Hyperoctahedral group, symmetric group representations and the moments of the real Wishart distribution

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Abstract

We compute all the moments of the Wishart distributed real symmetric matrix S. Our methods involve the use of the Gelfand pair $(\mathcal{S}_{2k}, H)$ where H is the hyperoctahedral group, and of its representation theory. Techniques based on graphs are also used.

1 Introduction

Let r be a positive integer, let M be the linear space of the (r,r) real matrices m, with unit e, let $V \subset M$ be the space of symmetric matrices s, let $\Omega \subset V$ be the cone of positive definite matrices and let $\overline{\Omega} \subset V$ be the cone of the non negative ones. If

$$p \in \Lambda = \{\frac{1}{2}, \frac{2}{2}, \dots \frac{r-1}{2}, \} \cup (\frac{r-1}{2}, \infty),$$

and if $\sigma \in \Omega$, the Wishart distribution $\gamma_{p,\sigma}$ on $\overline{\Omega}$ is defined by its Laplace transform: for $-\theta \in \Omega$:

$$\int_{\overline{\Omega}} \exp \mathrm{tr}\,(\theta s)\gamma_{p,\sigma}(ds) = (\det(e - \theta\sigma))^{-p}.$$

If π is an element of the group $\mathcal{S}_k$ of permutations of $\{1, \dots, k\}$, we denote by $m(\pi)$ the number of cycles of π and by $C(\pi)$ the set of these cycles. For $m_1, \dots, m_k$ in M we denote

$$r_\pi(\sigma)(m_1, \dots, m_k) = \prod_{c \in C(\pi)} \mathrm{tr}\,(\prod_{j \in c} \sigma m_j).$$

We have the following formula

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Proposition 1 *Let S be $\gamma_{p,\sigma}$ distributed and let $s_1,\dots,s_k\in V$ be $r\times r$ real symmetric matrices. Then*

$$\mathbb{E}(\mathrm{tr}\,(Ss_1)\mathrm{tr}\,(Ss_2)\dots\mathrm{tr}\,(Ss_k))=\sum_{\pi\in\mathcal{S}_k}p^{m(\pi)}r_\pi(\sigma)(s_1,\dots,s_k). \tag{1.1}$$

Proof: For $p\in\Lambda$ and $\sigma\in\Omega$, we write

$$\gamma_{p,\sigma}(dx)=\exp(-\mathrm{tr}\,(x\sigma^{-1}))\,(\det\sigma)^{-p}\,\mu_p(dx)$$

where the Laplace transform of μ_p is

$$L(\theta)=(\det(-\theta))^{-p},\quad \theta\in-\Omega.$$

Denote $\sigma(\theta)=(-\theta)^{-1}$ for simplicity. Standard arguments using induction show that the k-th differential of the map

$$\theta\ \to\ (\det\sigma(\theta))^p$$

is given by

$$(h_1,\dots,h_k)\to(\det\sigma(\theta))^p\sum_{r\in\mathcal{S}_k}p^{m(r)}r_\pi(\sigma)(h_1,\dots,h_k).$$

Since

$$\mathbb{E}(\mathrm{tr}\,(Sh_1)\dots\mathrm{tr}\,(Sh_k))=\frac{1}{L(\theta)}\,L^{(k)}(\theta)(h_1,\dots,h_k)$$

we get the result. More details can be found in [5]. ∎

The formula (1.1) has been obtained in [5] when S is complex Wishart distribution and $s_1,\dots,s_k$ are not only in the space $V_{\mathbf{C}}$ of hermitian $r\times r$ matrices but also in the space $M_{\mathbf{C}}$ of complex matrices. Let us point out that in the complex case, replacing $V_{\mathbf{C}}$ by the space $M_{\mathbf{C}}$ of complex matrices is a simple matter. Just write

$$m_j=s_j+it_j$$

where $s_j,t_j\in V_{\mathbf{C}}$ and use multilinearity in order to obtain

$$\mathbb{E}(\mathrm{tr}\,(Sm_1)\mathrm{tr}\,(Sm_2)\dots\mathrm{tr}\,(Sm_k))=\sum_{\pi\in\mathcal{S}_k}p^{m(\pi)}r_\pi(\sigma)(m_1,\dots,m_k),\quad m_1,\dots,m_k\in M_{\mathbf{C}}.$$

The reason is that the dimension r^2 of the **real** space $V_{\mathbf{C}}$ is equal to the **complex** dimension r^2 of $M_{\mathbf{C}}$. Therefore the complexification of $V_{\mathbf{C}}$ is simply $M_{\mathbf{C}}$. There is no such phenomenon in the real case.

Coming back to the real case, the formula (1.1) is simply false for S real Wishart and for $m_1,\dots,m_k\in M$ not necessarily symmetric (see the Corollary 11 for the exact formula which involves $(2k)!/(k!\,2^k)$ summands instead of $k!$ in (1.1)).

Also in the complex case, we have found the surprising formula

$$\mathbb{E}(r_\pi(S)(m_1,\dots,m_k))=\sum_{\pi'\in\mathcal{S}_k}p^{m(\pi\circ\pi'^{-1})}r_{\pi'}(\sigma)(m_1,\dots,m_k)\ =\ p^{m(\cdot)}*r_\cdot(\pi). \tag{1.2}$$

This leads naturally to the question whether in the real case we can compute

$$\mathbb{E}(r_\pi(S)(m_1,\dots,m_k)) \ \text{ for } \ m_1,\dots,m_k \in M$$

or at least

$$\mathbb{E}(r_\pi(S)(s_1,\dots,s_k)) \ \text{ for } \ s_1,\dots,s_k \in V.$$

There are 2 strong motivations for the computation of $\mathbb{E}(r_\pi(S)(s_1,\dots,s_k))$. The first is obviously the search for an analog of (1.2). The second one is more subtle.

Denote by $\mathcal{P}_k$ the space of functions P on V^{k+1}:

$$(x, s_1,\dots,s_k) \to P(x, s_1,\dots,s_k)$$

such that

(1) for fixed x the map $(s_1,\dots,s_k) \to P(x,s_1,\dots,s_k)$ is a k-multilinear form on V^k

(2) for fixed $(s_1,\dots,s_k)$, the function $x \to P(x,s_1,\dots,s_k)$ is a homogeneous polynomial of degree k on V

(3) P is invariant in the following sens: if $a \in M$ is invertible then

$$P(axa^*, s_1,\dots,s_k) = P(x, a^*s_1a,\dots,a^*s_ka).$$

The application $r_\pi(x,s_1,\dots,s_k) = r_\pi(x)(s_1,\dots,s_k)$ provides an example of an element of $\mathcal{P}_k$. Actually, we prove in the Appendix, that the linear space $\mathcal{P}_k$ is generated by $(r_\pi)_{\pi\in\mathcal{S}_k}$.

Now, if $\gamma_{p,\sigma}$ is a real Wishart distribution, introduce informally the operator T_p on real functions on Ω by

$$T_p(f)(\sigma) = \int_\Omega f(x)\gamma_{p,\sigma}(dx).$$

In other words, if S has distribution $\gamma_{p,\sigma}$,

$$\mathbb{E}(f(S)) = T_p(f)(\sigma).$$

It is not difficult to check that $T_p(f) \in \mathcal{P}_k$ if $f \in \mathcal{P}_k$. Therefore T_p is an endomorphism of $\mathcal{P}_k$ and $\mathbb{E}(r_\pi(S)(s_1,\dots,s_k))$ is a linear combination of the $r_\pi(\sigma)(s_1,\dots,s_k))$, when $r \in \mathcal{S}_k$.

To be more specific, if the dimension r is large enough with respect to k, then $(r_\pi)_{\pi\in\mathcal{S}_k}$ is a genuine basis of $\mathcal{P}_k$ and the exact computation of the coefficients of the $r_\pi(\sigma)(s_1,\dots,s_k)$ in $\mathbb{E}(r_\pi(S))$ is the computation of the representative matrix of T_p in the basis (r_π).

Now, why to compute $\mathbb{E}(r_\pi(S)(m_1,\dots,m_k))$ for $m_1,\dots,m_k \in M$ instead of V? Simply because the direct computation for V seems impossible. We proceed as follows. We express first

$$\mathbb{E}(r_\pi(S)(m_1,\dots,m_k))$$

as a linear combination of $(2k!)/(2^kk!)$ functions taken among the $2^kk!$ functions

$$\mathbb{E}(r_\pi(\sigma)(m_1^{(\epsilon_1)},\dots,m_k^{(\epsilon_k)}))$$

where $(\epsilon_1, \dots, \epsilon_k) \in \{-1,1\}^k$, with the convention $m^{(1)} = m$ and $m^{(-1)} = m^t$, the transposed matrix. Then we specify such a formula to symmetric $m_j = s_j$. Certainly, computing directly $\mathbb{E}(r_\pi(S)(s_1, \dots, s_k))$ by the methods used to compute $\mathbb{E}(r_\pi(S)(m_1, \dots, m_k))$ would be hard to follow.

Furthermore, the computation of $\mathbb{E}(r_\pi(S)(m_1, \dots, m_k))$ for $m_j \in M$ instead of V gives insight into the structure of the problem, actually governed not by the symmetric group $\mathcal{S}_k$ but by the Gelfand pair $(\mathcal{S}_{2k}, H)$ where H is the subgroup of the permutations $\pi \in \mathcal{S}_{2k}$ such that for all $j = 1, \dots, k$

$$|\pi(j) - \pi(j+k)| = k.$$

H is sometimes denoted in the literature by B_k and called the hyperoctahedral group. Note that H is not a normal subgroup, so $\mathcal{S}_{2k}/H$ is not a group.

