On inverse problems of Fourier synthesis

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Outline

- Introduction
- A reminder on ill-posed problems
- Fourier synthesis
- Asymptotic analysis
- A dual algorithm

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Fourier Synthesis

Recover a function from a partial and approximate knowledge of its Fourier transform.

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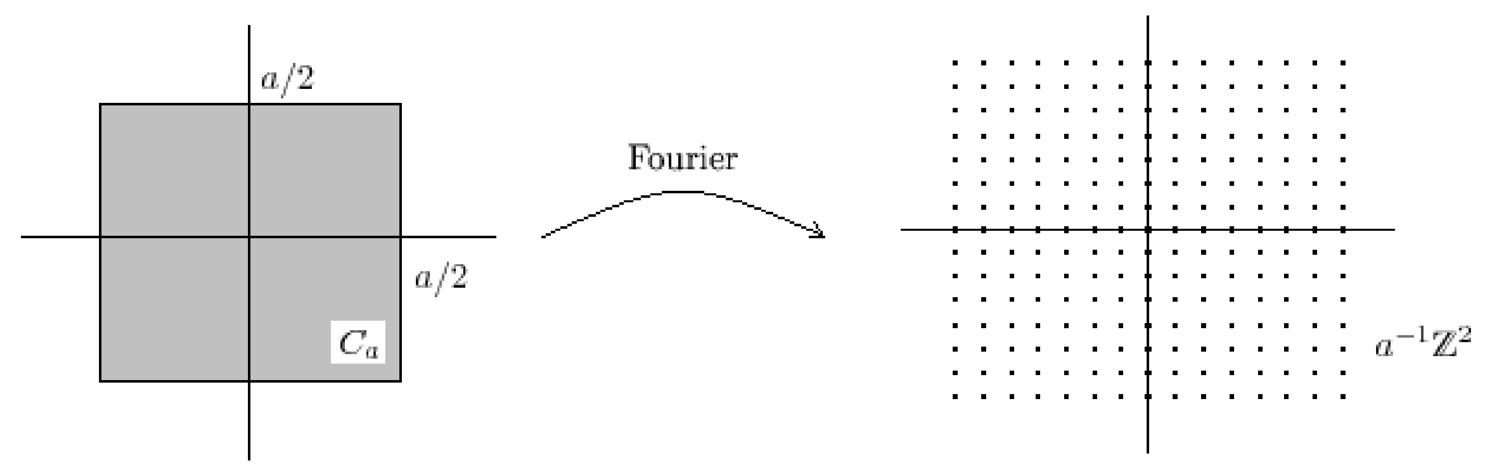

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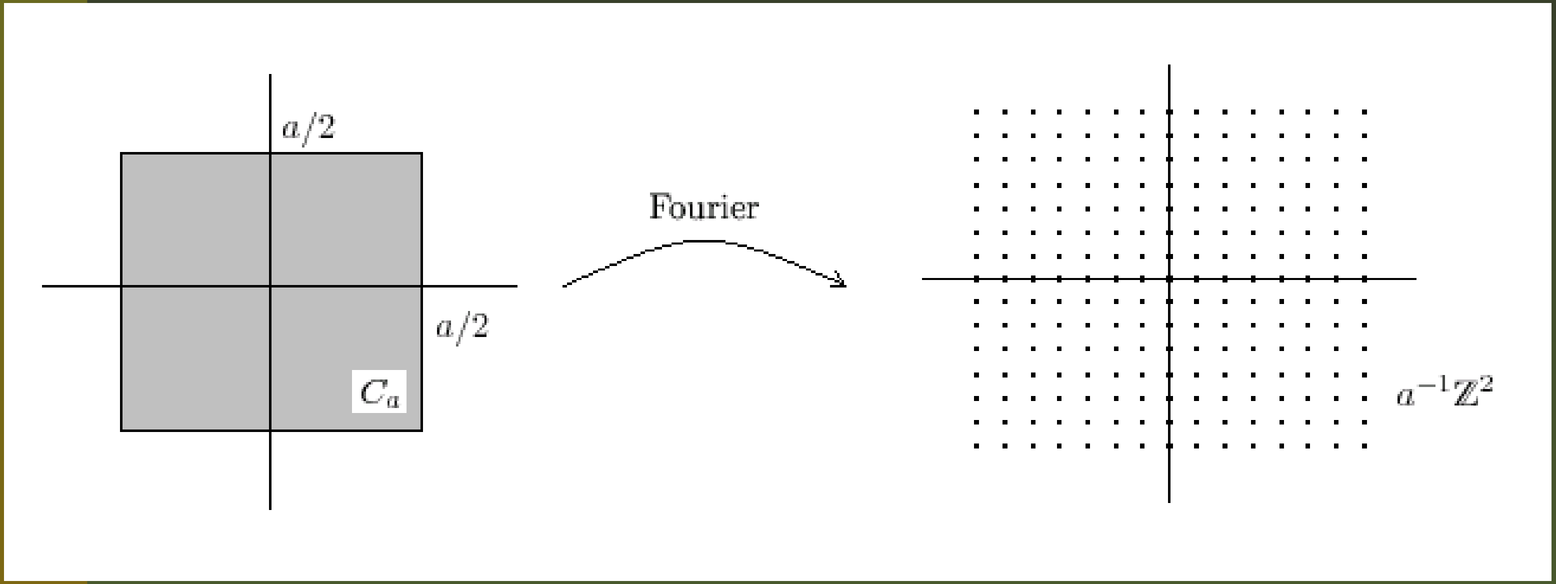

$$f(\mathbf{x}) = \frac{1}{a^n} \sum_{\mathbf{k} \in \mathbb{Z}^n} \hat{f}\left(\frac{\mathbf{k}}{a}\right) \exp\left[2i\pi \left\langle \frac{\mathbf{k}}{a}, \mathbf{x} \right\rangle\right] \mathbb{1}_{C_a}(\mathbf{x})$$

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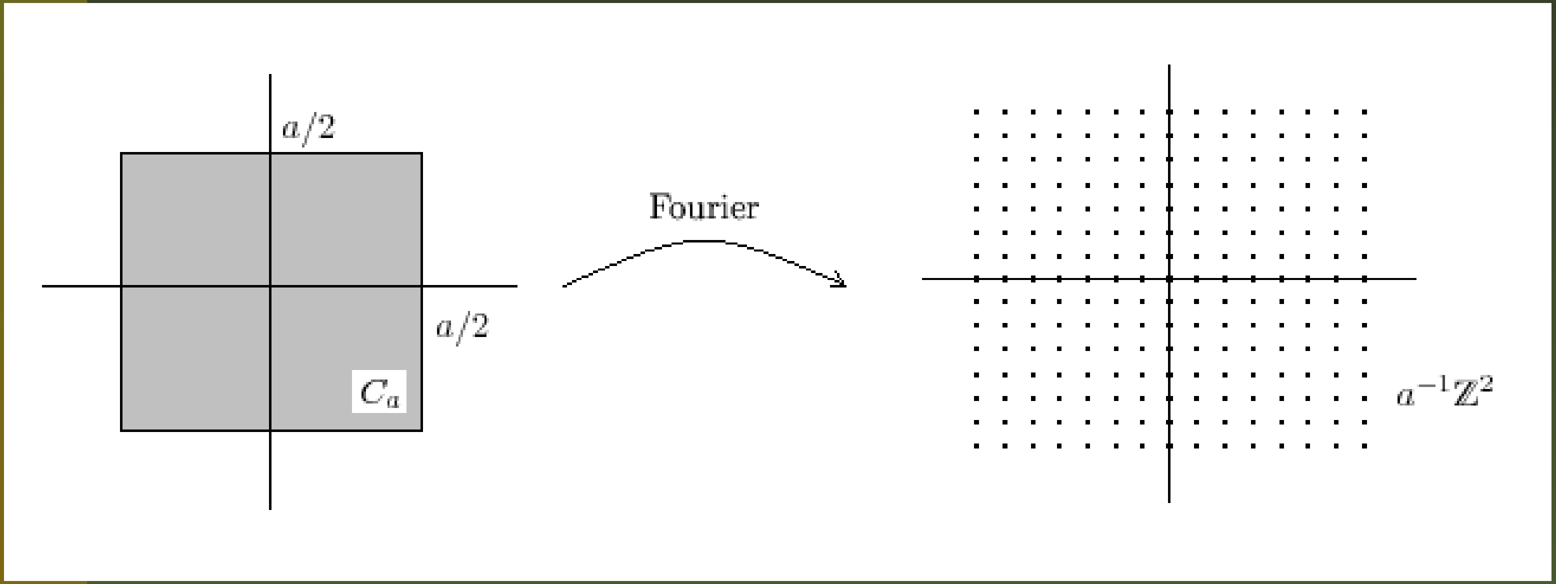
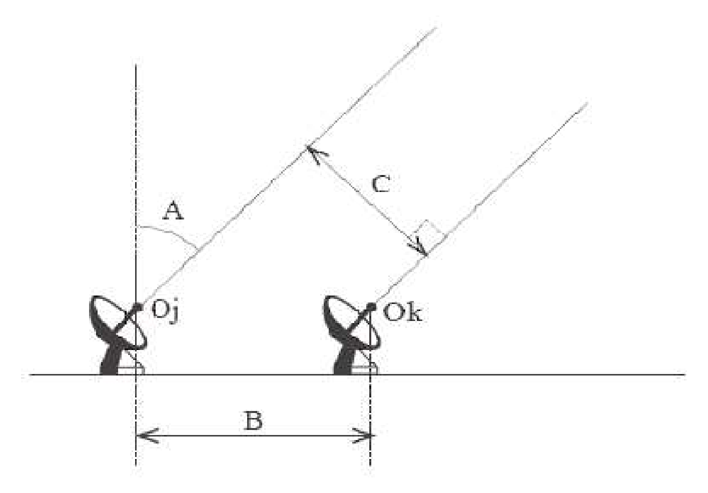
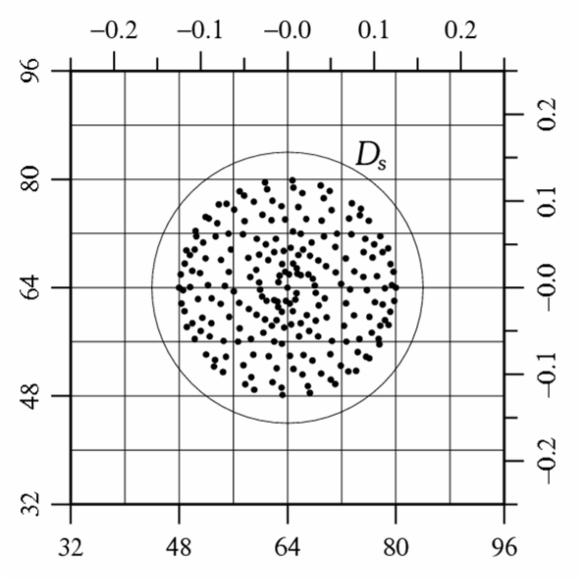


$$\hat{f}(\boldsymbol{\xi}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \hat{f}\left(\frac{\mathbf{k}}{a}\right) \operatorname{sinc} \pi a \left(\boldsymbol{\xi} - \frac{\mathbf{k}}{a}\right)$$

