star (|x|^\delta |g(f)|) \right).$$

By Lemma 7, $|x|k \in L^q(\mathbf{R}^n)$ for $q \in]1, \frac{n}{n-2}[$ if $n \geq 3$ (respectively for $q \in]1, \infty[$ if $n = 2$ and $q \in]1, \infty[$ if $n = 1$). Lemma 10 implies that $g(f) \in L^r(\mathbf{R}^n)$ for $r \in]\frac{n}{\gamma}, \infty[$, so we get $(|x|k) \star |g(f)| \in L^\infty(\mathbf{R}^n)$. Similarly, $k \in L^q(\mathbf{R}^n)$ for $q \in [1, \frac{n}{n-1}[$ and $|x|g(f) \in L^r(\mathbf{R}^n)$ for $r \in]\frac{n}{\gamma}, \infty[$ by Lemma 10, hence $k \star (|x|g(f)|) \in L^\infty(\mathbf{R}^n)$. Using (19) we get $|x|f(x) \in L^\infty(\mathbf{R}^n)$.

Suppose that $|x|^{\alpha}f(x) \in L^{\infty}(\mathbf{R}^n)$ and $\alpha\gamma \leq n+1$. Obviously $|x|^{\alpha\gamma}|g(f)| \in L^{\infty}(\mathbf{R}^n)$ and $k \in L^1(\mathbf{R}^n)$, hence $k \star (|x|^{\alpha\gamma}|g(f)|) \in L^{\infty}(\mathbf{R}^n)$. Observe that $|g(f)(x)| \leq \frac{C}{(1+|x|)^{\alpha\gamma}}$, so $g(f) \in L^q(\mathbf{R}^n)$ for all q verifying $q > \frac{n}{\alpha\gamma}$, $q \geq 1$. Using Lemma 7 and Young's theorem, we find that $(|x|^{\alpha\gamma}k) \star |g(f)| \in L^{\infty}(\mathbf{R}^n)$ and from (19) it follows that $|x|^{\alpha\gamma}f(x) \in L^{\infty}(\mathbf{R}^n)$. Hence $|x|^{\alpha}f \in L^{\infty}(\mathbf{R}^n)$ and $\alpha\gamma \leq n+1$ imply that $|x|^{\alpha\gamma}f \in L^{\infty}(\mathbf{R}^n)$. This clearly leads to the conclusion of the theorem. \square

Remark 12. Suppose that g is C^m and $|g^{(i)}|(x) \leq C_i|x|^{\gamma-i}$, $0 \leq i \leq m$ and f satisfies the hypothesis of Theorem 11. Then $|f(x)| \leq \frac{C}{(1+|x|)^{n+1}}$, in particular $f \in L^1(\mathbf{R}^n)$. Arguing as in the proof of Theorem 4 we obtain that $f \in W^{m+1,q}(\mathbf{R}^n)$ for all $q \in [1, \infty[$. As in Theorem 11 it can be proved that the derivatives of f of order $\leq m$ decay at infinity at least as $\frac{1}{|x|^{n+1}}$.

Remark 13. Suppose in addition that g is differentiable and there exists $\beta > 0$ such that $|g'(x)| \leq C|x|^{\beta}$. If $f \in L^p(\mathbf{R}^n)$, $1 < p \leq \infty$ satisfies (10) and $\int_{\mathbf{R}^n} g(f(x)) dx \neq 0$, then the decay rate of f given by Theorem 11 is optimal. More precisely, $|x|f$ cannot belong to $L^1(\mathbf{R}^n)$.

In particular, the solutions of equation (3) in \mathbf{R}^2 decay at infinity as $\frac{1}{|x|^3}$ and this algebraic rate is optimal.

Indeed, $|x|f \in L^1(\mathbf{R}^n)$ would imply that $x_j f$, $g(f)$ and $x_j g(f)$ are L^1 -functions, hence their Fourier transforms are continuous. But

$$(20) \quad \begin{aligned} -i\widehat{x_j f}(\xi) &= \partial_{\xi_j} \widehat{f}(\xi) = \partial_{\xi_j} \left(\frac{1}{1+|\xi|} \widehat{g(f)}(\xi) \right) (\xi) \\ &= -\frac{\xi_j}{(1+|\xi|)^2|\xi|} \widehat{g(f)}(\xi) + \frac{1}{1+|\xi|} \mathcal{F}(-ix_j g(f))(\xi). \end{aligned}$$

Take $\xi_j = s$ and $\xi_i = 0$ if $i \neq j$ in (20). For $s \downarrow 0$ we get

$$-i\widehat{x_j f}(0) = -\widehat{g(f)}(0) + \mathcal{F}(-ix_j g(f))(0),$$

while for $s \uparrow 0$ we obtain

$$-i\widehat{x_j f}(0) = \widehat{g(f)}(0) + \mathcal{F}(-ix_j g(f))(0),$$

Hence $\widehat{g(f)}(0) = 0$, i. e. $\int_{\mathbf{R}^n} g(f(x)) dx = 0$, contrary to our assumption.

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