Generalized perspectives and applications

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Multiplicative barrier

\( \mathcal{K}_n(x, y) := \frac{x^{1+\alpha}}{y^{\beta}} = \frac{x^{1+\alpha}}{y_1^{\beta_1} \cdots y_n^{\beta_n}}, \quad x \geq 0, \quad y > 0 \)
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(Computation of the bi-conjugate)
**LEMMA.** Let \( f : \mathbb{R}^n \rightarrow [-\infty, \infty] \). Then, the function \( \tilde{f} \) defined by

\[
\tilde{f}(x, y) = yf\left(\frac{x}{y}\right), \quad x \in \mathbb{R}^n, \; y \in \mathbb{R}^*_+
\]

is convex if and only if \( f \) is convex.
**Lemma.** Let \( f : \mathbb{R}^n \to [-\infty, \infty] \). Then, the function \( \tilde{f} \) defined by

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The function \( \tilde{f} \) is referred to as the *perspective* of \( f \).
An inductive argument

\[ \mathcal{K}_{n+1}(x, y_1, \ldots, y_{n+1}) = \frac{x^{1+\alpha}}{y_1^{\beta_1} \cdots y_{n+1}^{\beta_{n+1}}} \]

(positively homogeneous of degree
\[ \gamma := 1 + \alpha - |\beta| \geq 1 \])
An inductive argument

\[ H_{n+1}(x, y_1, \ldots, y_{n+1}) = \frac{x^{1+\alpha}}{y_1^{\beta_1} \cdots y_{n+1}^{\beta_{n+1}}} \]

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\[ H_{n+1}(x, y_1, \ldots, y_{n+1}) = \left( \frac{x^{1+\alpha}}{y_1^\gamma \cdots y_{n+1}^\gamma} \right)^{\gamma} \]

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An inductive argument

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\mathcal{K}_{n+1}(x, y_1, \ldots, y_{n+1}) = \frac{x^{1+\alpha}}{y_1^{\beta_1} \cdots y_{n+1}^{\beta_{n+1}}}
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(positively homogeneous of degree \( \gamma := 1 + \alpha - |\beta| \geq 1 \))

\[
\mathcal{K}_{n+1}(x, y_1, \ldots, y_{n+1}) = \begin{pmatrix}
\frac{x^{1+\alpha}}{\gamma} \\
\frac{\beta_1}{\gamma} & \frac{\beta_{n+1}}{\gamma} \\
\frac{y_1^{\gamma}}{y_1^{\gamma}} & \cdots & \frac{y_{n+1}^{\gamma}}{y_{n+1}^{\gamma}}
\end{pmatrix}^{\gamma}
\]

\[
y_{n+1} = \frac{x}{y_{n+1}}^{1+\alpha} \left( \frac{y_1}{y_{n+1}} \right)^{\beta_1} \cdots \left( \frac{y_n}{y_{n+1}} \right)^{\beta_n}
\]
Remark

Imai also proved the convexity, for $\alpha \geq |\beta|$, of

$$\frac{x^\alpha}{y^{\beta}} = \frac{x^\alpha}{y_1^{\beta_1} \cdots y_n^{\beta_n}}, \quad x \geq 0, \quad y \in \Sigma_n$$

where $\Sigma_n := \{y \in \mathbb{R}_+^n \mid y_1 + \cdots + y_n = 1\}$. 
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$$\frac{x^2}{y(1-y)} = y(1-y) \left(\frac{x}{y(1-y)}\right)^2, \quad x \geq 0, \quad y \in (0,1)$$
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$$g(y) f \left(\frac{x}{g(y)}\right)$$
Outline

- Introduction
- Generalized perspectives
- Conjugacy
- Other properties
- Applications
Classes of convex sets

$\Gamma(\mathbb{R}^n)$ nonempty convex subsets of $\mathbb{R}^n$

$\bar{\Gamma}(\mathbb{R}^n)$ nonempty closed convex subsets of $\mathbb{R}^n$
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\[ \Gamma_0(\mathbb{R}^n) = \{ C \in \Gamma(\mathbb{R}^n) \mid 0^+ C \subset C \} \]

\[ \bar{\Gamma}_0(\mathbb{R}^n) = \{ C \in \bar{\Gamma}(\mathbb{R}^n) \mid 0^+ C \subset C \} \]
\( C \subset 0^+C \)
$0^+ C \subset C$
A fundamental and easy result

\[ \alpha \cdot C := \begin{cases} \alpha C' & \text{if } \alpha > 0, \\ 0^+ C' & \text{if } \alpha = 0 \end{cases} \]
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**Proposition.** Let \( C \) be a convex subset of \( \mathbb{R}^n \). Then

1. the mapping \( \alpha \mapsto \alpha \cdot C \) is increasing if and only if \( 0^+ C \subset C \);
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PROPOSITION. Let $C \in \Gamma_0(R^n)$ and let $g : R^m \rightarrow \{-\infty\} \cup [0, \infty)$ be a proper concave function. Then, the set

$$C_g := \bigcup_{y \in \text{dom } g} (g(y) \cdot C \times \{y\})$$

is a convex subset of $R^{n+m}$. 
Building $C_g$

\[ \bigcup_{y \in \text{dom } g} \left( g(y) \cdot C \times \{y\} \right) \]
Proof of the proposition

\((x_1, y_1), (x_2, y_2) \in C_g \text{ and } \lambda \in (0, 1)\)
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\[(\lambda g(y_1) + (1 - \lambda)g(y_2)) \cdot C \geq g(\lambda y_1 + (1 - \lambda)y_2)\]

\[\lambda x_1 + (1 - \lambda)x_2 \in g(\lambda y_1 + (1 - \lambda)y_2) \cdot C\]

\[\lambda (x_1, y_1) + (1 - \lambda)(x_2, y_2) \in C_g\]
(1) A similar reasoning shows that $C_g$ is convex if $C \subset 0^+ C$ and $g : \mathbb{R}^m \to [0, \infty]$ is a proper convex function.
Remarks

(1) A similar reasoning shows that $C_g$ is convex if $C \subseteq 0^+C$ and $g : \mathbb{R}^m \rightarrow [0, \infty]$ is a proper convex function.

