On $SO(n)$-invariant functions

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Definition

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A function $f\colon M_n(\mathbb{R}) \to \bar{\mathbb{R}}$ is said to be $\mathrm{O}(n)$-invariant if

$$\forall \xi \in M_n(\mathbb{R}), \quad \forall U, V \in \mathrm{O}(n), \quad f(U\xi V) = f(\xi).$$

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Notation

The vector of singular values of A is denoted by $\sigma(A) = (\sigma_1(A), \ldots, \sigma_n(A))$. They are ordered increasingly.

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$$\begin{array}{rccl} \sigma\colon & M_n(\mathbb{R}) & \longrightarrow & K := \{x \in \mathbb{R}^n | 0 \le x_1 \le \ldots \le x_n\} \\ & A & \longmapsto & \sigma(A) \end{array}$$

O(*n*)-invariance and SVD

$\mathrm{O}(n)$-invariance and SVD

Proposition

Let $f\colon M_n(\mathbb{R}) \to \bar{\mathbb{R}}$. The following are equivalent:

(a) f is $\mathrm{O}(n)$-invariant;

(b) $f = f \circ \mathrm{diag} \circ \sigma$.

In this case, $g := f \circ \mathrm{diag}$ is the unique $\Pi(n)$-invariant function such that $f = g \circ \sigma$.

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$$\mathrm{diag}\, x = \begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & x_n \end{bmatrix}$$

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$\Pi(n)$ is the group of signed permutation $(n \times n)$-matrices

O(n)-invariance and convexity

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Let $f\colon M_n(\mathbb{R}) \to \bar{\mathbb{R}}$ be $\mathrm{O}(n)$-invariant, and let $g\colon \mathbb{R}^n \to \bar{\mathbb{R}}$ be the unique $\Pi(n)$-invariant function such that $f = g \circ \sigma$. Then the following are equivalent:

(i) f is closed proper convex;

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$D_n(\mathbb{R})$ is the subspace of $M_n(\mathbb{R})$ of diagonal matrices

Elements of a proof (I)

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Theorem (Von Neumann)

Let $\xi, \eta \in M_n(\mathbb{R})$. Then

$$\max_{Q,R \in \mathrm{O}(n)} \{\mathrm{tr}(Q\xi R^\top \eta^\top)\} = \sum_{j=1}^{n} \sigma_j(\xi)\sigma_j(\eta).$$

Consequently, $\mathrm{tr}(\xi\eta^\top) \leq \sum_{j=1}^{n} \sigma_j(\xi)\sigma_j(\eta)$.

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$$f^\star = g^\star \circ \sigma.$$

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- g closed proper convex $\Rightarrow g^{\star\star} = g$
- $f^{\star\star} = g^{\star\star} \circ \sigma = g \circ \sigma = f$, thus f is closed proper convex.

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The signed singular values of A are defined as follows:

- $\mu_1(A) = \mathrm{sgn}(\det A) \cdot \sigma_1(A)$;
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$\Pi_e(n)$ is the group of signed permutation $(n \times n)$-matrices having an even number of entries equal to -1

SO(n)-invariance and convexity

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Let $f\colon M_n(\mathbb{R}) \to \bar{\mathbb{R}}$ be $\mathrm{SO}(n)$-invariant, and let $g\colon \mathbb{R}^n \to \bar{\mathbb{R}}$ be the unique $\Pi_e(n)$-invariant function such that $f = g \circ \mu$. Then the following are equivalent:

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- B. Kostant (1973)

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$$f \text{ lower semi-continuous} \Leftrightarrow \operatorname{epi} f \text{ closed}$$

$$\operatorname{epi}\left(\sup_\alpha f_\alpha\right) = \bigcap_\alpha \operatorname{epi} f_\alpha$$

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Definition

A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be proper convex if

- f is convex;
- $f \not\equiv \infty$;
- $f > -\infty$.

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Definition

The convex conjugate of a function $f\colon \mathbb{R}^n \to \bar{\mathbb{R}}$ is the function $f^\star$ defined on $\mathbb{R}^n$ by

$$f^\star(\xi) = \sup_{x\in\mathbb{R}^n} \{\langle x,\xi\rangle - f(x)\}\,.$$

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$f^\star$ is convex lower semi-continuous

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$f^{\star\star}$ is the supremum of all affine functions that are below f

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Theorem

Let $f\colon \mathbb{R}^n \to \bar{\mathbb{R}}$ be proper convex and lower semi-continuous. Then $f^{\star\star} = f$. Conversely, if $f^{\star\star} = f$, then either f is proper convex and lower semi-continuous or $f \equiv \infty$ or $f \equiv -\infty$.

