Nonexistence of supersonic travelling-waves for nonlinear Schrödinger equations with nonzero conditions at infinity

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Abstract

We prove that the non-existence of supersonic finite-energy travelling-waves for nonlinear Schrödinger equations with nonzero conditions at infinity is a general phenomenon, which holds for a large class of equations. The same is true for sonic travelling-waves in dimension two. In higher dimensions we prove that sonic travelling-waves, if they exist, must approach their limit at infinity in a very rigid way. In particular, we infer that there are no sonic travelling-waves with finite energy and finite momentum.

Keywords. nonlinear Schrödinger equation, nonzero conditions at infinity, travelling-wave, integral identities, Gross-Pitaevskii equations and systems, cubic-quintic NLS.

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1 Introduction

The aim of this paper is to study travelling-wave solutions for nonlinear Schrödinger equations

$$i\frac{\partial \Phi}{\partial t} + \Delta\Phi + F(x, |\Phi|^2)\Phi = 0 \qquad \text{in } \mathbf{R}^N, \tag{1.1}$$

where F is a real-valued function defined on $\mathbf{R}^N \times \mathbf{R}_+$, Φ is a complex-valued function on $\mathbf{R}^N$ satisfying the "boundary condition" $|\Phi| \longrightarrow r_0$ as $|x| \longrightarrow \infty$, and r_0 is a positive constant verifying $\lim\limits_{|x|\to\infty,\ s\to r_0^2} F(x,s) = 0$.

The above equation with the considered non-zero conditions at infinity arise in a large variety of physical problems, such as superconductivity, superfluidity in Helium II, phase transitions and Bose-Einstein condensate. Two important particular cases of (1.1) have been extensively studied both by physicists and by mathematicians : the Gross-Pitaevskii equation (where $F(x,s) = 1 - s$) and the so-called "cubic-quintic" Schrödinger equation (where $F(x,s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, α_1, α_3, α_5 are positive and $\frac{3}{16} < \frac{\alpha_1\alpha_5}{\alpha_3^2} < \frac{1}{4}$).

Equation (1.1) has a Hamiltonian structure: denoting $V(x,s) = \int_s^{r_0^2} F(x,\tau)\,d\tau$, it is easy to see that, at least formally, the "energy"

$$E(\Phi) = \int_{\mathbf{R}^N} |\nabla\Phi|^2\,dx + \int_{\mathbf{R}^N} V(x, |\Phi|^2)\,dx \tag{1.2}$$

is a conserved quantity. There is another important (vector) quantity associated to (1.1), namely the momentum. It is given by

$$P(\Phi) = (P_1(\Phi), \dots, P_N(\Phi)), \qquad \text{where } P_k(\Phi) = \int_{\mathbf{R}^N} (i\frac{\partial\Phi}{\partial x_k}, \Phi)\,dx = \int_{\mathbf{R}^N} \mathrm{Re}(i\frac{\partial\Phi}{\partial x_k}\overline{\Phi})\,dx. \tag{1.3}$$

Note that, in general, the momentum is not well-defined for any solution Φ of finite energy. In the case where F does not depend on the variable x_k, the momentum with respect to the x_k-direction, P_k, is conserved by those solutions of (1.1) for which it can be well-defined.

It is worth to note that equation (1.1) can be put into a hydrodynamical form by using Madelung's transformation $\Phi(x,t) = \sqrt{\rho(x,t)}e^{i\theta(x,t)}$ (which is singular when $\Phi = 0$). A straightforward computation shows that, in the region where $\Phi \neq 0$, the functions $\rho = |\Phi|^2$ and θ satisfy the system

$$\rho_t + 2\mathrm{div}(\rho\nabla\theta) = 0, \tag{1.4}$$

$$\theta_t + |\nabla\theta|^2 - \frac{\Delta\rho}{2\rho} + \frac{|\nabla\rho|^2}{4\rho} - F(x,\rho) = 0. \tag{1.5}$$

Equation (1.4) and the derivatives with respect to $x_1, \dots, x_N$ of (1.5) are, respectively, the equation of conservation of mass and Euler's equations for a compressible inviscid fluid of density ρ and velocity $2\nabla\theta$.

Let us assume that F admits a partial derivative with respect to the last variable (in the sequel, this derivative will be denoted by $\partial_{N+1}F$ or by $\frac{\partial F}{\partial s}$) and that $\lim\limits_{|x|\to\infty,\ \rho\to r_0^2} \partial_{N+1}F(x,\rho) = -L$, where L is a positive constant. Taking the derivative with respect to t of (1.5) and substituting ρ_t from (1.4) we obtain

$$\theta_{tt} + 2\partial_{N+1}F(x,\rho)(\rho\Delta\theta + \nabla\rho.\nabla\theta) + \frac{\partial}{\partial t}\left(|\nabla\theta|^2 - \frac{\Delta\rho}{2\rho} + \frac{|\nabla\rho|^2}{4\rho}\right) = 0. \tag{1.6}$$

For a small oscillatory motion (i.e. a sound wave), all nonlinear terms in (1.6), except $2\rho\Delta\theta$, may be neglected. In view of the behavior of ρ and $\partial_{N+1}F(x,\rho)$ for large $|x|$, we find that in a neighborhood of infinity, the velocity potential θ essentially obeys the wave equation $\theta_{tt} - 2r_0^2L\Delta\theta = 0$. It is well-known that the solutions of the wave equation propagate with a finite speed; in the present situation, we infer that the velocity of sound waves at infinity is $r_0\sqrt{2L}$. In what follows we will always assume that $\partial_{N+1}F(x,\rho) \longrightarrow -L$ as $|x| \longrightarrow \infty$ and $\rho \longrightarrow r_0^2$ (the convergence being in a sense to be defined) and we will denote by $v_s = r_0\sqrt{2L}$ the sound velocity at infinity.

For a fixed $y \in S^{N-1}$, a travelling-wave for (1.1) moving with velocity c in direction y is a solution of the form $\Phi(x,t) = \psi(x - cty)$. Without loss of generality we will assume that $y = (1, 0, \dots, 0)$, i.e. travelling-waves move in the x_1-direction. The travelling-wave profile satisfies the equation

$$-ic\frac{\partial\psi}{\partial x_1} + \Delta\psi + F(x, |\psi|^2)\psi = 0 \qquad \text{in } \mathbf{R}^N. \tag{1.7}$$

In a series of papers, J. Grant, C.A. Jones, S.J. Putterman, P.H. Roberts et al. studied formally and numerically travelling-waves for the Gross-Pitaevskii equation and related systems (see, e.g., [16], [19], [21], [22], [7] and references therein). In particular, they conjectured that such solutions exist if and only if their speed c belongs to the interval $(-v_s, v_s)$. For the cubic-quintic nonlinear Schrödinger equation, the existence of subsonic travelling-waves in one dimension has been proved in [2] and their stability has been studied in [1]. The non-existence of such solutions for sonic and supersonic speeds has also been conjectured in any space dimension. In the case of the Gross-Pitaevskii equation, it has been shown in [17] that any travelling-wave of finite energy and speed $c > v_s$ must be constant. It has also been proved in [18] that the same result is true if $N = 2$ and $c^2 = v_s^2$. The proofs in [17], [18] strongly depend on the special algebraic structure of the nonlinearity in the Gross-Pitaevskii equation. In the present paper we show that the nonexistence of finite energy travelling-waves moving faster than the sound velocity is a general phenomenon, which holds for a large class of equations and systems of the form (1.1). We also prove that there are no finite energy sonic travelling-waves in space dimension two. In higher dimensions we show that any finite-energy sonic travelling-wave ψ must satisfy $|\psi|^2 - r_0^2 \in L^p(\mathbf{R}^N)$ for any $p > \frac{2N-1}{2N-3}$. On the other hand, if a sonic travelling-wave satisfies $|\psi|^2 - r_0^2 \in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$, then it must be constant.

This article is organized as follows: in the next section we prove that travelling-waves, whenever they exist, are smooth functions. If their speed is supersonic (or sonic, provided they converge sufficiently fast at infinity), then they must satisfy a special integral identity. This will be proved in Section 3. In section 4 we show how this identity implies, under general assumptions, the non-existence of travelling-waves with finite energy. We apply our results to the Gross-Pitaevskii equation, to the cubic-quintic Schrödinger equation and to a Gross-Pitaevskii-Schrödinger system which describes the motion of an uncharged impurity in a Bose condensate. In the last section we describe all supersonic and sonic travelling-waves (with finite or infinite energy) for one-dimensional equations with nonlinearities independent on the space variable.

2 Basic properties of travelling-waves

We keep the previous notation and we consider the following set of assumptions:

- **(H1)** $F : \mathbf{R}^N \times [0, \infty) \longrightarrow \mathbf{R}$ is a measurable function which has the following properties:
  - a) for any $s \in [0, \infty)$, $F(\cdot, s)$ is measurable;
  - b) for any $x \in \mathbf{R}^N$, $F(x, \cdot)$ is continuous;
  - c) F is bounded on bounded subsets of $\mathbf{R}^N \times [0, \infty)$.
- **(H2)** There exist $\alpha > 0$, $C > 0$ and $r_* > 0$ such that for any $x \in \mathbf{R}^N$ and for any $s \geq r_*$ we have $F(x, s) \leq -Cs^\alpha$.
- **(H3)** $\lim\limits_{|x| \to \infty} F(x, r_0^2) = 0$ and $F(\cdot, r_0^2) \in L^1(\mathbf{R}^N)$.
- **(H4)** F admits a partial derivative with respect to the last variable and $\partial_{N+1}F$ is bounded on bounded subsets of $\mathbf{R}^N \times [0, \infty)$. Moreover, $\lim\limits_{|x| \to \infty} \partial_{N+1}F(x, r_0^2) = -L$, where $L > 0$ and $\partial_{N+1}F(\cdot, r_0^2) + L \in L^{p_0}(\mathbf{R}^N)$ for some $p_0 \in [1, 2]$.
- **(H5)** There are some positive constants R_0, η, M such that $\partial_{N+1}^2 F$ exists on $(\mathbf{R}^N \setminus \overline{B}(0, R_0)) \times (r_0^2 - \eta, r_0^2 + \eta)$ and
$$|\partial_{N+1}^2 F(x, s)| \leq M \qquad \text{for all } (x, s) \in (\mathbf{R}^N \setminus \overline{B}(0, R_0)) \times (r_0^2 - \eta, r_0^2 + \eta).$$

Definition 2.1 A travelling-wave (of speed c) for (1.1) is a function $\psi \in L^1_{loc}(\mathbf{R}^N)$ that satisfies (1.7) in $\mathcal{D}'(\mathbf{R}^N)$ together with the "boundary condition" $|\psi| \longrightarrow r_0$ as $|x| \longrightarrow \infty$.

In view of (1.2), we say that a travelling-wave ψ has finite energy if $\nabla\psi \in L^2(\mathbf{R}^N)$ and $V(\cdot, |\psi|^2) \in L^1(\mathbf{R}^N)$.

We have the following result concerning the regularity of tavelling-waves:

Proposition 2.2 *Let ψ be a finite-energy travelling-wave for (1.1).*

i) Assume that $F : \mathbf{R}^N \times \mathbf{R}_+ \longrightarrow \mathbf{R}$ is measurable and satisfies **(H1a)**, **(H1b)**, **(H2)**, *the function $x \longmapsto \displaystyle\int_{r_0^2}^{r_*} F(x, \tau)\, d\tau$ belongs to $L^1_{loc}(\mathbf{R}^N)$ (where r_* is given by* **(H2)***) and $F(\cdot, |\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. Then $\psi \in L^\infty(\mathbf{R}^N)$.*

If, in addition, F satisfies **(H1c)**, *then $\psi \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1,\infty)$. In particular, $\psi \in C^{1,\alpha}(\mathbf{R}^N)$ for any $\alpha \in [0,1)$.*

ii) Suppose that $F \in C^k(\mathbf{R}^N \times [0,\infty))$ for some $k \in \mathbf{N}^*$, **(H2)** *holds, and $F(\cdot, |\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. Then $\psi \in W^{k+2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1,\infty)$. In particular, if F is C^∞, then $\psi \in C^\infty(\mathbf{R}^N)$.*

Proof. i) The proof relies upon the ideas and methods developed by A. Farina in [13, 14]. By **(H2)** we have

$$V(x,s) = -\int_{r_0^2}^{s} F(x,\tau)\,d\tau \geq -\int_{r_0^2}^{r_*} F(x,\tau)\,d\tau + \int_{r_*}^{s} C\tau^\alpha\,d\tau = -\int_{r_0^2}^{r_*} F(x,\tau)\,d\tau + \frac{C}{\alpha+1}(s^{\alpha+1} - r_*^{\alpha+1}).$$

Consequently, for any $s \geq r_*$ we get $s^{\alpha+1} \leq r_*^{\alpha+1} + \frac{\alpha+1}{C}\Big(V(x,s) + \displaystyle\int_{r_0^2}^{r_*} F(x,\tau)\,d\tau\Big)$, so that

$$|\psi|^{2\alpha+2}(x) \leq \max\Big(r_*^{\alpha+1}, r_*^{\alpha+1} + \frac{\alpha+1}{C}\Big(V(x,|\psi|^2(x)) + \int_{r_0^2}^{r_*} F(x,\tau)\,d\tau\Big)\Big).$$

Since $V(\cdot,|\psi|^2)$ and $\displaystyle\int_{r_0^2}^{r_*} F(\cdot,\tau)\,d\tau$ belong to $L^1_{loc}(\mathbf{R}^N)$, we infer that $\psi \in L^{2\alpha+2}_{loc}(\mathbf{R}^N)$.

We will use a well-known inequality of T. Kato (see Lemma A p. 138 in [23]):

If $u \in L^1_{loc}(\mathbf{R}^N)$ is a real-valued function and $\Delta u \in L^1_{loc}(\mathbf{R}^N)$, then

$$\Delta(u^+) \geq \operatorname{sgn}^+(u)\Delta u \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{2.1}$$

Let $\varphi(x) = e^{-\frac{icx_1}{2}}\psi(x)$. Then $\varphi \in L^{2\alpha+2}_{loc}(\mathbf{R}^N) \subset L^1_{loc}(\mathbf{R}^N)$ and an easy computation shows that φ satisfies

$$\Delta\varphi + \Big(F(x,|\varphi|^2) + \frac{c^2}{4}\Big)\varphi = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{2.2}$$

It is clear that $F(\cdot,|\varphi|^2)\varphi \in L^1_{loc}(\mathbf{R}^N)$ (because $F(x,|\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$ by hypothesis) and it follows from (2.2) that $\Delta\varphi \in L^1_{loc}(\mathbf{R}^N)$. Choose $\tilde{r} \geq r_*$ and $C_1 > 0$ such that $Cs^{2\alpha} - \frac{c^2}{4} \geq C_1(s-\tilde{r})^{2\alpha}$ for any $s \geq \tilde{r}$. Denoting $\varphi_1 = \mathrm{Re}(\varphi)$, $\varphi_2 = \mathrm{Im}(\varphi)$ and using Kato's inequality for $\varphi_i - \tilde{r}$, $i = 1,2$, then using (2.2) and **(H2)** we get

$$\begin{aligned}
\Delta(\varphi_i - \tilde{r})^+ &\geq \operatorname{sgn}^+(\varphi_i - \tilde{r})\Delta(\varphi_i - \tilde{r}) = \operatorname{sgn}^+(\varphi_i - \tilde{r})[-(F(x,|\varphi|^2) + \tfrac{c^2}{4})\varphi_i] \\
&\geq \operatorname{sgn}^+(\varphi_i - \tilde{r})[C|\varphi|^{2\alpha} - \tfrac{c^2}{4}]\varphi_i \geq \operatorname{sgn}^+(\varphi_i - \tilde{r})[C|\varphi_i|^{2\alpha} - \tfrac{c^2}{4}]\varphi_i \\
&\geq C_1\operatorname{sgn}^+(\varphi_i - \tilde{r})(\varphi_i - \tilde{r})^{2\alpha+1} = C_1[(\varphi_i - \tilde{r})^+]^{2\alpha+1}.
\end{aligned} \tag{2.3}$$

Next we use the following result of H. Brézis (Lemma 2 p. 273 in [9]):

Lemma 2.3 ([9]) *Let $p \in (1,\infty)$. Assume that $u \in L^p_{loc}(\mathbf{R}^N)$ satisfies*

$$-\Delta u + |u|^{p-1}u \leq 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N).$$

Then $u \leq 0$ a.e. on $\mathbf{R}^N$.

It follows from (2.3) that the function $u_i = (C_1)^{\frac{1}{2\alpha}}(\varphi_i - \tilde{r})^+$ satisfies $-\Delta u_i + |u_i|^{2\alpha}u_i \leq 0$ in $\mathcal{D}'(\mathbf{R}^N)$. Since $u_i \in L^{2\alpha+1}_{loc}(\mathbf{R}^N)$, we may use Lemma 2.3 and we get $u_i \leq 0$ a.e. in $\mathbf{R}^N$, that is $\varphi_i \leq \tilde{r}$ a.e. in $\mathbf{R}^N$.