(2) A desirable property:

$$C, g \text{ closed} \implies C_g \text{ closed.}$$

This can fail only in the case where $C = 0^+C$ and $g$ is closed proper convex.
Example

\[ C = [0, \infty) \subset \mathbb{R} \quad \text{and} \quad g(y) = \frac{1}{y}, \quad y > 0 \]
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\[ C = [0, \infty) \subset \mathbb{R} \quad \text{and} \quad g(y) = \frac{1}{y}, \quad y > 0 \]
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Let $C \in \Gamma(R^n)$ and let $g$ be an extended real valued function on $R^m$.

(1) If $C \in \Gamma_0(R^n)$ and $g$ is proper concave and non-negative on its domain, we denote by $C \bigtriangleup g$ the subset of $R^{n+m}$ defined by

$$C \bigtriangleup g := \bigcup_{y \in \text{dom } g} (g(y) \cdot C \times \{y\}).$$
Let $C \in \Gamma(\mathbb{R}^n)$ and let $g$ be an extended real valued function on $\mathbb{R}^m$.

(2) If $C \in \Gamma_+(\mathbb{R}^n)$ and $g$ is proper convex and non-negative, we denote by $C \triangle g$ the subset of $\mathbb{R}^{m+n}$ defined by

$$C \triangle g := \begin{cases} \bigcup_{y \in \text{dom } g} (g(y) \cdot C \times \{y\}) & \text{if } C \neq 0^+ C, \\ C \times \overline{\text{dom } g} & \text{if } C = 0^+ C. \end{cases}$$
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C \times \text{cl dom } g & \text{if } C = 0^+C.
\end{cases}$$

With this definition, $C \bigtriangleup g$ is closed whenever $C$ and $g$ are closed.
Characterizing \( \bar{\Gamma}(\mathbb{R}^n) \)

\[
C^\circ := \{ \xi \in \mathbb{R}^n \mid \forall x \in C, \langle \xi, x \rangle \leq 1 \}
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Characterizing $\bar{\Gamma}(\mathbb{R}^n)$

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(if $C$ is a cone, $C^\circ = \{ \xi \in \mathbb{R}^n \mid \forall x \in C, \langle \xi, x \rangle \leq 0 \}$)
Characterizing $\overline{\Gamma}(\mathbb{R}^n)$

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**Theorem [PM, 2005].** Let $C \in \Gamma(\mathbb{R}^n)$.

(1) If $C \subset 0^+ C$, then $C^\circ = (0^+ C)^\circ$, so that $C^\circ$ is a convex cone containing the origin.
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**Theorem** [PM, 2005]. Let $C \in \bar{\Gamma}(\mathbb{R}^n)$.

1. If $C \subset 0^+ C$, then $C^\circ = (0^+ C)^\circ$, so that $C^\circ$ is a convex cone containing the origin.

2. If $C$ is closed, then $C \subset 0^+ C$ if and only if $C^\circ$ is a cone.
Characterizing $\bar{\Gamma}(\mathbb{R}^n)$
Remark

One can show that, if $C$ is closed convex, then

$$0^+ C \subset C \iff 0 \in C.$$
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Let $C \subset \mathbb{R}^n$ be closed convex.

(1) [Rockafellar, 1970] If $0 \in C$, then

$$0^+ C = \bigcap_{\varepsilon > 0} \varepsilon C.$$
Remark

One can show that, if $C$ is closed convex, then

$$0^+ C \subset C \iff 0 \in C.$$  

Let $C \subset \mathbb{R}^n$ be closed convex.

(1) [Rockafellar, 1970] If $0 \in C$, then

$$0^+ C = \bigcap_{\varepsilon > 0} \varepsilon C.$$  

(2) [PM, 2005] If $C \subset 0^+ C$, then

$$0^+ C = \text{cl} \bigcup_{\varepsilon > 0} \varepsilon C = C^\circ.$$
From sets to functions
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\[ \text{epi} (f \triangle g) := (\text{epi } f) \triangle g = \bigcup_{y \in \text{dom } g} (g(y) \cdot \text{epi } f \times \{y\}) \]
From sets to functions

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\[ \alpha \cdot \text{epi } f = \text{epi } f \cdot \alpha := \begin{cases} \text{epi } f \alpha & \text{if } \alpha > 0 \\ \text{epi } f0^+ & \text{if } \alpha = 0 \end{cases} \]
From sets to functions

\[ C \leftarrow epi \, f \]

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\[ epi (f \, \Delta \, g) := (epi \, f) \, \Delta \, g = \bigcup_{y \in \text{dom} \, g} (g(y) \cdot epi \, f \times \{y\}) \]

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\[ (f\alpha)(x) := \alpha f \left( \frac{x}{\alpha} \right) \]
Definition 1
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- $f : \mathbb{R}^n \to (-\infty, \infty]$ proper convex with $f(0) \leq 0$

- $g : \mathbb{R}^m \to \{-\infty\} \cup [0, \infty)$ proper concave
Definition 1

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We say that the pair $(f, g)$ is of type I
Definition 1