Example 2: Aperture synthesis





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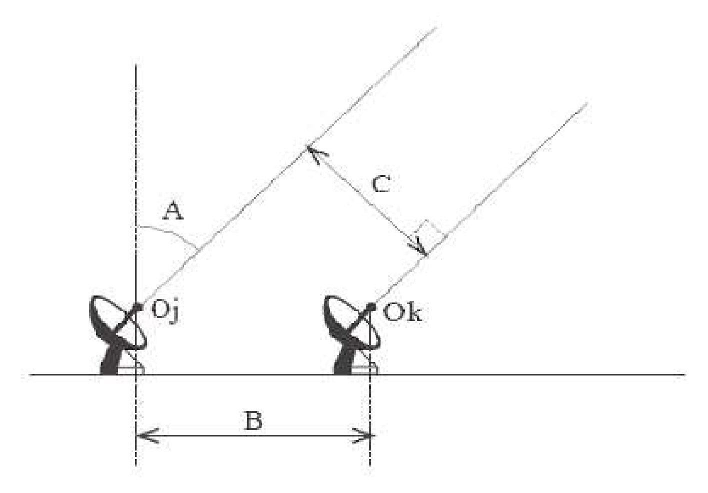
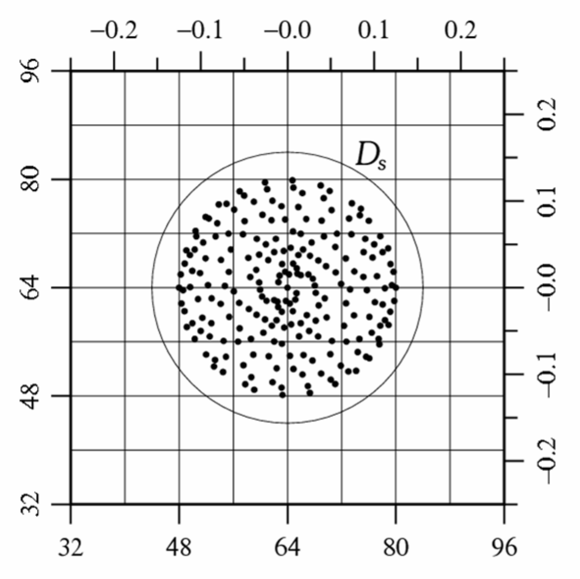
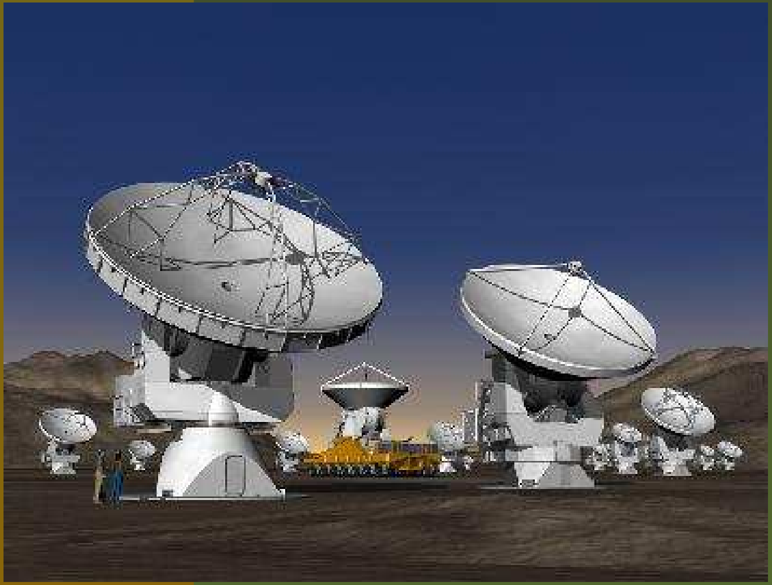
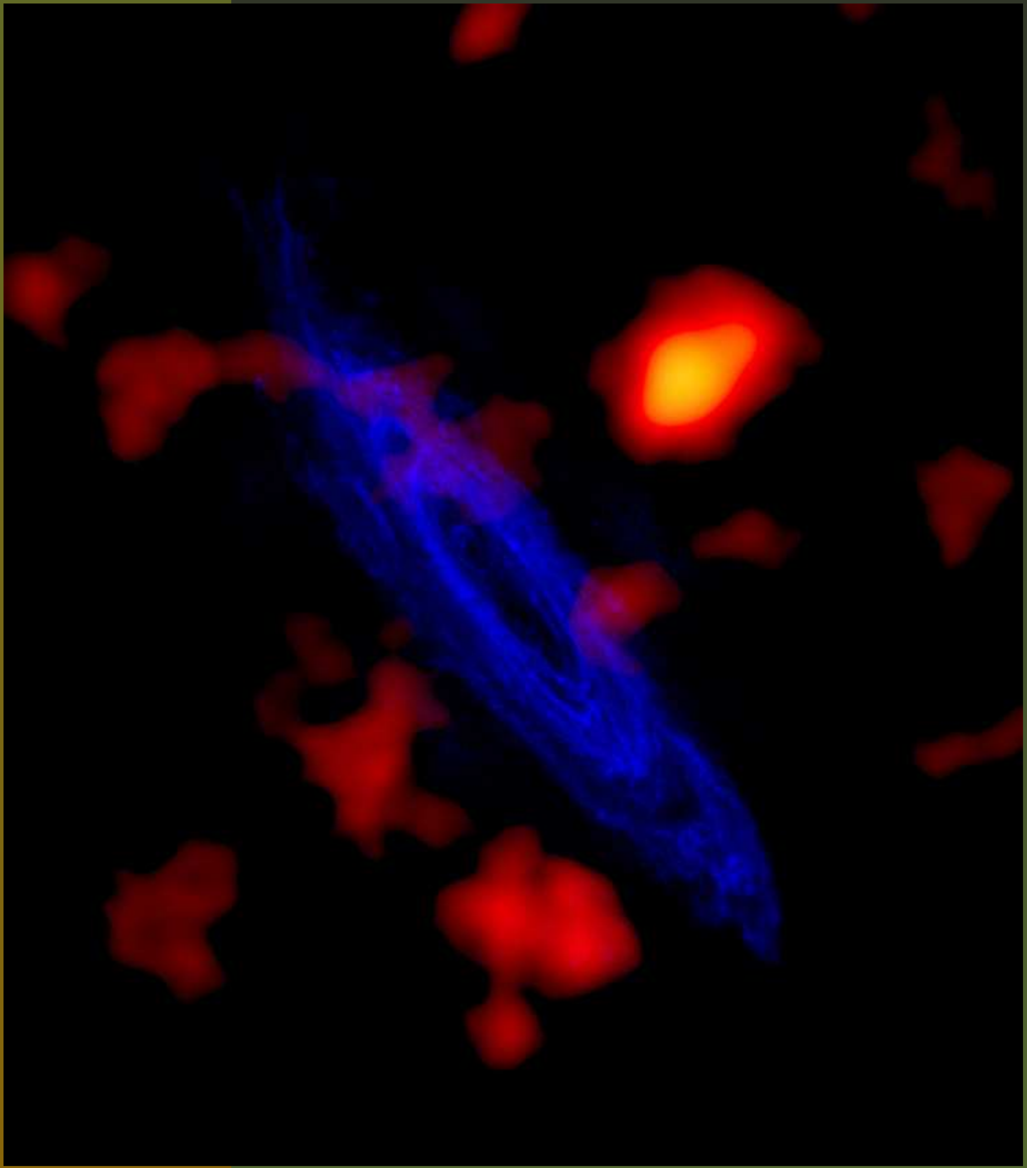






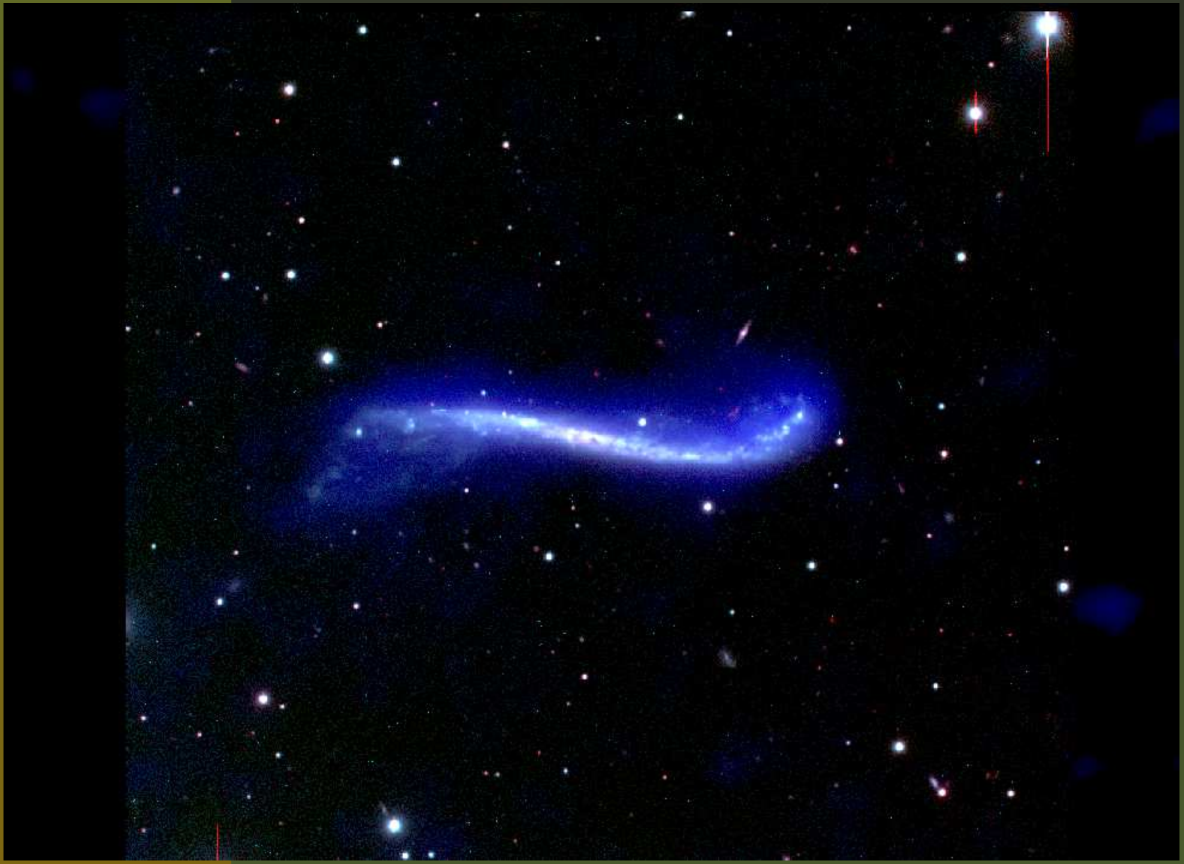
Andromeda Galaxy



WSRT

Courtesy of *National Radio Astronomy Observatory / Associated Universities, Inc. / National Science Foundation*

Integral Sign Galaxy ?



VLA

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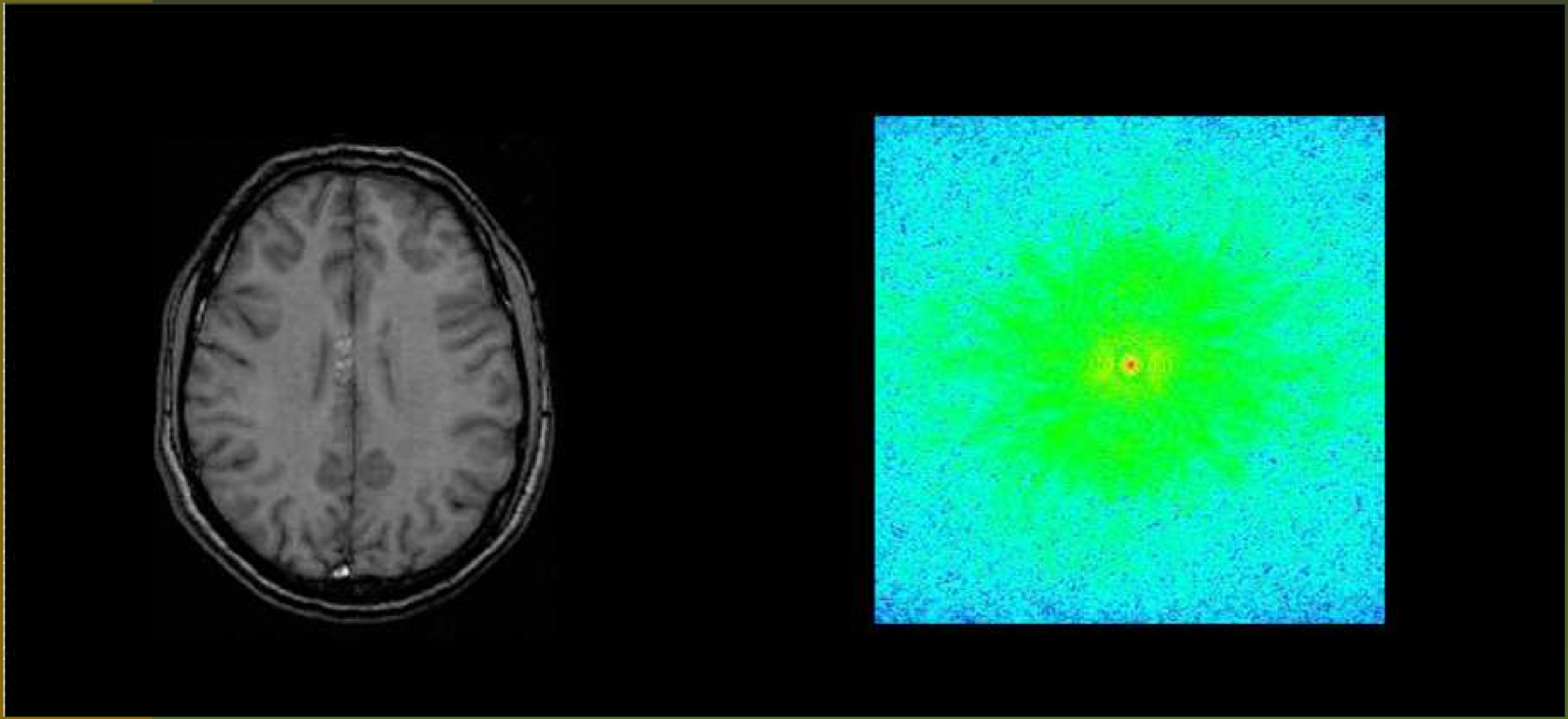
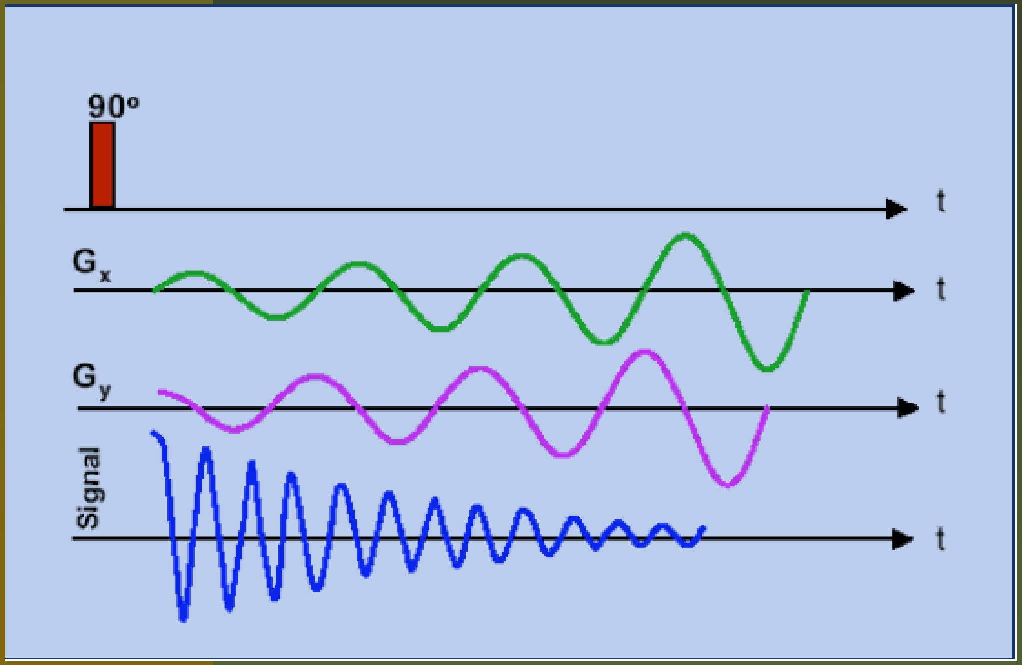
Example 3: MRI