It is obvious that both φ and $-\varphi$ satisfy (2.2). Repeating the above argument for $-\varphi$, we infer that $-\varphi_i \leq \tilde{r}$ a.e. on $\mathbf{R}^N$. Therefore we have $|\varphi_i| \leq \tilde{r}$ a.e. on $\mathbf{R}^N$, $i = 1, 2$, which implies that $\varphi \in L^\infty(\mathbf{R}^N)$. Since $|\varphi| = |\psi|$, we have proved that $\psi \in L^\infty(\mathbf{R}^N)$.

Using **(H1c)** and (2.2) we infer that $\Delta\varphi \in L^\infty(B(x, 2R)) \subset L^p(B(x, 2R))$ for any $x \in \mathbf{R}^N$, $R > 0$ and $p \geq 1$. By standard elliptic estimates we obtain $\varphi \in W^{2,p}(B(x, R))$ for any $x \in \mathbf{R}^N$, $R > 0$ and $p \in (1, \infty)$. Thus $\psi = e^{\frac{icx_1}{2}}\varphi \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in (1, \infty)$, consequently ψ belongs to $C^{1,\alpha}_{loc}(\mathbf{R}^N)$ for any $\alpha \in [0, 1)$ by the Sobolev embedding theorem.

ii) Assume $F \in C^1(\mathbf{R}^N \times [0, \infty))$. Differentiating (1.7) with respect to x_k we get

$$-ic\psi_{x_1 x_k} + \Delta\psi_{x_k} + \frac{\partial F}{\partial x_k}(x, |\psi|^2)\psi + 2\partial_{N+1}F(x, |\psi|^2)(\psi.\frac{\partial\psi}{\partial x_k})\psi + F(x, |\psi|^2)\frac{\partial\psi}{\partial x_k} = 0 \tag{2.4}$$

in $\mathcal{D}'(\mathbf{R}^N)$. Hence $\Delta\psi_{x_k} \in L^p_{loc}(\mathbf{R}^N)$ for $1 \leq p < \infty$. By standard elliptic regularity theory we get $\psi_{x_k} \in W^{2,p}_{loc}(\mathbf{R}^N)$ for $1 < p < \infty$, $1 \leq k \leq N$, therefore $\psi \in W^{3,p}_{loc}(\mathbf{R}^N)$ for $1 \leq p < \infty$. If $F \in C^k(\mathbf{R}^N \times [0, \infty))$ we may differentiate (2.4) further and repeat the above arguments. After an easy induction, we get $\psi \in W^{k+2,p}_{loc}(\mathbf{R}^N)$ for any $p \in (1, \infty)$. $\square$

Lemma 2.4 *Assume that* **(H1), (H3), (H4), (H5)** *hold and* $u \in L^4_{loc}(\mathbf{R}^N, \mathbf{C})$ *satisfies* $|u(x)| \longrightarrow r_0$ *as* $|x| \longrightarrow \infty$ *and* $V(\cdot, |u|^2) \in L^1(\mathbf{R}^N)$.

Then $|u|^2 - r_0^2 \in L^2(\mathbf{R}^N)$.

Proof. Let R_0, η, M be as in **(H5)**. From **(H4)** and the fact that $|u(x)| \longrightarrow r_0$ as $|x| \longrightarrow \infty$ it follows that there exists $R_1 > R_0$ such that

$$\partial_{N+1}F(x, r_0^2) < -\frac{L}{2} \qquad \text{and} \qquad |u(x)|^2 \in (r_0^2 - \eta, r_0^2 + \eta) \qquad \text{for any } x \text{ satisfying } |x| \geq R_1.$$

For $(x, s) \in (\mathbf{R}^N \setminus B(0, R_1)) \times (r_0^2 - \eta, r_0^2 + \eta)$ we get, by Taylor's formula with respect to the $(N+1)^{th}$ variable,

$$V(x, s) = -(s - r_0^2)F(x, r_0^2) - \frac{1}{2}(s - r_0^2)^2\partial_{N+1}F(x, r_0^2) - \frac{1}{2}\int_{r_0^2}^{s}(s - \tau)^2\partial^2_{N+1}F(x, \tau)\, d\tau.$$

In particular, for $s = |u(x)|^2$ we obtain

$$\begin{aligned} &-\tfrac{1}{2}(|u(x)|^2 - r_0^2)^2\partial_{N+1}F(x, r_0^2) \\ &= V(x, |u(x)|^2) + (|u(x)|^2 - r_0^2)F(x, r_0^2) + \tfrac{1}{2}\int_{r_0^2}^{|u(x)|^2}(|u(x)|^2 - \tau)^2\partial^2_{N+1}F(x, \tau)\, d\tau. \end{aligned} \tag{2.5}$$

For $x \in \mathbf{R}^N \setminus B(0, R_1)$ we get by **(H5)**

$$\left|\int_{r_0^2}^{|u(x)|^2}(|u(x)|^2 - \tau)^2\partial^2_{N+1}F(x, \tau)\, d\tau\right| \leq M\left|\int_{r_0^2}^{|u(x)|^2}(|u(x)|^2 - \tau)^2\, d\tau\right| = \frac{M}{3}|(|u(x)|^2 - r_0^2)|^3.$$

It is clear that there exists $R_2 \geq R_1$ such that $\frac{M}{3}\left||u(x)|^2 - r_0^2\right| \leq \frac{L}{4}$ on $\mathbf{R}^N \setminus B(0, R_2)$. Using **(H4)** and (2.5) we infer that

$$\begin{aligned} &\tfrac{L}{4}(|u(x)|^2 - r_0^2)^2 \leq -\tfrac{1}{2}(|u(x)|^2 - r_0^2)^2\partial_{N+1}F(x, r_0^2) \\ &\leq V(x, |u(x)|^2) + (|u(x)|^2 - r_0^2)F(x, r_0^2) + \tfrac{1}{2}\cdot\tfrac{M}{3}\left|\,|u(x)|^2 - r_0^2\right|^3 \\ &\leq V(x, |u(x)|^2) + (|u(x)|^2 - r_0^2)F(x, r_0^2) + \tfrac{L}{8}\left|\,|u(x)|^2 - r_0^2\right|^2 \qquad \text{on } \mathbf{R}^N \setminus B(0, R_2). \end{aligned}$$

Consequently

$$\frac{L}{8}(|u(x)|^2 - r_0^2)^2 \le V(x, |u(x)|^2) + (|u(x)|^2 - r_0^2)F(x, r_0^2) \qquad \text{on } \mathbf{R}^N \setminus B(0, R_2). \tag{2.6}$$

Since $F(\cdot, r_0^2) \in L^1(\mathbf{R}^N)$ by **(H3)**, $V(\cdot, |u|^2) \in L^1(\mathbf{R}^N)$ and $|\,|u(x)|^2 - r_0^2| \le \frac{3L}{4M}$ on $\mathbf{R}^N \setminus B(0, R_2)$, using (2.6) we get $(|u|^2 - r_0^2)^2 \in L^1(\mathbf{R}^N \setminus B(0, R_2))$. It is obvious that $(|u|^2 - r_0^2)^2 \in L^1(B(0, R_2))$ because $u \in L^4_{loc}(\mathbf{R}^N)$. Hence $(|u|^2 - r_0^2)^2 \in L^1(\mathbf{R}^N)$ and Lemma 2.4 is proved. □

Proposition 2.5 *Assume that* **(H1)-(H5)** *hold and let ψ be a finite-energy travelling-wave for (1.1) (in the sense of Definition 2.1) such that $F(\cdot, |\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. Then:*

i) $\nabla\psi \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$.

ii) Let $R_* \ge 0$ be such that $|\psi(x)| \ge \frac{r_0}{2}$ for $|x| \ge R_*$. There exists a real-valued function θ such that $\theta \in W^{2,p}_{loc}(\mathbf{R}^N \setminus \overline{B}(0, R_*))$ for any $p < \infty$, $\nabla\theta \in W^{1,p}(\mathbf{R}^N \setminus \overline{B}(0, R_*))$ for any $p \in [2, \infty)$ and

$$\psi(x) = |\psi(x)|e^{i\theta(x)} \qquad \text{on } \mathbf{R}^N \setminus B(0, R_*).$$

Proof. i) We already know by Proposition 2.2 i) and Lemma 2.4 that ψ is bounded, $\psi \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1, \infty)$ and $|\psi|^2 - r_0^2 \in L^2(\mathbf{R}^N)$.

Let R_0, η, M be as in **(H5)**. Choose $R_1 > R_0$ such that $|\psi|^2(x) \in (r_0^2 - \eta, r_0^2 + \eta)$ for $x \in \mathbf{R}^N \setminus B(0, R_1)$.

By using Taylor's formula with respect to the last variable for the function F we get

$$F(x, s) = F(x, r_0^2) + (s - r_0^2)\partial_{N+1}F(x, r_0^2) + \int_{r_0^2}^{s} (s - \tau)\partial^2_{N+1}F(x, \tau)\, d\tau \tag{2.7}$$

if $(x, s) \in (\mathbf{R}^N \setminus \overline{B}(0, R_0)) \times (r_0^2 - \eta, r_0^2 + \eta)$, hence

$$\begin{aligned} F(x, |\psi|^2(x))\psi(x) &= F(x, r_0^2)\psi(x) + (|\psi|^2(x) - r_0^2)\partial_{N+1}F(x, r_0^2)\psi(x) \\ &\quad + \psi(x)\int_{r_0^2}^{|\psi|^2(x)} (|\psi|^2(x) - \tau)\partial^2_{N+1}F(x, \tau)\, d\tau \quad \text{for any } |x| \ge R_1. \end{aligned} \tag{2.8}$$

We analyze the three terms in the right-hand side of (2.8). Assumptions **(H1)** and **(H3)** imply $F(\cdot, r_0^2) \in L^1 \cap L^\infty(\mathbf{R}^N)$. Since $\psi \in L^\infty(\mathbf{R}^N)$, it follows that $F(\cdot, r_0^2)\psi \in L^1 \cap L^\infty(\mathbf{R}^N)$.

We may write $(|\psi|^2 - r_0^2)\partial_{N+1}F(\cdot, r_0^2)\psi = -L(|\psi|^2 - r_0^2)\psi + (|\psi|^2 - r_0^2)(L + \partial_{N+1}F(\cdot, r_0^2))\psi$. We know that $\psi \in L^\infty(\mathbf{R}^N)$, $|\psi|^2 - r_0^2 \in L^2 \cap L^\infty(\mathbf{R}^N)$ and by **(H4)** we have $L + \partial_{N+1}F(\cdot, r_0^2) \in L^{p_0} \cap L^\infty(\mathbf{R}^N)$ for some $p_0 \in [1, 2]$, so we infer that $(|\psi|^2 - r_0^2)\partial_{N+1}F(\cdot, r_0^2)\psi \in L^2 \cap L^\infty(\mathbf{R}^N)$.

As in the proof of Lemma 2.4, for $x \in \mathbf{R}^N \setminus B(0, R_1)$ we have

$$\left| \int_{r_0^2}^{|\psi|^2(x)} (|\psi|^2(x) - \tau)\partial^2_{N+1}F(x, \tau)\, d\tau \right| \le M \left| \int_{r_0^2}^{|\psi|^2(x)} \left| |\psi|^2(x) - \tau \right| d\tau \right| = \frac{M}{2}(|\psi|^2(x) - r_0^2)^2. \tag{2.9}$$

Consequently the function $x \longmapsto \displaystyle\int_{r_0^2}^{|\psi|^2(x)} (|\psi|^2(x) - \tau)\partial^2_{N+1}F(x, \tau)\, d\tau$ belongs to $L^1 \cap L^\infty(\mathbf{R}^N \setminus B(0, R_1))$.

Summing up, we have proved that $F(\cdot, |\psi|^2)\psi \in L^2 \cap L^\infty(\mathbf{R}^N \setminus B(0, R_1))$. From **(H1)** and the fact that ψ is bounded on $\mathbf{R}^N$ it follows that $F(\cdot, |\psi|^2)\psi$ is bounded on $B(0, R_1)$, hence $F(\cdot, |\psi|^2)\psi \in L^2 \cap L^\infty(\mathbf{R}^N)$.

We have $\frac{\partial\psi}{\partial x_k} \in L^2(\mathbf{R}^N)$ because ψ has finite energy. Coming back to (1.7), we get

$$\Delta\psi = ic\frac{\partial\psi}{\partial x_1} - F(\cdot, |\psi|^2)\psi \in L^2(\mathbf{R}^N).$$

It is well-known that $\Delta\psi \in L^p(\mathbf{R}^N)$ with $1 < p < \infty$ implies $\frac{\partial^2\psi}{\partial x_j \partial x_k} \in L^p(\mathbf{R}^N)$ for any $j, k \in \{1, \dots, N\}$ (this follows, e.g., from the fact that $\frac{\xi_j \xi_k}{|\xi|^2}$ is a Fourier multiplier on $L^p(\mathbf{R}^N)$ if $1 < p < \infty$; see Theorem 3 p. 96 in [27]). Therefore all second derivatives of ψ are in $L^2(\mathbf{R}^N)$, so that $\frac{\partial\psi}{\partial x_k} \in H^1(\mathbf{R}^N) = W^{1,2}(\mathbf{R}^N)$ for $k = 1, \dots, N$.

The rest of the proof is an easy bootstrap argument. Assume that $\nabla\psi \in W^{1,p}(\mathbf{R}^N)$ for some $p \geq 2$. In case $p < N$, it follows from the Sobolev embedding theorem that $\nabla\psi \in L^{p^*}(\mathbf{R}^N)$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$. From (1.7) we have $\Delta\psi = ic\frac{\partial\psi}{\partial x_1} - F(\cdot, |\psi|^2)\psi \in L^{p^*}(\mathbf{R}^N)$ and we infer as previously that $\nabla\psi \in W^{1,p^*}(\mathbf{R}^N)$. Repeating this argument if necessary, after a finite number of steps we get $\nabla\psi \in W^{1,q}(\mathbf{R}^N)$ for some $q \geq N$. Then by Sobolev embedding we get $\nabla\psi \in L^r(\mathbf{R}^N)$ for any $r \in [q, \infty)$. From (1.7) we obtain $\Delta\psi \in L^p(\mathbf{R}^N)$ for $p \in [2, \infty)$ and we infer that $\nabla\psi \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$.

ii) Take $R_* > 0$ such that $|\psi(x)| \geq \frac{r_0}{2}$ on $\mathbf{R}^N \setminus B(0, R_*)$ and denote $\tilde{\psi}(x) = \frac{\psi(x)}{|\psi(x)|}$. It is then standard to prove that $\tilde{\psi} \in W^{2,p}_{loc}(\mathbf{R}^N \setminus \overline{B}(0, R_*))$ for $p \in [1, \infty)$ and $\nabla\tilde{\psi} \in W^{1,p}(\mathbf{R}^N \setminus \overline{B}(0, R_*))$ for any $p \in [2, \infty)$ (see, e.g., Lemma C1 p. 66 in [10]).

Let us consider first the case $N \geq 3$. For $R_* \leq R_1 < R_2$, the domain $\Omega_{R_1,R_2} = B(0, R_2) \setminus \overline{B}(0, R_1)$ is simply connected in $\mathbf{R}^N$. It follows from Theorem 3 p. 38 in [10] that there exists a real-valued function $\theta_{R_1,R_2} \in W^{2,p}(\Omega_{R_1,R_2})$ $(1 < p < \infty)$ such that $\tilde{\psi} = e^{i\theta_{R_1,R_2}}$ on Ω_{R_1,R_2}. If $R_* \leq R_1 < R_2$, $R_* \leq R_3 < R_4$ and $(R_1, R_2) \cap (R_3, R_4) \neq \emptyset$, then $\tilde{\psi} = e^{i\theta_{R_1,R_2}} = e^{i\theta_{R_3,R_4}}$ on $\Omega_{R_1,R_2} \cap \Omega_{R_3,R_4}$, thus $\theta_{R_3,R_4} - \theta_{R_1,R_2} \in 2\pi\mathbf{Z}$ on $\Omega_{R_1,R_2} \cap \Omega_{R_3,R_4}$. Since functions in $W^{s,p}(\Omega_{R_1,R_2} \cap \Omega_{R_3,R_4})$ with values in $\mathbf{Z}$ are constant when $sp \geq 1$ (see Theorem B1 p. 65 in [10]), there exists $k \in \mathbf{Z}$ such that $\theta_{R_3,R_4} - \theta_{R_1,R_2} = 2\pi k$ on $\Omega_{R_1,R_2} \cap \Omega_{R_3,R_4}$. Let $(R_n)_{n\geq 1}$ be an increasing sequence such that $R_* < R_1$ and $R_n \longrightarrow \infty$. Let $k_n \in \mathbf{Z}$ be such that $\theta_{R_*,R_n} = \theta_{R_*,R_1} + 2\pi k_n$ on Ω_{R_*,R_1}. Define $\theta(x) = \theta_{R_*,R_n}(x) - 2\pi k_n$ for $x \in \Omega_{R_*,R_n}$. It is clear that θ is well-defined on $\mathbf{R}^N \setminus \overline{B}(0, R_*)$, $\tilde{\psi} = e^{i\theta}$ and $\theta \in W^{2,p}_{loc}(\mathbf{R}^N \setminus \overline{B}(0, R_*))$ for any $p \in [1, \infty)$.