The paper is organized as follows. Section 2 contains needed combinatorial properties of the pair $(\mathcal{S}_{2k}, H)$. In Section 3 we compute the polynomial coefficients $A(p)$ and $B(p)$ in the formulas:

$$\mathbb{E}(r_\pi(S)(m_1^{\epsilon_1}, \dots, m_k^{\epsilon_k}))) = \sum_{(\pi',\epsilon')} A_{(\pi,\epsilon),(\pi',\epsilon')}(p) r_{\pi'}(\sigma)(m_1^{\epsilon'_1}, \dots, m_k^{\epsilon'_k}) \tag{1.3}$$

$$\mathbb{E}(r_\pi(S)(s_1, \dots, s_k)) = \sum_{\pi' \in \mathcal{S}_k} B_{\pi,\pi'}(p) r_{\pi'}(\sigma)(s_1, \dots, s_k). \tag{1.4}$$

in terms of a certain operation $(a,b) \to a \otimes b = c$ on the classes $a, b \in \mathcal{S}_{2k}/H$ and the inverse operation $c \oslash a$.

In Section 4 we design a method to compute $a \otimes b = c$ and to solve $a \otimes x = c$ when a and c are given. This method uses graphs.

Section 5 presents a group-theoretical interpretation of the matrix A by using the representation theory of the symmetric group $\mathcal{S}_{2k}$. One can say that whereas in the complex case only the convolution on $\mathcal{S}_k$ is used through the formula (1.2), in the real case one needs the convolution on $\mathcal{S}_{2k}/H$, in the sens of Gelfand pair. We are led to use the H-spherical representations of $\mathcal{S}_{2k}$, associated to convenient Young tableaux, a much more elaborated tool than the character table of $\mathcal{S}_{2k}$.

In the Appendix we justify the observations on the space $\mathcal{P}_k$ given above and add some more interesting information in that direction.

Let us mention that in [12] a formula for the moments of arbitrary polynomials in the entries of S was obtained as an application of MacMahon's Master Theorem. However it does not allow one to obtain formulas for $\mathbb{E}(r_\pi(S)(m_1, \dots, m_k))$.

2 The homogeneous space $\mathcal{S}_{2k}/H$

In this section

- we define a subgroup H of $\mathcal{S}_{2k}$, isomorphic to a hyperoctahedral group, and a homogeneous space $\mathcal{S}_{2k}/H$

• we motivate it by the applications to overcoming the difficulty consisting in the fact that the map

$$(\pi, \epsilon) \mapsto r_\pi(\sigma)(m_1^{\epsilon_1}, \ldots, m_k^{\epsilon_k}) \tag{2.5}$$

is by no means one to one.

• we next study the properties of $\mathcal{S}_{2k}/H$ useful in the computation of the moments $\mathbb{E}(r_\pi(S)(m_1^{\epsilon_1}, \ldots, m_k^{\epsilon_k}))$.

We introduce a very useful notation

$$j^+ = j + k, \quad j = 1, \ldots, k.$$

If $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{-1, 1\}^k = \mathbf{Z}_2^k$, denote by τ_ϵ the element of $\mathcal{S}_{2k}$ defined as follows: for all $j = 1, \ldots, k$, one sets

$$\begin{array}{lll} \tau_\epsilon(j) = j & \tau_\epsilon(j^+) = j^+ & \text{if } \epsilon_j = 1, \\ \tau_\epsilon(j) = j^+ & \tau_\epsilon(j^+) = j & \text{if } \epsilon_j = -1. \end{array}$$

If we write the elements $\{1, \ldots, 2k\}$ as

$$\begin{pmatrix} 1 & 2 & \ldots & j & \ldots & k \\ 1^+ & 2^+ & \ldots & j^+ & \ldots & k^+ \end{pmatrix}$$

then τ_ϵ acts by transposing the elements of some columns, e.g. for $k = 5$

$$\tau_{(1,-1,-1,1,-1)} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1^+ & 2^+ & 3^+ & 4^+ & 5^+ \end{pmatrix} = \begin{pmatrix} 1 & 2^+ & 3^+ & 4 & 5^+ \\ 1^+ & 2 & 3 & 4^+ & 5 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 8 & 4 & 10 \\ 6 & 2 & 3 & 9 & 5 \end{pmatrix}.$$

The map $\epsilon \mapsto \tau_\epsilon$ is an injective homomorphism of the commutative group $\mathbf{Z}_2^k$ (equipped with the product $\epsilon\epsilon' = (\epsilon_1\epsilon_1', \ldots, \epsilon_k\epsilon_k')$) into $\mathcal{S}_{2k}$. Its image is the subgroup $\mathcal{S}_{\{1,1^+\}} \times \ldots \mathcal{S}_{\{k,k^+\}}$ of $\mathcal{S}_{2k}$.

If $\pi \in \mathcal{S}_k$, denote by t_π and s_π the elements of $\mathcal{S}_{2k}$ defined as follows: for all $j = 1, \ldots, k$, one has

$$t_\pi(j) = j, \quad t_\pi(j^+) = \pi(j)^+,$$
$$s_\pi(j) = \pi(j), \quad s_\pi(j^+) = \pi(j)^+.$$

The permutations t_π are simply the elements of the subgroup $\mathcal{S}_{\{1^+,\ldots,k^+\}} = \mathcal{S}_{\{1+k,\ldots,2k\}}$ of $\mathcal{S}_{2k}$.

The permutations s_π act on $\begin{pmatrix} 1 & 2 & \ldots & k \\ 1^+ & 2^+ & \ldots & k^+ \end{pmatrix}$ by permutations of columns, e.g.

$$s_{(123)(45)} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1^+ & 2^+ & 3^+ & 4^+ & 5^+ \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 & 5 & 4 \\ 2^+ & 3^+ & 1^+ & 5^+ & 4^+ \end{pmatrix}.$$

Both $\pi \mapsto t_\pi$ and $\pi \mapsto s_\pi$ are injective homomorphisms of $\mathcal{S}_k$ into $\mathcal{S}_{2k}$. Note that $t(\mathcal{S}_k) \cup \tau(\mathbf{Z}_2^k)$ generates $\mathcal{S}_{2k}$.

Definition 2 *Let H be the subgroup of $\mathcal{S}_{2k}$ generated by all the permutations τ_ϵ, $\epsilon \in \mathbf{Z}_2^k$ and s_π, $\pi \in \mathcal{S}_k$.*

H is called a *wreath product* of $\mathcal{S}_2$ and $\mathcal{S}_k$ (the image of τ to be identified with $\mathcal{S}_2^{\otimes k}$ and the image of s with $\mathcal{S}_k$). It is also studied in the literature as the *hyperoctahedral* group B_k. Note that $(\mathcal{S}_{2k}, H)$ is a Gelfand pair (cf.[6]).

The following proposition, which is very easy to verify, recalls the structure of H.

Proposition 3 *For (ϵ, π) in $\mathbf{Z}_2^k \times \mathcal{S}_k$, define*

$$b_\pi(\epsilon) = (\epsilon_{\pi^{-1}(1)}, \ldots, \epsilon_{\pi^{-1}(k)}).$$

Then we have

(i) $\pi \mapsto b_\pi(\epsilon)$ is a homomorphism of $\mathcal{S}_k$ into the group of automorphisms of $\mathbf{Z}_2^k$.

(ii) The following commutation rule holds : $\tau_{b_\pi(\epsilon)} s_\pi = s_\pi \tau_\epsilon$.

(iii) The subgroup H is equal to

$$H = \{s_\pi \tau_\epsilon, \ (\pi, \epsilon) \in \mathcal{S}_k \times \mathbf{Z}_2^k\} \tag{2.6}$$

The group H is isomorphic to the semi-direct product of the groups $\mathcal{S}_k$ and $\mathbf{Z}_2^k$ induced by the action $\pi \mapsto b_\pi(\epsilon)$ of $\mathcal{S}_k$ on $\mathbf{Z}_2^k$.

The subgroup H has $2^k k!$ elements. It is not normal, and we denote by $\mathcal{S}_{2k}/H = \{C_1 \ldots, C_N\}$, with $N = \frac{(2k)!}{2^k k!} = 1.3.5 \ldots (2k-1)$, the family of its right cosets, i.e. the family of the subsets of $\mathcal{S}_{2k}$ of the form gH. We write

$$g \sim g'$$

if there exists $h \in H$ such that $g' = gh$, i.e. if g and g' are in the same coset.

Proposition 4 *Let (ϵ, π) and (ϵ', π') be in $\mathbf{Z}_2^k \times \mathcal{S}_k$. Then $\tau_\epsilon t_\pi \sim \tau_{\epsilon'} t_{\pi'}$ if and only if the following conditions hold*

1. *The fonction $j \mapsto \epsilon_j \epsilon'_j$ is constant on the cycles of π.*
2. *If $\epsilon_j \epsilon'_j = 1$ on a cycle c of π, then π and π' coincide on c.*
3. *If $\epsilon_j \epsilon'_j = -1$ on a cycle c of π, then π^{-1} and π' coincide on c.*

Proof: By Proposition 3(iii) $\tau_\epsilon t_\pi \sim \tau_{\epsilon'} t_{\pi'}$ if and only if there exist π'' and ϵ'' such that

$$\tau_\epsilon t_\pi = \tau_{\epsilon'} t_{\pi'} \tau_{\epsilon''} s_{\pi''},$$

or equivalently

$$t_\pi = \tau_{\epsilon\epsilon'} t_{\pi'} \tau_{\epsilon''} s_{\pi''}. \tag{2.7}$$

Thus, without loss of generality, we may assume that ϵ is the identity of $\mathbf{Z}_2^k$. Let us fix j in $\{1, 2, \ldots, k\}$. We apply (2.7)(with $\epsilon = id$) to j and j^+ and we discuss according to the value of $\epsilon''_{\pi''(j)}$.

