Generalized perspectives and applications

Pierre Maréchal

Institut de Mathématiques de Toulouse

France

Outline

- Introduction
- Generalized perspectives
- Conjugacy
- Other properties
- Applications

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Multiplicative barrier

$$\mathscr{K}_n(x, \mathbf{y}) := \frac{x^{1+\alpha}}{\mathbf{y}^{\boldsymbol{\beta}}} = \frac{x^{1+\alpha}}{y_1^{\beta_1} \cdots y_n^{\beta_n}}, \quad x \geq 0, \quad \mathbf{y} > \mathbf{0}$$

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H. Imai (1988): *On the convexity of the multiplicative version of Karmarkar's potential function.* Mathematical Programming **40**, 29-32

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J. Borwein (1998): *A generalization of Young's Inequality.* Mathematical inequalities and Applications **1**, 131-136

(Computation of the bi-conjugate)

Perspectives

LEMMA. Let $f\colon \mathbb{R}^n \to [-\infty, \infty]$. Then, the function $\breve{f}$ defined by

$$\breve{f}(\mathbf{x}, y) = yf\left(\frac{\mathbf{x}}{y}\right), \quad \mathbf{x} \in \mathbb{R}^n,\ y \in \mathbb{R}^*_+$$

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The function $\breve{f}$ is referred to as the *perspective* of f.

An inductive argument

$$\mathscr{K}_{n+1}(x, y_1, \ldots, y_{n+1}) = \frac{x^{1+\alpha}}{y_1^{\beta_1} \cdots y_{n+1}^{\beta_{n+1}}}$$

(positively homogeneous of degree $\gamma := 1 + \alpha - |\boldsymbol{\beta}| \geq 1$)

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Remark

Imai also proved the convexity, for $\alpha \geq |\beta|$, of

$$\frac{x^{\alpha}}{\mathbf{y}^{\boldsymbol{\beta}}} = \frac{x^{\alpha}}{y_1^{\beta_1} \cdots y_n^{\beta_n}}, \quad x \geq 0, \quad \mathbf{y} \in \Sigma_n$$

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$$g(y) f\left(\frac{x}{g(y)}\right) ?$$

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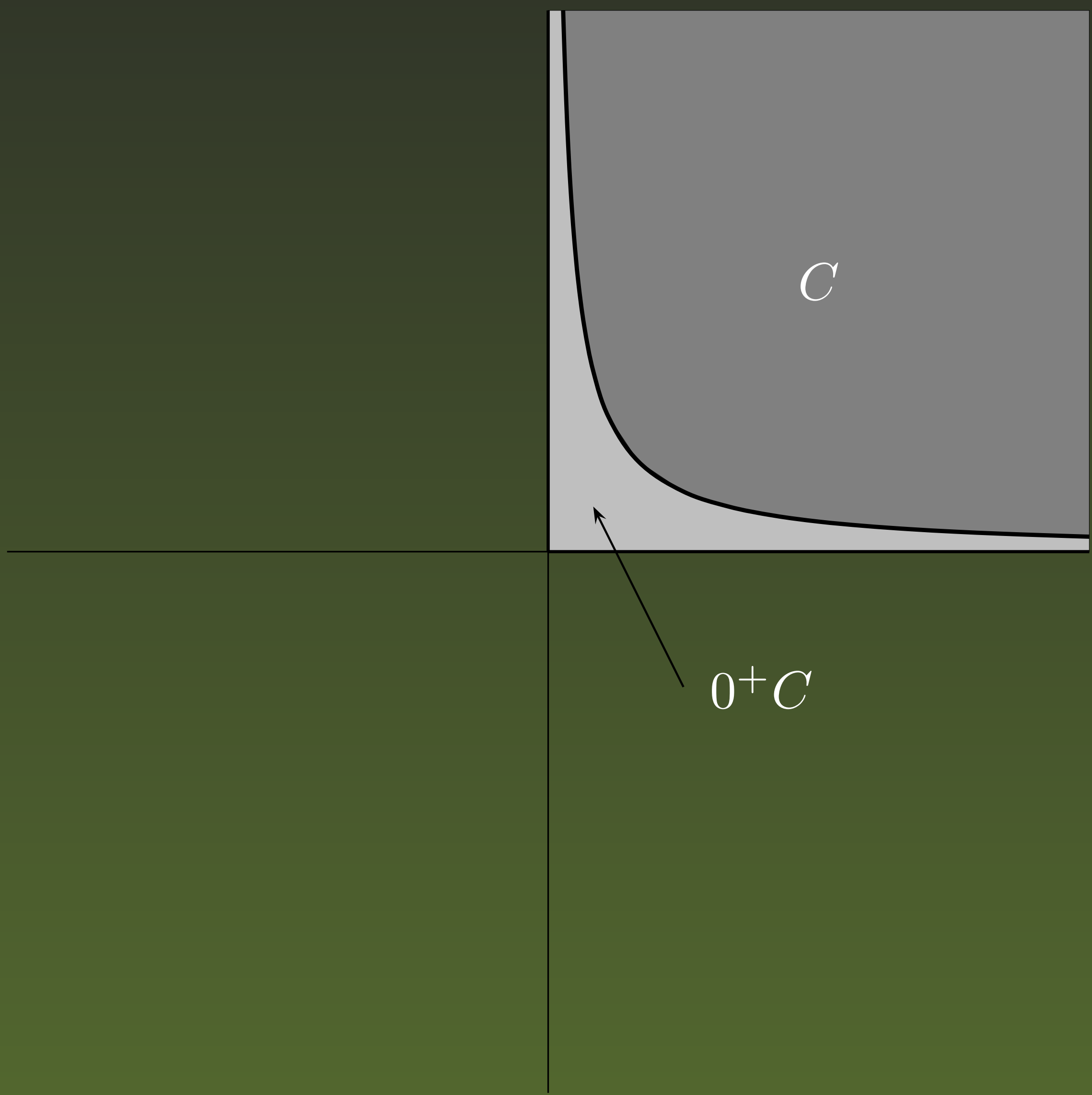