Next we consider the case $N = 2$. Since ψ is C^1 and $|\psi| \geq \frac{r_0}{2}$ on $\mathbf{R}^2 \setminus \overline{B}(0, R_*)$, the topological degree $deg(\psi, \partial B(0, R))$ is well-defined for any $R \geq R_*$ and does not depend on R. It is well-known that ψ admits a C^1 lifting θ (i.e. $\psi = |\psi|e^{i\theta}$) on $\mathbf{R}^2 \setminus \overline{B}(0, R_*)$ if and only if $deg(\psi, \partial B(0, R)) = 0$ for $R \geq R_*$. Denoting by $\tau = (-\sin\zeta, \cos\zeta)$ the unit tangent vector at $\partial B(0, R)$ at a point $Re^{i\zeta}$, we get

$$
\begin{aligned}
|deg(\psi, \partial B(0, R))| &= \left| \frac{1}{2i\pi} \int_0^{2\pi} \frac{\frac{\partial}{\partial\zeta}(\psi(Re^{i\zeta}))}{\psi(Re^{i\zeta})} \, d\zeta \right| = \left| \frac{R}{2i\pi} \int_0^{2\pi} \frac{\frac{\partial\psi}{\partial\tau}(Re^{i\zeta})}{\psi(Re^{i\zeta})} \, d\zeta \right| \\
&\leq \frac{R}{2\pi} \int_0^{2\pi} \frac{2}{r_0} |\nabla\psi(Re^{i\zeta})| \, d\zeta \leq \frac{R}{\pi r_0} \sqrt{2\pi} \left(\int_0^{2\pi} |\nabla\psi(Re^{i\zeta})|^2 \, d\zeta \right)^{\frac{1}{2}}.
\end{aligned}
\tag{2.10}
$$

On the other hand,

$$
\int_{\mathbf{R}^2 \setminus \overline{B}(0,R_*)} |\nabla\psi(x)|^2 \, dx = \int_{R_*}^{\infty} R \int_0^{2\pi} |\nabla\psi(Re^{i\zeta})|^2 \, d\zeta \, dR.
$$

We have $\displaystyle\int_{\mathbf{R}^2 \setminus \overline{B}(0,R_*)} |\nabla\psi(x)|^2 \, dx < \infty$ (because ψ has finite energy) and we infer that there exists $R_1 > R_*$ such that $\displaystyle R_1 \int_0^{2\pi} |\nabla\psi(R_1 e^{i\zeta})|^2 \, d\zeta < \frac{\pi r_0^2}{8} \frac{1}{R_1}$. From (2.10) we get

$$
|deg(\psi, \partial B(0, R_1))| < \frac{R_1}{\pi r_0} \sqrt{2\pi} \left(\frac{\pi r_0^2}{8} \frac{1}{R_1^2} \right)^{\frac{1}{2}} = \frac{1}{2}.
$$

Since the topological degree is an integer, we have necessarily $deg(\psi, \partial B(0, R_1)) = 0$. Consequently $deg(\psi, \partial B(0, R)) = 0$ for any $R \geq R_*$ and ψ admits a C^1 lifting θ. In fact, $\theta \in W^{2,p}_{loc}(\mathbf{R}^2 \setminus \overline{B}(0, R_*))$ because $\psi \in W^{2,p}_{loc}(\mathbf{R}^2 \setminus \overline{B}(0, R_*))$ (see Theorem 3 p. 38 in [10]).

If $N = 1$, the existence of a lifting $\psi = |\psi| e^{i\theta}$ follows immediately from Theorem 1 p. 27 in [10].

Finally, it is easy to see that $|\frac{\partial \tilde{\psi}}{\partial x_j}| = |\frac{\partial \theta}{\partial x_j}|$ and $|\frac{\partial^2 \tilde{\psi}}{\partial x_j \partial x_k}|^2 = |\frac{\partial^2 \theta}{\partial x_j \partial x_k}|^2 + |\frac{\partial \theta}{\partial x_j}|^2 |\frac{\partial \theta}{\partial x_k}|^2 \geq |\frac{\partial^2 \theta}{\partial x_j \partial x_k}|^2$, and i) implies $\nabla \theta \in W^{1,p}(\mathbf{R}^N \setminus \overline{B}(0, R_*))$ for any $p \in [2, \infty)$. $\square$

3 An integral identity

The main result of this section is given by the next theorem.

Theorem 3.1 *Assume that* **(H1) - (H5)** *hold. Let $\psi = \psi_1 + i\psi_2$ be a finite-energy travelling-wave for (1.1) such that $F(\cdot, |\psi|^2) \in L^1_{loc}(\mathbf{R}^N)$. Let R_* be sufficiently big, so that $|\psi| \geq \frac{r_0}{2}$ on $\mathbf{R}^N \setminus B(0, R_*)$ and let θ be the lifting given by Proposition 2.5 ii). Let $\chi \in C^\infty(\mathbf{R}^N)$ be a cut-off function such that $\chi = 0$ on $B(0, 2R_*)$ and $\chi = 1$ on $\mathbf{R}^N \setminus B(0, 3R_*)$. Then:*

i) The functions $F(\cdot, |\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2)$ and $G_j = \psi_1 \frac{\partial \psi_2}{\partial x_j} - \psi_2 \frac{\partial \psi_1}{\partial x_j} - r_0^2 \frac{\partial}{\partial x_j}(\chi\theta)$, $j = 1, \dots, N$, belong $L^1 \cap L^\infty(\mathbf{R}^N)$. (We always extend $\chi\theta$ by zero on $\overline{B}(0, R_*)$).

ii) If $N \geq 2$ and $c^2 > v_s^2$ we have the following identity:

$$\begin{aligned} &\int_{\mathbf{R}^N} |\nabla \psi|^2 - F(x, |\psi|^2)|\psi|^2 - \frac{v_s^2}{2}(|\psi|^2 - r_0^2)\, dx \\ &= c\left(1 - \frac{v_s^2}{c^2}\right) \int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - r_0^2 \frac{\partial}{\partial x_1}(\chi\theta)\, dx. \end{aligned} \tag{3.1}$$

iii) Identity (3.1) holds if $c^2 = v_s^2$ and

- *either $N = 2$*
- *or $N \geq 3$ and we assume in addition that $\psi_1 \dfrac{\partial \psi_2}{\partial x_1} - \psi_2 \dfrac{\partial \psi_1}{\partial x_1} \in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$.*

Proof. i) Let R_0, η, M be as in **(H5)** and take $R_1 > R_0$ such that $|\psi|^2(x) \in (r_0^2 - \eta, r_0^2 + \eta)$ for $x \in \mathbf{R}^N \setminus B(0, R_1)$. Using (2.7) and the fact that $v_s^2 = 2Lr_0^2$ we get

$$\begin{aligned} &F(x, |\psi|^2(x))|\psi|^2(x) + \tfrac{v_s^2}{2}(|\psi|^2(x) - r_0^2) = F(x, r_0^2)|\psi|^2(x) \\ &+ (|\psi|^2(x) - r_0^2)[\partial_{N+1} F(x, r_0^2) + L]|\psi|^2(x) - L\,(|\psi|^2(x) - r_0^2)^2 \\ &+ |\psi|^2(x) \int_{r_0^2}^{|\psi|^2(x)} (|\psi|^2(x) - \tau)\partial_{N+1}^2 F(x, \tau)\, d\tau \qquad \text{for any } |x| \geq R_1. \end{aligned} \tag{3.2}$$

Since $\psi \in L^\infty(\mathbf{R}^N)$ by Proposition 2.2 i) and $F(\cdot, r_0^2) \in L^1 \cap L^\infty(\mathbf{R}^N)$ by **(H1)** and **(H3)**, we infer that $F(\cdot, r_0^2)|\psi|^2 \in L^1 \cap L^\infty(\mathbf{R}^N)$.

We have $\psi \in L^\infty(\mathbf{R}^N)$, $\partial_{N+1} F(\cdot, r_0^2) + L \in L^{p_0} \cap L^\infty(\mathbf{R}^N)$ by **(H4)** and $|\psi|^2 - r_0^2 \in L^2 \cap L^\infty(\mathbf{R}^N)$ by Lemma 2.4, hence $(|\psi|^2 - r_0^2)[\partial_{N+1} F(\cdot, r_0^2) + L]|\psi|^2 \in L^1 \cap L^\infty(\mathbf{R}^N)$.

From Proposition 2.2 i), Lemma 2.4 and (2.9) it follows that the last two terms in the right-hand side of (3.2) are in $L^1 \cap L^\infty(\mathbf{R}^N \setminus \overline{B}(0, R_1))$. Hence $F(\cdot, |\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) \in L^1 \cap L^\infty(\mathbf{R}^N \setminus \overline{B}(0, R_1))$. Clearly, the function $F(\cdot, |\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2)$ is bounded on $\overline{B}(0, R_1)$, therefore this function belongs to $L^1 \cap L^\infty(\mathbf{R}^N)$.

Since $\psi_1 = |\psi| \cos\theta$ and $\psi_2 = |\psi| \sin\theta$, a straightforward computation gives

$$\psi_1 \frac{\partial \psi_2}{\partial x_j} - \psi_2 \frac{\partial \psi_1}{\partial x_j} = (\psi_1^2 + \psi_2^2)\frac{\partial \theta}{\partial x_j} \qquad \text{on } \mathbf{R}^N \setminus \overline{B}(0, R_*). \tag{3.3}$$

Therefore

$$\psi_1\frac{\partial\psi_2}{\partial x_j}-\psi_2\frac{\partial\psi_1}{\partial x_j}-r_0^2\frac{\partial}{\partial x_j}(\chi\theta)=(|\psi|^2-r_0^2)\frac{\partial\theta}{\partial x_j}\qquad\text{on } \mathbf{R}^N\setminus\overline{B}(0,3R_*). \tag{3.4}$$

From Lemma 2.4, Proposition 2.5 ii) and the Sobolev embedding theorem we have $|\psi|^2-r_0^2\in L^2\cap L^\infty(\mathbf{R}^N)$ and $\frac{\partial\theta}{\partial x_j}\in L^2\cap L^\infty(\mathbf{R}^N\setminus\overline{B}(0,R_*))$, respectively. Identity (3.4) implies $G_j\in L^1\cap L^\infty(\mathbf{R}^N\setminus\overline{B}(0,3R_*))$. Since G_j is continuous on $\mathbf{R}^N$, we conclude that $G_j\in L^1\cap L^\infty(\mathbf{R}^N)$.

ii) Equation (1.7) is equivalent to the system

$$c\frac{\partial\psi_2}{\partial x_1}+\Delta\psi_1+F(x,|\psi|^2)\psi_1=0\qquad\text{in } \mathcal{D}'(\mathbf{R}^N), \tag{3.5}$$

$$-c\frac{\partial\psi_1}{\partial x_1}+\Delta\psi_2+F(x,|\psi|^2)\psi_2=0\qquad\text{in } \mathcal{D}'(\mathbf{R}^N). \tag{3.6}$$

In view of Proposition 2.2 i), equalities (3.5) and (3.6) hold in $L^p_{loc}(\mathbf{R}^N)$ for $1\le p<\infty$. Multiplying (3.5) by ψ_2 and (3.6) by ψ_1, then substracting the resulting equalities we get

$$\frac{c}{2}\frac{\partial}{\partial x_1}(|\psi|^2-r_0^2)=\operatorname{div}(\psi_1\nabla\psi_2-\psi_2\nabla\psi_1). \tag{3.7}$$

We multiply (3.5) by ψ_1 and (3.6) by ψ_2, then we add the corresponding equalities to obtain

$$|\nabla\psi_1|^2+|\nabla\psi_2|^2-F(x,|\psi|^2)|\psi|^2-c(\psi_1\frac{\partial\psi_2}{\partial x_1}-\psi_2\frac{\partial\psi_1}{\partial x_1})=\frac{1}{2}\Delta(|\psi|^2-r_0^2). \tag{3.8}$$

From (3.7) and (3.8) we get

$$\frac{c}{2}\frac{\partial}{\partial x_1}(|\psi|^2-r_0^2)=\operatorname{div}(\psi_1\nabla\psi_2-\psi_2\nabla\psi_1-r_0^2\nabla(\chi\theta))+r_0^2\Delta(\chi\theta), \tag{3.9}$$

respectively

$$\begin{aligned}\frac{1}{2}\Delta(|\psi|^2-r_0^2)-\frac{v_s^2}{2}(|\psi|^2-r_0^2)&=|\nabla\psi_1|^2+|\nabla\psi_2|^2-F(x,|\psi|^2)|\psi|^2-\frac{v_s^2}{2}(|\psi|^2-r_0^2)\\&-c(\psi_1\frac{\partial\psi_2}{\partial x_1}-\psi_2\frac{\partial\psi_1}{\partial x_1}-r_0^2\frac{\partial}{\partial x_1}(\chi\theta))-cr_0^2\frac{\partial}{\partial x_1}(\chi\theta).\end{aligned} \tag{3.10}$$

Since $\psi\in W^{2,p}_{loc}(\mathbf{R}^N)$, equalities (3.7)-(3.10) hold in $L^p_{loc}(\mathbf{R}^N)$ for $1\le p<\infty$. We denote

$$H=|\nabla\psi_1|^2+|\nabla\psi_2|^2-F(x,|\psi|^2)|\psi|^2-\frac{v_s^2}{2}(|\psi|^2-r_0^2)-c(\psi_1\frac{\partial\psi_2}{\partial x_1}-\psi_2\frac{\partial\psi_1}{\partial x_1}-r_0^2\frac{\partial}{\partial x_1}(\chi\theta)).$$

We take the derivative of (3.9) with respect to x_1 (in $\mathcal{D}'(\mathbf{R}^N)$) and we multiply it by c, then we take the Laplacian of (3.10) (in $\mathcal{D}'(\mathbf{R}^N)$). Summing up the resulting equalities we obtain

$$\frac{1}{2}\Big(\Delta^2-v_s^2\Delta+c^2\frac{\partial^2}{\partial x_1^2}\Big)(|\psi|^2-r_0^2)=\Delta H+c\frac{\partial}{\partial x_1}(\operatorname{div}(G))\qquad\text{in } \mathcal{D}'(\mathbf{R}^N). \tag{3.11}$$

From i) we have $H,G_1,\dots,G_N\in L^1\cap L^\infty(\mathbf{R}^N)$ and we know from Lemma 2.4 that $|\psi|^2-r_0^2\in L^2\cap L^\infty(\mathbf{R}^N)$. Therefore $H,G_1,\dots,G_N,|\psi|^2-r_0^2\in\mathcal{S}'(\mathbf{R}^N)$ and we infer that, in fact, equality (3.11) holds in $\mathcal{S}'(\mathbf{R}^N)$. Taking the Fourier transform of (3.11) we get

$$\frac{1}{2}(|\xi|^4+v_s^2|\xi|^2-c^2\xi_1^2)\mathcal{F}(|\psi|^2-r_0^2)=-|\xi|^2\widehat{H}-c\sum_{k=1}^N\xi_1\xi_k\widehat{G_k}\qquad\text{in } \mathcal{S}'(\mathbf{R}^N). \tag{3.12}$$

We have $\widehat{H},\ \widehat{G_k} \in L^\infty \cap C^0(\mathbf{R}^N)$ because $H,\ G_k \in L^1(\mathbf{R}^N)$. Thus the right-hand side of (3.12) is a continuous function on $\mathbf{R}^N$. Since $|\psi|^2 - r_0^2 \in L^2(\mathbf{R}^N)$, we have $\mathcal{F}(|\psi|^2 - r_0^2) \in L^2(\mathbf{R}^N)$ and we infer that the left-hand side of (3.12) belongs to $L^2_{loc}(\mathbf{R}^N)$ and (3.12) holds a.e. on $\mathbf{R}^N$.

We denote

$$\Gamma = \{\xi \in \mathbf{R}^N \mid |\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2 = 0\}.$$

If $c^2 \le v_s^2$ we have $\Gamma = \{0\}$. If $c^2 > v_s^2$, it is easy to see that Γ is a nontrivial submanifold of $\mathbf{R}^N$. In the latter case, we claim that

$$|\xi|^2 \widehat{H}(\xi) + c\sum_{k=1}^N \xi_1\xi_k\widehat{G_k}(\xi) = 0 \qquad \text{for any } \xi \in \Gamma. \tag{3.13}$$

To prove this claim, we argue by contradiction and suppose that there exists $\xi^0 \in \Gamma$ such that $|\xi^0|^2\widehat{H}(\xi^0) + c\sum_{k=1}^N \xi_1^0\xi_k^0\widehat{G_k}(\xi^0) \neq 0$. By continuity, there exist $m > 0$ and a neighborhood U of ξ_0 such that $\left| |\xi|^2\widehat{H} + c\sum_{k=1}^N \xi_1\xi_k\widehat{G_k}\right| \ge m$ on U. From (3.12) we infer that

$$|\mathcal{F}(|\psi|^2 - r_0^2)(\xi)| \ge \frac{2m}{|\,|\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2|} \qquad \text{a.e. on } U \setminus \Gamma.$$

Since 0 and $(\sqrt{c^2 - v_s^2}, 0, \dots, 0)$ are not isolated points of Γ, we may assume that $\xi^0 \neq 0$ and $\xi^0 \neq (\sqrt{c^2 - v_s^2}, 0, \dots, 0)$. A straightforward computation (details can be found in [17], p. 98 in the case $v_s^2 = 2$; the general case is similar) shows that

$$\int_{U\setminus\Gamma} \frac{1}{|\,|\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2|^2}\, d\xi = \infty,$$

consequently $\int_{U\setminus\Gamma} |\mathcal{F}(|\psi|^2 - r_0^2)(\xi)|^2\, d\xi = \infty$. But this is in contradiction with $\mathcal{F}(|\psi|^2 - r_0^2) \in L^2(\mathbf{R}^N)$ and the claim is proved.