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We say that the pair $(f, g)$ is of type I and

$$(f \triangle g)(x, y) := \begin{cases} 
  g(y)f\left(\frac{x}{g(y)}\right) & \text{if } g(y) \in (0, \infty), \\
  f0^+(x) & \text{if } g(y) = 0, \\
  \infty & \text{if } g(y) = -\infty.
\end{cases}$$
Definition 2
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- $f : \mathbb{R}^n \rightarrow (\neg \infty, \infty]$ is proper convext with $f \geq f_0^+$
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Definition 2

- \( f : \mathbb{R}^n \rightarrow (-\infty, \infty] \) is proper convex with \( f \geq f^0 + \)
- \( g : \mathbb{R}^m \rightarrow [0, \infty] \) proper convex

We say that the pair \((f, g)\) is of type II
Definition 2

- $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$ is proper convex with $f \geq f^0$
- $g : \mathbb{R}^m \rightarrow [0, \infty]$ proper convex

We say that the pair $(f, g)$ is of type II and, if $f \neq f^0$, 

$$(f \triangle g)(x, y) := \begin{cases} 
  g(y) f \left( \frac{x}{g(y)} \right) & \text{if } g(y) \in (0, \infty), \\
  f^0(x) & \text{if } g(y) = 0, \\
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\end{cases}$$
Definition 2

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We say that the pair $(f, g)$ is of type II and, if $f \neq f^0^+$,

$$(f \triangle g)(x, y) := \begin{cases} g(y)f \left( \frac{x}{g(y)} \right) & \text{if } g(y) \in (0, \infty), \\ f^0^+(x) & \text{if } g(y) = 0, \\ \infty & \text{if } g(y) = \infty \end{cases}$$

and if $f = f^0^+$,

$$(f \triangle g)(x, y) := \begin{cases} f(x) & \text{if } y \in \text{cl dom } g, \\ \infty & \text{if } y \notin \text{cl dom } g. \\ 

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Theorem

(1) Assume that \((f, g)\) is of type I, and that \(f\) and \(g\) are closed. Then \(((\neg g)^*, f^*)\) is of type II, and

\[
(f \triangle g)^*(\xi, \eta) = ((\neg g)^* \triangle f^*)(\eta, \xi),
\]

\[
((\neg g)^* \triangle f^*)^*(y, x) = (f \triangle g)(x, y).
\]
Theorem

(1) Assume that \((f, g)\) is of type \(I\), and that \(f\) and \(g\) are closed. Then \(((−g)^*, f^*)\) is of type \(II\), and

\[
(f \triangle g)^*(ξ, η) = ((−g)^* \triangle f^*)(η, ξ),
\]
\[
((−g)^* \triangle f^*)^*(y, x) = (f \triangle g)(x, y).
\]

(2) Assume that \((f, g)\) is of type \(II\), and that \(f\) and \(g\) are closed. Then \((g^*, −f^*)\) is of type \(I\), and

\[
(f \triangle g)^*(ξ, η) = (g^* \triangle (−f^*))(η, ξ),
\]
\[
(g^* \triangle (−f^*))^*(y, x) = (f \triangle g)(x, y).
\]
Left and right scalar multiplications
Left and right scalar multiplications

\((\alpha f)(x) := \alpha f(x)\) and \((f \alpha)(x) := \alpha f\left(\frac{x}{\alpha}\right)\)
Left and right scalar multiplications

$$\left(\alpha f\right)(x) := \alpha f(x)$$ \quad and \quad $$\left(f \alpha\right)(x) := \alpha f \left(\frac{x}{\alpha}\right)$$

**Lemma.** Let $f : [-\infty, \infty]$ be any function. Then

$$\left(\alpha f\right)^*(\xi) = \left(f^* \alpha\right)(\xi),$$

$$\left(f \alpha\right)^*(\xi) = \left(\alpha f^*\right)(\xi).$$
Left and right scalar multiplications

\((\alpha f)(x) := \alpha f(x)\) and \((f \alpha)(x) := \alpha f \left( \frac{x}{\alpha} \right)\)

**Lemma.** Let \(f : [-\infty, \infty]\) be any function. Then

\[(\alpha f)^\star(\xi) = (f^\star \alpha)(\xi),\]

\[(f \alpha)^\star(\xi) = (\alpha f^\star)(\xi).\]

\[(\alpha f)^\star(\xi) := \sup \{ \langle \xi, x \rangle - \alpha f(x) \} \]

\[= \sup \{ \alpha (\langle \alpha^{-1} \xi, x \rangle - f(x)) \} \]

\[= \alpha \sup \{ \langle \alpha^{-1} \xi, x \rangle - f(x) \} \]

\[= \alpha f^\star(\alpha^{-1} \xi)\]
Sketch of a proof
Sketch of a proof

\[(f \triangle g)^*(\xi, \eta) = \sup_{x, y} \left\{ \langle \xi, x \rangle + \langle \eta, y \rangle - g(y)f\left(\frac{x}{g(y)}\right) \right\}\]
Sketch of a proof

\[(f \triangle g)^*(\xi, \eta) = \sup_{x,y} \left\{ \langle \xi, x \rangle + \langle \eta, y \rangle - g(y) f \left( \frac{x}{g(y)} \right) \right\} \]

\[= \sup_{y} \left\{ \langle \eta, y \rangle + \sup_{x} \left\{ \langle \xi, x \rangle - (fg(y))(x) \right\} \right\} \]
Sketch of a proof

\[(f \triangle g)^*(\xi, \eta) = \sup_{x,y} \left\{ \langle \xi, x \rangle + \langle \eta, y \rangle - g(y) f \left( \frac{x}{g(y)} \right) \right\} \]

\[= \sup_{y} \left\{ \langle \eta, y \rangle + \sup_{x} \left\{ \langle \xi, x \rangle - (f g(y))(x) \right\} \right\} \]