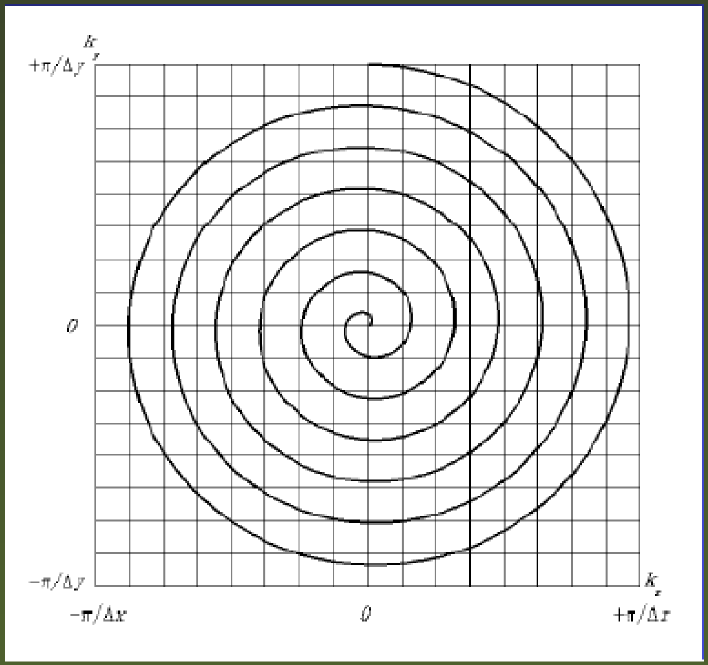
Standard acquisitions:



Example 3: MRI

Non-Cartesian and sparse acquisitions:





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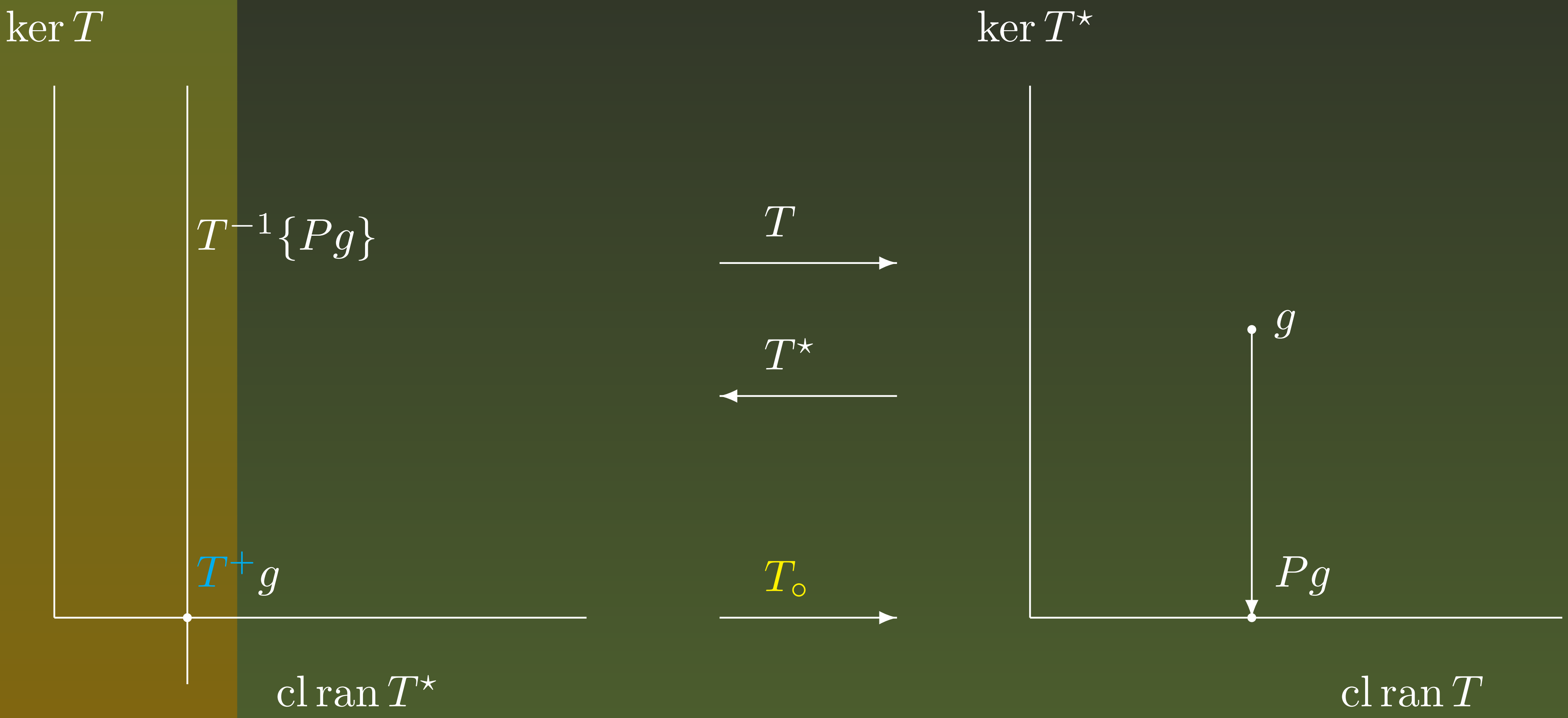
- Well-posedness in the *LS sense*: T^+ is continuous (with domain G)

Pseudo-inverse



$\ker T$

$\ker T^\star$

$T^{-1}\{Pg\}$

T

$T^\star$

g

$T^+ g$

$T_\circ$

Pg

$\operatorname{cl}\operatorname{ran} T^\star$

$\operatorname{cl}\operatorname{ran} T$

$T_\circ$: restriction of T to $(\ker T)^\perp = \operatorname{cl}\operatorname{ran} T^\star$

$$T^+ = T_\circ^{-1} \circ P$$

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(iii) $\operatorname{ran} T$ is not closed, so that

$$\mathcal{D}(T^+) = \operatorname{ran} T + (\operatorname{ran} T)^\perp \subsetneq G$$

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(iii) If $\operatorname{ran} T$ were closed, it would be a Banach space on its own, and the Open Mapping Theorem would imply continuity of T^{-1}. This would contradict Point (ii). ■

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Proposition For all $g \in \mathcal{D}(T^+)$,

$$T^+ g = \sum_{k\in\mathbb{N}^*} \frac{1}{\sqrt{\lambda_k}} \langle g, g_k \rangle f_k.$$

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Theorem The Tikhonov's solution f_α converges strongly to $f^+ := T^+ g$ as $\alpha \downarrow 0$.

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Truncated Fourier operator:

$$\begin{aligned} T_W : \quad L^2(V) &\longrightarrow L^2(W) \\ f &\longmapsto T_W f := \mathbb{1}_W \hat{f} = \mathbb{1}_W \mathbb{U} f. \end{aligned}$$

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$$(T_W f)(\xi) = \int_{\mathbb{R}^d} \underbrace{e^{-2i\pi\langle x,\xi\rangle}\mathbb{1}_V(x)\mathbb{1}_W(\xi)}_{\alpha(x,\xi)\in L^2(\mathbb{R}^d\times\mathbb{R}^d)} f(x)\,\mathrm{d}x.$$

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$\mathcal{D}(T_W^+)$ is a dense subset of $L^2(W)$

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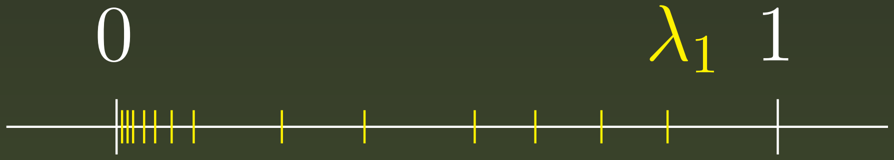

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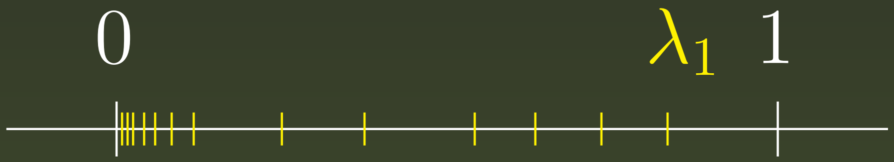

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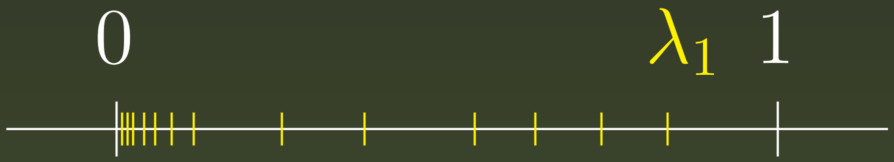

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$$\lambda_1^2 \;=\; \|T_W^\star T_W f_1\|^2_{L^2(V)}$$

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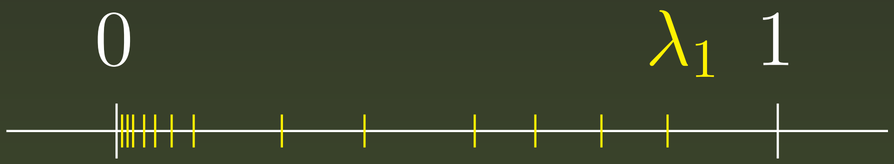

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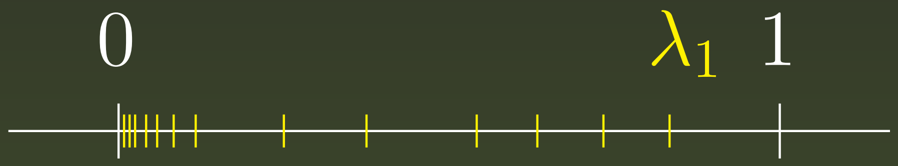

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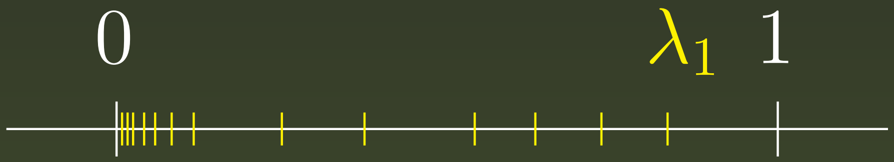

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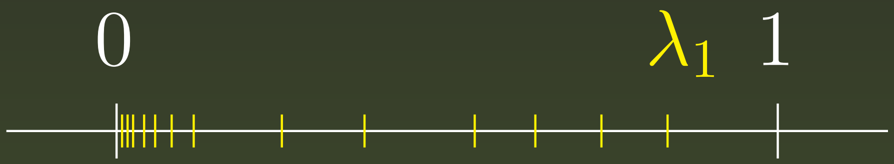

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&= \int_W \left| U f_1 \right|^2 \quad (\leq 1) \\
&< 1 \quad \text{(otherwise, } U f_1 \equiv 0 \text{ on } W^c\text{)} \quad \blacksquare
\end{aligned}$$

Fourier interpolation

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Proposition Assume that $\Omega \subseteq \mathbb{R}^d$ is such that Ω^c is bounded. Then,

(i) T_Ω is bounded and injective;

(ii) $\operatorname{ran} T_\Omega$ is closed;

(iii) $T_\Omega^{-1} \colon \operatorname{ran} T_\Omega \to L^2(V)$ is bounded.