It is not hard to see that $\Gamma = \{(\xi_1, \xi') \in \mathbf{R} \times \mathbf{R}^{N-1} \mid |\xi'|^2 = \frac{1}{2}(-v_s^2 - 2\xi_1^2 + \sqrt{v_s^4 + 4c^2\xi_1^2})\}$. Let $f(t) = \sqrt{\frac{1}{2}\left(-v_s^2 - 2t^2 + \sqrt{v_s^4 + 4c^2t^2}\right)}$. The function f is well-defined for $t \in [-\sqrt{c^2 - v_s^2}, \sqrt{c^2 - v_s^2}]$, $f(0) = 0$ and $\lim_{t\to 0}\frac{f^2(t)}{t^2} = -1 + \frac{c^2}{v_s^2}$. Fix $j \in \{2, \dots, N\}$. For $t \in (0, \sqrt{c^2 - v_s^2}]$, let $\xi(t) = (t, 0, \dots, 0, f(t), 0, \dots, 0)$ and $\tilde{\xi}(t) = (t, 0, \dots, 0, -f(t), 0, \dots, 0)$, where $f(t)$, respectively $-f(t)$, stand at the j^{th} place. It is obvious that $\xi(t), \tilde{\xi}(t) \in \Gamma$. From (3.13) we obtain

$$(t^2 + f^2(t))\widehat{H}(\xi(t)) + ct^2\widehat{G_1}(\xi(t)) + ctf(t)\widehat{G_j}(\xi(t)) = 0, \qquad \text{respectively} \tag{3.14}$$

$$(t^2 + f^2(t))\widehat{H}(\tilde{\xi}(t)) + ct^2\widehat{G_1}(\tilde{\xi}(t)) - ctf(t)\widehat{G_j}(\tilde{\xi}(t)) = 0. \tag{3.15}$$

We multiply (3.14) and (3.15) by $\frac{1}{t^2}$, then pass to the limit as $t \downarrow 0$ to obtain

$$\frac{c^2}{v_s^2}\widehat{H}(0) + c\widehat{G_1}(0) + c\sqrt{-1 + \frac{c^2}{v_s^2}}\widehat{G_j}(0) = 0, \text{ respectively} \tag{3.16}$$

$$\frac{c^2}{v_s^2}\widehat{H}(0) + c\widehat{G_1}(0) - c\sqrt{-1 + \frac{c^2}{v_s^2}}\widehat{G_j}(0) = 0. \tag{3.17}$$

From (3.16) and(3.17) we infer that $\frac{c^2}{v_s^2}\widehat{H}(0) + c\widehat{G_1}(0) = 0$ and $\widehat{G_j}(0) = 0$, that is $\int_{\mathbf{R}^N} H(x) + \frac{v_s^2}{c} G_1(x)\,dx = 0$ and $\int_{\mathbf{R}^N} G_j(x)\,dx = 0$. The first of these integral identities is exactly (3.1) and the latter can be written as

$$\int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_j} - \psi_2 \frac{\partial \psi_1}{\partial x_j} - r_0^2 \frac{\partial}{\partial x_j}(\chi\theta)\,dx = 0 \qquad \text{for } j = 2, \dots, N. \tag{3.18}$$

iii) Assume that $c^2 = v_s^2$. Then (3.1) is equivalent to $\widehat{H}(0) + c\widehat{G_1}(0) = 0$. Denoting $\xi = (\xi_1, \xi')$, where $\xi' = (\xi_2, \dots, \xi_N)$, identity (3.12) implies

$$\begin{aligned} \mathcal{F}(|\psi|^2 - r_0^2)(\xi) &= -2\frac{\xi_1^2}{|\xi|^4 + c^2|\xi'|^2}(\widehat{H}(\xi) + c\widehat{G_1}(\xi)) \\ &-2c\sum_{k=2}^{N} \frac{\xi_1\xi_k}{|\xi|^4 + c^2|\xi'|^2}\widehat{G_k}(\xi) - 2\frac{|\xi'|^2}{|\xi|^4 + c^2|\xi'|^2}\widehat{H}(\xi) \qquad \text{a.e. } \xi \in \mathbf{R}^N. \end{aligned} \tag{3.19}$$

For $\varepsilon \in (0, 1]$, we denote $\Omega_\varepsilon = \{(\xi_1, \xi') \in \mathbf{R} \times \mathbf{R}^{N-1} \mid \xi_1 \in [0, \varepsilon],\ 0 \le |\xi'| \le \xi_1\}$. We will use the following

Lemma 3.2 *Let $N \ge 2$ and $k \in \{2, \dots, N\}$.*

i) The function $\xi \longmapsto \frac{\xi_1^2}{\xi_1^4 + c^2|\xi'|^2}$ belongs to $L^p(\Omega_\varepsilon)$ if and only if $p < N - \frac{1}{2}$.

ii) The function $\xi \longmapsto \frac{\xi_1\xi_k}{\xi_1^4 + c^2|\xi'|^2}$ belongs to $L^p(\Omega_\varepsilon)$ for any $p \in [1, 2N-1)$.

Proof of Lemma 3.2. i) Using Fubini's theorem for positive functions, then passing to spherical coordinates in $\mathbf{R}^{N-1}$ and making the change of variables $r = \xi_1^2 t$ we get

$$\begin{aligned} \int_{\Omega_\varepsilon} \left(\frac{\xi_1^2}{\xi_1^4 + c^2|\xi'|^2}\right)^p d\xi &= \int_0^\varepsilon \xi_1^{2p} \int_{\{|\xi'| \le \xi_1\}} \frac{1}{(\xi_1^4 + c^2|\xi'|^2)^p}\,d\xi'\,d\xi_1 \\ &= \int_0^\varepsilon \xi_1^{2p} |S^{N-2}| \int_0^{\xi_1} \frac{r^{N-2}}{(\xi_1^4 + c^2 r^2)^p}\,dr\,d\xi_1 \\ &= |S^{N-2}| \int_0^\varepsilon \xi_1^{2p} \int_0^{\frac{1}{\xi_1}} \frac{(\xi_1^2 t)^{N-2}}{(\xi_1^4 + c^2\xi_1^4 t^2)^p}\xi_1^2\,dt\,d\xi_1 \qquad \text{(change of variables } r = \xi_1^2 t) \\ &= |S^{N-2}| \int_0^\varepsilon \xi_1^{2(N-1-p)} \int_0^{\frac{1}{\xi_1}} \frac{t^{N-2}}{(1 + c^2t^2)^p}\,dt\,d\xi_1. \end{aligned} \tag{3.20}$$

Assume that $p < N - \frac{1}{2}$. Obviously $\frac{t^{N-2}}{(1+c^2t^2)^p} \le 1$ for $t \in [0, 1]$ and $\frac{t^2}{1+c^2t^2} \le \frac{1}{c^2}$, thus we have

$$\int_0^{\frac{1}{\xi_1}} \frac{t^{N-2}}{(1 + c^2t^2)^p}\,dt \le 1 + \frac{1}{c^{2p}} \int_1^{\frac{1}{\xi_1}} t^{N-2p-2}\,dt = \begin{cases} C_1 + \frac{C_2}{\xi_1^{N-2p-1}} & \text{if } p \ne \frac{N-1}{2}, \\ C_3 + C_4 \ln \xi_1 & \text{if } p = \frac{N-1}{2}, \end{cases}$$

where C_j are some positive constants. This estimate implies that the right-hand side of (3.20) is finite if $p < N - \frac{1}{2}$.

If $p \ge N - \frac{1}{2}$, denote $c_p = \int_0^1 \frac{t^{N-2}}{(1 + c^2t^2)^p}\,dt > 0$. Since $\frac{1}{\xi_1} > 1$ for $\xi_1 \in (0, \varepsilon)$, the right-hand side of (3.20) is greater than $|S^{N-2}| c_p \int_0^\varepsilon \xi_1^{2(N-1-p)}\,d\xi_1 = \infty$.

ii) Proceeding as above, we have

$$\int_{\Omega_\varepsilon} \left| \frac{\xi_1 \xi_k}{\xi_1^4 + c^2 |\xi'|^2} \right|^p d\xi \le \int_{\Omega_\varepsilon} \frac{\xi_1^p |\xi'|^p}{(\xi_1^4 + c^2|\xi'|^2)^p} d\xi = \int_0^\varepsilon \xi_1^p |S^{N-2}| \int_0^{\xi_1} \frac{r^{p+N-2}}{(\xi_1^4 + c^2 r^2)^p} dr\, d\xi_1$$

$$(3.21) \qquad = |S^{N-2}| \int_0^\varepsilon \xi_1^p \int_0^{\frac{1}{\xi_1}} \frac{(\xi_1^2 t)^{p+N-2}}{(\xi_1^4 + c^2 \xi_1^4 t^2)^p} \xi_1^2 \, dt\, d\xi_1 \qquad \text{(change of variables } r = \xi_1^2 t)$$

$$= |S^{N-2}| \int_0^\varepsilon \xi_1^{2N-p-2} \int_0^{\frac{1}{\xi_1}} \frac{t^{p+N-2}}{(1+c^2t^2)^p} dt\, d\xi_1.$$

As previously,

$$\int_0^{\frac{1}{\xi_1}} \frac{t^{p+N-2}}{(1+c^2t^2)^p} dt < \frac{1}{c^{2p}} \int_0^{\frac{1}{\xi_1}} t^{N-p-2}\, dt = \frac{1}{c^{2p}(N-p-1)} \frac{1}{\xi_1^{N-p-1}} \qquad \text{if } N-p-1>0.$$

Therefore in the case $p < N-1$, the right-hand side of (3.21) is less than $C\int_0^\varepsilon \xi_1^{N-1}\, d\xi_1 < \infty$. If $p > N-1$, the integral $\displaystyle\int_0^\infty \frac{t^{p+N-2}}{(1+c^2t^2)^p} dt$ converges. Let a_p be its value. If $N-1 < p < 2N-1$, by (3.21) we get $\displaystyle\int_{\Omega_\varepsilon} \left| \frac{\xi_1 \xi_k}{\xi_1^4 + c^2|\xi'|^2} \right|^p d\xi \le |S^{N-2}| a_p \int_0^\varepsilon \xi_1^{2N-2-p} d\xi_1 < \infty.$ □

Remark. It can be proved that the function $\xi \longmapsto \frac{\xi_1\xi_k}{\xi_1^4 + c^2|\xi'|^2}$ does not belong to $L^p(\Omega_\varepsilon)$ if $p \ge 2N-1$, but we will not make use of this fact here.

Now we come back to the proof of Theorem 3.1. All we have to do is to show that $\widehat{H}(0) + c\widehat{G_1}(0) = 0$. We argue by contradiction and assume that $\widehat{H}(0) + c\widehat{G_1}(0) \neq 0$. Since the functions $\widehat{H}$ and $\widehat{G_j}$ are continuous, there exists $\varepsilon \in (0,1)$ such that $|\widehat{H}(\xi) + c\widehat{G_1}(\xi)| \ge \frac{1}{2}|\widehat{H}(0) + c\widehat{G_1}(0)|$ for any $\xi \in \Omega_\varepsilon$. Taking a smaller ε if necessary, we may also assume that $|\xi|^4 + c^2|\xi'|^2 \le 2(\xi_1^4 + c^2|\xi'|^2)$ for any $\xi \in \Omega_\varepsilon$. By (3.19) we have

$$\frac{1}{2}\frac{\xi_1^2}{\xi_1^4 + c^2|\xi'|^2}|\widehat{H}(0) + c\widehat{G_1}(0)| \le 2\frac{\xi_1^2}{|\xi|^4 + c^2|\xi'|^2}|\widehat{H}(\xi) + c\widehat{G_1}(\xi)|$$

$$(3.22) \qquad \le |\mathcal{F}(|\psi|^2 - r_0^2)(\xi)| + 2|c|\sum_{k=2}^N \frac{|\xi_1\xi_k|}{\xi_1^4 + c^2|\xi'|^2}|\widehat{G_k}(\xi)| + 2\frac{|\xi'|^2}{|\xi|^4 + c^2|\xi'|^2}|\widehat{H}(\xi)| \quad \text{a.e. on } \Omega_\varepsilon.$$

Consider first the case $N = 2$. We know that $\mathcal{F}(|\psi|^2 - r_0^2) \in L^2(\mathbf{R}^2)$, consequently $\mathcal{F}(|\psi|^2 - r_0^2) \in L^p(\Omega_\varepsilon)$ for any $p \in [1,2]$. Since $\widehat{G_k}$ are continuous and bounded, by Lemma 3.2 ii) we infer that the functions $\xi \longmapsto \frac{\xi_1\xi_k}{\xi_1^4 + c^2|\xi'|^2}\widehat{G_k}(\xi)$ belong to $L^p(\Omega_\varepsilon)$ for any $p \in [1,3)$. It is obvious that $\frac{|\xi'|^2}{|\xi|^4 + c^2|\xi'|^2}|\widehat{H}(\xi)| \le \frac{1}{c^2}|\widehat{H}(\xi)|$ and $\widehat{H}$ is continuous and bounded on $\mathbf{R}^N$. We conclude that the right-hand side of (3.22) belongs to $L^p(\Omega_\varepsilon)$ for any $p \in [1,2]$. Then (3.22) implies that $\xi \longmapsto \frac{\xi_1^2}{\xi_1^4 + c^2|\xi'|^2}$ belongs to $L^2(\Omega_\varepsilon)$, which contradicts Lemma 3.2 i). This contradiction proves that $\widehat{H}(0) + c\widehat{G_1}(0) = 0$.

Next we assume that $N \ge 3$ and $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$. Equation (3.8) can be written as

$$-\frac{1}{2}\Delta(|\psi|^2 - r_0^2) + \frac{v_s^2}{2}(|\psi|^2 - r_0^2)$$

$$(3.23) \qquad = -|\nabla\psi_1|^2 - |\nabla\psi_2|^2 + F(x, |\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) + c(\psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1}).$$

We have already proved that $F(\cdot,|\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) \in L^1 \cap L^\infty(\mathbf{R}^N)$. From Proposition 2.5 i) we have $|\nabla\psi|^2 \in L^p(\mathbf{R}^N)$ for any $p \in [1,\infty]$. Using the assumption $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$, we infer that the right-hand side of (3.23) belongs to $L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$. By the Hausdorff-Young inequality, for any function $f \in L^p(\mathbf{R}^N)$ with $1 \le p \le 2$ we have $\mathcal{F}(f) \in L^{p'}(\mathbf{R}^N)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ (see, e.g., Theorem 1.2.1 p. 6 in [4]). Passing to Fourier transforms in (3.23) we get

$$
\begin{aligned}
\mathcal{F}(|\psi|^2 - r_0^2)(\xi) \;=\;& \tfrac{2}{|\xi|^2+v_s^2}\mathcal{F}\Big[-|\nabla\psi|^2 + (F(\cdot,|\psi|^2)|\psi|^2 + \tfrac{v_s^2}{2}(|\psi|^2 - r_0^2)) \\
& + c(\psi_1 \tfrac{\partial\psi_2}{\partial x_1} - \psi_2 \tfrac{\partial \psi_1}{\partial x_1})\Big](\xi) \qquad \text{a.e. } \xi \in \mathbf{R}^N.
\end{aligned}
\tag{3.24}
$$

We obtain from (3.24) that $\mathcal{F}(|\psi|^2 - r_0^2) \in L^{N-\frac{1}{2}}(\mathbf{R}^N)$. Combined with the fact that $\widehat{H}$, $\widehat{G_j}$ and $\xi \longmapsto \frac{|\xi'|^2}{|\xi|^4 + c^2|\xi'|^2}$ are bounded and Lemma 3.2 ii), this implies that the last expression in (3.22) is in $L^{N-\frac{1}{2}}(\Omega_\varepsilon)$. We infer that the function $\xi \longmapsto \frac{\xi_1^2}{\xi_1^4 + c^2|\xi'|^2}|\widehat{H}(0) + c\widehat{G_1}(0)|$ must be in $L^{N-\frac{1}{2}}(\Omega_\varepsilon)$ for any sufficiently small ε. If $\widehat{H}(0) + c\widehat{G_1}(0) \neq 0$, this contradicts Lemma 3.2 i). Thus necessarily $\widehat{H}(0) + c\widehat{G_1}(0) = 0$ and the proof of Theorem 3.1 is complete. □

It is an open problem whether any finite energy travelling-wave ψ of (1.1) moving with speed $c = \pm v_s$ satisfies $\psi_1 \frac{\partial\psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$. Even for very particular cases of (1.1), such as the Gross-Pitaevskii equation, the answer to this question is not known. However, we have the following:

Proposition 3.3 *Assume that* **(H1) - (H5)** *hold and let $\psi = \psi_1 + i\psi_2$ be a finite-energy travelling-wave for (1.1) such that $F(\cdot,|\psi|^2) \in L^1_{loc}(\mathbf{R}^N)$. Let R_* be sufficiently big, so that $|\psi| \ge \frac{r_0}{2}$ on $\mathbf{R}^N \setminus B(0,R_*)$, let θ be the lifting given by Proposition 2.5 ii) and let $\chi \in C^\infty(\mathbf{R}^N)$ be a cut-off function as in Theorem 3.1. Then:*

i) Let $p \in (1,\infty)$. The following assertions are equivalent:

a) $\nabla(\chi\theta) \in L^p(\mathbf{R}^N)$;

b) $\psi_1 \frac{\partial\psi_2}{\partial x_j} - \psi_2 \frac{\partial\psi_1}{\partial x_j} \in L^p(\mathbf{R}^N)$ for any $j \in \{1,\dots,N\}$;

c) $\psi_1 \frac{\partial\psi_2}{\partial x_1} - \psi_2 \frac{\partial\psi_1}{\partial x_1} \in L^p(\mathbf{R}^N)$;

d) $|\psi|^2 - r_0^2 \in W^{2,p}(\mathbf{R}^N)$;

e) $|\psi|^2 - r_0^2 \in L^p(\mathbf{R}^N)$.

ii) If $N \ge 3$, there exists $\theta_0 \in \mathbf{R}$ such that $\chi\theta - \theta_0 \in W^{2,q}(\mathbf{R}^N)$ for any $q \in [\frac{2N}{N-2},\infty)$.