\[= \sup_{y} \left\{ \langle \eta, y \rangle + (g(y) f^*)(\xi) \right\} \]
Sketch of a proof

\[(f \triangle g)^*(\xi, \eta) = \sup_{x,y} \left\{ \langle \xi, x \rangle + \langle \eta, y \rangle - g(y) f \left( \frac{x}{g(y)} \right) \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle + \sup_x \left\{ \langle \xi, x \rangle - (fg(y))(x) \right\} \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle + (g(y)f^*)(\xi) \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle + f^*(\xi)g(y) \right\} \]
Sketch of a proof

\[(f \triangledown g)^*(\xi, \eta) = \sup_{x,y} \left\{ \langle \xi, x \rangle + \langle \eta, y \rangle - g(y)f\left(\frac{x}{g(y)}\right) \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle + \sup_x \left\{ \langle \xi, x \rangle - (fg(y))(x) \right\} \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle + (g(y)f^*)(\xi) \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle + f^*(\xi)g(y) \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle - (f^*(\xi)(-g))(y) \right\} \]
Sketch of a proof

$$(f \triangle g)^*(\xi, \eta) = \sup_{x,y} \left\{ \langle \xi, x \rangle + \langle \eta, y \rangle - g(y) f \left( \frac{x}{g(y)} \right) \right\}$$

$$= \sup_y \left\{ \langle \eta, y \rangle + \sup_x \left\{ \langle \xi, x \rangle - (fg(y))(x) \right\} \right\}$$

$$= \sup_y \left\{ \langle \eta, y \rangle + (g(y)f^*)(\xi) \right\}$$

$$= \sup_y \left\{ \langle \eta, y \rangle + f^*(\xi)g(y) \right\}$$

$$= \sup_y \left\{ \langle \eta, y \rangle - (f^*(\xi)(-g))(y) \right\}$$

$$\left[ f^*(\xi) \geq 0 \text{ if } f(0) \leq 0 \right]$$
Sketch of a proof

\[(f \triangle g)^*(\xi, \eta) = \sup_{x,y} \left\{ \langle \xi, x \rangle + \langle \eta, y \rangle - g(y)f \left( \frac{x}{g(y)} \right) \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle + \sup_x \left\{ \langle \xi, x \rangle - (fg(y))(x) \right\} \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle + (g(y)f^*)(\xi) \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle + f^*(\xi)g(y) \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle - (f^*(\xi)(-g))(y) \right\} \]

\[= \left( (-g)^* f^*(\xi) \right)(\eta) \]
Sketch of a proof

\[(f \triangle g)^*(\xi, \eta) = \sup_{x,y} \left\{ \langle \xi, x \rangle + \langle \eta, y \rangle - g(y)f \left(\frac{x}{g(y)}\right) \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle + \sup_x \left\{ \langle \xi, x \rangle - (f g(y))(x) \right\} \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle + (g(y)f^*)(\xi) \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle + f^*(\xi)g(y) \right\} \]

\[= \sup_y \left\{ \langle \eta, y \rangle - (f^*(\xi)(-g))(y) \right\} \]

\[= (-g)^* f^*(\xi) (\eta) \]

\[= f^*(\xi)(-g)^* \left(\frac{\eta}{f^*(\xi)}\right) \]
Definition 1’
Definition 1’

- $f : \mathbb{R}^n \rightarrow [-\infty, \infty)$ proper concave with $f(0) \geq 0$
Definition 1’

- $f : \mathbb{R}^n \rightarrow [\,{-\infty, \infty})$ proper concave with $f(0) \geq 0$

- $g : \mathbb{R}^m \rightarrow \{\,{-\infty}\} \cup [0, \infty)$ proper concave
Definition 1’

- \( f : \mathbb{R}^n \rightarrow [\pm \infty) \) proper concave with \( f(0) \geq 0 \)

- \( g : \mathbb{R}^m \rightarrow \{-\infty\} \cup [0, \infty) \) proper concave

We say that the pair \((f, g)\) is of type I’
Definition 1’

- \( f : \mathbb{R}^n \rightarrow [-\infty, \infty) \) proper concave with \( f(0) \geq 0 \)
- \( g : \mathbb{R}^m \rightarrow \{-\infty\} \cup [0, \infty) \) proper concave

We say that the pair \((f, g)\) is of type I’ and

\[
(f \nabla g)(x, y) := \begin{cases} 
  g(y)f \left( \frac{x}{g(y)} \right) & \text{if } g(y) \in (0, \infty), \\
  f0^+(x) & \text{if } g(y) = 0, \\
  -\infty & \text{if } g(y) = -\infty.
\end{cases}
\]
Definition 2’

- $f : \mathbb{R}^n \rightarrow [-\infty, \infty)$ is proper concave with $f \leq f_0^+$
Definition 2’

- $f : \mathbb{R}^n \rightarrow [\neg \infty, \infty)$ is proper concave with $f \leq f_0^+$
- $g : \mathbb{R}^m \rightarrow [0, \infty]$ proper convex
Definition 2’

- $f : \mathbb{R}^n \to [-\infty, \infty)$ is proper concave with $f \leq f_0^+$
- $g : \mathbb{R}^m \to [0, \infty]$ proper convex

We say that the pair $(f, g)$ is of type II’
Definition 2’

- $f : \mathbb{R}^n \to [-\infty, \infty)$ is proper concave with $f \leq f^0^+ 
- g : \mathbb{R}^m \to [0, \infty]$ proper convex

We say that the pair $(f, g)$ is of type II’ and, if $f \neq f^0^+$, 

$$(f \nabla g)(x, y) := \begin{cases} 
g(y)f \left( \frac{x}{g(y)} \right) & \text{if } g(y) \in (0, \infty), 
\quad f^0^+(x) & \text{if } g(y) = 0, 
-\infty & \text{if } g(y) = \infty \end{cases}$$
Definition 2’