First case: suppose that $\epsilon''_{\pi''(j)} = 1$. Apply first both sides of (2.7) to j. One gets $\tau_{\epsilon'}(\pi'')(j) = j$ which implies that $\epsilon'_{\pi''(j)} = 1$ and $\pi''(j) = j$.

From the equality $\pi''(j)=j$ and the application of (2.7) on j^+ we get $\tau_{\epsilon'}(\pi'(j)^+)=\pi(j)^+$, hence $\epsilon'_{\pi'(j)}=1$ and $\pi(j)=\pi'(j)$. Thus, in this case, $\epsilon'_j=\epsilon'_{\pi(j)}=1$ and $\pi(j)=\pi'(j)$.

Second case: suppose that $\epsilon''_{\pi''(j)}=-1$. From the action on j^+, we have $\pi(j)^+=\tau_{\epsilon'}(\pi''(j))$, so that $\epsilon'_{\pi''(j)}=-1$ and $\pi(j)=\pi''(j)$. From the action on j, we obtain $j=\tau_{\epsilon'}(\pi'\pi(j)^+)$ and we deduce that $\epsilon'_{\pi'\pi(j)}=-1$ and $\pi'\pi(j)=j$. Thus, in this case, $\epsilon'_j=\epsilon'_{\pi(j)}=-1$ and $\pi(j)=\pi'^{-1}(j)$.

We infer that the function $j\mapsto\epsilon'_j$ is constant on the cycles of π and the conditions 2 and 3 now follow.

It is easy to check the converse part of the proposition. ∎

We explain now, in order to justify its introduction, in which way the space $\mathcal{S}_{2k}/H$ appears in relation to (2.5).

For fixed k, denote by D the set of sequences $B=(c_1,\ldots,c_{2k})$ of elements of $\{1,\ldots,r\}$. If $g\in\mathcal{S}_{2k}$, denote by l_g the permutation of D defined by $l_g(B)=(c_{g(1)},\ldots,c_{g(2k)})$. We will adopt a useful notation

$$l_g(B)=\begin{pmatrix} c_{g(1)} & \cdots & c_{g(k)}\\ c_{g(k+1)} & \cdots & c_{g(2k)}\end{pmatrix}.$$

The map $g\mapsto l_g$ is an injective antihomomorphism of $\mathcal{S}_{2k}$: we have $l_g\circ l_{g'}=l_{g'g}$.

Elements of D will be also frequently denoted by $B=\begin{pmatrix} a_1 & \cdots & a_k\\ b_1 & \cdots & b_k\end{pmatrix}$. For such a B, and for a matrix $m=(m_{ij})_{1\le i,j\le r}$, we denote

$$m(B)=m_{a_1b_1}\ldots m_{a_kb_k}.$$

Thus $m(l_g(B))$ makes sense. We want to compute $m(l_g(B))$ for $g=\tau_\epsilon t_\pi$.

For this we introduce the following basis $\{e(a,b)\}_{1\le a,b\le r}$ of M. If $1\le a,b\le r$, denote by $e(a,b)$ the matrix such that $e(a,b)_{b,a}=1$ and $e(a,b)_{t,s}=0$ if $(t,s)\neq(b,a)$.

It has the property that $\mathrm{tr}\,(me(a,b))=m_{ab}$ for $m\in M$, and for $B=\begin{pmatrix} a_1 & \cdots & a_k\\ b_1 & \cdots & b_k\end{pmatrix}$ we have

$$m(B)=\mathrm{tr}\,(me(a_1,b_1))\ldots\mathrm{tr}\,(me(a_k,b_k)). \tag{2.8}$$

If $\pi\in\mathcal{S}_k$ is a circular permutation, say $\pi(j)=j+1$ for $j=1,\ldots,k-1$ and $\pi(k)=1$, then([5], (4.28))

$$m(l_{t_\pi}(B))=\mathrm{tr}\,(me(a_1,b_1)\ldots me(a_k,b_k)).$$

More generally, for $\pi\in\mathcal{S}_k$ we have([5], (4.29))

$$m(l_{t_\pi}(B))=r_\pi(m)(e(a_1,b_1),\ldots,e(a_k,b_k)) \tag{2.9}$$

Recall the notation $m^{(-1)}$ for the transpose. Thus $e(a,b)^{(-1)}=e(b,a)$. With this remark, we can write

$$m(l_{\tau_\epsilon t_\pi}(B))=m(l_{t_\pi}l_{\tau_\epsilon}(B))=r_\pi(m)(e(a_1,b_1)^{\epsilon_1},\ldots,e(a_k,b_k)^{\epsilon_k}). \tag{2.10}$$

Proposition 5 *If $g \sim g'$ then for all symmetric $s \in V$ and for all $h \in H$ one has*

$$s(l_h(B)) = s(B). \tag{2.11}$$

Proof: Let us observe that for all $\pi \in \mathcal{S}_k$ one has

$$m(l_{s_\pi}(B)) = m(B)$$

and for all $\epsilon \in \mathbf{Z}_2^k$ and for all symmetric s one has

$$s(l_{\tau_\epsilon}(B)) = s(B).$$

From the last two formulas, we infer that for all symmetric $s \in V$ and for all $h \in H$ one has

$$s(l_h(B)) = s(B). \tag{2.12}$$

Remark now that if $g \sim g'$, i.e. if there exists h in H such that $g = g'h$, then for all B in D and for all s in V we have

$$s(l_g(B)) = s(l_{g'}(B)). \tag{2.13}$$

To see this, introduce $B_1 = l_{g'}(B)$. Thus $B = l_{g'^{-1}}(B_1)$,

$$l_g(B) = l_g(l_{g'^{-1}}(B_1)) = l_{g'^{-1}g}(B_1) = l_h(B_1).$$

Now $s(l_h(B_1)) = s(B_1)$ from (2.11). ∎

Corollary 6

$$r_\pi(\sigma)(m_1^{\epsilon_1}, \ldots, m_k^{\epsilon_k}) = r_{\pi'}(\sigma)(m_1^{\epsilon'_1}, \ldots, m_k^{\epsilon'_k}) \tag{2.14}$$

for all symmetric matrices $\sigma \in V$ and all real matrices $m_1, \ldots, m_k$ if and only if

$$\tau_\epsilon t_\pi \sim \tau_{\epsilon'} t_{\pi'}.$$

Proof: The sufficiency follows from Proposition 5 and from (2.10), by multilinearity of $r_\pi(\sigma)$. To prove the necessity, observe that (2.11) holds for all $s \in V$ if and only if $h \in H$. Let $\tau_\epsilon t_\pi \approx \tau_{\epsilon'} t_{\pi'}$ if and only if (2.14) holds. It follows that this equivalence relation on $\{\tau_\epsilon t_\pi \mid (\epsilon, \pi) \in \mathbf{Z}_2^k \times \mathcal{S}_k\}$ is the same as $\sim$: in fact, the equivalence classes of $((1, 1, \ldots, 1), id)$ are equal to $H \cap \{\tau_\epsilon t_\pi \mid (\epsilon, \pi) \in \mathbf{Z}_2^k \times \mathcal{S}_k\}$ for both relations. ∎

From now on, if C is in $\mathcal{S}_{2k}/H$, B is in D and s is in V, we define $s(C(B))$ as equal to $s(l_g(B))$ where g is an arbitrary element of C.

Now we are coming back to the study of properties of the space $\mathcal{S}_{2k}/H$ which will be useful in the computation of the moments of the Wishart law. The objective is to show that in each coset C_i of $\mathcal{S}_{2k}/H$ one may choose a representative of the form $\tau_\epsilon t_\pi$.

The following theorem analyses the position of the set of permutations of this form with respect to $\mathcal{S}_{2k}/H$. A special role is played by such permutations for the pairs (ϵ, π) that we will call particular.

Definition 7 *We will say that a pair* $T = (\epsilon, \pi) \in \mathbf{Z}_2^k \times \mathcal{S}_k$ *is* **particular** *if, for any cycle* c *of* π*, we have* $\epsilon_j = 1$ *when* j *is the smallest element of* c*. We will call the corresponding permutation* $T' = \tau_\epsilon t_\pi$ **particular**.

Theorem 8 *The class of* $\mathcal{S}_{2k}/H$ *which contains* $\tau_\epsilon t_\pi$ *contains exactly* $2^{m(\pi)}$ *elements of type* $\tau_{\epsilon'} t_{\pi'}$*. For a given* $\pi \in \mathcal{S}_k$*, there are* $2^{k-m(\pi)}$ *distinct classes of* $\mathcal{S}_{2k}/H$ *containing an element of the form* $\tau_\epsilon t_\pi$.

Every class of $\mathcal{S}_{2k}/H$ *contains exactly one particular permutation of the form* $\tau_\epsilon t_\pi$.

Proof: It follows from the Proposition 4 that for given (ϵ, π), the number of ϵ' such that $\epsilon\epsilon'$ is constant on the cycles of π is exactly $2^{m(\pi)}$. This implies the first two statements.