Moreover, if $c^2 = v_s^2$ we have:

iii) $|\psi|^2 - r_0^2 \in L^p(\mathbf{R}^N)$ and $\psi_1 \frac{\partial\psi_2}{\partial x_j} - \psi_2 \frac{\partial\psi_1}{\partial x_j} \in L^p(\mathbf{R}^N)$ for any $p > \frac{2N-1}{2N-3}$ and $j \in \{1,\dots,N\}$.

iv) $\nabla(|\psi|^2 - r_0^2) \in L^p(\mathbf{R}^N)$ for any $p > \frac{2N-1}{2N-2}$.

v) $\partial^2_{j,k}(|\psi|^2 - r_0^2) \in L^p(\mathbf{R}^N)$ for any $p \in (1,\infty)$.

Proof. i) Since $\psi \in L^\infty(\mathbf{R}^N)$ and (3.3) holds, the equivalence a) ⇔ b) is clear. It is also obvious that b) ⇒ c).

From the classical Marcinkiewicz Theorem (see Theorem 3 p. 96 in [27]) it follows that the functions $\frac{1}{|\xi|^2+v_s^2}$, $\frac{\xi_j}{|\xi|^2+v_s^2}$ and $\frac{\xi_j\xi_k}{|\xi|^2+v_s^2}$ are L^p-multipliers for $1 < p < \infty$. Assume that $\psi_1 \frac{\partial\psi_2}{\partial x_1} - \psi_2 \frac{\partial\psi_1}{\partial x_1} \in L^p(\mathbf{R}^N)$. Since $|\nabla\psi|^2 \in L^1 \cap L^\infty(\mathbf{R}^N)$ and $F(\cdot,|\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) \in L^1 \cap L^\infty(\mathbf{R}^N)$

by Theorem 3.1 i), we have $-|\nabla\psi|^2+(F(\cdot,|\psi|^2)|\psi|^2+\frac{v_s^2}{2}(|\psi|^2-r_0^2))+c(\psi_1\frac{\partial\psi_2}{\partial x_1}-\psi_2\frac{\partial\psi_1}{\partial x_1}) \in L^p(\mathbf{R}^N)$ and we infer from (3.24) that $|\psi|^2 - r_0^2 \in W^{2,p}(\mathbf{R}^N)$. Hence c) $\Rightarrow$ d). It is obvious that d) $\Rightarrow$ e).

It follows from Proposition 2.5 ii) that $\partial_k(\chi\theta) \in \mathcal{S}'(\mathbf{R}^N)$. It is then clear that all terms appearing in (3.9) belong to $\mathcal{S}'(\mathbf{R}^N)$. We take the derivative of (3.9) with respect to x_k (in $\mathcal{S}'(\mathbf{R}^N)$), then we take the Fourier transform of the resulting equality to obtain

$$\mathcal{F}(\frac{\partial}{\partial x_k}(\chi\theta)) = -\sum_{j=1}^{N}\frac{\xi_j\xi_k}{|\xi|^2}\widehat{G_j} + \frac{c}{2}\frac{\xi_1\xi_k}{|\xi|^2}\mathcal{F}(|\psi|^2 - r_0^2)$$

or equivalently

$$\frac{\partial}{\partial x_k}(\chi\theta) = \sum_{j=1}^{N} R_j R_k(G_j) - \frac{c}{2}R_1 R_k(|\psi|^2 - r_0^2), \tag{3.25}$$

where R_j is the Riesz transform, $R_j\phi = \mathcal{F}^{-1}(i\frac{\xi_j}{|\xi|}\widehat{\phi})$. It is well-known that the Riesz transform maps continuously $L^p(\mathbf{R}^N)$ into $L^p(\mathbf{R}^N)$ for $1 < p < \infty$ (see, e.g., Theorem 3 p. 96 and Example (iii) p. 95 in [27]). From Theorem 3.1 i) we have $G_j \in L^1 \cap L^\infty(\mathbf{R}^N)$, therefore $R_jR_k(G_j) \in L^q(\mathbf{R}^N)$ for any $q \in (1,\infty)$. Assume that $|\psi|^2 - r_0^2 \in L^p(\mathbf{R}^N)$ for some $p \in (1,\infty)$. Then $R_1R_k(|\psi|^2 - r_0^2) \in L^p(\mathbf{R}^N)$ and from (3.25) we infer that $\frac{\partial}{\partial x_k}(\chi\theta) \in L^p(\mathbf{R}^N)$ for any $k \in \{1,\dots,N\}$. Thus e) $\Rightarrow$ a) and i) is proved.

ii) It is well-known that for any function ϕ satisfying $\nabla\phi \in L^p(\mathbf{R}^N)$ with $p < N$, there exists a constant λ such that $\phi - \lambda \in L^{p^*}(\mathbf{R}^N)$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$ (see Theorem 4.5.9 in [20] or Lemma 7 and Remark 4.2 in [15] p. 774-775 for a different proof). From Proposition 2.5 ii) we have $\nabla(\chi\theta) \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2,\infty)$. If $N \geq 3$, we infer that there exists $\theta_0 \in \mathbf{R}$ such that $\chi\theta - \theta_0 \in L^q(\mathbf{R}^N)$ for $q \in [\frac{2N}{N-2},\infty)$. Therefore $\chi\theta - \theta_0 \in W^{2,q}(\mathbf{R}^N)$ for any $q \in [\frac{2N}{N-2},\infty)$ and, in particular, $\chi\theta - \theta_0 \longrightarrow 0$ as $|x| \longrightarrow \infty$.

iii) We will use the following result due to Lizorkin (see Theorem 8 p. 288 in [24]):

Theorem 3.4 ([24]) *Let $\beta \in [0,1)$ and let $K \in L^\infty(\mathbf{R}^N) \cap C^N(\mathbf{R}^N \setminus \{0\})$. Assume that*

$$\Big(\prod_{j=1}^{N}\xi_j^{k_j+\beta}\Big)\partial_1^{k_1}\dots\partial_N^{k_N}K \in L^\infty(\mathbf{R}^N) \qquad \textit{for any } k_1,\dots,k_N \in \{0,1\}.$$

Then K is a Fourier multiplier from $L^p(\mathbf{R}^N)$ to $L^{\frac{p}{1-\beta p}}(\mathbf{R}^N)$ for any $p \in (1,\frac{1}{\beta})$.

Let $K(\xi) = \frac{|\xi|^2}{|\xi|^4+c^2|\xi'|^2}$, where $\xi' = (\xi_2,\dots,\xi_N)$. A straightforward but tedious computation shows that K satisfies the assumptions of Lizorkin's theorem for $\beta = \frac{1}{2N-1}$. From (3.19) we obtain

$$|\psi|^2 - r_0^2 = 2R_1^2\left(\mathcal{F}^{-1}\left(K(\widehat{H}+c\widehat{G_1})\right)\right) + 2c\sum_{j=2}^{N}R_1R_j\left(\mathcal{F}^{-1}(K\widehat{G_j})\right) + 2\sum_{j=2}^{N}R_j^2\left(\mathcal{F}^{-1}(K\widehat{H})\right), \tag{3.26}$$

where R_j's denote Riesz transforms. Since $H, G_1,\dots,G_N \in L^1 \cap L^\infty(\mathbf{R}^N)$, by (3.26) and Lizorkin's theorem we infer that $|\psi|^2 - r_0^2 \in L^p(\mathbf{R}^N)$ for any $p \in (\frac{2N-1}{2N-3},\infty)$. The rest of iii) follows from part i), b) $\Leftrightarrow$ e).

iv) and v) From iii) and i), d) $\Leftrightarrow$ e) it follows immediately that $|\psi|^2 - r_0^2 \in W^{2,p}(\mathbf{R}^N)$ for any $p \in (\frac{2N-1}{2N-3}, \infty)$. Using (3.19) we obtain

$$
\begin{aligned}
\partial^2_{k\ell}\left(|\psi|^2 - r_0^2\right) &= 2R_k R_\ell R_1^2 \left(\mathcal{F}^{-1}\left(|\xi|^2 K(\widehat{H} + c\widehat{G_1})\right)\right) \\
&+2c\sum_{j=2}^{N} R_k R_\ell R_1 R_j \left(\mathcal{F}^{-1}(|\xi|^2 K\widehat{G_j})\right) \\
&+2\sum_{j=2}^{N} R_k R_\ell R_j^2 \left(\mathcal{F}^{-1}(|\xi|^2 K\widehat{H})\right) \qquad \text{in } \mathcal{S}'(\mathbf{R}^N).
\end{aligned} \tag{3.27}
$$

It can be proved by direct computation that the function $|\xi|^2 K$ satisfies the assumptions of Lizorkin's theorem for $\beta = 0$. Consequently $|\xi|^2 K$ is an L^p-multiplier for $1 < p < \infty$. Since $H, G_j \in L^1 \cap L^\infty(\mathbf{R}^N)$, it follows from (3.27) that $\partial^2_{k\ell}\left(|\psi|^2 - r_0^2\right) \in L^p(\mathbf{R}^N)$ for $1 < p < \infty$.

By using the Gagliardo-Nirenberg inequality

$$||\nabla\phi||^2_{L^p} \le C||\phi||_{L^q}||\nabla^2\phi||_{L^r} \qquad \text{if} \quad \frac{1}{p} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{r}\right),$$

we infer that $\nabla(|\psi|^2 - r_0^2) \in L^p(\mathbf{R}^N)$ for any $p > \frac{2N-1}{2N-2}$. □

Corollary 3.5 *Under the assumptions of Theorem 3.1, assume that $N \ge 3$, $c^2 = v_s^2$ and the momentum of ψ with respect to the x_1-direction is well-defined, that is $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^1(\mathbf{R}^N)$. Then ψ satisfies (3.1).*

Proof. From Proposition 3.4 iii) and i) we have $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^p(\mathbf{R}^N)$ for $p \in (\frac{2N-1}{2N-3}, \infty)$. Then the assumption $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^1(\mathbf{R}^N)$ implies $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^p(\mathbf{R}^N)$ for any $p \in [1, \infty)$. Now the conclusion follows from Theorem 3.1 iii). □

4 Nonexistence results

In this section we show how Theorem 3.1 may be used to prove nonexistence of supersonic and sonic travelling-waves with finite energy for some equations of type (1.1).

1. We consider the equation

$$i\frac{\partial \Phi}{\partial t} + \Delta\Phi + G(|\Phi|^2)\Phi = 0 \qquad \text{in } \mathbf{R}^N. \tag{4.1}$$

We assume that the function $G : [0, \infty) \longrightarrow \mathbf{R}$ satisfies the following asumptions:

- **(A1)** $G \in C^2([0,\infty), \mathbf{R})$ and there exists $r_0 > 0$ such that $G(r_0^2) = 0$ and $G'(r_0^2) < 0$.
- **(A2)** There exists $\alpha > 0$ such that $\limsup\limits_{s\to\infty} \frac{G(s)}{s^\alpha} < 0$.

Obviously, equation (4.1) is of the form (1.1). As previously, we associate to (4.1) the "boundary condition" $|\Phi| \longrightarrow r_0^2$ as $|x| \longrightarrow \infty$. In this context, the sound velocity at infinity is $v_s = r_0\sqrt{-2G'(r_0^2)}$. The energy corresponding to (4.1) is $E(\Phi) = \displaystyle\int_{\mathbf{R}^N} |\nabla\Phi|^2\,dx + \int_{\mathbf{R}^N} V(|\Phi|^2)\,dx$, where $V(s) = \displaystyle\int_s^{r_0^2} G(\tau)\,d\tau$. Let ψ be a finite-energy travelling-wave for (4.1) (in the sense of Definition 2.1) moving with speed c. Then ψ satisfies the equation

$$-ic\frac{\partial\psi}{\partial x_1} + \Delta\psi + G(|\psi|^2)\psi = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N), \qquad |\psi| \longrightarrow r_0 \quad \text{as } |x| \longrightarrow \infty. \tag{4.2}$$

If G satisfies **(A1)-(A2)**, it is easy to see that $F(x,s) := G(s)$ satisfies the assumptions **(H1)-(H5)** in section 2 (with $L = -G'(r_0^2)$). It is then clear that the conclusions of Propositions 2.2, 2.5 and Theorem 3.1 i) are valid for ψ. Moreover, we have:

Proposition 4.1 *(Pohozaev identities) Let ψ be as above. Choose $R_* > 0$ such that $|\psi| \geq \frac{r_0}{2}$ on $\mathbf{R}^N \setminus B(0,R_*)$. Let θ be the lifting of $\frac{\psi}{|\psi|}$ on $\mathbf{R}^N \setminus B(0,R_*)$ (as given by Proposition 2.5 ii)) and let χ be a cut-off function as in Theorem 3.1. The following identities hold:*

$$-\int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_1}\right|^2 dx + \int_{\mathbf{R}^N} \sum_{j=2}^{N} \left|\frac{\partial \psi}{\partial x_j}\right|^2 dx + \int_{\mathbf{R}^N} V(|\psi|^2)\,dx = 0 \qquad \textit{and} \tag{4.3}$$

$$\begin{aligned} &-\int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_k}\right|^2 dx + \int_{\mathbf{R}^N} \sum_{j=1,\, j\neq k}^{N} \left|\frac{\partial \psi}{\partial x_j}\right|^2 dx + \int_{\mathbf{R}^N} V(|\psi|^2)\,dx \\ &\quad -c\int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - r_0^2 \frac{\partial}{\partial x_1}(\chi\theta)\,dx = 0 \qquad \textit{for } k = 2,\dots,N. \end{aligned} \tag{4.4}$$

It is worth to note that Proposition 4.1 is valid for any speed $c \in \mathbf{R}$.

Proof. Since the arguments are rather classical, we only sketch the proof.

Formally, travelling-waves are critical points of the functional $E_c = E + cP_1$, where E is the energy and P_1 is the momentum with respect to the x_1-direction (see (1.3)). Identities (4.3) and (4.4) are simple consequences of the behavior of E_c with respect to dilations in $\mathbf{R}^N$. To be more precise, define $\psi_{k,t}(x) = \psi(x_1,\dots,x_{k-1},tx_k,x_{k+1},\dots,x_N)$ and $g_k(t) = E_c(\psi_{k,t})$. If ψ is a critical point of E_c, one would expect that $g_k'(1) = \frac{d}{dt}(E_c(\psi_{k,t}))|_{t=1} = 0$ and this is precisely (4.3) if $k = 1$, respectively (4.4) if $k \geq 2$. However, this argument is not rigorous for at least two reasons. First, it is not clear what function space one should consider to define E_c (and this could not be a vector space because of the boundary conditions at infinity). Second, even if an appropriate function space is found, we do not know whether $\frac{d}{dt}(\psi_{k,t})|_{t=1} = x_k \frac{\partial \psi}{\partial x_k}$ belong to the tangent space at ψ of the considered function space.

The most convenient way to prove Pohozaev identities is to use a truncation argument. Fix a function $\eta \in C_c^\infty(\mathbf{R}^N)$ such that $\eta = 1$ on $B(0,1)$ and $\eta = 0$ on $\mathbf{R}^N \setminus B(0,2)$. For $n \geq 1$, define $\eta_n(x) = \eta(\frac{x}{n})$. We take the scalar product of (4.2) by $x_k\eta_n(x)\frac{\partial \psi}{\partial x_k}$ and we integrate by parts the resulting equality. It is standard (see, e.g., Proposition 1 p. 320 in [3] or Lemma 2.4 p. 104 in [11]) to prove that

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} (\Delta\psi, x_k\eta_n(x)\frac{\partial \psi}{\partial x_k})\,dx = -\int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_k}\right|^2 dx + \frac{1}{2}\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx \qquad \text{and} \tag{4.5}$$

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} (G(|\psi|^2)\psi, x_k\eta_n(x)\frac{\partial \psi}{\partial x_k})\,dx = \frac{1}{2}\int_{\mathbf{R}^N} V(|\psi|^2)\,dx. \tag{4.6}$$

It is obvious that $(ic\frac{\partial \psi}{\partial x_1}, \eta_n(x)x_1\frac{\partial \psi}{\partial x_1}) = c\eta_n(x)x_1(i\frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_1}) = 0$. Thus taking the scalar product of (4.2) by $x_1\eta_n(x)\frac{\partial \psi}{\partial x_1}$, integrating and using (4.5) and (4.6) we get (4.3).