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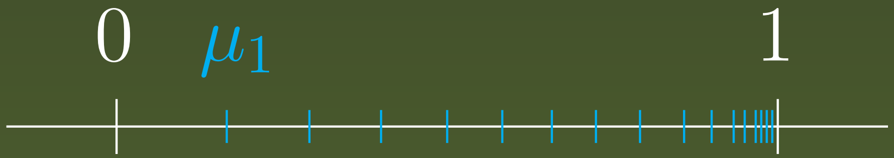

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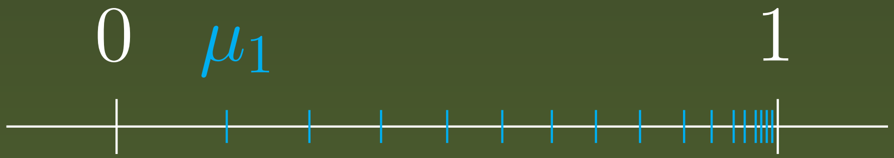

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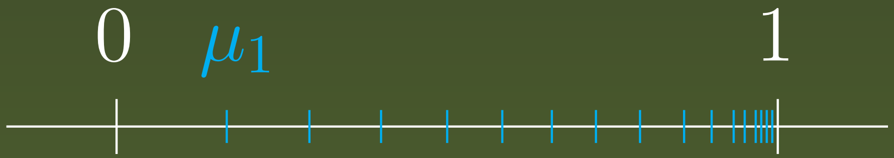

↪ $T^\star_\Omega T_\Omega \colon L^2(V) \to \operatorname{ran}(T^\star_\Omega T_\Omega)$ and $T_\Omega \colon L^2(V) \to \operatorname{ran} T_\Omega$ have continuous inverses, and $\operatorname{ran} T_\Omega$ is closed ■

Regularization

Regularization

Minimize $$\frac{1}{2}\,\|\,g - T_W f\,\|^2_{L^2(W)} + \frac{\alpha}{2}\,\|\,\mathbb{1}_{W_\beta} U f\,\|^2_{L^2(W_\beta)}$$

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energy of f in the *high frequency domain*

Apodized version

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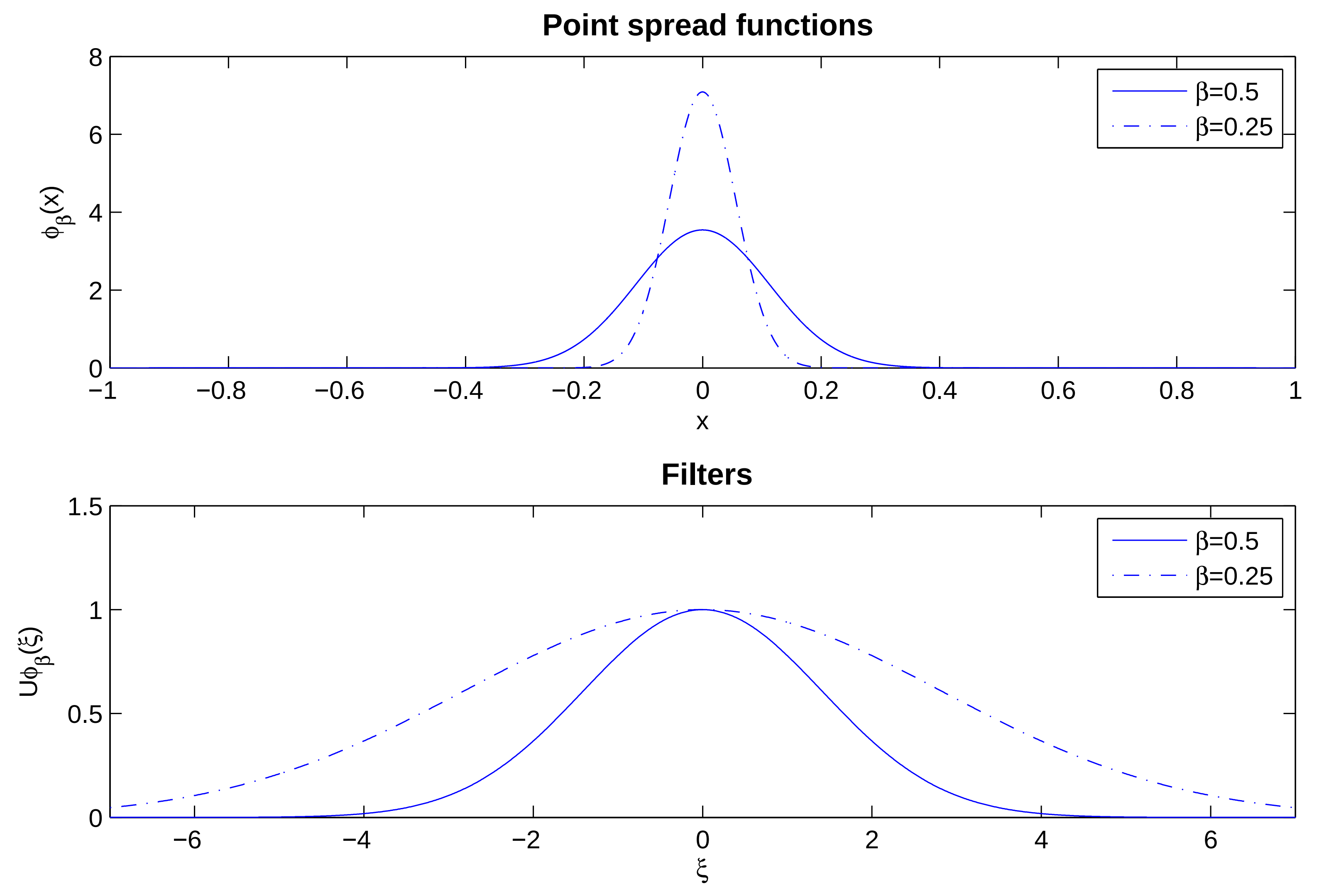
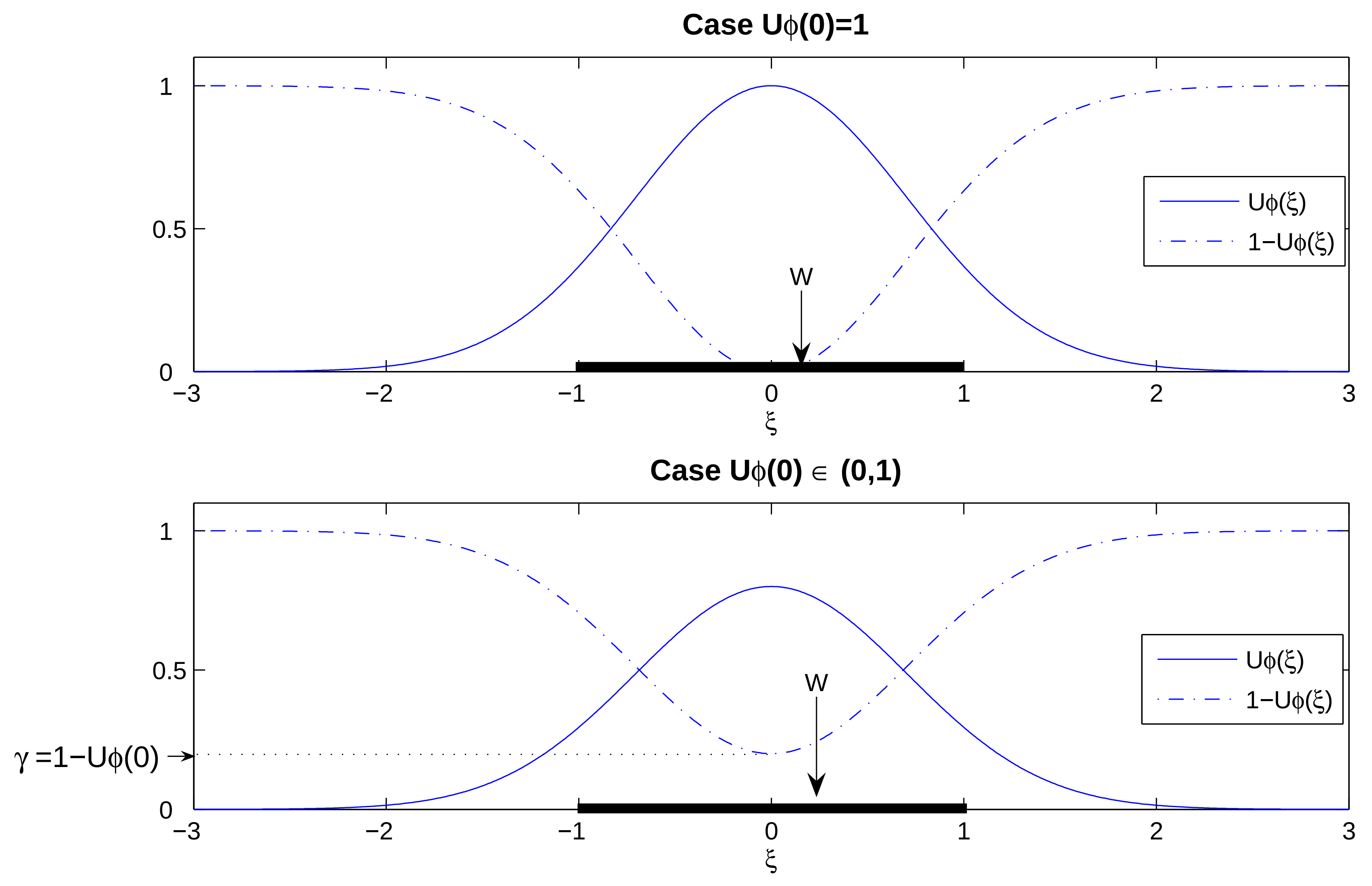
$$\phi_\beta(x) = \frac{1}{\beta^d}\phi\left(\frac{x}{\beta}\right) \qquad \hat{\phi}_\beta(\xi) = \hat{\phi}(\beta\xi)$$

Filters and associated PSF



Filters and their *complement*



General framework

$$\left|\begin{array}{rl} \text{Minimize} & \frac{1}{2}\left\| g - T_W f \right\|^2 + \alpha \mathcal{H}(f) \\ \text{s.t.} & f \in L^2(V) \end{array}\right.$$

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Main issues

Well-posedness

Asymptotic behavior (α and/or β tend to zero)

Well-posedness

$$(\mathcal{P}_{\alpha,\beta}) \quad \left|\begin{array}{rl} \text{Minimize} & \dfrac{1}{2}\left\|\hat{\phi}_\beta g - T_W f\right\|^2 + \dfrac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\hat{f}\right\|^2 \\ \text{s.t.} & f \in L^2(V) \end{array}\right.$$

Well-posedness

Definition $\langle f_1, f_2 \rangle_\beta := \int_{\mathbb{R}^d} |1 - \hat{\phi}_\beta|^2 U f_1 \overline{U f_2}$

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Lemma $\langle \cdot, \cdot \rangle_\beta$ is an inner product which turns $L^2(V)$ into a Hilbert space. The corresponding norm $\| \cdot \|_\beta$ is equivalent to $\| \cdot \|_{L^2(V)}$.

Proposition Let $\alpha, \beta > 0$ be fixed. Then $(\mathcal{P}_{\alpha,\beta})$ has a unique solution $f_{\alpha,\beta}$, which depends continuously on $g \in L^2(W)$.

$$(\mathcal{P}_{\alpha,\beta}) \quad \left| \begin{array}{ll} \text{Minimize} & \frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_\beta) \hat{f} \right\|^2 \\ \text{s.t.} & f \in L^2(V) \end{array} \right.$$

Proof

$(\mathcal{P}_{\alpha,\beta})$ can be rewritten as:

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The conclusion then follows from Tikhonov’s theory and the continuity of the multiplication $g \mapsto \hat{\phi}_\beta g$. ■

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Theorem Let $\beta > 0$ be fixed and let $g \in \mathcal{D}(T_W^+)$.

(i) If $\hat{\phi}_\beta g \in \mathcal{D}(T_W^+)$, then $f_{\alpha,\beta} \longrightarrow T_W^+(\hat{\phi}_\beta g)$ as $\alpha \downarrow 0$.

(ii) If $\hat{\phi}_\beta g \notin \mathcal{D}(T_W^+)$, then $\| f_{\alpha,\beta} \|_{L^2(V)} \longrightarrow \infty$ as $\alpha \downarrow 0$.

Tikhonov-like regularization $(\alpha \downarrow 0)$

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Unique solution: $T_W^+(\hat{\phi}_\beta g)$

Proposition Assume that $\phi \in L^1(\mathbb{R}^d)$ is such that $\hat{\phi}$ is analytic, and let $\beta > 0$ be fixed and $g \in \mathcal{D}(T_W^+)$. Then, the following are equivalent:

(i) $\hat{\phi}_\beta g \in \mathcal{D}(T_W^+)$;

(ii) $\operatorname{supp}(\phi_\beta * T_W^+ g) \subseteq V$.

Outline

- Introduction
- A reminder on ill-posed problems
- Fourier synthesis
- Asymptotic analysis (joint work with N. Alibaud and Y. Saesor)
- A dual algorithm

Main Result

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If $g \in T_W(L^2(V) \cap H^s(\mathbb{R}^d))$, then $f_{\alpha,\beta} \to T_W^+ g$ strongly as $\beta \downarrow 0$.