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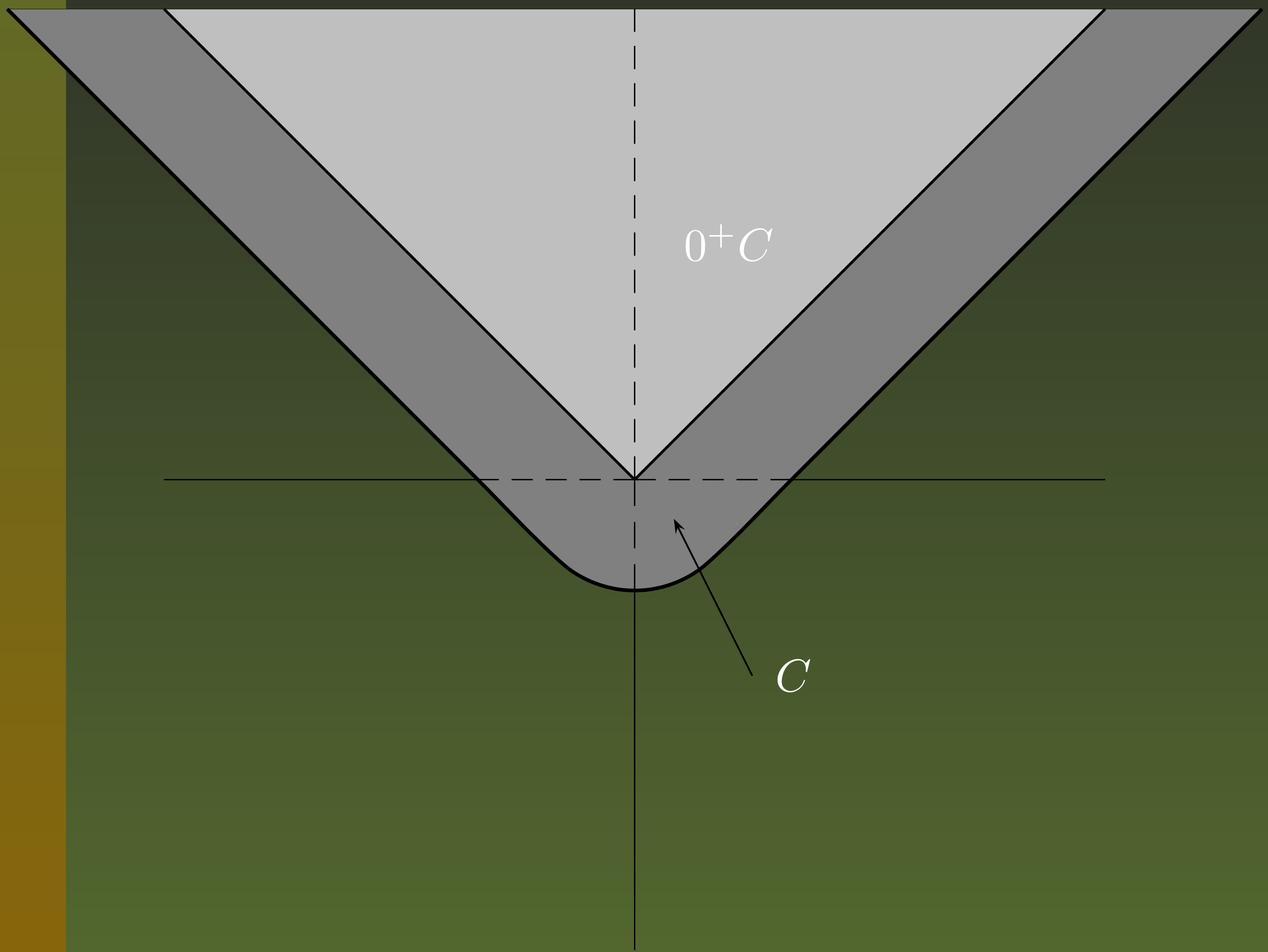