By (3.3) we have $(-i\frac{\partial \psi}{\partial x_j}, \psi) = \psi_1\frac{\partial \psi_2}{\partial x_j} - \psi_2\frac{\partial \psi_1}{\partial x_j} = |\psi|^2\frac{\partial \theta}{\partial x_j}$ on $\mathbf{R}^N \setminus \overline{B}(0,R_*)$. Using the convention $\partial^\alpha(\chi\theta) = 0$, $(\partial^\alpha\chi)\theta = 0$ on $B(0,2R_*)$, we have

$$\begin{aligned} (-i\tfrac{\partial \psi}{\partial x_j}, \psi) &= (1-\chi)(-i\tfrac{\partial \psi}{\partial x_j}, \psi) + \chi|\psi|^2\tfrac{\partial \theta}{\partial x_j} \\ &= (1-\chi)(-i\tfrac{\partial \psi}{\partial x_j}, \psi) + |\psi|^2\tfrac{\partial(\chi\theta)}{\partial x_j} - |\psi|^2\theta\tfrac{\partial \chi}{\partial x_j} \qquad \text{on } \mathbf{R}^N. \end{aligned} \tag{4.7}$$

Therefore we get for $k = 2, \dots, N$:

$$\int_{\mathbf{R}^N} (-ic\frac{\partial\psi}{\partial x_1}, x_k\eta_n(x)\frac{\partial\psi}{\partial x_k})\,dx$$

$$= \frac{c}{2}\int_{\mathbf{R}^N} x_k\eta_n(x)\Big[\frac{\partial}{\partial x_1}(-i\psi, \frac{\partial\psi}{\partial x_k}) + \frac{\partial}{\partial x_k}(-i\frac{\partial\psi}{\partial x_1}, \psi)\Big]\,dx$$

$$= -\frac{c}{2}\int_{\mathbf{R}^N} x_k\frac{\partial\eta_n}{\partial x_1}(x)(-i\psi, \frac{\partial\psi}{\partial x_k}) + \Big(\eta_n(x) + x_k\frac{\partial\eta_n}{\partial x_k}(x)\Big)(-i\frac{\partial\psi}{\partial x_1}, \psi)\,dx$$

$$\begin{aligned}(4.8)\qquad &= \frac{c}{2}\int_{\mathbf{R}^N} x_k\frac{\partial\eta_n}{\partial x_1}(x)\Big[(1-\chi)(-i\frac{\partial\psi}{\partial x_k}, \psi) + |\psi|^2\frac{\partial(\chi\theta)}{\partial x_k} - |\psi|^2\theta\frac{\partial\chi}{\partial x_k}\Big]\,dx\\ &\quad -\frac{c}{2}\int_{\mathbf{R}^N} \eta_n(x)(-i\frac{\partial\psi}{\partial x_1}, \psi)\,dx\\ &\quad -\frac{c}{2}\int_{\mathbf{R}^N} x_k\frac{\partial\eta_n}{\partial x_k}(x)\Big[(1-\chi)(-i\frac{\partial\psi}{\partial x_1}, \psi) + |\psi|^2\frac{\partial(\chi\theta)}{\partial x_1} - |\psi|^2\theta\frac{\partial\chi}{\partial x_1}\Big]\,dx\end{aligned}$$

$$= \frac{c}{2}\int_{\mathbf{R}^N} x_k|\psi|^2\Big(\frac{\partial\eta_n}{\partial x_1}\frac{\partial(\chi\theta)}{\partial x_k} - \frac{\partial\eta_n}{\partial x_k}\frac{\partial(\chi\theta)}{\partial x_1}\Big) - \eta_n(x)\Big(-i\frac{\partial\psi}{\partial x_1}, \psi\Big)\,dx \qquad \text{if } n > 3R_*$$

because $\text{supp}(1-\chi) \subset \overline{B}(0, 3R_*)$ and $\text{supp}\nabla\eta_n \subset \overline{B}(0, 2n)\backslash B(0, n)$, consequently $(1-\chi)\frac{\partial\eta_n}{\partial x_j} = 0$ and $\frac{\partial\chi}{\partial x_\ell}\frac{\partial\eta_n}{\partial x_j} = 0$ on $\mathbf{R}^N$ for $n > 3R_*$.

It is obvious that

$$\begin{aligned}(4.9)\qquad &\int_{\mathbf{R}^N} x_k\Big(\frac{\partial\eta_n}{\partial x_1}\frac{\partial(\chi\theta)}{\partial x_k} - \frac{\partial\eta_n}{\partial x_k}\frac{\partial(\chi\theta)}{\partial x_1}\Big)\,dx\\ &= \int_{\mathbf{R}^N} x_k\Big[\frac{\partial}{\partial x_1}\Big(\eta_n\frac{\partial(\chi\theta)}{\partial x_k}\Big) - \frac{\partial}{\partial x_k}\Big(\eta_n\frac{\partial(\chi\theta)}{\partial x_1}\Big)\Big]\,dx = \int_{\mathbf{R}^N} \eta_n\frac{\partial(\chi\theta)}{\partial x_1}\,dx.\end{aligned}$$

Since $|\psi|^2 - r_0^2$ and $\nabla(\chi\theta)$ belong to $L^2(\mathbf{R}^N)$, using the Dominated Convergence Theorem we obtain

$$\begin{aligned}(4.10)\qquad &\Big|\int_{\mathbf{R}^N} x_k(|\psi|^2 - r_0^2)\Big(\frac{\partial\eta_n}{\partial x_1}\frac{\partial(\chi\theta)}{\partial x_k} - \frac{\partial\eta_n}{\partial x_k}\frac{\partial(\chi\theta)}{\partial x_1}\Big)\,dx\Big|\\ &\le 2||\nabla\eta||_{L^\infty(\mathbf{R}^N)}\int_{B(0,2n)\backslash B(0,n)} |\,|\psi|^2 - r_0^2|\,\Big(\Big|\frac{\partial(\chi\theta)}{\partial x_1}\Big| + \Big|\frac{\partial(\chi\theta)}{\partial x_k}\Big|\Big)\,dx \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned}$$

Recall that $\psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - r_0^2\frac{\partial(\chi\theta)}{\partial x_1} \in L^1(\mathbf{R}^N)$ by Theorem 3.1 i) and by dominated convergence we get

$$\begin{aligned}(4.11)\qquad &\int_{\mathbf{R}^N} \eta_n\Big[\Big(-i\frac{\partial\psi}{\partial x_1}, \psi\Big) - r_0^2\frac{\partial(\chi\theta)}{\partial x_1}\Big]\,dx = \int_{\mathbf{R}^N} \eta_n\Big[\psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - r_0^2\frac{\partial(\chi\theta)}{\partial x_1}\Big]\,dx\\ &\longrightarrow \int_{\mathbf{R}^N} \psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - r_0^2\frac{\partial(\chi\theta)}{\partial x_1}\,dx \qquad \text{as } n \longrightarrow \infty.\end{aligned}$$

Combining (4.8)-(4.11) we find

$$(4.12)\qquad \lim_{n\to\infty}\int_{\mathbf{R}^N} (-ic\frac{\partial\psi}{\partial x_1}, x_k\eta_n(x)\frac{\partial\psi}{\partial x_k})\,dx = -\frac{c}{2}\int_{\mathbf{R}^N} \psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - r_0^2\frac{\partial(\chi\theta)}{\partial x_1}\,dx.$$

Taking the scalar product of (4.2) by $\eta_n(x)x_k \frac{\partial \psi}{\partial x_k}$, integrating over $\mathbf{R}^N$ and using (4.5), (4.6) and (4.12) we obtain (4.4). $\square$

Theorem 4.2 *Assume that $N \geq 2$,* **(A1), (A2)** *hold and let ψ be a finite-energy travelling-wave for (3.1) such that $G(|\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. Suppose that*

- *either $c^2 > v_s^2$, where $v_s = r_0\sqrt{-2G'(r_0^2)}$ is the sound velocity at infinity,*
- *or $N = 2$ and $c^2 = v_s^2$,*
- *or $N \geq 3$ and $c^2 = v_s^2$ and $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$.*

Moreover, assume that G satisfies

- **(A3)** *there exists $\alpha \in [-1 + \frac{N-3}{N-1}(1 - \frac{v_s^2}{c^2}), \frac{v_s^2}{c^2}]$ such that*

$$sG(s) + \frac{v_s^2}{2}(s - r_0^2) + (1 - \alpha - \frac{v_s^2}{c^2})V(s) \leq 0 \qquad \text{for any } s \geq 0.$$

Then ψ is constant.

Proof. It follows from Propositions 2.2 and 2.5 that ψ is smooth and Proposition 4.1 implies that ψ satisfies (4.3) and (4.4). Summing up the identities (4.4) for $k = 2, \ldots, N$ we get

$$\begin{aligned} &\int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_1}\right|^2 + \frac{N-3}{N-1}\sum_{k=2}^N \left|\frac{\partial \psi}{\partial x_k}\right|^2 dx + \int_{\mathbf{R}^N} V(|\psi|^2)\,dx \\ &-c\int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - r_0^2 \frac{\partial(\chi\theta)}{\partial x_1}\,dx = 0. \end{aligned} \tag{4.13}$$

On the other hand, from Theorem 3.1 we have

$$\begin{aligned} &\int_{\mathbf{R}^N} |\nabla \psi|^2 - G(|\psi|^2)|\psi|^2 - \frac{v_s^2}{2}(|\psi|^2 - r_0^2)\,dx \\ &-c(1 - \frac{v_s^2}{c^2})\int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - r_0^2 \frac{\partial}{\partial x_1}(\chi\theta)\,dx = 0. \end{aligned} \tag{4.14}$$

We multiply (4.13) by $-1 + \frac{v_s^2}{c^2}$ and we add the resulting equality to (4.14) to get

$$\begin{aligned} &\int_{\mathbf{R}^N} \frac{v_s^2}{c^2}\left|\frac{\partial \psi}{\partial x_1}\right|^2 + \Big(1 - (1 - \frac{v_s^2}{c^2})\frac{N-3}{N-1}\Big)\sum_{k=2}^N \left|\frac{\partial \psi}{\partial x_k}\right|^2 dx \\ &-\int_{\mathbf{R}^N} G(|\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) + (1 - \frac{v_s^2}{c^2})V(|\psi|^2)\,dx = 0. \end{aligned} \tag{4.15}$$

Let α satisfy **(A3)**. Multiplying (4.3) by α and adding it to (4.15) we obtain

$$\begin{aligned} &\int_{\mathbf{R}^N} \Big(\frac{v_s^2}{c^2} - \alpha\Big)\left|\frac{\partial \psi}{\partial x_1}\right|^2 + \Big(\alpha + 1 - (1 - \frac{v_s^2}{c^2})\frac{N-3}{N-1}\Big)\sum_{k=2}^N \left|\frac{\partial \psi}{\partial x_k}\right|^2 dx \\ &= \int_{\mathbf{R}^N} G(|\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) + (1 - \alpha - \frac{v_s^2}{c^2})V(|\psi|^2)\,dx. \end{aligned} \tag{4.16}$$

By **(A3)**, the right-hand side of (4.16) is less than or equal to zero. If $\alpha \in (-1+(1-\frac{v_s^2}{c^2})\frac{N-3}{N-1}, \frac{v_s^2}{c^2})$, it follows from (4.16) that $\int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_k}\right|^2 dx = 0$ for $k = 1, \dots, N$, which implies $\nabla \psi = 0$ on $\mathbf{R}^N$, i.e. ψ is constant. If $\alpha = -1 + (1 - \frac{v_s^2}{c^2})\frac{N-3}{N-1}$, we infer from (4.16) that $\int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_1}\right|^2 dx = 0$, consequently $\frac{\partial \psi}{\partial x_1} = 0$ on $\mathbf{R}^N$ which implies that ψ does not depend on x_1. Since $\int_{\mathbf{R}^N} |\nabla \psi|^2 \, dx$ is finite, we have necessarily $\nabla \psi = 0$ on $\mathbf{R}^N$, which means that is ψ is constant. A similar argument shows that ψ is constant in the case $\alpha = \frac{v_s^2}{c^2}$. $\square$

Remark. Let α, C_1 and $\tilde{r}$ be positive constants satisfying $G(s^2) + \frac{c^2}{4} \leq -C_1(s - \tilde{r})^{2\alpha}$ for any $s \geq \tilde{r}$ (such constants exist by assumption **(A2)**). Let ψ be as in Theorem 4.2. It follows from the proof of Proposition 2.2 i) that $|\psi(x)| \leq \tilde{r}\sqrt{2}$ for any x. Therefore the proof of Theorem 4.2 is still valid if the inequality in **(A3)** only holds for all $s \in [0, 2\tilde{r}^2]$.

If $c^2 = v_s^2$, $N \geq 3$ and ψ is as above, we already know from Proposition 3.3 iii) that $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^p(\mathbf{R}^N)$ for any $p \in (\frac{2N-1}{2N-3}, \infty)$. Therefore we have:

Corollary 4.3 *Assume that* **(A1), (A2), (A3)** *hold, $N \geq 3$ and $c^2 = v_s^2$. Let ψ be a travelling-wave for (4.1) having finite energy, finite momentum with respect to the x_1-direction (i.e. $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^1(\mathbf{R}^N)$) and such that $G(|\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. Then ψ is constant.*

Example 4.4 The Gross-Pitaevskii equation is of type (4.1) with $G(s) = 1 - s$. In this case we have $r_0 = 1$, $V(s) = \frac{1}{2}(s-1)^2$ and $v_s = \sqrt{2}$. For any finite-energy function ψ we have $\int_{\mathbf{R}^N} (|\psi|^2 - 1)^2 \, dx < \infty$, hence $\psi \in L^4_{loc}(\mathbf{R}^N)$ and consequently $G(|\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. Assumptions **(A1)** and **(A2)** are clearly satisfied. We find $sG(s) + \frac{v_s^2}{2}(s - r_0^2) + (1 - \alpha - \frac{v_s^2}{c^2})V(s) = -(\frac{1}{2} + \alpha + \frac{v_s^2}{c^2})(1-s)^2$. The last expression is nonpositive for any s if $\alpha \geq -\frac{1}{2} - \frac{v_s^2}{c^2}$, thus assumption **(A3)** is also satisfied. Hence the conclusion of Theorem 4.2 holds for the Gross-Pitaevskii equation. In particular, we recover the non-existence results in [17], [18].

Example 4.5 The cubic-quintic Schrödinger equation is of the form (4.1) with $G(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, where α_1, α_3, α_5 are positive and $\frac{3}{16} < \frac{\alpha_1\alpha_5}{\alpha_3^2} < \frac{1}{4}$. The nonlinearity G can be written as $G(s) = -\alpha_5(s - r_1^2)(s - r_0^2)$, where $0 < r_1 < r_0$. In this case we have $v_s^2 = -2r_0^2 G'(r_0) = 2\alpha_5 r_0^2(r_0^2 - r_1^2)$ and $V(s) = \frac{\alpha_5}{3}(s - r_0^2)^2(s + \frac{1}{2}r_0^2 - \frac{3}{2}r_1^2)$. For any function ψ with finite energy we have $V(|\psi|^2) \in L^1(\mathbf{R}^N)$, which implies $\psi \in L^6_{loc}(\mathbf{R}^N)$ and consequently $G(|\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. It is obvious that G satisfies **(A1)** and **(A2)**. If $c^2 \geq v_s^2$ we have $-\frac{v_s^2}{c^2} \in [-1 + \frac{N-3}{N-1}(1 - \frac{v_s^2}{c^2}), \frac{v_s^2}{c^2}]$ and an easy computation shows that $sG(s) + \frac{v_s^2}{2}(s - r_0^2) + V(s) = -\frac{\alpha_5}{6}(4s + 5r_0^2 - 3r_1^2) \leq 0$ for any $s \geq 0$. Hence assumption **(A3)** holds for $\alpha = -\frac{v_s^2}{c^2}$, therefore the conclusion of Theorem 4.2 is valid for the cubic-quintic Schrödinger equation.