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$$\left.\begin{array}{r} (f_n)\ \text{bounded} \\ \displaystyle\lim_{R\to\infty}\sup_n \int_{\|x\|>R} |f_n(x)|^2\, \mathrm{d}x = 0 \\ \displaystyle\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0 \text{ as } \|h\| \to 0 \end{array}\right\} \Rightarrow (f_n)\ \text{precompact}$$

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$\tilde{g} := UT_W^+ g$ (analytic extension of g)

Step 1

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$$g \in T_W\left(L^2(V) \cap H^s(\mathbb{R}^d)\right) \implies \int_{\mathbb{R}^d} \left\|\xi\right\|^{2s} \left|\tilde{g}(\xi)\right|^2 \mathrm{d}\xi < \infty$$

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$$T_W T_W^+ g = g \quad \text{and} \quad U T_W^+ g = \tilde{g}$$

$$\left\|\hat{\phi}_\beta g - T_W f_{\alpha,\beta}\right\|^2 \geq 0$$

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$$\hat{\phi}_\beta(\xi)=\hat{\phi}(\beta\xi)$$

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$$\left\|(1-\hat{\phi}_\beta)\hat{f}_{\alpha,\beta}\right\|^2 \le \frac{1+\alpha}{\alpha}\left\|(1-\hat{\phi}_\beta)\tilde{g}\right\|^2$$

$$\int_{\mathbb{R}^d}\left|1-\hat{\phi}(\beta\xi)\right|^2\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \le \frac{1+\alpha}{\alpha}\int_{\mathbb{R}^d}\left|1-\hat{\phi}(\beta\xi)\right|^2\left|\tilde{g}(\xi)\right|^2 \mathrm{d}\xi$$

Step 1 $\big(f_{\alpha,\beta}\big)_{\beta\in(0,1]}$ is bounded

$$\Big\|(1-\hat\phi_\beta)\hat f_{\alpha,\beta}\Big\|^2 \le \frac{1+\alpha}{\alpha}\Big\|(1-\hat\phi_\beta)\tilde g\Big\|^2$$

$$\int_{\mathbb{R}^d}\Big|1-\hat\phi(\beta\xi)\Big|^2\Big|\hat f_{\alpha,\beta}(\xi)\Big|^2\,\mathrm{d}\xi \le \frac{1+\alpha}{\alpha}\int_{\mathbb{R}^d}\Big|1-\hat\phi(\beta\xi)\Big|^2\Big|\tilde g(\xi)\Big|^2\,\mathrm{d}\xi$$

$$m_\beta\int_{\mathbb{R}^d}\frac{|1-\hat\phi(\beta\xi)|^2}{|1-\hat\phi(\beta\xi/\|\xi\|)|^2}\Big|\hat f_{\alpha,\beta}(\xi)\Big|^2\,\mathrm{d}\xi$$

$$\le\frac{1+\alpha}{\alpha}M_\beta\int_{\mathbb{R}^d}\frac{|1-\hat\phi(\beta\xi)|^2}{|1-\hat\phi(\beta\xi/\|\xi\|)|^2}\Big|\tilde g(\xi)\Big|^2\,\mathrm{d}\xi$$

$m_\beta := \min\limits_{\|\xi\|=1}\Big|1-\hat\phi(\beta\xi)\Big|^2 > 0$ and $M_\beta := \max\limits_{\|\xi\|=1}\Big|1-\hat\phi(\beta\xi)\Big|^2 > 0$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\left\|(1-\hat{\phi}_\beta)\hat{f}_{\alpha,\beta}\right\|^2 \leq \frac{1+\alpha}{\alpha}\left\|(1-\hat{\phi}_\beta)\tilde{g}\right\|^2$$

$$\int_{\mathbb{R}^d}\left|1-\hat{\phi}(\beta\xi)\right|^2\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2\mathrm{d}\xi \leq \frac{1+\alpha}{\alpha}\int_{\mathbb{R}^d}\left|1-\hat{\phi}(\beta\xi)\right|^2\left|\tilde{g}(\xi)\right|^2\mathrm{d}\xi$$

$$\int_{\mathbb{R}^d}\frac{|1-\hat{\phi}(\beta\xi)|^2}{|1-\hat{\phi}(\beta\xi/\|\xi\|)|^2}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2\mathrm{d}\xi$$

$$\leq \frac{1+\alpha}{\alpha}\frac{M_\beta}{m_\beta}\int_{\mathbb{R}^d}\frac{|1-\hat{\phi}(\beta\xi)|^2}{|1-\hat{\phi}(\beta\xi/\|\xi\|)|^2}\left|\tilde{g}(\xi)\right|^2\mathrm{d}\xi$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

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$$\int_{\mathbb{R}^d} \frac{|1-\hat{\phi}(\beta\xi)|^2}{|1-\hat{\phi}(\beta\xi/\|\xi\|)|^2} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

$$\leq \frac{1+\alpha}{\alpha}\,\frac{M_\beta}{m_\beta} \int_{\mathbb{R}^d} \frac{|1-\hat{\phi}(\beta\xi)|^2}{|1-\hat{\phi}(\beta\xi/\|\xi\|)|^2} \left|\tilde{g}(\xi)\right|^2 \mathrm{d}\xi$$

Lemma $\exists\, \nu_0, C_0 > 0$ such that $\forall \beta \in (0,1]$, $\forall \xi \in \mathbb{R}^d \setminus \{0\}$,

$$\nu_0 \left(\|\xi\|^{2s} \mathbb{1}_{B_{1/\beta}}(\xi) + \frac{1}{M_\beta} \mathbb{1}_{B^c_{1/\beta}}(\xi) \right) \leq \frac{|1-\hat{\phi}(\beta\xi)|^2}{|1-\hat{\phi}(\beta\xi/|\xi|)|^2} \leq C_0 \|\xi\|^{2s}$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\int_{\mathbb{R}^d} \frac{|1-\hat{\phi}(\beta\xi)|^2}{|1-\hat{\phi}(\beta\xi/\|\xi\|)|^2}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

$$\leq \frac{1+\alpha}{\alpha}\,\frac{M_\beta}{m_\beta}\int_{\mathbb{R}^d} \frac{|1-\hat{\phi}(\beta\xi)|^2}{|1-\hat{\phi}(\beta\xi/\|\xi\|)|^2}\left|\tilde{g}(\xi)\right|^2 \mathrm{d}\xi$$

Lemma $\exists\,\nu_0, C_0 > 0$ such that $\forall\beta \in (0,1]$, $\forall\xi \in \mathbb{R}^d \setminus \{0\}$,

$$\nu_0\left(\|\xi\|^{2s}\mathbb{1}_{B_{1/\beta}}(\xi) + \frac{1}{M_\beta}\mathbb{1}_{B^c_{1/\beta}}(\xi)\right) \leq \frac{|1-\hat{\phi}(\beta\xi)|^2}{|1-\hat{\phi}(\beta\xi/|\xi|)|^2} \leq C_0\|\xi\|^{2s}$$

$$\nu_0\int_{\|\xi\|\leq 1/\beta}\|\xi\|^{2s}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi + \frac{\nu_0}{M_\beta}\int_{\|\xi\|>1/\beta}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

$$\leq \frac{1+\alpha}{\alpha}\,\frac{M_\beta}{m_\beta}C_0\int_{\mathbb{R}^d}\|\xi\|^{2s}\left|\tilde{g}(\xi)\right|^2 \mathrm{d}\xi$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\nu_0 \int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi + \frac{\nu_0}{M_\beta} \int_{\|\xi\|>1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

$$\leq \frac{1+\alpha}{\alpha} \frac{M_\beta}{m_\beta} C_0 \int_{\mathbb{R}^d} \|\xi\|^{2s} \left|\tilde{g}(\xi)\right|^2 \mathrm{d}\xi$$

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$$\leq \frac{1+\alpha}{\alpha} \frac{M_\beta}{m_\beta} C_0 \underbrace{\int_{\mathbb{R}^d} \|\xi\|^{2s} \left|\tilde{g}(\xi)\right|^2 \mathrm{d}\xi}_{\leq C_1}$$

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$$\leq \frac{1+\alpha}{\alpha} \frac{M_\beta}{m_\beta} C_0 \underbrace{\int_{\mathbb{R}^d} \|\xi\|^{2s} \left|\tilde{g}(\xi)\right|^2 \mathrm{d}\xi}_{\leq C_1}$$

$$\left.\begin{array}{l} \dfrac{M_\beta}{m_\beta} \to 1 \quad \text{as} \quad \beta \to 0 \ \text{ and } \ \beta \to \infty \\ \qquad\qquad \dfrac{M_\beta}{m_\beta} \quad \text{is continuous} \end{array}\right\} \Longrightarrow \frac{M_\beta}{m_\beta} \quad \text{is bounded}$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\nu_0 \int_{\|\xi\|\le 1/\beta} \|\xi\|^{2s} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi + \frac{\nu_0}{M_\beta} \int_{\|\xi\|>1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$
$$\le \frac{1+\alpha}{\alpha} \frac{M_\beta}{m_\beta} C_0 \underbrace{\int_{\mathbb{R}^d} \|\xi\|^{2s} \left|\tilde{g}(\xi)\right|^2 \mathrm{d}\xi}_{\le C_1}$$

$$\left.\begin{array}{l} \dfrac{M_\beta}{m_\beta} \to 1 \quad \text{as} \quad \beta \to 0 \ \text{and} \ \beta \to \infty \\ \dfrac{M_\beta}{m_\beta} \quad \text{is continuous} \end{array}\right\} \Longrightarrow \frac{M_\beta}{m_\beta} \quad \text{is bounded}$$

$$\nu_0 \int_{\|\xi\|\le 1/\beta} \|\xi\|^{2s} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi + \frac{\nu_0}{M_\beta} \int_{\|\xi\|>1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$
$$\le \frac{1+\alpha}{\alpha} C_2 C_0 C_1 = K$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi + \frac{1}{M_\beta}\int_{\|\xi\|>1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq \frac{K}{\nu_0}$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\underbrace{\int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi}_{} + \frac{1}{M_\beta}\int_{\|\xi\|>1/\beta}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq \frac{K}{\nu_0}$$

$$\geq \int_{1\leq\|\xi\|\leq 1/\beta} \|\xi\|^{2s}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\underbrace{\int_{\|\xi\|\le 1/\beta} \|\xi\|^{2s} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi}_{} + \frac{1}{M_\beta} \int_{\|\xi\|>1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \le \frac{K}{\nu_0}$$

$$\ge \int_{1\le\|\xi\|\le 1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

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$$\underbrace{\int_{\|\xi\|\le 1/\beta} \|\xi\|^{2s} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi}_{\ge \int_{1\le\|\xi\|\le 1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi} + \frac{1}{M_\beta}\int_{\|\xi\|>1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \le \frac{K}{\nu_0}$$

$$M_\beta := \max_{\|\xi\|=1} \left|1-\hat{\phi}(\beta\xi)\right|^2$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\underbrace{\int_{\|\xi\|\le 1/\beta}\|\xi\|^{2s}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2\,\mathrm{d}\xi}_{} + \frac{1}{M_\beta}\int_{\|\xi\|>1/\beta}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2\,\mathrm{d}\xi \le \frac{K}{\nu_0}$$

$$\ge \int_{1\le\|\xi\|\le 1/\beta}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2\,\mathrm{d}\xi$$

$$M_\beta := \max_{\|\xi\|=1}\left|1-\hat{\phi}(\beta\xi)\right|^2$$

$$\le \left(1+\max_{\|\xi\|=1}\left|\hat{\phi}(\beta\xi)\right|\right)^2$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\underbrace{\int_{\|\xi\|\le 1/\beta} \|\xi\|^{2s} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi}_{\ge \int_{1\le\|\xi\|\le 1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi} + \frac{1}{M_\beta}\int_{\|\xi\|>1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \le \frac{K}{\nu_0}$$

$$\begin{aligned} M_\beta &:= \max_{\|\xi\|=1} \left|1-\hat{\phi}(\beta\xi)\right|^2 \\ &\le \left(1+\max_{\|\xi\|=1}\left|\hat{\phi}(\beta\xi)\right|\right)^2 \\ &\le \left(1+\|\phi\|_1\right)^2 \end{aligned}$$

Step 1

$\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\underbrace{\int_{\|\xi\|\leq 1/\beta}\|\xi\|^{2s}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2\,\mathrm{d}\xi}_{\geq \int_{1\leq\|\xi\|\leq 1/\beta}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2\,\mathrm{d}\xi} + \frac{1}{M_\beta}\int_{\|\xi\|>1/\beta}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2\,\mathrm{d}\xi \leq \frac{K}{\nu_0}$$

$$\begin{aligned} M_\beta := \max_{\|\xi\|=1}\left|1-\hat{\phi}(\beta\xi)\right|^2 &\leq \left(1+\max_{\|\xi\|=1}\left|\hat{\phi}(\beta\xi)\right|\right)^2 \\ &\leq \left(1+\|\phi\|_1\right)^2 \end{aligned}$$

$$\int_{1\leq\|\xi\|\leq 1/\beta}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2\,\mathrm{d}\xi$$

$$+\frac{1}{\left(1+\|\phi\|_1\right)^2}\int_{\|\xi\|>1/\beta}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2\,\mathrm{d}\xi \leq \frac{K}{\nu_0}$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\underbrace{\int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi}_{} + \frac{1}{M_\beta}\int_{\|\xi\|>1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq \frac{K}{\nu_0}$$

$$\geq \int_{1\leq\|\xi\|\leq 1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

$$\begin{aligned} M_\beta &:= \max_{\|\xi\|=1} \left|1-\hat{\phi}(\beta\xi)\right|^2 \\ &\leq \left(1+\max_{\|\xi\|=1}\left|\hat{\phi}(\beta\xi)\right|\right)^2 \\ &\leq \left(1+\|\phi\|_1\right)^2 \end{aligned}$$

$$\frac{1}{\left(1+\|\phi\|_1\right)^2}\int_{1\leq\|\xi\|\leq 1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

$$+\frac{1}{\left(1+\|\phi\|_1\right)^2}\int_{\|\xi\|>1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq \frac{K}{\nu_0}$$