To prove the last one, we recall that the *portrait* of a permutation $\pi \in \mathcal{S}_k$ is the sequence $\mathbf{i} = (i_1, \ldots, i_k)$ of non negative integers such that i_j is the number of cycles of π with length j. This implies $m(\pi) = i_1 + \cdots + i_k$ and $1i_1 + 2i_2 + \cdots + ki_k = k$. The number of $\pi \in \mathcal{S}_k$ with given portrait $\mathbf{i}$ is

$$\sharp \mathbf{i} = \frac{k!}{\prod_{j=1}^k i_j! j^{i_j}}.$$

We now prove the following equality

$$\sum_{\pi \in \mathcal{S}_k} 2^{k-m(\pi)} = N = \frac{(2k)!}{2^k k!},$$

which is equivalent to

$$\sum_{i_1,\ldots,i_k;\ 1i_1+2i_2+\cdots+ki_k=k} \frac{k!}{\prod_{j=1}^k i_j! j^{i_j}} 2^{k-i_1-\cdots-i_k} = \frac{(2k)!}{2^k k!}. \tag{2.15}$$

To prove this equality, let us denote by a_k its left hand side and by b_k its right hand side, and let us show that $\sum_{k=0}^\infty a_k z^k/k! = \sum_{k=0}^\infty b_k z^k/k!$ for $|z|$ small enough. It is standard to check that $\sum_{k=0}^\infty b_k z^k/k! = (1-2z)^{-1/2}$. For the a_k we write

$$\begin{aligned}
\sum_{k=0}^\infty \frac{a_k z^k}{k!} &= \sum_{i_1,\ldots,i_k,\ldots} \prod_{j=1}^\infty \frac{2^{i_j(j-1)} z^{i_j}}{i_j! j^{i_j}} \\
&= \sum_{i_1,\ldots,i_k,\ldots} \exp\left(\frac{z}{1} + \frac{2^{2-1}z^2}{2} + \cdots + \frac{2^{j-1}z^j}{j} + \cdots\right) \\
&= (1-2z)^{-1/2}.
\end{aligned}$$

Here the infinite sum $\sum_{i_1,\ldots,i_k,\ldots}$ is taken on the countable set of sequences $i_1, \ldots, i_k, \ldots$ such that $i_k = 0$ for k large enough.

It follows from the Definition 7 that given π in $\mathcal{S}_k$, the number of particular pairs (ϵ, π) is $2^{k-m(\pi)}$. We are going to show that

(i) If (ϵ, π) and (ϵ', π) are two distinct particular pairs, then $\tau_\epsilon t_\pi \sim \tau_{\epsilon'} t_{\pi'}$ is impossible.

(ii) For all $(\epsilon, \pi) \in \mathbf{Z}_2^k \times \mathcal{S}_k$, there exists a particular pair (ϵ', π') such that $\tau_\epsilon t_\pi \sim \tau_{\epsilon'} t_{\pi'}$.

Suppose that (i) is false. Then $\epsilon\epsilon'$ is a constant function on the cycles of π. Since the pairs are particular, there exists on each cycle a j such that $\epsilon_j = \epsilon'_j = 1$, therefore $\epsilon = \epsilon'$. This implies $\pi = \pi'$ and the pairs are equal: contradiction.

For the assertion (ii), given (ϵ, π), we define ϵ' such that $\epsilon\epsilon'$ is a constant function on the cycles of π and such that $\epsilon'_j = 1$ when j is the smallest element of a cycle. Finally we define π' as equal to π on the cycles on which $\epsilon\epsilon' = 1$, and as equal to π^{-1} on the others. Clearly the pair $(\epsilon,' \pi)$ is particular and $\tau_\epsilon t_\pi \sim \tau_{\epsilon'} t_{\pi'}$.

This shows that the set of particular pairs is in a 1-1 correspondence with the elements of $\mathcal{S}_{2k}/H$ containing some $\tau_\epsilon t_\pi$. The identity (2.15) shows that the number of particular pairs is N and the proof of Theorem 8 is complete. ∎

We denote

$$c_i = \{(\epsilon, \pi) \in \mathbf{Z}_2^k \times \mathcal{S}_k \mid \tau_\epsilon t_\pi \in C_i\} \neq \emptyset, \tag{2.16}$$

and

$$T_i \in c_i$$

the only particular pair T_i belonging, according to Theorem 8, to the set c_i.

Examples: For $k = 2$ we have $N = 3$ classes. Let e be the identity and $d = (12)$, the two elements of $\mathcal{S}_2$.

The sets c_i defined by (2.16) corresponding to the three classes C_1, C_2, C_3, are:

$$\begin{aligned} c_1 &= \{(\epsilon, e) \mid \epsilon \in \mathbf{Z}_2^2\} \\ c_2 &= \{((1,1), d),\ ((-1,-1), d)\} \\ c_3 &= \{((1,-1), d),\ ((-1,1), d)\} \end{aligned}$$

The associated particular pairs are $T_1 = ((1,1), e),\ T_2 = ((1,1), d), T_3 = ((1,-1), d)$.

For $k = 3$ we have $N = 15$ classes.

The particular pairs are:

$$\begin{array}{ll} T_1 = ((1,1,1), e), & \\ T_2 = ((1,1,1), (1)(23)), & T_3 = ((1,1,-1), (1)(23)), \\ T_4 = ((1,1,1), (2)(13)), & T_5 = ((1,1,-1), (2)(13)), \\ T_6 = ((1,1,1), (3)(12)), & T_7 = ((1,-1,1), (3)(12)), \\ T_8 = ((1,1,1), (123)), & T_9 = ((1,1,-1), (123)), \\ T_{10} = ((1,-1,1), (123)), & T_{11} = ((1,-1,-1), (123)), \\ T_{12} = ((1,1,1), (132)), & T_{13} = ((1,1,-1), (132)), \\ T_{14} = ((1,-1,1), (132)), & T_{15} = ((1,-1,-1), (132)). \end{array}$$

The set $c_1 = \{(\epsilon, e);\ \epsilon \in \mathbf{Z}_2^3\}$ has $2^3 = 8$ elements.

Each of the 6 sets $c_2, \ldots, c_7$ has $2^2 = 4$ elements, for example

$$c_2 = \{((\pm 1, \pm(1,1)), (1)(23))\}, \quad c_3 = \{(((\pm 1, \pm(1,-1)), (1)(23))\}$$

and c_4, c_5, c_6, c_7 are constructed in an analogous manner.

Each of the 8 sets $c_8, \ldots, c_{15}$ contains $2^1 = 2$ elements and they are all of the form

$$\{(\epsilon, (123)),\ (-\epsilon, (132))\},\ \epsilon \in \mathbf{Z}_2^3$$

(note that $(123)^{-1} = (132)$). The $2^k k! = 48$ elements $t_\pi \tau_\epsilon$ are distributed in the above way in the sets $c_1, \ldots, c_{15}$.

3 Basic formulas

Proposition 9 *Let B be in D and S be $\gamma_{p,\sigma}$ distributed on $\bar{\Omega}$. Then*

$$I\!E(S(B)) = \frac{1}{2^k} \sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} \sum_{\epsilon \in \mathbf{Z}_2^k} \sigma(l_{\tau_\epsilon t_\pi}(B)).$$

Proof: Introduce the symmetric matrix $h(a,b) = \frac{1}{2}(e(a,b) + e(b,a))$. We can now write for $B = (a_1, b_1, \ldots, a_k, b_k)$

$$\begin{aligned}
\mathbb{E}(S(B)) &\stackrel{(1)}{=} \mathbb{E}(S_{a_1 b_1} \ldots S_{a_k b_k}) \\
&\stackrel{(2)}{=} \mathbb{E}(\mathrm{tr}\,(Se(a_1,b_1)) \ldots \mathrm{tr}\,(Se(a_k,b_k))) \\
&\stackrel{(3)}{=} \mathbb{E}(\mathrm{tr}\,(Sh(a_1,b_1)) \ldots \mathrm{tr}\,(Sh(a_k,b_k))) \\
&\stackrel{(4)}{=} \sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)(h(a_1,b_1), \ldots, h(a_k,b_k)) \\
&\stackrel{(5)}{=} \sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} \frac{1}{2^k} \sum_{\epsilon \in \mathbf{Z}_2^k} r_\pi(\sigma)(e(a_1,b_1)^{\epsilon_1}, \ldots, e(a_k,b_k)^{\epsilon_k}) \\
&\stackrel{(6)}{=} \frac{1}{2^k} \sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} \sum_{\epsilon \in \mathbf{Z}_2^k} \sigma(l_{\tau_\epsilon t_\pi}(B)).
\end{aligned}$$

In this list of equalities, (1) comes from the definition of $S(B)$, (2) comes from (2.8), (3) comes from the symmetry of S, (4) comes from Proposition 1, (5) from linearity, and (6) from (2.10). ∎

The formula in Proposition 9 can be improved by observing that the numbers $\sigma(l_{\tau_\epsilon t_\pi}(B))$ are not all distinct: there are $2^k k!$ pairs (ϵ, π) and only $N < 2^k k!$ possible values $\sigma(C_i(B))$ of these numbers. By a slight abuse of notation we will write

$$\sigma(T(B)) = \sigma(l_{\tau_\epsilon t_\pi}(B)) = \sigma(l_{T'}(B)) = \sigma(C(B))$$

for a particular pair $T = (\epsilon, \pi)$ such that $T' = \tau_\epsilon t_\pi \in C \in \mathcal{S}_{2k}/H$. We will also write

$$m(j) = m(\pi_j)$$

for a particular pair $T_j = (\epsilon_j, \pi_j)$.