Remark. The proof of nonexistence of supersonic and sonic travelling-waves for equations of type (1.1) relies on identity (3.1), combined with Pohozaev identities. We have proved (3.1) in an "indirect" way, starting from (3.11), using the Fourier transform and analyzing the behavior near the origin of the symbols of the differential operators involved. A natural question is whether (3.1) could be proved "directly", by multiplying the equations by appropriate functions and integrating by parts (and it is very tempting to try to do so because of the form of equations (3.7) and (3.8)!). We suspect that it is not possible to find such a proof, a heuristical reason being the following: if a "direct" proof of (3.1) could be found, it should be valid for any value of c. Since Pohozaev identities are also valid for any c, one could infer that, for any c, equation (4.1) and the system (4.17)-(4.18) below do not admit nontrivial finite-energy travelling-waves. However, in the case of the Gross-Pitaevskii equation the existence of non-trivial, finite-energy travelling-waves moving with sufficiently small speed c has been proved in [7] in dimension $N = 2$, respectively in [6] and [12] in dimension $N = 3$. In a recent work [5], existence of travelling-waves has been proved in space dimensions $N = 2$ and $N = 3$ for a wider range of speeds, including speeds c close to (and less than) v_s if $N = 2$. For Schrödinger equations of cubic-quintic type, the existence of small velocity travelling-waves has been proved in [25] in any space dimension $N \geq 4$. Even for these particular cases, the question whether such solutions exist for any speed $c \in (-v_s, v_s)$ is, to our knowledge, still open.

2. Our second application concerns the system

$$i\frac{\partial \Psi}{\partial t} + \Delta\Psi - \frac{1}{\varepsilon^2}(|\Psi|^2 + \frac{1}{\varepsilon^2}|\Phi|^2 - 1)\Psi = 0 \qquad \text{in } \mathbf{R}^N, \tag{4.17}$$

$$i\delta\frac{\partial \Phi}{\partial t} + \Delta\Phi - \frac{1}{\varepsilon^2}(q^2|\Psi|^2 - \varepsilon^2 k^2)\Phi = 0 \qquad \text{in } \mathbf{R}^N, \tag{4.18}$$

which describes the motion of an uncharged impurity in a Bose condensate (see [16]). Here Ψ and Φ are the wavefunctions for bosons, respectively for the impurity, and ε, δ, q, k are dimensionless physical constants. Assuming that the condensate is at rest at infinity, the functions Ψ and Φ must satisfy the "boundary conditions" $|\Psi| \longrightarrow 1$ and $|\Phi| \longrightarrow 0$ as $|x| \longrightarrow \infty$.

The system (4.17)-(4.18) has a Hamiltonian structure, the associated energy is

$$E(\Psi, \Phi) = \int_{\mathbf{R}^N} |\nabla\Psi|^2 + \frac{1}{\varepsilon^2 q^2}|\nabla\Phi|^2 + \frac{1}{2\varepsilon^2}(|\Psi|^2 - 1)^2 + \frac{1}{\varepsilon^4}|\Psi|^2|\Phi|^2 - \frac{k^2}{\varepsilon^2 q^2}|\Phi|^2\, dx. \tag{4.19}$$

We are interested in travelling-wave solutions for (4.17)-(4.18), i.e. solutions of the form $\Psi(x,t) = \psi(x_1 - ct, x_2, \dots, x_N)$, $\Phi(x,t) = \varphi(x_1 - ct, x_2, \dots, x_N)$. Such solutions must satisfy the equations

$$-ic\frac{\partial \psi}{\partial x_1} + \Delta\psi - \frac{1}{\varepsilon^2}(|\psi|^2 + \frac{1}{\varepsilon^2}|\varphi|^2 - 1)\psi = 0, \tag{4.20}$$

$$-ic\delta\frac{\partial \varphi}{\partial x_1} + \Delta\varphi - \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k^2)\varphi = 0, \tag{4.21}$$

together with the boundary conditions $|\psi| \longrightarrow 1$ and $|\varphi| \longrightarrow 0$ as $|x| \longrightarrow \infty$.

Equation (4.17) is of type (1.1). In view of the analysis in the Introduction, the associated sound velocity at infinity is $\frac{\sqrt{2}}{\varepsilon}$.

In space dimension one, the system (4.20)-(4.21) with the considered boundary conditions has been studied in [26]. It was proved that it admits nontrivial solutions if c is less than the sound velocity at infinity; in this case the structure of the set of travelling-waves has been investigated and it was proved that it contains global subcontinua in appropriate (weighted) Sobolev spaces.

Here we study the finite energy travelling-waves for (4.17)-(4.18) in dimension $N \geq 2$. In view of (4.19), by *finite energy travelling-wave* we mean a couple of functions $(\psi, \varphi) \in L^1_{loc}(\mathbf{R}^N) \times L^1_{loc}(\mathbf{R}^N)$ which satisfy (4.20)-(4.21) in $\mathcal{D}'(\mathbf{R}^N)$, the boundary conditions $|\psi| \longrightarrow 1$, $\varphi \longrightarrow 0$ as $|x| \longrightarrow \infty$ and such that $\nabla\psi$, $\nabla\varphi$, $\varphi \in L^2(\mathbf{R}^N)$, $(|\psi|^2 - 1)^2 + \frac{2}{\varepsilon^2}|\psi|^2|\varphi|^2 \in L^1(\mathbf{R}^N)$. As before, we denote $\psi_1 = \mathrm{Re}(\psi)$, $\psi_2 = \mathrm{Im}(\psi)$, $\varphi_1 = \mathrm{Re}(\varphi)$, $\varphi_2 = \mathrm{Im}(\varphi)$. We have:

Proposition 4.6 *Let $c \in \mathbf{R}$ and let (ψ, φ) be a finite energy travelling wave for (4.17)-(4.18). Then:*

i) The function ψ is bounded and C^∞ and $\varphi, \nabla\psi \in W^{k,p}(\mathbf{R}^N)$ for any $k \in \mathbf{N}$ and $p \geq 2$.

ii) There exist $R_* \geq 0$ and a real-valued function θ such that $\psi = |\psi|e^{i\theta}$ on $\mathbf{R}^N \setminus B(0, R_*)$ and $\nabla\theta \in W^{k,p}(\mathbf{R}^N \setminus B(0, R_*))$ for any $k \in \mathbf{N}$ and $p \geq 2$.

iii) Let $\chi \in C^\infty(\mathbf{R}^N)$ be a cut-off function such that $\chi = 0$ on $B(0, 2R_*)$ and $\chi = 1$ on $\mathbf{R}^N \setminus B(0, 3R_*)$. We have $\psi_1 \frac{\partial\psi_2}{\partial x_1} - \psi_2 \frac{\partial\psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\theta) \in L^1(\mathbf{R}^N)$ and the following Pohozaev-type identities hold:

$$
\begin{aligned}
&\int_{\mathbf{R}^N} -\Big|\frac{\partial\psi}{\partial x_1}\Big|^2 - \frac{1}{\varepsilon^2 q^2}\Big|\frac{\partial\varphi}{\partial x_1}\Big|^2 + \sum_{j=2}^N \Big(\Big|\frac{\partial\psi}{\partial x_j}\Big|^2 + \frac{1}{\varepsilon^2 q^2}\Big|\frac{\partial\varphi}{\partial x_j}\Big|^2\Big)\,dx \\
&+ \int_{\mathbf{R}^N} \frac{1}{2\varepsilon^2}(|\psi|^2 - 1)^2 + \frac{1}{\varepsilon^4}|\psi|^2|\varphi|^2 - \frac{k^2}{\varepsilon^2 q^2}|\varphi|^2\,dx = 0,
\end{aligned}
\tag{4.22}
$$

and for any $k \in \{2, \dots, N\}$,

$$
\begin{aligned}
&\int_{\mathbf{R}^N} -\Big|\frac{\partial\psi}{\partial x_k}\Big|^2 - \frac{1}{\varepsilon^2 q^2}\Big|\frac{\partial\varphi}{\partial x_k}\Big|^2 + \sum_{j=1,\, j\neq k}^N \Big(\Big|\frac{\partial\psi}{\partial x_j}\Big|^2 + \frac{1}{\varepsilon^2 q^2}\Big|\frac{\partial\varphi}{\partial x_j}\Big|^2\Big)\,dx \\
&+ \int_{\mathbf{R}^N} \frac{1}{2\varepsilon^2}(|\psi|^2 - 1)^2 + \frac{1}{\varepsilon^4}|\psi|^2|\varphi|^2 - \frac{k^2}{\varepsilon^2 q^2}|\varphi|^2\,dx \\
&-c\int_{\mathbf{R}^N} \psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\theta)\,dx - \frac{2c\delta}{\varepsilon^2 q^2}\int_{\mathbf{R}^N} \varphi_1\frac{\partial\varphi_2}{\partial x_1}\,dx = 0.
\end{aligned}
\tag{4.23}
$$

Proof. Putting $F(x,s) = -\frac{1}{\varepsilon^2}(s + \frac{1}{\varepsilon^2}|\varphi(x)|^2 - 1)$, equation (4.20) is a particular case of (1.7). Clearly, in this case we have $r_0 = 1$.

It is obvious that F satisfies the assumptions **(H1a)** and **(H1b)** in Section 2. Clearly, $F(x,s) \leq -\frac{1}{\varepsilon^2}(s-1) \leq -\frac{1}{2\varepsilon^2}s$ for any $s \geq 2$ and $x \in \mathbf{R}^N$, hence F satisfies **(H2)** for $r_* = 2$. Moreover, $\displaystyle\int_{r_0^2}^{r_*} F(x,\tau)\,d\tau = -\frac{1}{\varepsilon^2}(\frac{1}{2} + \frac{1}{\varepsilon^2}|\varphi(x)|^2)$ is a locally integrable function of x. We have $|\psi|^4 \leq 2(|\psi|^2 - 1)^2 + 2$ and $(|\psi|^2 - 1)^2 \in L^1(\mathbf{R})$ because (ψ, φ) has finite energy, hence $\psi \in L^4_{loc}(\mathbf{R}^N)$. We also have $|\,|\varphi|^2\psi| \leq \frac{1}{2}(|\varphi|^2 + |\varphi|^2|\psi|^2)$ and $|\varphi|^2,\ |\varphi|^2|\psi|^2 \in L^1(\mathbf{R})$. It is then clear that $F(\cdot, |\psi|^2)\psi = -\frac{1}{\varepsilon^2}|\psi|^2\psi - \frac{1}{\varepsilon^4}|\varphi|^2\psi + \frac{1}{\varepsilon^2}\psi$ belongs to $L^1_{loc}(\mathbf{R}^N)$. Hence we may use Proposition 2.2 i) and we infer that $\psi \in L^\infty(\mathbf{R}^N)$.

By hypothesis we have $\varphi \in L^2(\mathbf{R}^N)$ and $\nabla\varphi \in L^2(\mathbf{R}^N)$, that is $\varphi \in W^{1,2}(\mathbf{R}^N)$. Assume that $\varphi \in W^{1,p}(\mathbf{R}^N)$ for some $p \in (1,\infty)$. Since ψ is bounded, by (4.21) we find $\Delta\varphi \in L^p(\mathbf{R}^N)$, and we infer that $\varphi \in W^{2,p}(\mathbf{R}^N)$. If $p < N$, by the Sobolev embedding we have $\varphi \in L^{p^*}(\mathbf{R}^N)$ and $\nabla\varphi \in L^{p^*}(\mathbf{R}^N)$ (where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$), hence $\varphi \in W^{1,p^*}(\mathbf{R}^N)$. Repeating the above argument if necessary, after a finite number of steps we find $\varphi \in W^{2,q}(\mathbf{R}^N)$ for some $q \geq N$ and the Sobolev embedding implies $\varphi \in L^r(\mathbf{R}^N)$ and $\nabla\varphi \in L^r(\mathbf{R}^N)$ for any $r \in [q,\infty)$. Using (4.21) again, we conclude that $\Delta\varphi \in L^r(\mathbf{R}^N)$, hence $\varphi \in W^{2,r}(\mathbf{R}^N)$ for any $r \in [2,\infty)$.

It follows that $\varphi \in C^1(\mathbf{R}^N)$, which implies $F \in C^1(\mathbf{R}^N)$ (and consequently F satisfies **(H1c)**). By Proposition 2.2 ii) we get $\psi \in W^{3,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1,\infty)$. In particular, $\psi \in C^2(\mathbf{R}^N)$.

We have $F(x,1) = -\frac{1}{\varepsilon^4}|\varphi(x)|^2$ and F clearly satisfies assumption **(H3)**. It is obvious that $\partial_{N+1}F(x,s) = -\frac{1}{\varepsilon^2}$ and $\partial^2_{N+1}F(x,s) = 0$ on $\mathbf{R}^N \times \mathbf{R}_+$, therefore F satisfies **(H4)** and **(H5)**. Thus we may use Proposition 2.5 i) and we infer that $\nabla\psi \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2,\infty)$.

The rest of the proof is a very easy induction. For $k \in \mathbf{N}^*$, assume that $\nabla\psi \in W^{k,p}(\mathbf{R}^N)$ and $\varphi \in W^{k+1,p}(\mathbf{R}^N)$ for any $p \in [2,\infty)$. Consider $\alpha \in \mathbf{N}^N$ such that $|\alpha| = k$. Differentiating

(4.20) and (4.21) we obtain

$$\Delta(\partial^\alpha \psi) = ic\partial^\alpha \frac{\partial \psi}{\partial x_1} + \frac{1}{\varepsilon^2}\partial^\alpha \Big((|\psi|^2 + \frac{1}{\varepsilon^2}|\varphi|^2 - 1)\psi \Big), \qquad \text{respectively}$$

$$\Delta(\partial^\alpha \psi) = ic\delta\partial^\alpha \frac{\partial \psi}{\partial x_1} + \frac{1}{\varepsilon^2}\partial^\alpha \Big(q^2|\psi|^2 - \varepsilon^2 k^2)\varphi \Big).$$

We infer that $\Delta(\partial^\alpha \psi)$, $\Delta(\partial^\alpha \varphi) \in L^p(\mathbf{R}^N)$ for any $p \in [2,\infty)$. By hypothesis we have $\partial^\alpha \psi$, $\partial^\alpha \varphi \in L^p(\mathbf{R}^N)$, therefore $\partial^\alpha \psi$, $\partial^\alpha \varphi \in W^{2,p}(\mathbf{R}^N)$ for any $p \in [2,\infty)$. Since this is true for any α with $|\alpha| = k$, we have $\nabla\psi \in W^{k+1,p}(\mathbf{R}^N)$ and $\varphi \in W^{k+2,p}(\mathbf{R}^N)$. We conclude that $\nabla\psi$ and φ belong to $W^{k,p}(\mathbf{R}^N)$ for any $k \in \mathbf{N}$ and $p \in [2,\infty)$.

ii) is an immediate corollary of Proposition 2.5 ii).

iii) It follows directly from Theorem 3.1 i) that $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\theta) \in L^1(\mathbf{R}^N)$. The proof of (4.22) and (4.23) is similar to that of (4.3) and (4.4) (multiply (4.20) by $x_j \eta_n \frac{\partial \psi}{\partial x_j}$ and (4.21) by $\frac{1}{\varepsilon^2 q^2} x_j \eta_n \frac{\partial \varphi}{\partial x_j}$, where $\eta_n(x) = \eta(\frac{x}{n})$ is a cut-off function, add the resulting equalities, integrate by parts and pass to the limit as $n \longrightarrow \infty$). We omit the details. $\square$

We have the following result concerning the non-existence of supersonic travelling-waves for (4.17)-(4.18):

Theorem 4.7 *Let $N \geq 2$ and let (ψ, φ) be a finite energy travelling-wave for the system (4.17)-(4.18), moving with velocity c. Assume that:*

- *either $c^2 > \frac{2}{\varepsilon^2}$,*
- *or $N = 2$ and $c^2 = \frac{2}{\varepsilon^2}$,*
- *or $N \geq 3$ and $c^2 = \frac{2}{\varepsilon^2}$ and $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$.*

Then $\varphi = 0$ and ψ is constant on $\mathbf{R}^N$.