- \( f: \mathbb{R}^n \to [-\infty, \infty) \) is proper concave with \( f \leq f^0^+ \)
- \( g: \mathbb{R}^m \to [0, \infty] \) proper convex

We say that the pair \((f, g)\) is of type II’ and, if \( f \neq f^0^+ \),

\[
(f \nabla g)(x, y) := \begin{cases} 
  g(y) f \left( \frac{x}{g(y)} \right) & \text{if } g(y) \in (0, \infty), \\
  f^0^+(x) & \text{if } g(y) = 0, \\
  -\infty & \text{if } g(y) = \infty
\end{cases}
\]

and if \( f = f^0^+ \),

\[
(f \nabla g)(x, y) := \begin{cases} 
  f(x) & \text{if } y \in \text{cl dom } g, \\
  -\infty & \text{if } y \notin \text{cl dom } g.
\end{cases}
\]
Theorem
Theorem

(1) Assume that $(f, g)$ is of type I’, and that $f$ and $g$ are closed. Then $(g_\ast, -f_\ast)$ is of type II’, and

$$(f \triangledown g)_\ast(\xi, \eta) = (g_\ast \triangledown -f_\ast)(\eta, \xi),$$

$$(g_\ast \triangledown -f_\ast)_\ast(y, x) = (f \triangledown g)(x, y).$$
Theorem

(1) Assume that \((f, g)\) is of type I’, and that \(f\) and \(g\) are closed. Then \((g\star, -f\star)\) is of type II’, and

\[
(f \nabla g)_\star(\xi, \eta) = (g\star \nabla -f\star)(\eta, \xi),
\]

\[
(g\star \nabla -f\star)_\star(y, x) = (f \nabla g)(x, y).
\]

(2) Assume that \((f, g)\) is of type II’, and that \(f\) and \(g\) are closed. Then \(((g\star, f\star))\) is of type I’, and

\[
(f \nabla g)_\star(\xi, \eta) = ((-g)\star \nabla (f\star))(\eta, \xi),
\]

\[
((-g)\star \nabla (f\star))_\star(y, x) = (f \nabla g)(x, y).
\]
Outline

- Introduction
- Generalized perspectives
- Conjugacy
- Other properties
- Applications
Associativity

$(f, g)$ of type I
Associativity

\((f, g)\) of type I

\((f \bigtriangleup g)\) closed proper convex
Associativity

$(f, g)$ of type I

$(f \bigtriangleup g)$ closed proper convex

Additional assumption: $0 \in \text{dom } g$. Then:

$$(f \bigtriangleup g)(0, 0) \leq 0$$
(f, g) of type I

(f △ g) closed proper convex

Additional assumption: 0 ∈ dom g. Then:

(f △ g)(0, 0) ≤ 0

If h is closed proper concave nonnegative, we may form

(f △ g) △ h
Associativity

$(f, g)$ of type I

$(f \triangle g)$ closed proper convex

Additional assumption: $0 \in \text{dom } g$. Then:

$$(f \triangle g)(0, 0) \leq 0$$

If $h$ is closed proper concave nonnegative, we may form

$$(f \triangle g) \triangle h$$

Since $g(0) \geq 0$, $g \triangle h$ is closed proper concave.
Associativity

\[(f, g)\] of type I

\[(f \triangle g)\] closed proper convex

Additional assumption: \(0 \in \text{dom } g\). Then:

\[(f \triangle g)(0, 0) \leq 0\]

If \(h\) is closed proper concave nonnegative, we may form

\[ (f \triangle g) \triangle h \]

Since \(g(0) \geq 0\), \(g \nabla h\) is closed proper concave.

One can show that \(g \nabla h\) is actually nonnegative (on its domain), so we can also consider \(f \triangle (g \nabla h)\)
(f, g) of type I

(f △ g) closed proper convex

Additionnal assumption: 0 ∈ dom g. Then:

(f △ g)(0, 0) ≤ 0

If h is closed proper concave nonnegative, we may form

(f △ g) △ h

Since g(0) ≥ 0, g ▽ h is closed proper concave.

One can show that g ▽ h is actually nonnegative (on its domain), so we can also consider f △ (g ▽ h)

Both (f △ g) △ h and f △ (g ▽ h) are closed proper convex
**Theorem [PM, 2005].** With the above assumptions,

\[(f \triangle g) \triangle h = f \triangle (g \nabla h).\]
Theorem [PM, 2005]. With the above assumptions,

\[(f \triangle g) \triangle h = f \triangle (g \nabla h).\]

\[\left[(f \triangle g) \triangle h\right]((x, y), z) = h(z)(f \triangle g) \left(\frac{x}{h(z)}, \frac{y}{h(z)}\right)\]
Theorem [PM, 2005]. With the above assumptions,

\[(f \triangle g) \triangle h = f \triangle (g \nabla h)\].

\[[(f \triangle g) \triangle h]((x, y), z) = h(z)(f \triangle g)\left(\frac{x}{h(z)}, \frac{y}{h(z)}\right)\]
**Theorem** [PM, 2005]. With the above assumptions,

\[(f \triangle g) \triangle h = f \triangle (g \nabla h).\]

\[\left[(f \triangle g) \triangle h\right]((x, y), z) = h(z)(f \triangle g)\left(\frac{x}{h(z)}, \frac{y}{h(z)}\right)\]

\[= h(z)g\left(\frac{y}{h(z)}\right)f\left(\frac{x/h(z)}{g\left(\frac{y}{h(z)}\right)}\right)\]
**Theorem [PM, 2005].** With the above assumptions,

\[(f \triangle g) \triangle h = f \triangle (g \nabla h)\].