A fundamental and easy result

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(1) the mapping $\alpha \mapsto \alpha \cdot C$ is increasing if and only if $0^+C \subset C$;

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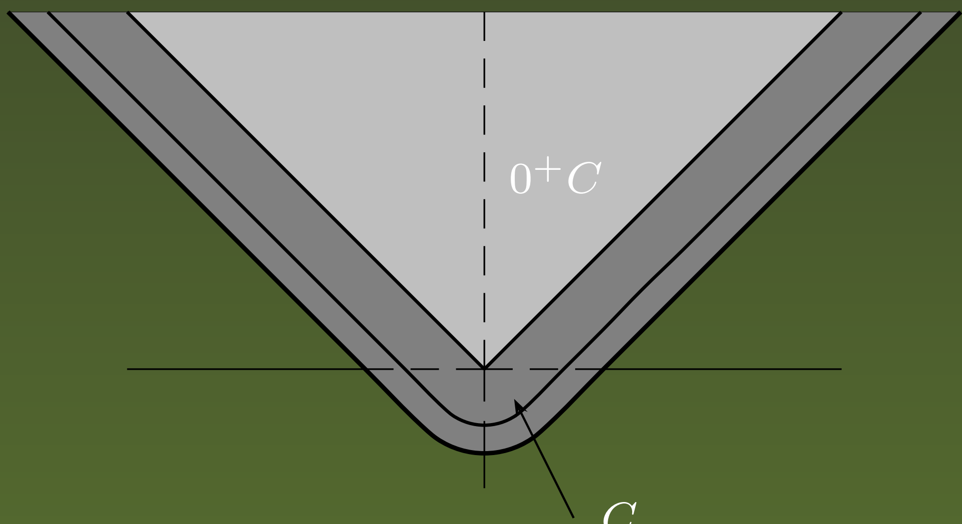

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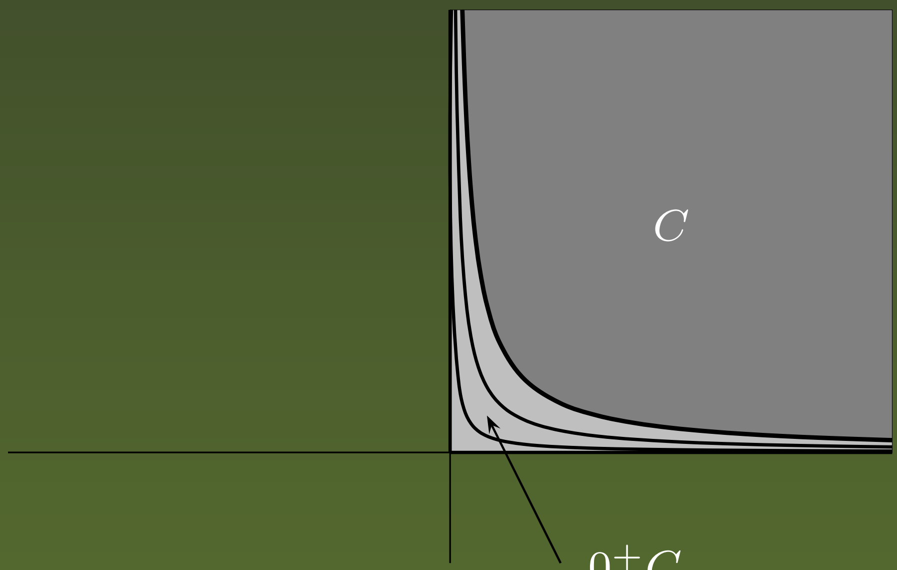