Theorem 10 *Let B be in D and S be $\gamma_{p,\sigma}$ distributed on $\bar{\Omega}$. Then*

$$I\!E(S(B)) = \sum_{j=1}^{N} \frac{p^{m(j)}}{2^{k-m(j)}} \sigma(T_j(B)). \tag{3.17}$$

Proof: Proposition 9 implies that

$$\mathbb{E}(S(B)) = \sum_{j=1}^{N} a_j \sigma(T_j(B)).$$

where, using the notation c_i defined in (2.16)

$$a_j = \sum_{(\epsilon,\pi)\in c_j} \frac{p^{m(\pi)}}{2^k}.$$

However, Proposition 4 shows that all the π such that there exists ϵ satisfying $(\epsilon, \pi) \in c_j$ have the same number of cycles $m(j)$. Furthermore, Theorem 8 implies that c_j has $2^{m(j)}$ elements. Thus finally

$$a_j = \frac{p^{m(j)}}{2^{k-m(j)}} \tag{3.18}$$

and the formula (3.17) is proved. ∎

Observe that Theorem 10 gives us a quick way of computing any moments of the form

$$\mathbb{E}(S_{a_1b_1}S_{a_2b_2}\dots S_{a_kb_k}).$$

We have $S_{ab} = \mathrm{tr}\,(Se_{ab})$ and (2.10) and Theorem 10 imply

Corollary 11 *For any matrices $m_1, \dots, m_k \in M$ we have*

$$I\!E(\mathrm{tr}\,(Sm_1)\dots\mathrm{tr}\,(Sm_k)) = \sum_{j=1}^{N} \frac{p^{m(j)}}{2^{k-m(j)}} r_{\pi_j}(\sigma)(m_1^{\epsilon_1(j)}, \dots m_k^{\epsilon_k(j)})$$

where $(\epsilon(j), \pi_j) = T_j$ are the particular pairs, $j = 1, \dots, N$.

Example. For $k = 3$ we made a list of particular pairs on page 8. An application of the Corollary 11 gives

$$\begin{aligned}\mathbb{E}(\mathrm{tr}\,(Sm_1)(\mathrm{tr}\,(Sm_2)\mathrm{tr}\,(Sm_3)) &= p^3\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_3)\\
+\frac{p^2}{2}[\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3) + \mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3^t) &+ \mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_1\sigma m_3)\\
+\mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_1\sigma m_3^t) + \mathrm{tr}\,(\sigma m_3)\mathrm{tr}\,(\sigma m_1\sigma m_2) &+ \mathrm{tr}\,(\sigma m_3)\mathrm{tr}\,(\sigma m_1\sigma m_2^t)]\\
+\frac{p}{4}[\mathrm{tr}\,(\sigma m_1\sigma m_2\sigma m_3) + \mathrm{tr}\,(\sigma m_1\sigma m_2\sigma m_3^t) + \mathrm{tr}\,(\sigma m_1\sigma m_2^t\sigma m_3) &+ \mathrm{tr}\,(\sigma m_1\sigma m_2^t\sigma m_3^t)\\
+\mathrm{tr}\,(\sigma m_1\sigma m_3\sigma m_2) + \mathrm{tr}\,(\sigma m_1\sigma m_3\sigma m_2^t) + \mathrm{tr}\,(\sigma m_1\sigma m_3^t\sigma m_2) &+ \mathrm{tr}\,(\sigma m_1\sigma m_3^t\sigma m_2^t)].\end{aligned}$$

Definition 12 *Let us fix an order of particular pairs $T_1, \ldots, T_N$ and denote by $T'_1, \ldots, T'_N$ the corresponding particular permutations $T'_j = \tau_{\epsilon(j)} t_{\pi_j}$. For $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, N\}$ denote by*

$$n = i \otimes j$$

the unique $n \in \{1, \ldots, N\}$ such that

$$T'_i T'_j \sim T'_n$$

i.e. $T'_i T'_j \in C_n$. Since for fixed $i \in \{1, \ldots, N\}$ the map $j \mapsto i \otimes j$ from $\{1, \ldots, N\}$ to itself is clearly injective, we denote by

$$n \oslash i = j$$

the unique j such that

$$n = i \otimes j.$$

We have now the following formulas for moments of the Wishart distribution:

Theorem 13 *Let a_j's be defined by (3.18). For all $i \in \{1, \ldots, N\}$ and for all $B \in D$*

$$\mathbb{E}(S(T_i(B)) = \sum_{n=1}^{N} a_{n \oslash i} \sigma(T_n(B)). \tag{3.19}$$

Proof: In Theorem 10 we replace B by $l_{T'_i}(B)$. We obtain

$$\mathbb{E}(S(T_i(B)) = \sum_{j=1}^{N} a_j \sigma(l_{T'_j} l_{T'_i}(B)) = \sum_{j=1}^{N} a_j \sigma(l_{T'_i T'_j}(B)) = \sum_{j=1}^{N} a_j \sigma(T_{i \otimes j}(B)) \tag{3.20}$$

and using Definition 12 we end the proof. ∎

In a similar way as to obtain Corollary 11 we can extend the formula (3.19) by linearity and we get

Corollary 14 *Let $a_j = a_j(p) = \frac{p^{m(j)}}{2^{k-m(j)}}$. For any matrices $m_1, \ldots, m_k \in M$ we have*

$$\mathbb{E}(r_{\pi_i}(S)(m_1^{\epsilon_1(i)}, \ldots, m_k^{\epsilon_k(i)})) = \sum_{j=1}^{N} a_j r_{\pi_{i \otimes j}}(\sigma)(m_1^{\epsilon_1(i \otimes j)}, \ldots, m_k^{\epsilon_k(i \otimes j)}) \tag{3.21}$$

or, equivalently,

$$\mathbb{E}(r_{\pi_i}(S)(m_1^{\epsilon_1(i)}, \ldots, m_k^{\epsilon_k(i)})) = \sum_{n=1}^{N} a_{n \oslash i} r_{\pi_n}(\sigma)(m_1^{\epsilon_1(n)}, \ldots, m_k^{\epsilon_k(n)}).$$

where $(\epsilon(n), \pi_n) = T_n$ are the particular pairs. The polynomials $A_{(\pi,\epsilon),(\pi',\epsilon')}$ in (1.3) are defined by

$$A_{(\pi_i,\epsilon(i)),(\pi_n,\epsilon(n))}(p) = a_{n \oslash i}(p) = \frac{p^{m_{n \oslash i}}}{2^{k-m_{n \oslash i}}}, \qquad i, n \in \{1, \ldots, N\}. \tag{3.22}$$

If in the Corollary 14 we take the m_i as symmetric matrices s_i, the role of the transpose vanishes and we get

Corollary 15 *For any symmetric matrices $s_1, \dots, s_k \in V$ we have*

$$\mathbb{E}(r_{\pi_i}(S)(s_1, \dots, s_k)) = \sum_{n=1}^{N} a_{n \oslash i} r_{\pi_n}(\sigma)(s_1, \dots, s_k). \tag{3.23}$$

Now, since we know that $i \mapsto \pi_i$ is surjective from $\{1, \dots, N\}$ on $\mathcal{S}_k$, we can get the polynomials $B_{\pi,\pi'}(p)$ in(1.4) as sums of suitable polynomials $a_i(p)$. Actually, a fixed permutation $\pi \in \mathcal{S}_k$ appears in $2^{k-m(\pi)}$ different particular pairs, so (3.23) becomes

$$\mathbb{E}(r_{\pi_i}(S)(s_1, \dots, s_k)) = \sum_{\pi' \in \mathcal{S}_k} B_{\pi_i,\pi'} r_{\pi_n}(\sigma)(s_1, \dots, s_k).$$

4 Applications of graphs

A main problem arising when we want to use practically the formulas of the preceding section is, given two particular permutations T_i' and T_j', how to identify a particular permutation T_n' such that $T_i'T_j' \sim T_n'$.

In the first part of this section we design a quick graph method allowing to identify the particular permutation $\tau_\epsilon t_\pi$ equivalent to a given permutation $g \in \mathcal{S}_{2k}$. Next we will illustrate its application to the Wishart distribution moment problem.

In the second part of this section we will give one more method based on graphs leading to practical computation of moments of the Wishart real laws.

Graph techniques in application to random matrices, permutations and spherical (zonal) polynomials have been fruitfully used eg. in [6], [7] and [10].

Definition 16 *Recall the notation $j^+ = j + k$. To a permutation $\pi \in \mathcal{S}_{2k}$ we associate a 1-regular graph*

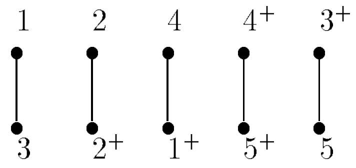
$$\eta(\pi) = \{\{\pi(1), \pi(1^+)\}, \{\pi(2), \pi(2^+)\}, \dots \{\pi(j), \pi(j^+)\}, \dots, \{\pi(k), \pi(k^+)\}\}.$$

In other words, $\eta(\pi)$ is the graph in which i is adjacent to j if and only if

$$|\pi^{-1}(i) - \pi^{-1}(j)| = k.$$

Definition 16 is analogous to that of [7] and [10].