Step 1

$\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\underbrace{\int_{\|\xi\|\le 1/\beta} \|\xi\|^{2s} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi}_{} + \frac{1}{M_\beta} \int_{\|\xi\|>1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \le \frac{K}{\nu_0}$$

$$\ge \int_{1\le\|\xi\|\le 1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

$$\begin{aligned} M_\beta &:= \max_{\|\xi\|=1} \left|1-\hat{\phi}(\beta\xi)\right|^2 \\ &\le \left(1+\max_{\|\xi\|=1}\left|\hat{\phi}(\beta\xi)\right|\right)^2 \\ &\le \left(1+\|\phi\|_1\right)^2 \end{aligned}$$

$$\int_{\|\xi\|\ge 1} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \le \frac{K}{\nu_0}\left(1+\|\phi\|_1\right)^2 =: K_0$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\int_{\|\xi\|\geq 1} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq K_0$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\int_{\|\xi\|\geq 1} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \leq K_0$$

$$\int_{\mathbb{R}^d} \left| \mathbb{1}_{B_1^c} \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \leq K_0$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\int_{\|\xi\|\geq 1}\left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq K_0$$

$$\int_{\mathbb{R}^d}\left|\mathbb{1}_{B_1^c}\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq K_0$$

$$\int_{\mathbb{R}^d}\left|T_{B_1^c}f_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq K_0$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\int_{\|\xi\|\geq 1} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq K_0$$

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$$\int_{\mathbb{R}^d} \left|T_{B_1^c}f_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq K_0$$

$$\left\| f_{\alpha,\beta} \right\|^2 = \left\| T_{B_1^c}^{-1} T_{B_1^c} f_{\alpha,\beta} \right\|^2 \leq \left\| T_{B_1^c}^{-1} \right\|^2 \left\| T_{B_1^c} f_{\alpha,\beta} \right\|^2$$

Step 1 $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

$$\int_{\|\xi\|\geq 1} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq K_0$$

$$\int_{\mathbb{R}^d} \left|\mathbb{1}_{B_1^c}\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq K_0$$

$$\int_{\mathbb{R}^d} \left|T_{B_1^c} f_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \leq K_0$$

$$\| f_{\alpha,\beta} \|^2 = \left\| T_{B_1^c}^{-1} T_{B_1^c} f_{\alpha,\beta} \right\|^2 \leq \left\| T_{B_1^c}^{-1} \right\|^2 \left\| T_{B_1^c} f_{\alpha,\beta} \right\|^2$$

$$\| f_{\alpha,\beta} \|^2 \leq \left\| T_{B_1^c}^{-1} \right\|^2 K_0$$

Overview of the proof

Step 1: $\big(f_{\alpha,\beta}\big)_{\beta\in(0,1]}$ is bounded

Step 2: $\big(f_{\alpha,\beta}\big)_{\beta\in(0,1]}$ converges weakly to $T_W^+ g$

$$\beta_n \downarrow 0,\ f_n := f_{\alpha,\beta_n}$$

- $\exists (f_{n_k}) \rightharpoonup T_W^+ g$

Step 3: The convergence is in fact strong

- (f_n) is bounded (Step 1)
- V bounded $\hookrightarrow$ $\displaystyle \lim_{R\to\infty} \sup_n \int_{\|x\|>R} |f_n(x)|^2 \,\mathrm{d}x = 0$
- $\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

Step 2

$\exists (f_{n_k}) \rightharpoonup T_W^+ g$

$$\beta_n \downarrow 0, \quad f_n := f_{\alpha,\beta_n}, \quad \phi_{n_k} := \phi_{\beta_{n_k}}$$

Step 2

$\exists (f_{n_k}) \rightharpoonup T_W^+ g$

$$\beta_n \downarrow 0, \quad f_n := f_{\alpha,\beta_n}, \quad \phi_{n_k} := \phi_{\beta_{n_k}}$$

$$\left.\begin{array}{r}\text{Step 1} \Longrightarrow (f_n) \ \text{ bounded} \\ \text{Weak Compactness Theorem}\end{array}\right\} \Longrightarrow \exists (f_{n_k}) \rightharpoonup f'$$

Step 2

$\exists (f_{n_k}) \rightharpoonup T_W^+ g$

$$\beta_n \downarrow 0, \quad f_n := f_{\alpha,\beta_n}, \quad \phi_{n_k} := \phi_{\beta_{n_k}}$$

$$\left.\begin{array}{r} \text{Step 1} \Longrightarrow (f_n) \ \text{bounded} \\ \text{Weak Compactness Theorem} \end{array}\right\} \Longrightarrow \exists (f_{n_k}) \rightharpoonup f'$$

To be shown: $f' = T_W^+ g$

Step 2

$$\exists (f_{n_k}) \rightharpoonup T_W^+ g$$

$$\beta_n \downarrow 0, \quad f_n := f_{\alpha,\beta_n}, \quad \phi_{n_k} := \phi_{\beta_{n_k}}$$

$$\left.\begin{array}{r}\text{Step 1} \Longrightarrow (f_n) \text{ bounded} \\ \text{Weak Compactness Theorem}\end{array}\right\} \Longrightarrow \exists (f_{n_k}) \rightharpoonup f'$$

To be shown: $f' = T_W^+ g$

$$\left\| \hat{\phi}_{n_k} g - T_W f_{n_k} \right\|^2 + \alpha \left\| (1 - \hat{\phi}_{n_k}) \hat{f}_{n_k} \right\|^2$$

$$\leq \left\| \hat{\phi}_{n_k} g - T_W f \right\|^2 + \alpha \left\| (1 - \hat{\phi}_{n_k}) \hat{f} \right\|^2$$

Step 2

$$\exists (f_{n_k}) \rightharpoonup T_W^+ g$$

$$\beta_n \downarrow 0, \quad f_n := f_{\alpha,\beta_n}, \quad \phi_{n_k} := \phi_{\beta_{n_k}}$$

$$\left.\begin{array}{r} \text{Step 1} \Longrightarrow (f_n) \text{ bounded} \\ \text{Weak Compactness Theorem} \end{array}\right\} \Longrightarrow \exists (f_{n_k}) \rightharpoonup f'$$

To be shown: $f' = T_W^+ g$

$$\left\| \hat{\phi}_{n_k} g - T_W f_{n_k} \right\|^2 + \alpha \left\| (1 - \hat{\phi}_{n_k}) \hat{f}_{n_k} \right\|^2$$

$$\leq \left\| \hat{\phi}_{n_k} g - T_W f \right\|^2 + \alpha \left\| (1 - \hat{\phi}_{n_k}) \hat{f} \right\|^2$$

Step 2

$$\exists (f_{n_k}) \rightharpoonup T_W^+ g$$

$$\beta_n \downarrow 0, \quad f_n := f_{\alpha,\beta_n}, \quad \phi_{n_k} := \phi_{\beta_{n_k}}$$

$$\left.\begin{array}{r}\text{Step 1} \Longrightarrow (f_n) \ \text{ bounded} \\ \text{Weak Compactness Theorem}\end{array}\right\} \Longrightarrow \exists (f_{n_k}) \rightharpoonup f'$$

To be shown: $f' = T_W^+ g$

$$\left\| \hat{\phi}_{n_k} g - T_W f_{n_k} \right\|^2 \quad \leq \quad \left\| \hat{\phi}_{n_k} g - T_W f \right\|^2 + \alpha \left\| (1 - \hat{\phi}_{n_k}) \hat{f} \right\|^2$$

Step 2

$$\exists (f_{n_k}) \rightharpoonup T_W^+ g$$

$$\beta_n \downarrow 0, \quad f_n := f_{\alpha,\beta_n}, \quad \phi_{n_k} := \phi_{\beta_{n_k}}$$

$$\left.\begin{array}{r}\text{Step 1} \implies (f_n) \;\text{ bounded} \\ \text{Weak Compactness Theorem}\end{array}\right\} \implies \exists (f_{n_k}) \rightharpoonup f'$$

To be shown: $f' = T_W^+ g$

$$\left\| \hat\phi_{n_k} g - T_W f_{n_k} \right\|^2 \quad \leq \quad \left\| \hat\phi_{n_k} g - T_W f \right\|^2 + \alpha \underbrace{\left\| (1 - \hat\phi_{n_k}) \hat f \right\|^2}$$

$$(1 - \hat\phi_{n_k}) \hat f \to \hat f - \hat\phi(0) \hat f = 0$$

Step 2

$\exists(f_{n_k}) \rightharpoonup T_W^+ g$

$$\beta_n \downarrow 0, \quad f_n := f_{\alpha,\beta_n}, \quad \phi_{n_k} := \phi_{\beta_{n_k}}$$

$$\left.\begin{array}{r}\text{Step 1} \Longrightarrow (f_n) \ \text{bounded}\\ \text{Weak Compactness Theorem}\end{array}\right\} \Longrightarrow \exists(f_{n_k}) \rightharpoonup f'$$

To be shown: $f' = T_W^+ g$

$$\left\|\hat{\phi}_{n_k} g - T_W f_{n_k}\right\|^2 \le \underbrace{\left\|\hat{\phi}_{n_k} g - T_W f\right\|^2}_{} + \alpha\left\|(1-\hat{\phi}_{n_k})\hat{f}\right\|^2$$

$$\hat{\phi}_{n_k} g - T_W f \to g - T_W f$$

Step 2

$$\exists (f_{n_k}) \rightharpoonup T_W^+ g$$

$$\beta_n \downarrow 0, \quad f_n := f_{\alpha,\beta_n}, \quad \phi_{n_k} := \phi_{\beta_{n_k}}$$

$$\left.\begin{array}{r} \text{Step 1} \Longrightarrow (f_n) \text{ bounded} \\ \text{Weak Compactness Theorem} \end{array}\right\} \Longrightarrow \exists (f_{n_k}) \rightharpoonup f'$$

To be shown: $f' = T_W^+ g$

$$\underbrace{\left\| \hat{\phi}_{n_k} g - T_W f_{n_k} \right\|^2}_{} \leq \left\| \hat{\phi}_{n_k} g - T_W f \right\|^2 + \alpha \left\| (1 - \hat{\phi}_{n_k}) \hat{f} \right\|^2$$

$$\left.\begin{array}{r} \hat{\phi}_{n_k} g \to \hat{\phi}(0) g = g \\ T_W f_{n_k} \rightharpoonup T_W f' \end{array}\right\} \Longrightarrow \hat{\phi}_{n_k} g - T_W f_{n_k} \rightharpoonup g - T_W f'$$

Step 2

$$\exists (f_{n_k}) \rightharpoonup T_W^+ g$$

$$\beta_n \downarrow 0, \quad f_n := f_{\alpha,\beta_n}, \quad \phi_{n_k} := \phi_{\beta_{n_k}}$$

$$\left.\begin{array}{r} \text{Step 1} \Longrightarrow (f_n) \ \text{ bounded} \\ \text{Weak Compactness Theorem} \end{array}\right\} \Longrightarrow \exists (f_{n_k}) \rightharpoonup f'$$

To be shown: $f' = T_W^+ g$

$$\underbrace{\left\| \hat{\phi}_{n_k} g - T_W f_{n_k} \right\|^2}_{} \leq \left\| \hat{\phi}_{n_k} g - T_W f \right\|^2 + \alpha \left\| (1 - \hat{\phi}_{n_k}) \hat{f} \right\|^2$$

$$\left\| g - T_W f' \right\|^2 \leq \liminf_{k \to \infty} \left\| \hat{\phi}_{n_k} g - T_W f_{n_k} \right\|^2 .$$

Step 2

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$\hookrightarrow f'$ minimizes $f \mapsto \| g - T_W f \|^2$

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$$\| g - T_W f' \|^2 \leq \| g - T_W f \|^2$$

$$\hookrightarrow f' \text{ minimizes } f \mapsto \| g - T_W f \|^2$$

$$\hookrightarrow \quad f' = T_W^+ g$$

Overview of the proof

Step 1: $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ is bounded

Step 2: $\left(f_{\alpha,\beta}\right)_{\beta\in(0,1]}$ converges weakly to $T_W^+ g$

$$\beta_n \downarrow 0,\ f_n := f_{\alpha,\beta_n}$$

- $\exists (f_{n_k}) \rightharpoonup T_W^+ g$

Step 3: The convergence is in fact strong

- (f_n) is bounded (Step 1)
- V bounded $\hookrightarrow$ $\lim\limits_{R\to\infty} \sup\limits_n \int_{\|x\|>R} |f_n(x)|^2\,\mathrm{d}x = 0$
- $\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

Step 3 $\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

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$$\left\| \mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta} \right\|^2 \;=\; \left\| U \left(\mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta} \right) \right\|^2$$

Step 3

$\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

$$\begin{aligned}\left\|\mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta}\right\|^2 &= \left\|U\left(\mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta}\right)\right\|^2 \\ &= \int_{\mathbb{R}^d} \left|e^{-2i\pi\langle h,\xi\rangle} - 1\right|^2 \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi\end{aligned}$$