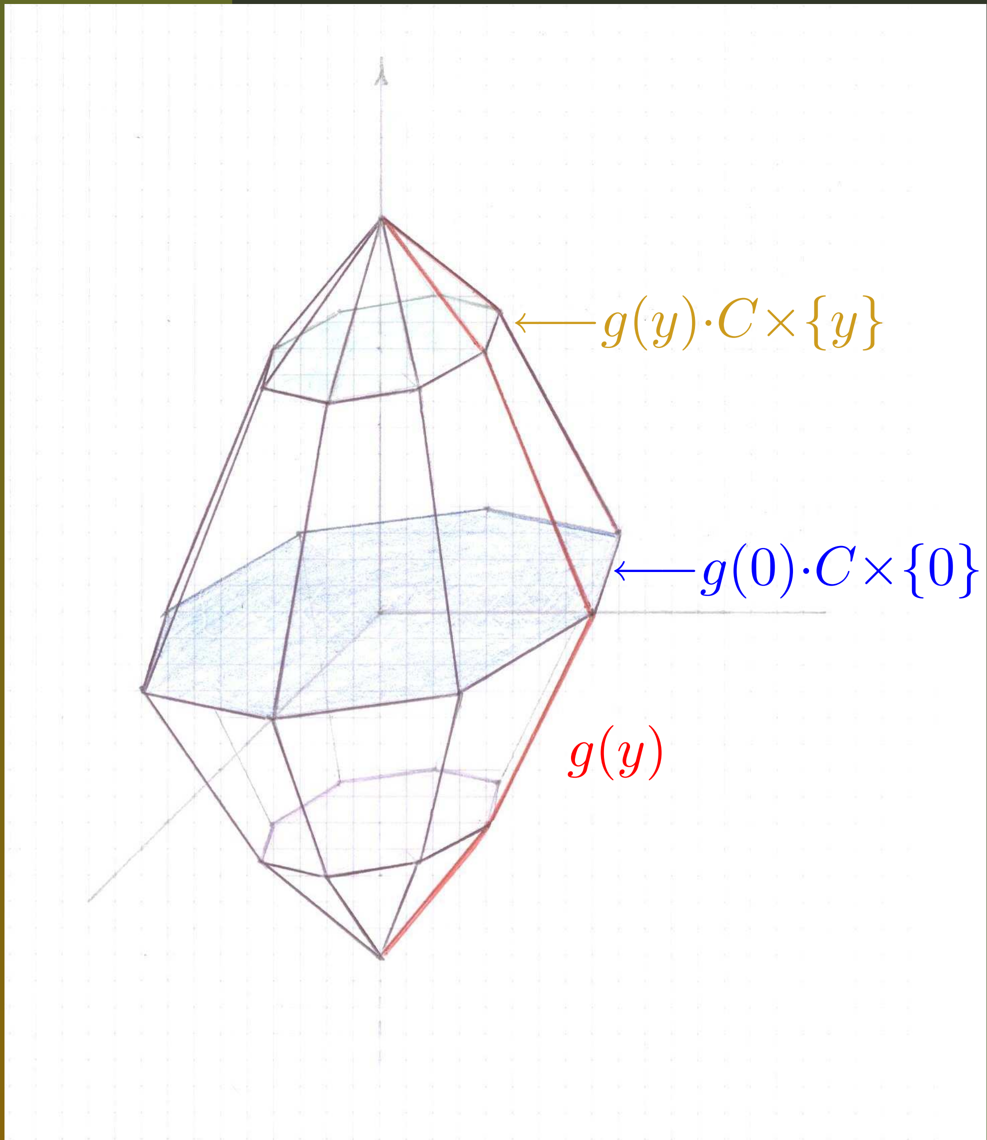
Building convex sets on $\mathbb{R}^n \times \mathbb{R}^n$

PROPOSITION. Let $C \in \Gamma_0(\mathbb{R}^n)$ and let $g\colon \mathbb{R}^m \to \{-\infty\} \cup [0, \infty)$ be a proper concave function. Then, the set

$$C_g := \bigcup_{\mathbf{y} \in \operatorname{dom} g} \big(g(\mathbf{y}) \cdot C \times \{\mathbf{y}\}\big)$$

is a convex subset of $\mathbb{R}^{n+m}$.