Example. Let $k = 5$ and $\pi \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1^+ & 2^+ & 3^+ & 4^+ & 5^+ \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 4^+ & 3^+ \\ 3 & 2^+ & 1^+ & 5^+ & 5 \end{pmatrix}$.
Then in order to draw the graph $\eta(\pi)$ we just connect the elements in the columns:

$\eta(\pi) =$ 

Like in [6], denote the set of all 1-regular graphs with vertices $\{1, 2, \dots, 2k\}$ by $\mathcal{F}_k$.

Proposition 17 *For any $\pi_1, \pi_2 \in \mathcal{S}_{2k}$*

$$\pi_1 \sim \pi_2 \quad \Leftrightarrow \quad \pi_1 H = \pi_2 H \quad \Leftrightarrow \quad \eta(\pi_1) = \eta(\pi_2).$$

It follows that $\mathcal{S}_{2k}/H \equiv \mathcal{F}_k$.

Proof: This is simply a reformulation for the right cosets of Lemma 3.1(a) of [6] saying that $Hg_1 = Hg_2$ is equivalent to $\eta(g_1^{-1}) = \eta(g_2^{-1})$: in fact $\pi_1 H = \pi_2 H$ if and only if $H\pi_1^{-1} = H\pi_2^{-1}$. ∎

We show now in which way, given a graph $\eta \in \mathcal{F}_k$, one can easily determine the particular pair (ϵ, π) such that $\eta = \eta(\tau_\epsilon t_\pi)$. The identity graph

$$\eta(e) = \{\{1, 1^+\}, \{2, 2^+\}, \dots, \{k, k^+\}\}$$

is used in this algorithm.

Proposition 18 *Let $\eta \in \mathcal{F}_k$. Let γ be the 2-regular graph(ie. all its cycles have even length):*

$$\gamma = \eta(e) \cup \eta.$$

Then $\eta = \eta(\tau_\epsilon t_\pi)$ with $(\epsilon, \pi) \in \mathbf{Z}_2^k \times \mathcal{S}_k$, a particular pair obtained in the following way:

Consider the even length cycles γ_j, $j = 1, \dots, J \leq k$, whose disjoint union is equal to γ.

(1) **Identification of π.** *Any cycle γ_j of length $2q_j$ is of the form*

$$\gamma_j = (i_{j1}^{\overset{+}{\circ}}, i_{j1}^{\overset{\circ}{+}}, \dots, i_{jq_j}^{\overset{+}{\circ}}, i_{jq_j}^{\overset{\circ}{+}}), \quad i_{jl} \in \{1, \dots, k\} \tag{4.24}$$

where the notation of an ordered couple $i^{\overset{+}{\circ}} i^{\overset{\circ}{+}}$ means that $i^{\overset{+}{\circ}}, i^{\overset{\circ}{+}} \in \{i, i^+\}$ with $i^{\overset{+}{\circ}} = i$ implying $i^{\overset{\circ}{+}} = i^+$ and $i^{\overset{+}{\circ}} = i^+$ implying $i^{\overset{\circ}{+}} = i$.

We adopt the convention that the "starting point" $i_{j1}^{\overset{+}{\circ}} = \min\{i_{jl}^{+} \mid l = 1, \dots, q_j\}$. Set $\pi_j = (i_1, i_2, \dots, i_{q_j})$. Then $\pi = \prod_{j=1}^J \pi_j$.

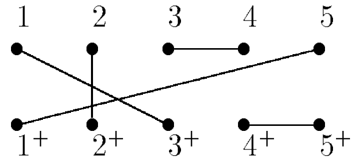
(2) **Identification of $\epsilon = (\epsilon_1, \dots, \epsilon_k)$.** *For a given $m \in \{1, \dots, k\}$ there exists one and only one i_{jl} in a cycle γ_j as in (4.24) above such that $m = i_{jl}$.*

Then $\epsilon_m = 1$ if the couple $i_{jl}^{\overset{+}{\circ}} i_{jl}^{\overset{\circ}{+}} = i_{jl}^{+} i_{jl}$ and $\epsilon_m = -1$ otherwise.

Before proving this Proposition, let us give an example illustrating it.

Example. Let us identify the particular pair corresponding to the following graph $\eta \in \mathcal{F}_k$:

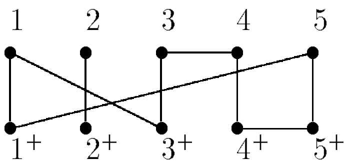
$\eta =$



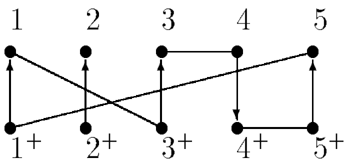
Let us follow the algorithm given in Proposition 18:
$\gamma_1 = (1^+, 1, 3^+, 3, 4, 4^+, 5^+, 5)$, $\gamma_2 = (2^+, 2)$
(1) Identification of π: $\pi_1 = (1,3,4,5)$, $\pi_2 = (2)$, $\pi = (1,3,4,5)(2)$
(2) Identification of ϵ: $\epsilon = (1,1,1,-1,1)$.

Let us note that the preceding steps may be done very quickly, simply by looking at graphs in the following way.

Add the graph $\eta(e)$ of the identity to the graph η:

$\gamma = \eta(e) \cup \eta =$ 

We immediately see the two cycles. We start going along the first cycle at $1^+ = \min\{1^+, 3^+, 4^+, 5^+\}$ and go to 1. We start the second cycle at 2^+. In the picture, this leads to

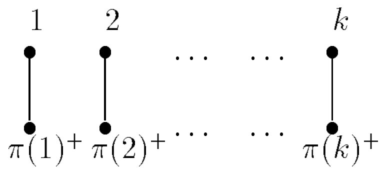


(we only put arrows when going along the edges $\{i, i^+\}$, which are vertical). The ϵ_i is equal to ± 1 if the edge $\{i, i^+\}$ is traced in the direction $i^+ \to i$ or $i \to i^+$ respectively (upwards or downwards respectively in the picture).

Therefore $\pi = (1,3,4,5)(2)$ and $\epsilon = (1,1,1,-1,1)$.

Proof: We know by Proposition 17 and Theorem 8 that all the graphs $\eta \in \mathcal{F}_k$ are obtained as graphs of particular permutations.

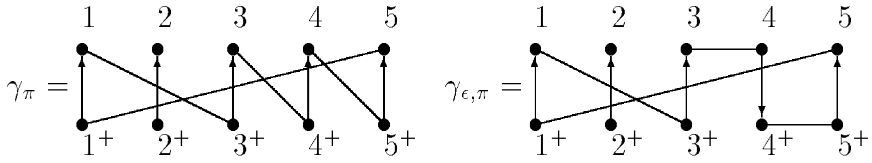
In the case $\eta = \eta(t_\pi)$ (ie. when $\epsilon = id$), the graph η is of the form

$\eta =$ 

and applying (1) of the Proposition, starting with an i^+ gives (4.24) equal to $(i^+, i, \pi(i)^+, \pi(i), \pi(\pi(i))^+, ...)$ and finally the cycle $(i, \pi(i), \pi(\pi(i)), ...)$ of π. This is equivalent to the decomposition of π into cycles. It is evident that the edges $\{m, m^+\}$ are all run over in the direction $m^+ \to m$. The proposition is true in this case.

The definition of τ_ϵ implies that the graph of $\tau_\epsilon t_\pi$ is obtained from the graph of t_π by interchanging the vertices m and m^+ whenever $\epsilon_m = -1$.

Consider first the case when only one $\epsilon_m = -1$. The couple m^+m in the decomposition (4.24) of the graph $\gamma_\pi = \eta(e) \cup \eta(t_\pi)$ is replaced by the couple mm^+ in the analogous decomposition of the graph $\gamma_{\epsilon,\pi} = \eta(e) \cup \eta(\tau_\epsilon t_\pi)$. It follows that the cycle π_j that we deduce from (4.24) is the same for γ_π and $\gamma_{\epsilon,\pi}$:*(1)* is proved valid and so is *(2)*.



Let us illustrate this part of the proof on an example. Take as in the last example $k = 5$, $\pi = (1, 3, 4, 5)$, $\epsilon = (1, 1, 1, -1, 1)$. Then

In the first case $\gamma_1 = (1^+, 1, 3^+, 3, 4^+, 4, 5^+, 5)$, $\gamma_2 = (2^+, 2)$, $\epsilon = id$.
In the second case $\gamma_1 = (1^+, 1, 3^+, 3, 4, 4^+, 5^+, 5)$, $\gamma_2 = (2^+, 2)$, $\epsilon = (1, 1, 1, -1, 1)$.
Geometrically, the horizontal edges $\{3, 4\}$ and $\{4^+, 5^+\}$ of the graph $\gamma_{\epsilon,\pi}$ are replaced by the oblique edges $\{3, 4^+\}$ and $\{4, 5^+\}$ of the graph γ_π. In order to pass from $\gamma_{\epsilon,\pi}$ to γ_π one performs a central reflection of the edge $\{4, 4^+\}$.