\[
[(f \triangle g) \triangle h]((x, y), z) = h(z)(f \triangle g) \left( \frac{x}{h(z)}, \frac{y}{h(z)} \right)
\]

\[
= h(z)g\left(\frac{y}{h(z)}\right)f\left(\frac{x}{h(z)}\right)
\]

\[
= h(z)g\left(\frac{y}{h(z)}\right)f\left(\frac{x}{h(z)g\left(\frac{y}{h(z)}\right)}\right)
\]
Theorem [PM, 2005]. With the above assumptions,

\[(f \triangle g) \triangle h = f \triangle (g \triangledown h).\]

\[
[(f \triangle g) \triangle h]((x, y), z) = h(z)(f \triangle g)\left(\frac{x}{h(z)}, \frac{y}{h(z)}\right)
\]

\[= h(z)g(y/h(z))f\left(\frac{x/h(z)}{g(y/h(z))}\right)\]

\[= h(z)g(y/h(z))f\left(\frac{x}{h(z)g(y/h(z))}\right)\]
Theorem [PM, 2005]. With the above assumptions,

\[(f \triangle g) \triangle h = f \triangle (g \nabla h).\]

\[
[(f \triangle g) \triangle h][(x, y), z] = h(z)(f \triangle g) \left( \frac{x}{h(z)}, \frac{y}{h(z)} \right)
\]

\[
= h(z)g\left(\frac{y}{h(z)}\right)f\left(\frac{x}{h(z)}\right)
\]

\[
= h(z)g\left(\frac{y}{h(z)}\right)f\left(\frac{x}{h(z)g\left(\frac{y}{h(z)}\right)}\right)
\]

\[
= (g \nabla h)(y, z)f\left(\frac{x}{(g \nabla h)(y, z)}\right)
\]
Theorem [PM, 2005]. With the above assumptions,

\[(f \triangle g) \triangle h = f \triangle (g \nabla h).\]

\[\left[(f \triangle g) \triangle h\right]((x, y), z) = h(z)(f \triangle g)\left(\frac{x}{h(z)}, \frac{y}{h(z)}\right)\]

\[= h(z)g\left(\frac{y}{h(z)}\right)f\left(\frac{x}{h(z)}\right)\]

\[= h(z)g\left(\frac{y}{h(z)}\right)f\left(\frac{x}{h(z)g\left(\frac{y}{h(z)}\right)}\right)\]

\[= (g \nabla h)(y, z)f\left(\frac{x}{(g \nabla h)(y, z)}\right)\]

\[= \left[f \triangle (g \nabla h)\right](x, (y, z))\]
Commutative laws
Commutative laws

Infimal convolution

\[(f_1 \vee f_2)(x) := \inf \{ f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x \}\]
Commutative laws

**Infimal convolution**

\[(f_1 \lor f_2)(x) := \inf \left\{ f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x \right\}\]

**Supremal convolution**

\[(g_1 \land g_2)(x) := \sup \left\{ g_1(x_1) + g_2(x_2) \mid x_1 + x_2 = x \right\}\]
Commutative laws

Infimal convolution

\[(f_1 \vee f_2)(x) := \inf \{ f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x \}\]

Supremal convolution

\[(g_1 \wedge g_2)(x) := \sup \{ g_1(x_1) + g_2(x_2) \mid x_1 + x_2 = x \}\]

\[-(g_1 \wedge g_2) = (-g_1) \vee (-g_2)\]
Commutative laws

**Infimal convolution**

\[(f_1 \vee f_2)(x) := \inf \left\{ f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x \right\}\]

**Supremal convolution**

\[(g_1 \wedge g_2)(x) := \sup \left\{ g_1(x_1) + g_2(x_2) \mid x_1 + x_2 = x \right\}\]

\[-(g_1 \wedge g_2) = (-g_1) \vee (-g_2)\]

Roughly speaking, addition and epigraphic sum are dual operations.
Theorem [PM, 2005]
Theorem [PM, 2005]

- \( f_1, f_2 : \mathbb{R}^n \rightarrow (-\infty, \infty] \) closed proper **convex** with
  \( f_1(0) \leq 0, f_2(0) \leq 0 \)
Theorem [PM, 2005]

- $f_1, f_2: \mathbb{R}^n \to (-\infty, \infty]$ closed proper convex with $f_1(0) \leq 0, f_2(0) \leq 0$

- $g: \mathbb{R}^m \to \{-\infty\} \cup [0, \infty)$ closed proper concave
Theorem [PM, 2005]

- \( f_1, f_2 : \mathbb{R}^n \rightarrow (-\infty, \infty] \) closed proper \textit{convex} with \( f_1(0) \leq 0, \ f_2(0) \leq 0 \)

- \( g : \mathbb{R}^m \rightarrow \{-\infty\} \cup [0, \infty) \) closed proper \textit{concave}

Then \((f_1 + f_2, g)\) is of type I, and

\[
(f_1 + f_2) \triangle g = (f_1 \triangle g) + (f_2 \triangle g).
\]
Theorem [PM, 2005]

- $f_1, f_2 : \mathbb{R}^n \rightarrow (-\infty, \infty]$ closed proper convex with $f_1 \geq f_1^0 +$, $f_2 \geq f_2^0 +$
Theorem [PM, 2005]

- \( f_1, f_2 : \mathbb{R}^n \rightarrow (-\infty, \infty] \) closed proper \textbf{convex} with \( f_1 \geq f_1^0 +, f_2 \geq f_2^0 + \)

- \( g : \mathbb{R}^m \rightarrow [0, \infty] \) closed proper \textbf{convex}
Theorem [PM, 2005]