Buiding C_g



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Proof of the proposition

$$(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2) \in C_g \quad \text{and} \quad \lambda \in (0, 1)$$

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(2) A desirable property:

$$C, g \text{ closed} \implies C_g \text{ closed.}$$

This can fail *only* in the case where $C = 0^+C$ and g is closed proper convex.

Example

$$C = [0, \infty) \subset \mathbb{R} \quad \text{and} \quad g(y) = \frac{1}{y}, \;\; y > 0$$

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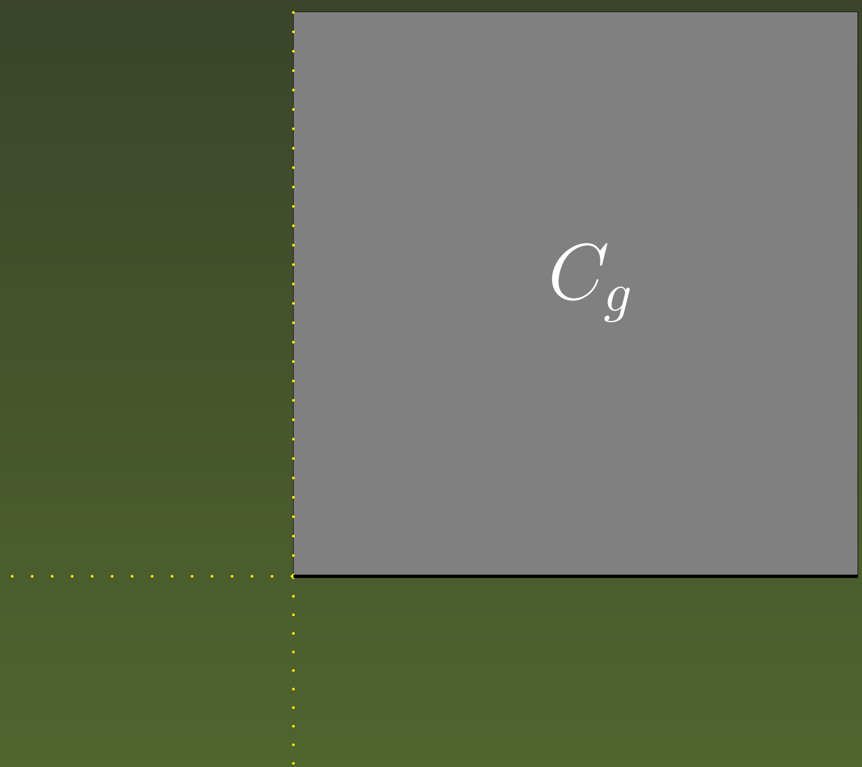

Final definition

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(1) If $C \in \Gamma_0(\mathbb{R}^n)$ and g is proper concave and non-negative on its domain, we denote by $C \triangle g$ the subset of $\mathbb{R}^{n+m}$ defined by

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(2) If $C \in \Gamma_+(\mathbb{R}^n)$ and g is proper convex and non-negative, we denote by $C \triangle g$ the subset of $\mathbb{R}^{m+n}$ defined by

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With this definition, $C \triangle g$ is closed whenever C and g are closed.

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(1) If $C \subset 0^+C$, then $C^\circ = (0^+C)^\circ$, so that C° is a convex cone containing the origin.

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(2) If C is closed, then $C \subset 0^+C$ if and only if C° is a cone.