Step 3

$\sup_n \left\| \mathcal{T}_h f_n - f_n \right\| \to 0$ as $\left\| h \right\| \to 0$

$$\begin{aligned}
\left\| \mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta} \right\|^2 &= \left\| U \left(\mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta} \right) \right\|^2 \\
&= \int_{\mathbb{R}^d} \left| e^{-2i\pi\langle h,\xi\rangle} - 1 \right|^2 \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \\
&= I_1 + I_2
\end{aligned}$$

Step 3 $\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

$$\begin{aligned}\left\|\mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta}\right\|^2 &= \left\|U\left(\mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta}\right)\right\|^2 \\ &= \int_{\mathbb{R}^d} \left|e^{-2i\pi\langle h,\xi\rangle} - 1\right|^2 \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \\ &= I_1 + I_2\end{aligned}$$

$$I_1 := \int_{\|\xi\|\leq 1/\beta} \left|e^{-2i\pi\langle h,\xi\rangle} - 1\right|^2 \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

$$I_2 := \int_{\|\xi\|> 1/\beta} \left|e^{-2i\pi\langle h,\xi\rangle} - 1\right|^2 \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

Step 3 $\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

$$s' := \min\{1, s\}$$

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$$I_1 \quad = \quad \int_{0<\|\xi\|\leq 1/\beta} \frac{|e^{-2i\pi\langle h,\xi\rangle} - 1|^2}{\|\xi\|^{2s'}} \|\xi\|^{2s'} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

Step 3

$\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

$$s' := \min\{1, s\}$$

$$\begin{aligned} I_1 &= \int_{0<\|\xi\|\leq 1/\beta} \frac{|e^{-2i\pi\langle h,\xi\rangle} - 1|^2}{\|\xi\|^{2s'}} \|\xi\|^{2s'} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \\ &\leq \sup_{\xi\neq 0} \frac{|e^{-2i\pi\langle h,\xi\rangle} - 1|^2}{\|\xi\|^{2s'}} \int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s'} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \end{aligned}$$

Step 3

$\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

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$$\begin{aligned}
I_1 &= \int_{0<\|\xi\|\leq 1/\beta} \frac{|e^{-2i\pi\langle h,\xi\rangle} - 1|^2}{\|\xi\|^{2s'}} \|\xi\|^{2s'} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \\
&\leq \sup_{\xi\neq 0} \frac{|e^{-2i\pi\langle h,\xi\rangle} - 1|^2}{\|\xi\|^{2s'}} \int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s'} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \\
&= \|h\|^{2s'} \sup_{\xi'\neq 0} \frac{|e^{-2i\pi\langle \|h\|^{-1}h,\xi'\rangle} - 1|^2}{\|\xi'\|^{2s'}} \int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s'} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi
\end{aligned}$$

$$\xi' = \|h\|\xi$$

Step 3 $\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

$$s' := \min\{1, s\}$$

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I_1 &= \int_{0<\|\xi\|\leq 1/\beta} \frac{|e^{-2i\pi\langle h,\xi\rangle} - 1|^2}{\|\xi\|^{2s'}} \|\xi\|^{2s'} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \\
&\leq \sup_{\xi\neq 0} \frac{|e^{-2i\pi\langle h,\xi\rangle} - 1|^2}{\|\xi\|^{2s'}} \int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s'} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \\
&= \|h\|^{2s'} \underbrace{\sup_{\xi'\neq 0} \frac{|e^{-2i\pi\langle \|h\|^{-1}h,\xi'\rangle} - 1|^2}{\|\xi'\|^{2s'}}}_{\leq K_1} \int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s'} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi
\end{aligned}$$

since $\left|e^{-2i\pi\langle \|h\|^{-1}h,\xi'\rangle} - 1\right| = \mathcal{O}(\|\xi'\|)$ near the origin

Step 3

$\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

$$s' := \min\{1, s\}$$

$$\begin{aligned}
I_1 &= \int_{0<\|\xi\|\leq 1/\beta} \frac{|e^{-2i\pi\langle h,\xi\rangle} - 1|^2}{\|\xi\|^{2s'}} \|\xi\|^{2s'} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \\
&\leq \sup_{\xi\neq 0} \frac{|e^{-2i\pi\langle h,\xi\rangle} - 1|^2}{\|\xi\|^{2s'}} \int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s'} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \\
&= \|h\|^{2s'} \sup_{\xi'\neq 0} \frac{|e^{-2i\pi\langle \|h\|^{-1}h,\xi'\rangle} - 1|^2}{\|\xi'\|^{2s'}} \int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s'} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi
\end{aligned}$$

$$I_1 \leq K_1 \|h\|^{2s'} \int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s'} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi$$

Step 3

$\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

$$I_1 \;\le\; K_1 \|h\|^{2s'} \int_{\|\xi\| \le 1/\beta} \|\xi\|^{2s'} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi$$

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$$s' := \min\{1, s\} \Longrightarrow \|\xi\|^{2s'} \leq 1 + \|\xi\|^{2s}$$

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$$I_1 \;\leq\; K_1 \|h\|^{2s'} \left(\int_{\|\xi\|\leq 1/\beta} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi + \int_{\|\xi\|\leq 1/\beta} \|\xi\|^{2s} \left|\hat{f}_{\alpha,\beta}(\xi)\right|^2 \mathrm{d}\xi \right)$$

Step 3 $\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

$$I_1 \;\le\; K_1 \|h\|^{2s'} \int_{\|\xi\| \le 1/\beta} \|\xi\|^{2s'} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi$$

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$$I_1 \;\le\; K_1 \|h\|^{2s'} \left(\underbrace{\int_{\|\xi\| \le 1/\beta} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi}_{\le \|\hat{f}_{\alpha,\beta}\|^2 = \|f_{\alpha,\beta}\|^2 \le K_2} + \int_{\|\xi\| \le 1/\beta} \|\xi\|^{2s} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \right)$$

Step 3 $\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

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$$\leq\; K_1 \|h\|^{2s'} \left(K_2 + \int_{\|\xi\| \leq 1/\beta} \|\xi\|^{2s} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \,\mathrm{d}\xi \right)$$

Step 3

$\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

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Reminder from Step 1:

$$\int_{\|\xi\| \leq 1/\beta} \|\xi\|^{2s} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi + \frac{1}{M_\beta} \int_{\|\xi\| > 1/\beta} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \leq \frac{K}{\nu_0}$$

Step 3 $\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

$$I_1 \;\leq\; K_1 \|h\|^{2s'} \left(K_2 + K_3\right) = K_4 \|h\|^{2s'}$$

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$$\begin{aligned} I_2 \;&:=\; \int_{\|\xi\| > 1/\beta} \left| e^{-2i\pi\langle h,\xi\rangle} - 1 \right|^2 \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \\ &\leq\; 4 \int_{\|\xi\| > 1/\beta} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \\ &\leq\; \frac{4K}{\nu_0} M_\beta \end{aligned}$$

Step 3 $\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

$$\left.\begin{aligned} I_1 &\le K_4 \|h\|^{2s'} \\ I_2 &\le \frac{4K}{\nu_0} M_\beta \end{aligned}\right\} \Longrightarrow I_1 + I_2 \le \max\left\{K_4, \frac{4K}{\nu_0}\right\}\left(\|h\|^{2s'} + M_\beta\right)$$

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$$\left\|\mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta}\right\|^2 \le K_5\left(\|h\|^{2s'} + M_\beta\right)$$

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$$\left\|\mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta}\right\|^2 \le K_5\left(\|h\|^{2s'} + M_\beta\right)$$

$$M_\beta := \max_{\|\xi\|=1} \left|1 - \hat{\phi}(\beta\xi\right|^2 \to 0 \quad \text{as} \quad \beta \to 0$$

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$$\hookrightarrow \quad \forall \varepsilon > 0, \quad \exists n_0 \in \mathbb{N}^* : \quad \forall n \ge n_0, \quad M_{\beta_n} \le \varepsilon$$

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$$\limsup_{h \to 0} \sup_{n \in \mathbb{N}^*} \left\|\mathcal{T}_h f_n - f_n\right\|^2 \le K_5\, \varepsilon$$

Examples: Lévy kernels

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$$\left|1 - \hat{\phi}(\xi)\right| \sim_{\xi \to 0} \left\| \xi \right\|^s$$

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Examples: Lévy kernels

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$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \hat{\phi}(\xi) \neq 1$$

$$\hat{\phi}\colon \xi \mapsto \exp\bigl(-\|\xi\|^s\bigr), \quad s \in [0,2]$$

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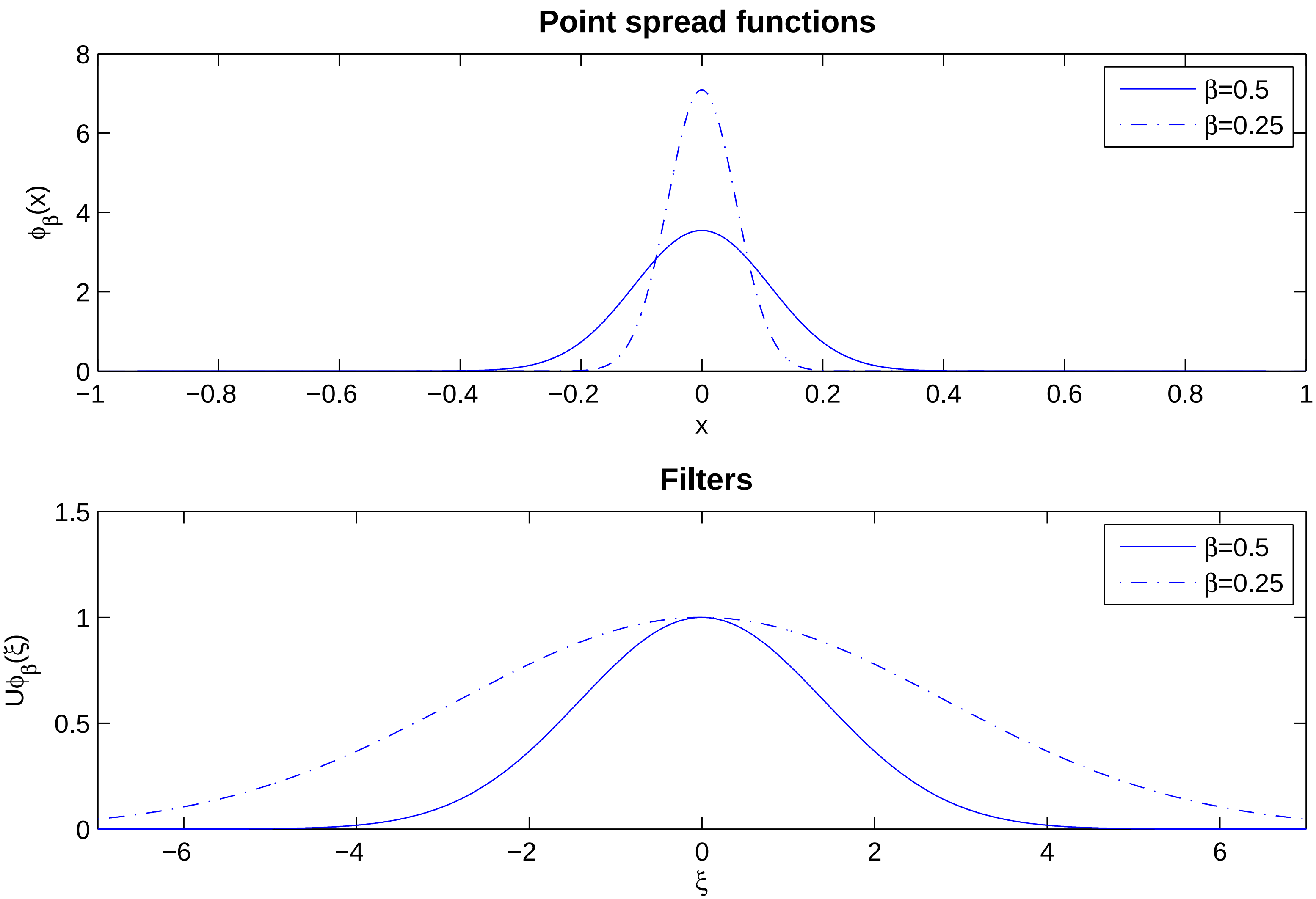
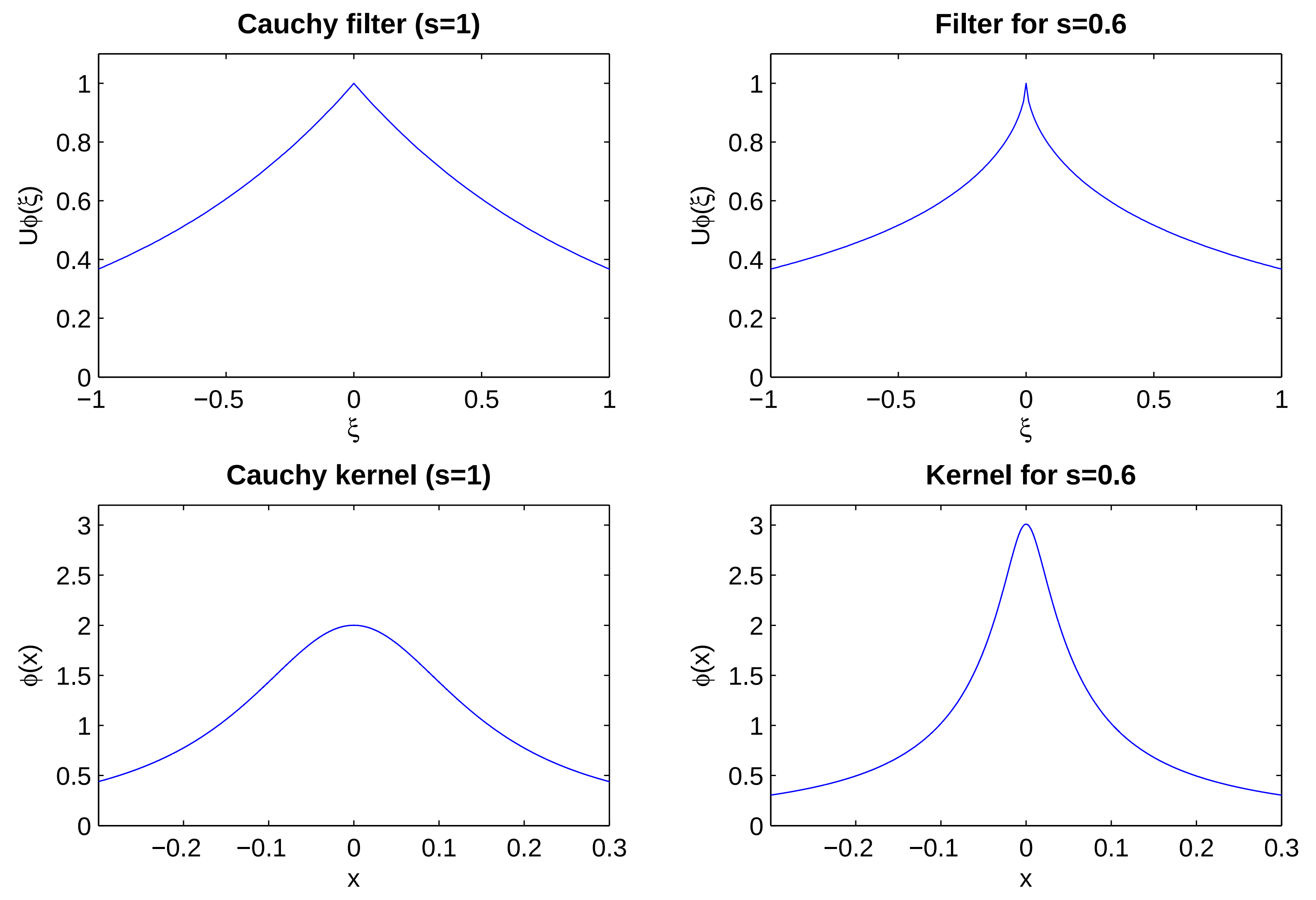
$\hookrightarrow$ ϕ is positive, isotropic, radially decreasing, C^∞