Proof. Let θ, χ be as in Proposition 4.6 and let $F(x,s) = -\frac{1}{\varepsilon^2}(s + \frac{1}{\varepsilon^2}|\varphi(x)|^2 - 1)$. We have already seen that F satisfies assumptions **(H1)-(H5)** and it follows that identity (3.1) holds. Taking into account the particular form of F, this identity can be written as

$$\tag{4.24} \begin{aligned} &\int_{\mathbf{R}^N} |\nabla\psi|^2 + \frac{1}{\varepsilon^2}(|\psi|^2 - 1)^2 + \frac{1}{\varepsilon^4}|\varphi|^2|\psi|^2 \, dx \\ &= c(1 - \frac{2}{\varepsilon^2 c^2}) \int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\theta) \, dx. \end{aligned}$$

We take the scalar product of (4.21) by φ, then we integrate the resulting equality to get

$$\tag{4.25} \int_{\mathbf{R}^N} |\nabla\varphi|^2 \, dx + \frac{q^2}{\varepsilon^2} \int_{\mathbf{R}^N} |\varphi|^2|\psi|^2 \, dx - k^2 \int_{\mathbf{R}^N} |\varphi|^2 \, dx - 2c\delta \int_{\mathbf{R}^N} \varphi_1 \frac{\partial \varphi_2}{\partial x_1} \, dx = 0.$$

Summing up the identities (4.23) for $k = 2, 3, \dots, N$, we find

$$\tag{4.26} \begin{aligned} &\int_{\mathbf{R}^N} \Big|\frac{\partial \psi}{\partial x_1}\Big|^2 + \frac{1}{\varepsilon^2 q^2}\Big|\frac{\partial \varphi}{\partial x_1}\Big|^2 + \frac{N-3}{N-1}\sum_{j=2}^N \Big(\Big|\frac{\partial \psi}{\partial x_j}\Big|^2 + \frac{1}{\varepsilon^2 q^2}\Big|\frac{\partial \varphi}{\partial x_j}\Big|^2 \Big) \, dx \\ &+ \int_{\mathbf{R}^N} \frac{1}{2\varepsilon^2}(|\psi|^2 - 1)^2 + \frac{1}{\varepsilon^4}|\psi|^2|\varphi|^2 - \frac{k^2}{\varepsilon^2 q^2}|\varphi|^2 \, dx \\ &-c\int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\theta) \, dx - \frac{2c\delta}{\varepsilon^2 q^2} \int_{\mathbf{R}^N} \varphi_1 \frac{\partial \varphi_2}{\partial x_1} \, dx = 0. \end{aligned}$$

Next we combine the equalities (4.24)-(4.26) in order to eliminate the terms $\int_{\mathbf{R}^N} \varphi_1 \frac{\partial \varphi_2}{\partial x_1}\, dx$ and $\int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\theta)\, dx$. We find:

$$\begin{aligned}
&\frac{2}{\varepsilon^2 c^2} \int_{\mathbf{R}^N} \left| \frac{\partial \psi}{\partial x_1} \right|^2 dx + \Big(1 - \Big(1 - \frac{2}{\varepsilon^2 c^2}\Big)\frac{N-3}{N-1}\Big) \int_{\mathbf{R}^N} \sum_{j=2}^{N} \left| \frac{\partial \psi}{\partial x_j} \right|^2 dx \\
&+ \frac{2}{(N-1)\varepsilon^2 q^2}\Big(1 - \frac{2}{\varepsilon^2 c^2}\Big) \int_{\mathbf{R}^N} \sum_{j=2}^{N} \left| \frac{\partial \varphi}{\partial x_j} \right|^2 dx \\
&+ \frac{1}{2\varepsilon^2}\Big(1 + \frac{2}{\varepsilon^2 c^2}\Big) \int_{\mathbf{R}^N} (|\psi|^2 - 1)^2\, dx + \frac{1}{\varepsilon^4} \int_{\mathbf{R}^N} |\varphi|^2 |\psi|^2\, dx = 0.
\end{aligned} \tag{4.27}$$

Obviously, all integrals in (4.27) are nonnegative. If $c^2 \geq \frac{2}{\varepsilon^2}$, all coefficients are also nonnegative, therefore each term in (4.27) must be zero. In particular, $\int_{\mathbf{R}^N} \left| \frac{\partial \psi}{\partial x_k} \right|^2 dx = 0$ for any $k \in \{1, \ldots, N\}$, which implies $\nabla \psi = 0$ on $\mathbf{R}^N$, i.e. ψ is constant. Since $\int_{\mathbf{R}^N} (|\psi|^2 - 1)^2\, dx = 0$, necessarily $|\psi| = 1$. We have also $0 = \int_{\mathbf{R}^N} |\varphi|^2 |\psi|^2\, dx = \int_{\mathbf{R}^N} |\varphi|^2\, dx$, hence $\varphi = 0$ on $\mathbf{R}^N$. $\square$

5 The one-dimensional case

Since most of the proofs in the preceding section are not valid in space dimension $N = 1$ (in particular, we do not have identities analogous to (4.4) and (4.23)), we treat separately the one-dimensional case. It turns out that some integrations can be performed explicitly and some of the results are stronger than in higher dimensions.

Let $G : [0, \infty) \longrightarrow \mathbf{R}$ be a function satisfying the following assumption:

- **(A)** $G \in C([0, \infty))$ and there exists $r_0 > 0$ such that $G(r_0^2) = 0$.
  Moreover, $G \in C^1([r_0^2 - \eta, r_0^2 + \eta])$ for some $\eta > 0$ and $G'(r_0^2) = -L < 0$.

We consider the Schrödinger equation

$$i \frac{\partial \Psi}{\partial t} + \Psi_{xx} + G(|\Psi|^2)\Psi = 0 \qquad \text{in } \mathbf{R}, \tag{5.1}$$

together with the "boundary condition" $|\Psi| \longrightarrow r_0$ as $x \longrightarrow \pm\infty$. We have seen in the Introduction that the sound velocity at infinity associated to (5.1) and to the considered boundary condition is $v_s = r_0\sqrt{2L}$. As usually, a travelling-wave moving with velocity c is a solution of the form $\Psi(x, t) = \psi(x - ct)$. It must satisfy

$$-ic\psi' + \psi'' + G(|\psi|^2)\psi = 0 \qquad \text{in } \mathbf{R}, \qquad |\psi(x)| \longrightarrow r_0 \qquad \text{as } x \longrightarrow \pm\infty. \tag{5.2}$$

We have the following result concerning supersonic and sonic travelling-waves:

Theorem 5.1 *Let $\psi \in L^1_{loc}(\mathbf{R})$ be a solution of (5.2) in $\mathcal{D}'(\mathbf{R})$ such that $G(|\psi|^2)\psi \in L^1_{loc}(\mathbf{R})$. Assume that G satisfies* **(A)** *and*

i) either $c^2 > v_s^2$, or

ii) $c^2 = v_s^2$ and, denoting $V(s) = \int_s^{r_0^2} G(\tau)\, d\tau$ and $W(s) = v_s^2 s^2 - 4(s + r_0^2)V(s + r_0^2)$, there exists $\varepsilon > 0$ such that one of he following conditions is verified:

a) $W(s) > 0$ on $(-\varepsilon, 0) \cup (0, \varepsilon)$;
b) $W(s) > 0$ on $(-\varepsilon, 0)$ and $W(s) < 0$ on $(0, \infty)$;
c) $W(s) > 0$ on $(0, \varepsilon)$ and $W(s) < 0$ on $[-r_0^2, 0)$.

Then either ψ is constant, or $\psi(x) = r_0 e^{i(cx+\theta_0)}$, where θ_0 is a real constant.

Remark. Theorem 5.1 gives *all* supersonic and sonic travelling-waves for equation (5.1), no matter whether their energy is finite or not (and we see that finite energy travelling-waves must be constant).

It is easy to see that W is C^2 near 0 and $W(0) = W'(0) = W''(0) = 0$. Condition ii) a) is satisfied, for instance, if G is C^3 near r_0^2 (this clearly implies that W is C^4 near 0) and $W'''(0) = 0$, $W^{(iv)}(0) > 0$, or equivalently $r_0^2 G''(r_0^2) = 3L$ and $4G''(r_0^2) + r_0^2 G'''(r_0^2) > 0$. The condition $W(s) > 0$ on $(-\varepsilon, 0)$ in ii) b), respectively $W(s) > 0$ on $(0, \varepsilon)$ in ii) c), is satisfied if G is C^3 near r_0^2 and $W'''(0) < 0$ (respectively $W'''(0) > 0$); however, in these cases only an information on the behavior of G in a neighborhood of r_0^2 is not sufficient to get the conclusion of Theorem 5.1.

Proof of Theorem 5.1. Let $\varphi(x) = e^{-\frac{icx}{2}}\psi(x)$. Then $\varphi \in L^1_{loc}(\mathbf{R})$ and it is easy to see that

$$\varphi'' + \Big(G(|\varphi|^2) + \frac{c^2}{4}\Big)\varphi = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}). \tag{5.3}$$

From (5.3) we get $\varphi'' \in L^1_{loc}(\mathbf{R})$. This implies that φ' is a continuous function on $\mathbf{R}$ (see, e.g., Lemma VIII.2 p. 123 in [8]). Thus $\varphi \in C^1(\mathbf{R})$. Since $|\varphi| \longrightarrow r_0$ as $x \longrightarrow \pm\infty$, we infer that φ is bounded on $\mathbf{R}$. Coming back to (5.3) we see that φ'' is continuous and bounded on $\mathbf{R}$. In particular $\varphi \in C^2(\mathbf{R})$ and this implies $\psi \in C^2(\mathbf{R})$.

Denoting $\psi_1 = \mathrm{Re}(\psi)$, $\psi_2 = \mathrm{Im}(\psi)$, equation (5.2) is equivalent to the system

$$c\psi_2' + \psi_1'' + G(|\psi|^2)\psi_1 = 0, \tag{5.4}$$

$$-c\psi_1' + \psi_2'' + G(|\psi|^2)\psi_2 = 0 \qquad \text{in } \mathbf{R}. \tag{5.5}$$

We multiply (5.4) by $2\psi_1'$ and (5.5) by $2\psi_2'$, then we add the resulting equalities to get $[(\psi_1')^2 + (\psi_2')^2]' - (V(|\psi|^2))' = 0$. Hence there exists $k_1 \in \mathbf{R}$ such that

$$|\psi'|^2(x) - V(|\psi|^2)(x) = k_1 \qquad \text{for any } x \in \mathbf{R}. \tag{5.6}$$

Multiplying (5.4) by ψ_2 and (5.5) by $-\psi_1$, then summing up the corresponding equations we obtain $\frac{c}{2}(|\psi|^2 - r_0^2)' - (\psi_1\psi_2' - \psi_2\psi_1')' = 0$. Consequently there is some $k_2 \in \mathbf{R}$ such that

$$\frac{c}{2}(|\psi|^2 - r_0^2) - (\psi_1\psi_2' - \psi_2\psi_1') = k_2 \qquad \text{in } \mathbf{R}. \tag{5.7}$$

Next we multiply (5.4) by $2\psi_1$ and (5.5) by $2\psi_2$, then we add the resulting equalities to find

$$2c(\psi_1\psi_2' - \psi_2\psi_1') + (|\psi|^2 - r_0^2)'' - 2|\psi'|^2 + 2G(|\psi|^2)|\psi|^2 = 0. \tag{5.8}$$

Taking into account (5.6) and (5.7), equation (5.8) can be written as

$$(|\psi|^2 - r_0^2)'' + c^2(|\psi|^2 - r_0^2) - 2V(|\psi|^2) + 2G(|\psi|^2)|\psi|^2 = 2k_1 + 2ck_2. \tag{5.9}$$

Denote $v(x) = |\psi|^2(x) - r_0^2$. Then v is real-valued, C^2 and tends to zero as $x \longrightarrow \pm\infty$, hence there exists a sequence $x_n \longrightarrow \infty$ such that $v''(x_n) \longrightarrow 0$. Writing (5.9) for x_n, then passing to the limit as $n \longrightarrow \infty$ we see that necessariy $k_1 + ck_2 = 0$ and v satisfies the equation

$$v'' + c^2 v - 2V(v + r_0^2) + 2(v + r_0^2)G(v + r_0^2) = 0 \qquad \text{in } \mathbf{R}. \tag{5.10}$$

Next we multiply (5.10) by $2v'$, then we integrate the resulting equation and we obtain $(v')^2 + c^2v^2 - 4(v + r_0^2)V(v + r_0^2) = k_3$ in $\mathbf{R}$, where k_3 is a constant. It is clear that there exists a sequence $y_n \longrightarrow \infty$ such that $v'(y_n) \longrightarrow 0$, consequently $k_3 = 0$ and we have

$$(v')^2(x) + c^2v^2(x) - 4(v + r_0^2)V(v + r_0^2)(x) = 0 \qquad \text{for any } x \in \mathbf{R}. \tag{5.11}$$

Our aim is to prove that, under the assumptions of Theorem 5.1, we have $v = 0$ on $\mathbf{R}$.

Suppose first that $c^2 > v_s^2 = 2Lr_0^2$. Since G satisfies **(A)**, it follows that $V \in C^2([r_0^2 - \eta, r_0^2 + \eta])$ and we have by Taylor's formula

$$V(r_0^2 + s) = V(r_0^2) + sV'(r_0^2) + \frac{1}{2}s^2V''(r_0^2) + s^2h(s) = \frac{1}{2}Ls^2 + s^2h(s) \qquad \text{for } s \in [-\eta, \eta],$$

where $h(s) \longrightarrow 0$ as $s \longrightarrow 0$. Take $\varepsilon_1 \in (0, \eta]$ such that $c^2 - v_s^2 - 2Ls - 4(s + r_0^2)h(s) > 0$ for any $s \in [-\varepsilon_1, \varepsilon_1]$. Suppose that $v(x_0) \in [-\varepsilon_1, 0) \cup (0, \varepsilon_1]$ for some $x_0 \in \mathbf{R}$. By (5.11) we obtain

$$0 = (v')^2(x_0) + v^2(x_0)[c^2 - v_s^2 - 2Lv(x_0) - 4(v(x_0) + r_0^2)h(v(x_0))] > 0,$$

a contradiction. Consequently we cannot have $v(x) \in [-\varepsilon_1, 0) \cup (0, \varepsilon_1]$. Since v is continuous and $v(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$, we infer that necessarily $v(x) = 0$ for any $x \in \mathbf{R}$.

Next assume that $c^2 = v_s^2$. Equation (5.11) can be written as

$$(v')^2(x) + W(v(x)) = 0 \qquad \text{on } \mathbf{R}. \tag{5.12}$$

If assumption ii) a) is verified, we cannot have $v(x) \in (-\varepsilon, 0) \cup (0, \varepsilon)$ and we infer, as above, that $v = 0$ on $\mathbf{R}$. In case ii) b), we cannot have $v(x) \in (-\varepsilon, 0)$ and we infer that $v(x) \geq 0$ for any $x \in \mathbf{R}$. Since $v(x) \longrightarrow 0$ as $x \longrightarrow \infty$, there is some x_0 such that v achieves a nonnegative maximum at x_0. Then $v'(x_0) = 0$ and from (5.12) we get $W(v(x_0)) = 0$. But $W(s) < 0$ for $s > 0$ by ii) b), hence $v(x_0) = 0$ and consequently $v = 0$ on $\mathbf{R}$. Similarly we have $v = 0$ in the case ii) c) (note that $v = |\psi|^2 - r_0^2 \geq -r_0^2$ and it suffices to know that $W < 0$ on $[-r_0^2, 0)$).

Thus we have always $v = 0$, that is $|\psi|^2 = r_0^2$ on $\mathbf{R}$. Consequently there exists a lifting $\theta \in C^2(\mathbf{R}, \mathbf{R})$ such that $\psi(x) = r_0e^{i\theta(x)}$ for any $x \in \mathbf{R}$. It is clear that $\psi_1\psi_2' - \psi_2\psi_1' = |\psi|^2\theta' = r_0^2\theta'$ (see (3.3)). On the other hand we have $\psi_1\psi_2' - \psi_2\psi_1' = -k_2$ by (5.7), hence $\theta' = -\frac{k_2}{r_0^2}$ is constant, therefore $\theta(x) = -\frac{k_2}{r_0^2}x + \theta_0$, where θ_0 is a real constant. Since $\psi = r_0e^{i(-\frac{k_2}{r_0^2}x + \theta_0)}$ satisfies equation (5.2), we find $-c\frac{k_2}{r_0^2} - \left(\frac{k_2}{r_0^2}\right)^2 = 0$, thus either $\frac{k_2}{r_0^2} = 0$ or $\frac{k_2}{r_0^2} = -c$. Finally we have either $\psi(x) = e^{i\theta_0}$ or $\psi(x) = e^{i(cx+\theta_0)}$ and the proof is complete. $\square$

Example 5.2 In the case of the Gross-Pitaevskii equation we have $G(s) = 1 - s$ and we obtain $W(s) = -2s^3$ (see Example 4.4). In the case of the cubic-quintic nonlinearity we have $G(s) = -\alpha_5(s - r_1^2)(s - r_0^2)$, where $\alpha_5 > 0$, $0 < r_1 < r_0$ (see Example 4.5) and a simple computation gives $W(s) = -2\alpha_5s^3(\frac{4}{3}r_0^2 - r_1^2 + \frac{1}{3}s)$. Therefore both the Gross-Pitaevskii and the cubic-quintic nonlinearities satisfy assumption ii) b) and Theorem 5.1 gives all sonic and supersonic travelling-waves for these equations.

Remark. The proof of Theorem 5.1 provides a method to find subsonic travelling-waves for (5.1). With the above notation, it follows from (5.11) that on any interval where $v' \neq 0$ we have $v'(x) = \pm\sqrt{4(v + r_0^2)V(v + r_0^2)(x) - c^2v^2(x)}$. In many interesting applications this equation can be integrated and we obtain explicitly $v = |\psi|^2 - r_0^2$. Then it is not hard to find (up to a constant) the corresponding phase θ.

Remark. Assume that $N = 1$ and let (ψ, φ) be a finite-energy travelling-wave for the system (4.17)-(4.18). It follows from the proof of Proposition 4.6 that ψ and φ are C^∞ functions and $\psi', \varphi \in W^{k,p}(\mathbf{R})$ for any $k \in \mathbf{N}$ and $p \geq 2$. If $c^2 \geq \frac{2}{\varepsilon^2}$ (recall that $\frac{\sqrt{2}}{\varepsilon}$ is the sound velocity at infinity associated to (3.21)-(3.22)) and if there is a lifting $\psi(x) = v(x)e^{i\alpha(x)}$, $\varphi(x) = u(x)e^{i\beta(x)}$, where v, u, α, β are real-valued functions of class C^2, Proposition 3.1 p. 1545 in [26] implies that $v = 1$, α is constant and $\varphi = 0$ on $\mathbf{R}$.

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