The general case now follows by decomposing $\epsilon \in \mathbf{Z}_2^k$ into a product of elements with only one entry equal to -1. It is clear by construction that (ϵ, π) is a particular pair. ∎

As an application of Proposition 18 let us compute the following moments in the case $k = 3$:

$$\mathbb{E}(\mathrm{tr}\,(Sm_1)\mathrm{tr}\,(Sm_2Sm_3))$$
$$\mathbb{E}(\mathrm{tr}\,(Sm_1Sm_2Sm_3.)$$

We apply the formula (3.21) from the Corollary 14 and we use the particular pairs listed on page 8. We get

$$\mathbb{E}(\mathrm{tr}\,(Sm_1)\mathrm{tr}\,(Sm_2Sm_3)) = p^3\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3)$$
$$+\frac{p^2}{2}\sum_{j=2}^{7} r_{\pi_{2\otimes j}}(\sigma)(m_1^{\epsilon_1(2\otimes j)}, m_2^{\epsilon_2(2\otimes j)}, m_3^{\epsilon_3(2\otimes j)}) + \frac{p}{4}\sum_{j=8}^{15} r_{\pi_{2\otimes j}}(\sigma)(m_1^{\epsilon_1(2\otimes j)}, m_2^{\epsilon_2(2\otimes j)}, m_3^{\epsilon_3(2\otimes j)}). \tag{4.25}$$

As the operation $i \mapsto i \otimes j$ is injective, we only have to compute the values of $2 \otimes j$ for $j = 2, \ldots, 7$, ie. first the compositions of permutations $T_2'T_j' = t_{(23)}\tau_{\epsilon_j}t_{\pi_j}$ and next the equivalent particular permutation $T_n' \sim T_2'T_j'$, using $t_\pi t_{\pi'} = t_{\pi\pi'}$ or, if needed, Proposition 18. For example, $T_2'T_2' = t_{(23)}t_{(23)} = e$, so $2 \otimes 2 = 1$.
$T_2'T_3' = t_{(23)}\tau_{(1,1,-1)}t_{(23)} = \begin{pmatrix} 1 & 2 & 2^+ \\ 1 & 3 & 3^+ \end{pmatrix} \sim T_3'$, so $2 \otimes 3 = 3$.
In a similar way we find $2 \otimes 4 = 8$, $2 \otimes 5 = 15$, $2 \otimes 6 = 12$ and $2 \otimes 7 = 11$.
The results of $2 \otimes 8, \ldots, 2 \otimes 15$ form the set $\{4, 5, 6, 7, 9, 10, 13, 14\}$. Finally

$$\begin{aligned}
& \mathbb{E}(\mathrm{tr}\,(Sm_1)\mathrm{tr}\,(Sm_2Sm_3)) = p^3\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3) \\
+ \quad & \frac{p^2}{2}\Big(\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_3) + \mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3^t) \\
+ \quad & \mathrm{tr}\,(\sigma m_1\sigma m_2\sigma m_3) + \mathrm{tr}\,(\sigma m_1\sigma m_2^t\sigma m_3^t) + \mathrm{tr}\,(\sigma m_1\sigma m_3\sigma m_2) + \mathrm{tr}\,(\sigma m_1\sigma m_3^t\sigma m_2^t)\Big) \\
+ \quad & \frac{p}{4}\Big(\mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_1\sigma m_3) + \mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_1\sigma m_3^t) \\
+ \quad & \mathrm{tr}\,(\sigma m_3)\mathrm{tr}\,(\sigma m_1\sigma m_2) + \mathrm{tr}\,(\sigma m_3)\mathrm{tr}\,(\sigma m_1\sigma m_2^t) \\
+ \quad & \mathrm{tr}\,(\sigma m_1\sigma m_2\sigma m_3^t) + \mathrm{tr}\,(\sigma m_1\sigma m_2^t\sigma m_3) + \mathrm{tr}\,(\sigma m_1\sigma m_3\sigma m_2^t) + \mathrm{tr}\,(\sigma m_1\sigma m_3^t\sigma m_2)\Big)
\end{aligned}$$

$$(4.26)$$

Similarly, we compute $8 \otimes 2 = 6,\ 8 \otimes 3 = 9,\ 8 \otimes 4 = 2,\ 8 \otimes 5 = 15,\ 8 \otimes 6 = 4$ and $8 \otimes 7 = 10$ and we deduce the formula

$$\begin{aligned}
& \mathbb{E}(\mathrm{tr}\,(Sm_1Sm_2Sm_3)) = p^3\mathrm{tr}\,(\sigma m_1\sigma m_2\sigma m_3) \\
+ \quad & \frac{p^2}{2}\Big(\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3) + \mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_1\sigma m_3) + \mathrm{tr}\,(\sigma m_3)\mathrm{tr}\,(\sigma m_1\sigma m_2) \\
+ \quad & \mathrm{tr}\,(\sigma m_1\sigma m_2\sigma m_3^t) + \mathrm{tr}\,(\sigma m_1\sigma m_2^t\sigma m_3) + \mathrm{tr}\,(\sigma m_1\sigma m_3^t\sigma m_2^t)\Big) \\
+ \quad & \frac{p}{4}\Big(\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_3) + \mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3^t) \\
+ \quad & \mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_1\sigma m_3^t) + \mathrm{tr}\,(\sigma m_3)\mathrm{tr}\,(\sigma m_1\sigma m_2^t) \\
+ \quad & \mathrm{tr}\,(\sigma m_1\sigma m_2^t\sigma m_3^t) + \mathrm{tr}\,(\sigma m_1\sigma m_3\sigma m_2) + \mathrm{tr}\,(\sigma m_1\sigma m_3\sigma m_2^t) + \mathrm{tr}\,(\sigma m_1\sigma m_3^t\sigma m_2)\Big).
\end{aligned} \tag{4.27}$$

It follows that for s_1, s_2, s_3 symmetric we have, using the additional fact that $\mathrm{tr}\,(\sigma s_1\sigma s_2\sigma s_3) = \mathrm{tr}\,(\sigma s_1\sigma s_3\sigma s_2)$:

$$\begin{aligned}
& \mathbb{E}(\mathrm{tr}\,(Ss_1)\mathrm{tr}\,(Ss_2Ss_3)) = (p^3 + \frac{p^2}{2})\mathrm{tr}\,(\sigma s_1)\mathrm{tr}\,(\sigma s_2\sigma s_3) \\
+ \quad & \frac{p^2}{2}\mathrm{tr}\,(\sigma s_1)\mathrm{tr}\,(\sigma s_2)\mathrm{tr}\,(\sigma s_3) + (2p^2 + p)\mathrm{tr}\,(\sigma s_1\sigma s_2\sigma s_3) \\
+ \quad & \frac{p}{2}\Big(\mathrm{tr}\,(\sigma s_2)\mathrm{tr}\,(\sigma s_1\sigma s_3) + \mathrm{tr}\,(\sigma s_3)\mathrm{tr}\,(\sigma s_1\sigma s_2)\Big)
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E}(\mathrm{tr}\,(Ss_1Ss_2Ss_3)) = (p^3 + \frac{3p^2}{2} + p)\mathrm{tr}\,(\sigma s_1\sigma s_2\sigma s_3) \\
+ \quad & (\frac{p^2}{2} + \frac{p}{4})\Big(\mathrm{tr}\,(\sigma s_1)\mathrm{tr}\,(\sigma s_2\sigma s_3) + \mathrm{tr}\,(\sigma s_2)\mathrm{tr}\,(\sigma s_1\sigma s_3) + \mathrm{tr}\,(\sigma s_3)\mathrm{tr}\,(\sigma s_1\sigma s_2)\Big) \\
+ \quad & \frac{p}{4}\mathrm{tr}\,(\sigma s_1)\mathrm{tr}\,(\sigma s_2)\mathrm{tr}\,(\sigma s_3).
\end{aligned} \tag{4.28}$$

We see in which way the polynomials $B_{\pi,\pi'}$ arise as sums of the monomials a_j equal in this case $p^3, p^2/2$ and $p/4$.

The above procedure, which we used to compute quickly "by hand" all the moments in the case $k = 3$, may be transformed into a **MAPLE** program computing the numbers $n = i \otimes j$ and $n \oslash i = j$. Already for $k = 4$ we have to deal with $N = 105$ particular pairs. When $k = 5$ the case of $s_1, \ldots, s_5$ symmetric can be treated in order to determine the matrix B (dimension 120×120).

More graphs' applications. We end this section by indicating one more possibility of using graph techniques in order to simplify and accelerate the computation of moments. This technique allows to avoid computation of compositions of permutations when applying Corollary 14. To this end we will use the double coset space

$$H\backslash\mathcal{S}_{2k}/H = \{\, HgH \mid g \in \mathcal{S}_{2k}\}.$$

Definition 19 *Let $\eta_1, \eta_2 \in \mathcal{F}_k$ be two 1-regular graphs. We denote by*

$$\Lambda(\eta_1, \eta_2)$$

the partition of the integer k whose parts are half the cycle-lengths of the graph $\eta_1 \cup \eta_2$.
If $g \in \mathcal{S}_{2k}$ then we write

$$\Lambda(g) = \Lambda(\eta(g), \eta(e)).$$

Let M_n be the set of all partitions of n. For a permutation $g \in \mathcal{S}_n$ denote by $\mathbf{m}(g) \in M_n$ the partition notation of the portrait of g. For example, if $g = 12(3,4)(5,6)(7,8,9,10) \in \mathcal{S}_{10}$ then $\mathbf{m}(g) = (1,1,2,2,4)$ is a partition of $n = 10$ determining uniquely the portrait of g.

Proposition 20 *(i) $g_1 \in \mathcal{S}_{2k}$ and $g_2 \in \mathcal{S}_{2k}$ belong to the same double class in $H \backslash \mathcal{S}_{2k} / H$ if and only if $\Lambda(g_1) = \Lambda(g_2)$.*
(ii) Let $g, w \in \mathcal{S}_{2k}$. Then $\Lambda(gw^{-1}) = \Lambda(\eta(g), \eta(w))$.
(iii) Let $\pi \in \mathcal{S}_k$ and $\epsilon \in \mathbf{Z}_2^k$. Then $\Lambda(\tau_\epsilon t_\pi) = \mathbf{m}(\pi)$.