Examples: Lévy kernels



Examples: Lévy kernels



Hybrid approach

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Theorem Assume that

- $\phi \in L^1(\mathbb{R}^d)$ and $\hat{\phi}(0) = \int \phi(x)\,\mathrm{d}x \in (0,1)$

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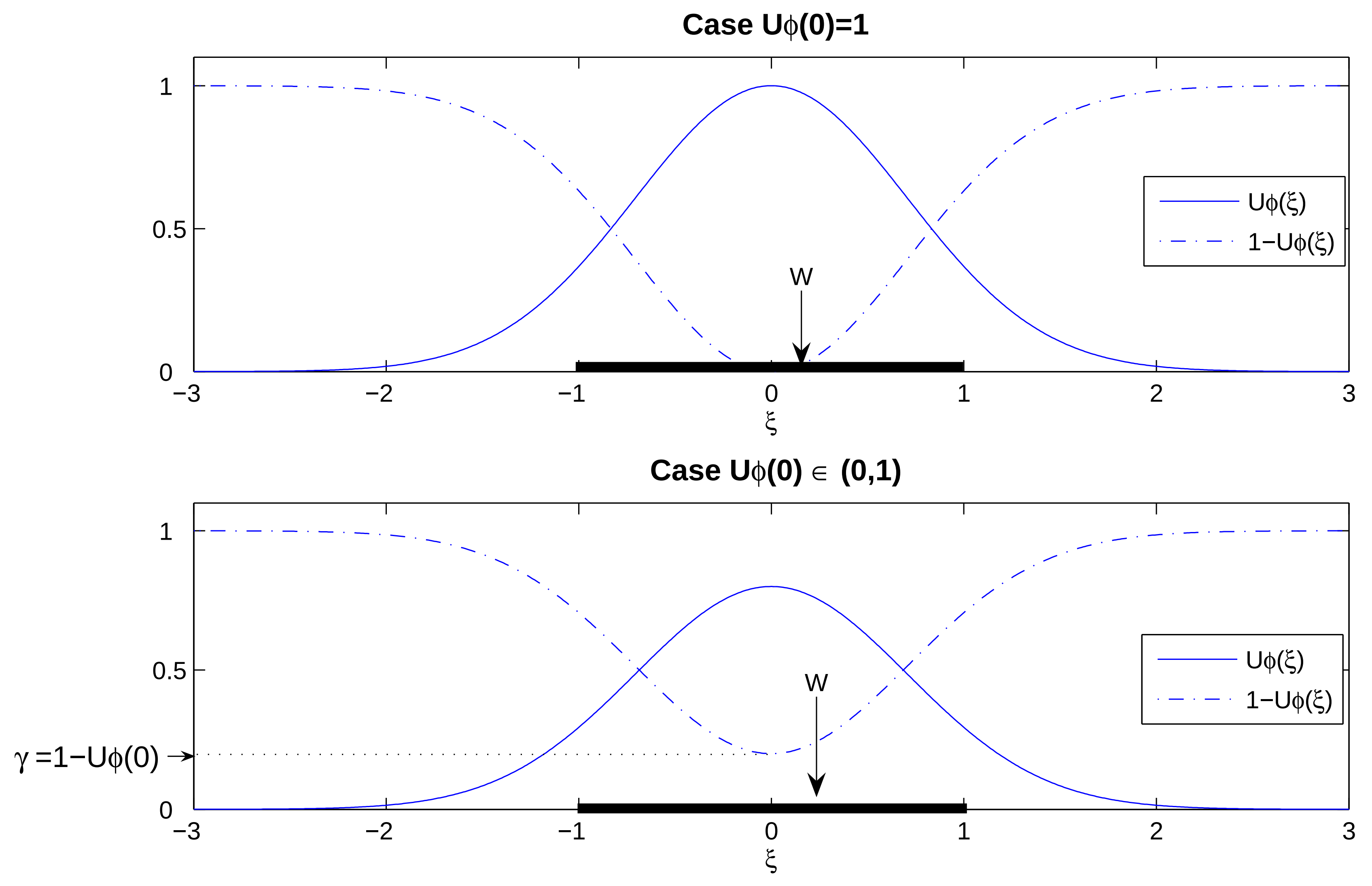
Let $f_n := f_{\alpha_n,\beta_n}$. If $g \in \mathcal{D}(T_W^+)$, then $\hat{\phi}(0)^{-1} f_n \to T_W^+ g$, as $n \to \infty$.

Hybrid approach



Outline

- Introduction
- A reminder on ill-posed problems
- Fourier synthesis
- Asymptotic analysis
- A dual algorithm (joint work with D. Wallach)

Finite dimensional setting

$$(P)\quad \left|\begin{array}{rl} \text{Minimize} & \dfrac{1}{2}\,\|\,\mathbf{g}-T\mathbf{f}\,\|^2_{\mathbb{C}^m}+\dfrac{\alpha}{2}\,\|\,H\mathbf{f}\,\|^2_{\mathbb{C}^{m_r}} \\ \text{s.t.} & \mathbf{f}\in\mathbb{R}^n \end{array}\right.$$

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$$g_k\simeq\hat{f}_0(\xi_k)=\int_V e^{-2i\pi\langle x,\xi_k\rangle}f_0(x)\,\mathrm{d}x,\qquad k=1,\ldots,m$$

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$$(P) \quad \left| \begin{array}{rl} \text{Minimize} & \dfrac{1}{2} \| \mathbf{g} - T\mathbf{f} \|^2_{\mathbb{C}^m} + \dfrac{\alpha}{2} \| H\mathbf{f} \|^2_{\mathbb{C}^{m_r}} \\ \text{s.t.} & \mathbf{f} \in \mathbb{R}^n \end{array} \right.$$

The components of $\mathbf{f} \in \mathbb{R}^n$ (or $\mathbb{C}^n$) are the coordinates of the approximation of f in the finite dimensional subspace generated by some *interpolation family* $\{e_1, \ldots, e_n\}$.

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$T \in M_{m\times n}(\mathbb{C})$ is the matrix whose entry (j,k) is the Fourier transform of e_k at ξ_j

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$H \in M_{m_r\times n}(\mathbb{C})$ is formed likewise, with ξ_j replaced by *regularizing sampling points.*

Finite dimensional setting

Framework:

$$\text{Minimize} \quad \Phi(\mathbf{f}) - \Psi(T\mathbf{f}) \quad \text{with} \quad \left|\begin{array}{ll} \Phi & \text{convex} \\ \Psi & \text{concave} \end{array}\right.$$

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Fenchel duality:

$$\inf_{\mathbf{f}\in\mathbb{R}^n} \{\Phi(\mathbf{f}) - \Psi(T\mathbf{f})\} = \max_{\boldsymbol{\lambda}\in\mathbb{R}^m} \{\Psi_\star(\boldsymbol{\lambda}) - \Phi^\star(T^\star\boldsymbol{\lambda})\}$$

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Primal-dual relationship:

$$\bar{\mathbf{f}} = \nabla\Phi^\star(T^\star\bar{\boldsymbol{\lambda}})$$

Partially infinite setting

$$(\mathcal{P}) \quad \left| \begin{array}{rl} \text{Minimize} & \dfrac{1}{2\alpha}\left\| \mathbf{g} - \mathcal{T} f \right\|^2_{\mathbb{C}^m} + \dfrac{1}{2}\left\| T_{W_\beta} f \right\|^2_{L^2(W_\beta)} \\ \text{s.t.} & f \in L^2(V) \end{array} \right.$$

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Conjugacy: real/complex

$$\Psi_\star(\xi) := \inf_x \{\langle \xi, x\rangle - \Psi(x)\} \quad \text{(convex)}$$

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Then,

$$\inf_{x\in X} \{\Phi(x) - \Psi(\mathcal{T}x)\} = \max_{\boldsymbol{\lambda}\in\mathbb{C}^d} \{D(\boldsymbol{\lambda}) := \Psi_\star(\boldsymbol{\lambda}) - \Phi^\star(\mathcal{T}^\star\boldsymbol{\lambda})\}.$$

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Remark the Constraint Qualification (CQ) is always satisfied by Problem $(\mathcal{P})$

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$$\begin{aligned}
\Phi^\star(\xi) &= \sup\left\{ \left\langle \xi\,, x \right\rangle - F(Tx) \,\middle|\, x \in X \right\} \\
&= \sup\left\{ \left\langle \xi\,, T^{-1}y \right\rangle - F(y) \,\middle|\, y \in Y \right\} \\
&= \sup\left\{ \left\langle (T^{-1})^\star\xi\,, y \right\rangle - F(y) \,\middle|\, y \in Y \right\} \\
&= F^\star\big((T^\star)^{-1}\xi\big) \;\blacksquare
\end{aligned}$$

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Lemma

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Then Φ is self-conjugate, that is, $\Phi^\star = \Phi$.

Computing $D(\boldsymbol{\lambda})$ and $\nabla D(\boldsymbol{\lambda})$

$$\operatorname{ran} T_{W_\beta} \text{ is closed}$$

Computing $D(\boldsymbol{\lambda})$ and $\nabla D(\boldsymbol{\lambda})$

$\operatorname{ran} T_{W_\beta}$ is closed

thus it is a Hilbert space on its own

Computing $D(\boldsymbol{\lambda})$ and $\nabla D(\boldsymbol{\lambda})$

ran T_{W_β} is closed

$$\begin{aligned}
\Phi^\star(\varphi) &= \frac{1}{2}\left\| (T^\star_{W_\beta})^{-1}\varphi \right\|^2_{L^2(W_\beta)} \\
&= \frac{1}{2}\left\langle (T^\star_{W_\beta})^{-1}\varphi\,,\ (T^{-1}_{W_\beta})^\star\varphi \right\rangle_{L^2(W_\beta)} \\
&= \frac{1}{2}\left\langle (T^\star_{W_\beta}T_{W_\beta})^{-1}\varphi\,,\ \varphi \right\rangle_{L^2(V)}
\end{aligned}$$

Computing $D(\boldsymbol{\lambda})$ and $\nabla D(\boldsymbol{\lambda})$

$$T^{\star}_{W_\beta} T_{W_\beta} = I - T^{\star}_{B_{1/\beta}} T_{B_{1/\beta}}$$

$$\| T^{\star}_{B_{1/\beta}} T_{B_{1/\beta}} \| < 1$$

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$$\Phi^\star(\varphi) = \frac{1}{2} \sum_{k=0}^{\infty} \left\langle (T^\star_{B_{1/\beta}} T_{B_{1/\beta}})^k \varphi \, , \, \varphi \right\rangle_{L^2(V)}$$

Illustration