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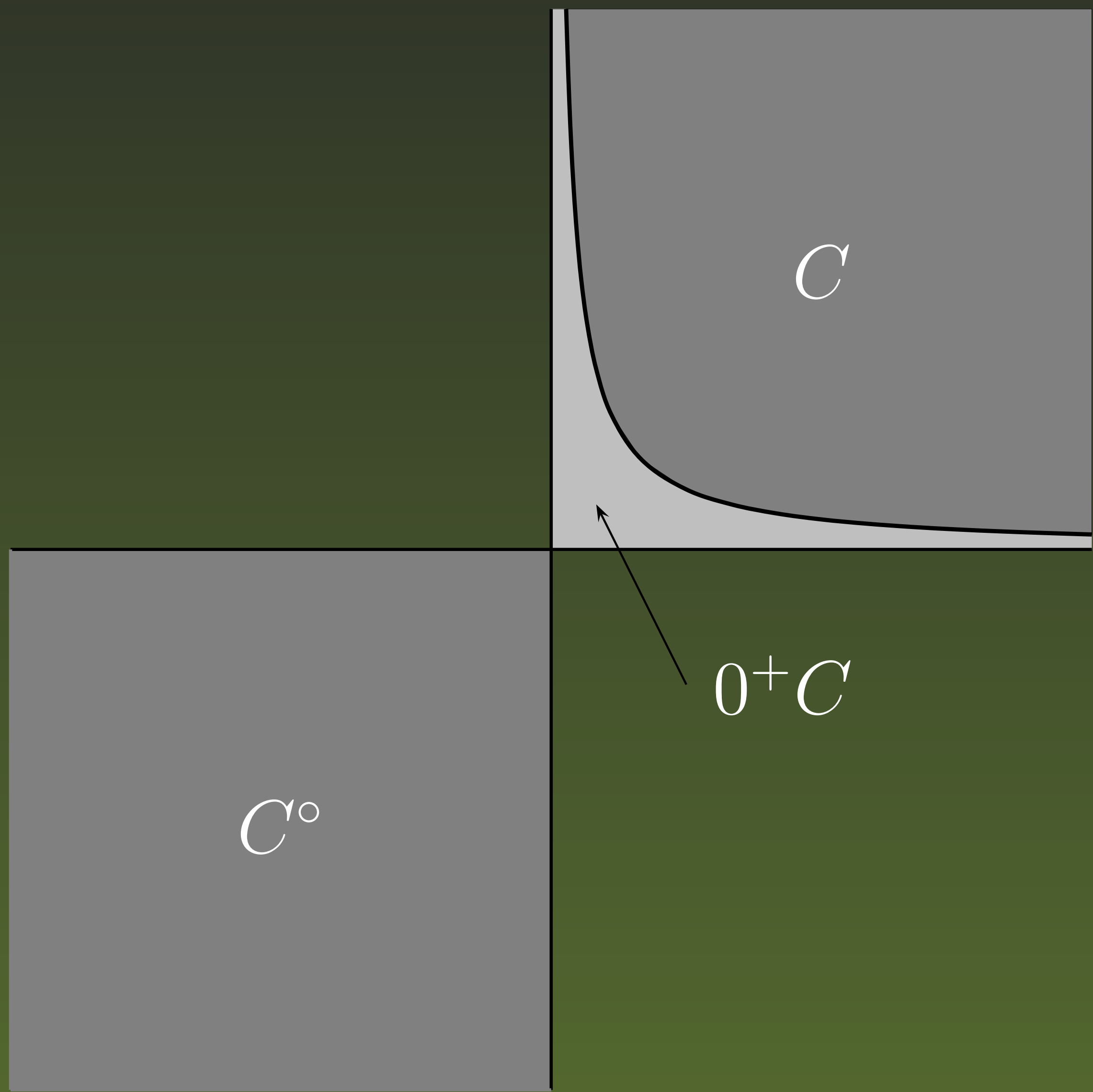

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$$(f\alpha)(\mathbf{x}) := \alpha f \left(\frac{\mathbf{x}}{\alpha} \right)$$

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Left and right scalar multiplications

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$$(\alpha f)(\mathbf{x}) := \alpha f(\mathbf{x}) \quad \text{and} \quad (f\alpha)(\mathbf{x}) := \alpha f\left(\frac{\mathbf{x}}{\alpha}\right)$$

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$$\begin{aligned} (\alpha f)^\star(\boldsymbol{\xi}) &:= \sup\big\{\langle \boldsymbol{\xi}, \mathbf{x}\rangle - \alpha f(\mathbf{x})\big\} \\ &= \sup\big\{\alpha\big(\big\langle \alpha^{-1}\boldsymbol{\xi}, \mathbf{x}\big\rangle - f(\mathbf{x})\big)\big\} \\ &= \alpha \sup\big\{\big\langle \alpha^{-1}\boldsymbol{\xi}, \mathbf{x}\big\rangle - f(\mathbf{x})\big\} \\ &= \alpha f^\star(\alpha^{-1}\boldsymbol{\xi}) \end{aligned}$$