- \( f_1, f_2 : \mathbb{R}^n \rightarrow (-\infty, \infty] \) closed proper convex with \( f_1 \geq f_1^{0+}, f_2 \geq f_2^{0+} \)

- \( g : \mathbb{R}^m \rightarrow [0, \infty] \) closed proper convex

- \( \text{dom } f_1 \cap \text{dom } f_2 \neq \emptyset \)
Theorem [PM, 2005]

- $f_1, f_2 : \mathbb{R}^n \to (-\infty, \infty]$ closed proper convex with $f_1 \geq f_1 0^+, f_2 \geq f_2 0^+$

- $g : \mathbb{R}^m \to [0, \infty]$ closed proper convex

- $\text{dom } f_1 \cap \text{dom } f_2 \neq \emptyset$

Then $(f_1 + f_2, g)$ is of type II and

$$(f_1 + f_2) \triangle g = (f_1 \triangle g) + (f_2 \triangle g).$$
Theorem [PM, 2005]
Theorem [PM, 2005]

- \( f : \mathbb{R}^n \rightarrow (-\infty, \infty] \) closed proper **convex** with \( f(0) \leq 0 \)
Theorem [PM, 2005]

- \( f : \mathbb{R}^n \to (-\infty, \infty] \) closed proper \textit{convex} with \( f(0) \leq 0 \)

- \( g_1, g_2 : \mathbb{R}^m \to \{-\infty\} \cup [0, \infty) \) closed proper \textit{concave}
Theorem [PM, 2005]

- \( f : \mathbb{R}^n \rightarrow (-\infty, \infty] \) closed proper \textit{convex} with \( f(0) \leq 0 \)

- \( g_1, g_2 : \mathbb{R}^m \rightarrow \{-\infty\} \cup [0, \infty) \) closed proper \textit{concave}

- \( \text{ri dom } g_1^* \cap \text{ri dom } g_2^* \neq \emptyset \)
Theorem [PM, 2005]

- \( f : \mathbb{R}^n \to (-\infty, \infty] \) closed proper convex with \( f(0) \leq 0 \)

- \( g_1, g_2 : \mathbb{R}^m \to \{-\infty\} \cup [0, \infty) \) closed proper concave

- \( \text{ri dom } g_1^\star \cap \text{ri dom } g_2^\star \neq \emptyset \)

Then \((f, g_1 \land g_2)\) is of type I and

\[
    f \triangledown (g_1 \land g_2) = (f \triangledown g_1) \lor (f \triangledown g_2).
\]
Theorem [PM, 2005]
Theorem [PM, 2005]

- $f: \mathbb{R}^n \rightarrow (-\infty, \infty]$ closed proper convex with $f \geq f^0+$
Theorem [PM, 2005]

- \( f : \mathbb{R}^n \rightarrow (-\infty, \infty] \) closed proper convex with \( f \geq f^0 + \)

- \( g_1, g_2 : \mathbb{R}^m \rightarrow [0, \infty] \) closed proper convex
Theorem [PM, 2005]

- \( f : \mathbb{R}^n \to (-\infty, \infty] \) closed proper convex with \( f \geq f_0^+ \)
- \( g_1, g_2 : \mathbb{R}^m \to [0, \infty] \) closed proper convex
- \( \text{ri dom } g_1^* \cap \text{ri dom } g_2^* \neq \emptyset \)
Theorem [PM, 2005]

- \( f : \mathbb{R}^n \rightarrow (-\infty, \infty] \) closed proper convex with \( f \geq f^0 \)
- \( g_1, g_2 : \mathbb{R}^m \rightarrow [0, \infty] \) closed proper convex
- \( \text{ri dom } g_1^* \cap \text{ri dom } g_2^* \neq \emptyset \)

If \( f \neq f^0 \), or if \( f = f^0 \) and the sets \( \text{ri dom } g_k^*0^+ \) have a point in common, then \( (f, g_1 \vee g_2) \) is of type II, and

\[
f \triangle (g_1 \vee g_2) = (f \triangle g_1) \vee (f \triangle g_2).
\]
Outline

- Introduction
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Minimizing condition numbers (with Jane Ye)

Minimize $\kappa(A)$

s.t. $A \in \Omega$
Minimizing condition numbers (with Jane Ye)

Minimize $\kappa(A)$

s.t. $A \in \Omega$

$\Omega$ compact convex subset of $S_n^+$
(not containing thenull matrix)
Minimizing condition numbers (with Jane Ye)

\[
\begin{align*}
(P) & \quad \text{Minimize } \kappa(A) \\
\text{s.t. } & \quad A \in \Omega \\
\Omega & \quad \text{compact convex subset of } S_n^+ \\
& \quad \text{(not containing the null matrix)} \\
\kappa(A) & \quad \text{condition number of } A
\end{align*}
\]
Minimizing condition numbers (with Jane Ye)

\( (P) \) | Minimize \( \kappa(A) \)
\[ \text{s.t. } A \in \Omega \]
\( \Omega \) compact convex subset of \( S_n^+ \)
(not containing the null matrix)

\( \kappa(A) \) condition number of \( A \)

\[ \kappa(A) = \begin{cases} 
\frac{\lambda_1(A)}{\lambda_n(A)} & \text{if } \lambda_n(A) > 0 \\
\infty & \text{if } \lambda_n(A) = 0 \text{ and } \lambda_1(A) > 0 \\
0 & \text{if } A = 0 
\end{cases} \]
Minimizing condition numbers (with Jane Ye)

\[
(\mathcal{P}) \quad \begin{align*}
\text{Minimize} & \quad \kappa(A) \\
\text{s.t.} & \quad A \in \Omega
\end{align*}
\]

\(\Omega\) compact convex subset of \(S_n^+\) (not containing the null matrix)