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Remark

The functions that are convex and lower semi-continuous are said to be convex closed.

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Theorem (Refined Von Neumann)

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Lemma

Let $D \in M_n(\mathbb{R})$ be diagonal, with diagonal entries whose absolute values are pairwise distinct. If $M \in M_n(\mathbb{R})$ is such that both MD and DM are symmetric, then M is diagonal.

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$\forall i, j \in \{1, \ldots, n\}$, $(MD^2)_{ij} = M_{ij} d_j^2$ and $(D^2 M)_{ij} = d_i^2 M_{ij}$

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$\forall i, j \in \{1, \ldots, n\}, (MD^2)_{ij} = M_{ij}d_j^2$ and $(D^2M)_{ij} = d_i^2 M_{ij}$

$i \neq j \Rightarrow d_i^2 \neq d_j^2 \Rightarrow M_{ij} = 0$

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$\zeta = UMV^\top$ with $M := \operatorname{diag}\big(\mu_1(\zeta), \ldots, \mu_n(\zeta)\big)$ and $U, V \in \mathrm{SO}(n)$

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$$\begin{aligned}
\max_{Q,R\in SO(n)} \{\mathrm{tr}(Q\xi R^\top \zeta^\top)\} &= \max_{Q_1,R_1\in SO(n)} \{\mathrm{tr}(Q_1\xi R_1^\top M)\} \\
&= \sum_j \mu_j(\xi)\mu_j(M) \\
&= \sum_j \mu_j(\xi)\mu_j(\zeta)
\end{aligned}$$

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(continuity argument)

Proof of the Refined Von Neumann Theorem

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STEP 2: w.l.o.g. $|\mu_1(\eta)| < \mu_2(\eta) < \dots < \mu_n(\eta)$

(continuity argument)

STEP 3:

$$\sup_{Q,R \in SO(n)} \left\{ \operatorname{tr}(Q\xi R^\top \eta^\top) \right\} = \operatorname{tr}(Q_0 \xi R_0^\top \eta^\top)$$

with $Q_0, R_0 \in \mathrm{SO}(n)$ such that $Q_0 \xi R_0^\top$ is diagonal

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If $|y_1| \leq y_2 \leq \ldots \leq y_n$ and $(x_1, \ldots, x_n) \in \mathbb{R}^n$, then the maximum value of $\langle Mx, y \rangle$ for $M \in \Pi_e(n)$ is obtained for M such that

$$|(Mx)_1| \leq (Mx)_2 \leq \ldots \leq (Mx)_n.$$

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$$\forall t\in\mathbb{R},\qquad Q(t) := e^{tA}Q_0 \quad \text{and} \quad R(t) := e^{tB}R_0$$

with A and B skew-symmetric

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$$0=\varphi'(0)=\mathrm{tr}(AQ_0\xi R_0^\top\eta^\top)+\mathrm{tr}(Q_0\xi R_0^\top B^\top\eta^\top)$$

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$Q_0\xi R_0^\top$ is diagonal by the lemma

($\eta^\top$ is diagonal, with entries pairwise distinct in absolute value)

From Von Neumann to the conjugacy relationship (I)

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Theorem

Let f be $\mathrm{SO}(n)$-invariant, that is, $f = g \circ \mu$ with $g = f \circ \mathrm{diag}$ $\Pi_e(n)$-invariant. Then $f^\star = g^\star \circ \mu$.

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A final remark (I)

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$\xi \leftarrow -\xi$ yields

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n is even $\Rightarrow$ $\det(-\xi) = \det\xi \Rightarrow \mu(-\xi) = \mu(\xi)$

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n is odd $\not\Rightarrow \det(-\xi) = \det \xi$

$|\operatorname{tr}(\xi\eta^\top)| \not\leq \langle \mu(\xi), \mu(\eta) \rangle$ in general

$$\xi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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$$\operatorname{tr}(\xi\eta^\top) = -3$$

$$\mu(\xi) = (-1, 1, 1)^\top \text{ and } \mu(\eta) = (1, 1, 1)^\top$$

$$\langle \mu(\xi), \mu(\eta) \rangle = 1$$

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