Sketch of a proof

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$$(f \triangle g)^\star(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sup_{\mathbf{x},\mathbf{y}} \left\{ \langle \boldsymbol{\xi}, \mathbf{x} \rangle + \langle \boldsymbol{\eta}, \mathbf{y} \rangle - g(\mathbf{y}) f\left(\frac{\mathbf{x}}{g(\mathbf{y})} \right) \right\}$$

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Associativity

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Additionnal assumption: $\mathbf{0} \in \operatorname{dom} g$. Then:

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One can show that $g \bigtriangledown h$ is actually nonnegative (on its domain), so we can also consider $f \bigtriangleup (g \bigtriangledown h)$

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Both $(f \bigtriangleup g) \bigtriangleup h$ and $f \bigtriangleup (g \bigtriangledown h)$ are closed proper convex

Associative laws

THEOREM [PM, 2005]. With the above assumptions,

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&= h(\mathbf{z})g\big(\mathbf{y}/h(\mathbf{z})\big) f\left(\frac{\mathbf{x}/h(\mathbf{z})}{g\big(\mathbf{y}/h(\mathbf{z})\big)}\right) \\
&= h(\mathbf{z})g\big(\mathbf{y}/h(\mathbf{z})\big) f\left(\frac{\mathbf{x}}{h(\mathbf{z})g\big(\mathbf{y}/h(\mathbf{z})\big)}\right) \\
&= (g \bigtriangledown h)(\mathbf{y},\mathbf{z}) f\left(\frac{\mathbf{x}}{(g \bigtriangledown h)(\mathbf{y},\mathbf{z})}\right) \\
&= \big[f \bigtriangleup (g \bigtriangledown h)\big]\big(\mathbf{x},(\mathbf{y},\mathbf{z})\big)
\end{aligned}$$

Commutative laws

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Infimal convolution

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Roughly speaking, addition and epigraphic sum are dual operations.

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Then $(f_1 + f_2, g)$ is of type I, and

$$(f_1 + f_2) \mathbin{\triangle} g = (f_1 \mathbin{\triangle} g) + (f_2 \mathbin{\triangle} g).$$

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Then $(f_1 + f_2, g)$ is of type II and

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If $f \neq f0^+$, or if $f = f0^+$ and the sets $\operatorname{ri}\operatorname{dom} {g_k}^\star 0^+$ have a point in common, then $(f, g_1 \vee g_2)$ is of type II, and

$$f \bigtriangleup (g_1 \vee g_2) = (f \bigtriangleup g_1) \vee (f \bigtriangleup g_2).$$

Outline

- Introduction
- Generalized perspectives
- Conjugacy
- Other properties
- Applications

Minimizing condition numbers (with Jane Ye)

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Take restriction to $\{A = B\}$

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κ_p and κ_p^p have the same minimizers !

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Replace the original quasiconvex problem

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and let $p \to \infty$.

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Question: can we approach a solution to Problem $(\mathscr{P})$ with solutions to the surrogate problems $(\mathscr{P}_p)$?

Main results [PM & Jane Ye, 2009]

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THEOREM 1 [Exact approximation]. Let $(p_k)_{k\in\mathbb{N}^\star} \subset [1,\infty)$ be a sequence which tends to infinity, and, for every $k \in \mathbb{N}^\star$, let $\bar{A}_{p_k}$ be a solution to problem $(\mathscr{P}_{p_k})$. Then every cluster point $\bar{A}$ of the sequence $(\bar{A}_{p_k})$ (there is at least one) is a global solution of problem $(\mathscr{P})$.

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$$\forall A \in \Omega, \quad \kappa(A) \geq \kappa(\bar{A}_k) - \varepsilon_k$$

Back to multiplicative functions

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$$\mathscr{K}_n(x, \mathbf{y}) := \frac{x^{1+\alpha}}{\mathbf{y}^{\beta}} = \frac{x^{1+\alpha}}{y_1^{\beta_1} \cdots y_n^{\beta_n}}, \qquad x \geq 0, \qquad \mathbf{y} > \mathbf{0}$$

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THEOREM [PM, 2010]. The function K_n is convex.

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$$(f \triangle g)(\mathbf{x}, \mathbf{y}) = \frac{g(\mathbf{x})^{1+|\boldsymbol{\beta}|}}{\mathbf{y}^{\boldsymbol{\beta}}}$$

Some references

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Thank you for your attention !