\(\kappa(A)\) condition number of \(A\)

\[
\kappa(A) = \begin{cases} 
\lambda_1(A)/\lambda_n(A) & \text{if } \lambda_n(A) > 0 \\
\infty & \text{if } \lambda_n(A) = 0 \text{ and } \lambda_1(A) > 0 \\
0 & \text{if } A = 0
\end{cases}
\]

\(\lambda_1(A) \geq \ldots \geq \lambda_n(A)\) eigenvalues of \(A\)
Properties of $\kappa$

- $\kappa$ is nonconvex
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- $\kappa$ is nonconvex
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- $\kappa$ is pseudoconvex [PM & Jane Ye, 2009]
Example: Markovitz model for portfolio selection
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\[(M) \quad \text{Minimize} \quad \langle x, Qx \rangle \\
\text{s.t.} \quad x \in \Delta_n, \quad \langle c, x \rangle \geq b\]
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\(Q\) is constrained to belong to some polytope \(P\)
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- $Q$ is a covariance matrix (to be inferred)
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\[ \begin{align*} \text{Minimize} & \quad \kappa(Q) \\ \text{s.t.} & \quad Q \in S_n^+ \cap P \end{align*} \]
An auxiliary function
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\[ \kappa(A) = \frac{\lambda_1(A)}{\lambda_n(A)}, \quad A \succ 0 \]
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\[ \kappa_p^p(A) := \begin{cases} 
\lambda_1^{p+1}(A)/\lambda_n^p(A) & \text{if } \lambda_n(A) > 0 \\
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**Theorem** [PM & Jane Ye, 2009]. $\kappa_p$ is convex on $S^+_n$. 
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Theorem [PM & Jane Ye, 2009]. $\kappa_p^p$ is convex on $S_n^+$. 

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$$\lambda_n(B)\lambda_1^{p+1}\left( \frac{A}{\lambda_n(B)} \right) = \frac{\lambda_1^{p+1}(A)}{\lambda_n(B)} \text{ convex}$$
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convex

Take restriction to $\{ A = B \}$
An auxiliary function

\[ \kappa_p(A) = \frac{\lambda_1^p(A)}{\lambda_n(A)} \]
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\( \kappa_p(A) \to \kappa(A) \) pointwise as \( p \to \infty \)
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\( \kappa_p \) and \( \kappa_p^p \) have the same minimizers!
An optimization strategy
An optimization strategy

Replace the original quasiconvex problem

\[ (P) \quad \text{Minimize} \quad \kappa(A) \]
\[ \text{s.t.} \quad A \in \Omega \]

by the surrogate convex problem

\[ (P_p) \quad \text{Minimize} \quad \kappa_p(A) \]
\[ \text{s.t.} \quad A \in \Omega \]

and let \( p \to \infty \).
An optimization strategy

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\[(\mathcal{P}) \quad \text{Minimize} \quad \kappa(A) \quad \text{s.t.} \quad A \in \Omega\]

by the surrogate convex problem

\[(\mathcal{P}_p) \quad \text{Minimize} \quad \kappa^p_p(A) \quad \text{s.t.} \quad A \in \Omega\]

and let \(p \to \infty\).

Question: can we approach a solution to Problem \((\mathcal{P})\) with solutions to the surrogate problems \((\mathcal{P}_p)\)?
Main results [PM & Jane Ye, 2009]
THEOREM 1 [Exact approximation]. Let \((p_k)_{k \in \mathbb{N}^*} \subset [1, \infty)\) be a sequence which tends to infinity, and, for every \(k \in \mathbb{N}^*\), let \(\bar{A}_{p_k}\) be a solution to problem \((P_{p_k})\). Then every cluster point \(\bar{A}\) of the sequence \((\bar{A}_{p_k})\) (there is at least one) is a global solution of problem \((P)\).
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**THEOREM 2 [Inexact approximation].** Let \(p_k \uparrow \infty\), and let \(\varepsilon_k \downarrow 0\). Let \(\bar{A}_k := \bar{A}^{\varepsilon_k}_{p_k}\) an \(\varepsilon_k\)-solution to Problem \((P_{p_k})\). Then every cluster point \(\bar{A}\) of the sequence \((\bar{A}_k)\) (there is at least one) is a global solution of Problem \((P)\).
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\[\forall A \in \Omega, \quad \kappa(A) \geq \kappa(\bar{A}_k) - \varepsilon_k\]
Back to multiplicative functions
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\[ \mathcal{K}_n(x, y) := \frac{x^{1+\alpha}}{y^\beta} = \frac{x^{1+\alpha}}{y_1^{\beta_1} \cdots y_n^{\beta_n}}, \quad x \geq 0, \quad y > 0 \]
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\[ K_n(x, y) := \frac{g(x)^{1+\alpha}}{y^\beta}, \quad x \in \mathbb{R}^n, \quad y > 0 \]

where \( g \) is any nonnegative convex function with \( g(0) = 0 \)
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**Theorem [PM, 2010].** The function \( K_n \) is convex.
The ultimate short proof (case $\alpha = |\beta|$)
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\[ f(y) := \frac{1}{y^\beta} = \frac{1}{y_1^{\beta_1} \cdots y_n^{\beta_n}}, \quad y > 0 \]
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f^0(y) = \delta(y|\mathbb{R}_+^n) \quad \text{thus} \quad f \geq f^0
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$(f, g)$ is of type II
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$f$ is log-convex, thus convex and nonnegative

$$f^0^+(y) = \delta(y|R^n_+) \quad \text{thus} \quad f \geq f^0^+$$

$(f, g)$ is of type II

$$(f \triangle g)(x, y) = \frac{g(x)^{1+|\beta|}}{y^{\beta}}$$
Some references


Thank you for your attention!