Proof: The properties (i) and (ii) go back to James[7], see also [10, page 46] and [6, Lemma 3.1(b)]. Proposition 18 implies (iii). ∎

Let us now describe a graph method of computation of moments of the real Wishart law based on Proposition 20.

Fix $i \in \{1, \ldots, N\}$. The idea is to compute the sets of results $i \otimes j$ (and not all the $i \otimes j$ separately), when T_j''s belong to the same double coset HgH, ie. when $\Lambda(T_j') = \mathbf{m}$ is fixed. We are looking for n_j's such that $i \otimes j = n_j$ which is equivalent to $T_i' T_j' = g_j$ with $g_j \sim T_{n_j}'$.

Proposition 21 *Let $\mathbf{m} \in M_n$ and $i \in \{1, \ldots, N\}$ be fixed. Then*

$$\Lambda(T_j') = \mathbf{m} \text{ and } T_i' T_j' = g_j \in \mathcal{S}_{2k} \quad \Leftrightarrow \quad \mathbf{m} = \Lambda(\delta(T_i'), \delta(g_j))$$

where we write $\delta(g) = \eta(g^{-1})$.

Proof: We have $T_i' T_j' = g_j \Leftrightarrow T_j' = T_i'^{-1} g_j$ and when $\Lambda(T_j') = \mathbf{m}$ is fixed this is equivalent to

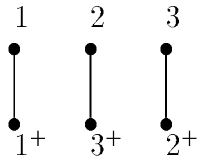
$$\mathbf{m} = \Lambda(T_j') = \Lambda(T_i'^{-1} g_j) = \Lambda(\eta(T_i'^{-1}), \eta(g_j^{-1})) = \Lambda(\delta(T_i'), \delta(g_j)).$$

∎

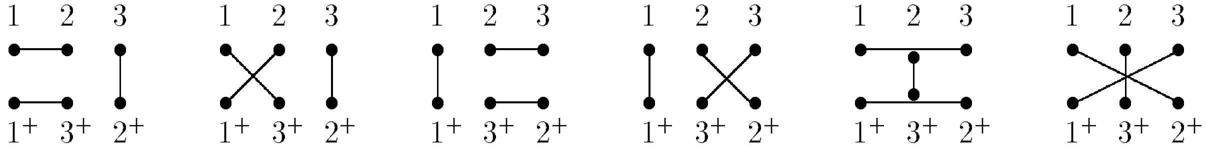
Remark. Note that $\delta(g)$ is what is called the graph of $g \in \mathcal{S}_{2k}$ in [6]).

Using graphs it is easy to determine the set of graphs $\delta(g_j)$ such that $\Lambda(\delta(T_i'), \delta(g_j)) = \mathbf{m}$ for i and $\mathbf{m}$ given. Next Proposition 18 may be used in order to determine the n_j's knowing the graphs $\delta(g_j) = \eta(g_j^{-1})$. Moreover one may easily reformulate Proposition 18 in order to determine the particular pair (ϵ, π) such that $\tau_\epsilon t_\pi \sim g$ knowing $\delta(g) = \eta(g^{-1})$. In part (1) of Proposition 18 just change the direction of running over the cycles by starting with $i \to i^+$ and in part (2) put $\epsilon_m = 1$ if the couple $i_{jl}^{\overset{+}{\circ}} i_{jl}^{\overset{\circ}{+}} = i_{jl} i_{jl}^{+}$ and $\epsilon_m = -1$ otherwise.

Let us illustrate this method for $k = 3$ and the computation of $\mathbb{E}(\mathrm{tr}\,(Sm_1)\mathrm{tr}\,(Sm_2Sm_3))$. In the formula (4.25) the coefficients $j = 2, \dots, 7$ correspond to $\Lambda(T'_j) = (2,1) \in M_3$. We have $T'_2 = t_{(23)} = T_2'^{-1}$ and the graph $\delta(T'_2) = \eta(T'_2)$ is

$\delta(T'_2) =$ 

The six 1-regular graphs $\delta(g_j)$ are



Next we quickly identify the particular permutations equivalent to the g_j's as $T'_{15}, T'_8, T'_3, T'_1, T'_{11}, T'_{12}$ respectively. As before, we deduce that the results of $2 \otimes 8, \dots, 2 \otimes 15$ form the set $\{4, 5, 6, 7, 9, 10, 13, 14\}$ and we obtain the formula (4.26).

When $k = 4$ the number of portraits is 5 and the computation of a moment formula boils down to identifying of 56 graphs, 56 being the number of permutations with portrait $\mathbf{m} = (2,1,1), (2,2)$ or $(3,1)$. Let us write two of these formulas, for s_1, s_2, s_3, s_4 symmetric (the "full" formulas for m_1, m_2, m_3, m_4 have 105 terms). We use the fact that $r_\pi(\sigma)(s_1, \dots, s_k) = r_{\pi^{-1}}(\sigma)(s_1, \dots, s_k)$ when $s_1, \dots, s_k$ are symmetric and we abbreviate the notation $r_\pi(\sigma)(s_1, \dots, s_k) = r_\pi$.

$$
\begin{aligned}
&\mathbb{E}(\mathrm{tr}\,(Ss_1)\mathrm{tr}\,(Ss_2)\mathrm{tr}\,(Ss_3Ss_4)) = (p^4 + \frac{p^3}{2})r_{12(34)} \\
&+\frac{p^3}{2}r_{id} + (p^3 + \frac{p^2}{2})r_{(12)(34)} + (2p^3 + p^2)(r_{1(234)} + r_{2(134)}) \\
&+\frac{p^2}{2}(r_{(12)34} + r_{14(23)} + r_{13(24)} + r_{23(14)} + r_{24(13)}) + 2p^2 r_{(1324)} + 4p^2 r_{(1342)} \\
&+\frac{p}{2}(r_{(13)(24)} + r_{(14)(23)}) + p(r_{3(124)} + r_{4(123)}) + 3p r_{(1234)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}(\mathrm{tr}\,(Ss_1Ss_2Ss_3Ss_4)) = (p^4 + 2p^3 + \frac{7}{4}p^2 + \frac{1}{2}p)r_{(1234)} + (\frac{p^3}{2} + \frac{p^2}{2} + \frac{p}{8})(r_{(12)(34)} + r_{(14)(23)}) \\
&+(\frac{p^3}{2} + \frac{3}{4}p^2 + \frac{p}{2})(r_{1(234)} + r_{2(134)} + r_{3(124)} + r_{4(123)}) + (\frac{p^3}{2} + \frac{7}{4}p^2 + p)(r_{(1324)} + r_{(1324)}) \\
&+(\frac{p^2}{4} + \frac{3p}{8})r_{(13)(24)} + (\frac{p^2}{4} + \frac{p}{8})(r_{12(34)} + r_{13(24)} + r_{14(23)} + \dots + r_{34(12)}) + \frac{p}{8}r_{id}.
\end{aligned}
$$

5 Applications of groups and representations

In Corollary 14 we saw that the matrix $(A_{ij})_{i,j=1,\dots,N}$ containing the coefficients of all the formulas for the moments of a Wishart law on real symmetric matrices, where we put

$$A_{ij} = A_{(\pi_i,\epsilon(i)),(\pi_j,\epsilon(j))}(p)$$

is determined by the formula (3.22), ie.

$$A_{ij}(p) = a_{j\oslash i}(p) = \frac{p^{m_{j\oslash i}}}{2^{k-m_{j\oslash i}}}, \qquad i,j \in \{1,\dots,N\}. \tag{5.29}$$

(recall that $N = |S_{2k}/H| = 1 \times 3 \times \dots (2k-1)$.)

Our goal in this section is to present a mathematical interpretation of the matrix A and to explain in which way the representation theory of the symmetric group may be used in order to determine it.

Definition 22 *Let $\lambda \in M_k$ be a partition of k. We define a square $N \times N$ matrix E_λ in the following way:*

$$E_\lambda = (e_{ij})$$

where $e_{ij} = 1$ if $\mathbf{m}(\pi_{j\oslash i}) = \lambda$ and $e_{ij} = 0$ otherwise.

Note that $E_{1,1,\dots,1} = \mathrm{Id}$, the identity matrix. We also observe that

$$\sum_{\lambda \in M_k} E_\lambda = \mathbf{1}$$

where $\mathbf{1}$ denotes a $N \times N$ matrix with all coefficients equal to 1.

Proposition 23 *The matrices E_λ are symmetric.*

Proof: Fix $\lambda \in M_k$. By Propositions 18(1) and 20 and using the fact that $\Lambda(\pi) = \Lambda(\pi^{-1})$ we have $e_{ij} = 1 \Leftrightarrow \Lambda(T_i'^{-1}T_j') = \lambda \Leftrightarrow \Lambda(T_j'^{-1}T_i') = \lambda \Leftrightarrow e_{ji} = 1$. ∎

For $\lambda \in M_k$ let us denote by m_λ the number of non-zero terms of the partition λ. If a permutation $\pi \in \mathcal{S}_k$ has a portrait indexed by the partition λ then the cycles' number is $m(\pi) = m_\lambda$. Note that the sum of coefficients in any column of the matrix E_λ is equal to the number n_λ of particular pairs (ϵ, π) with $\mathbf{m}(\pi) = \lambda$.

The following theorem follows from (5.29).

Theorem 24 *The matrix A can be represented as a following weighted sum of matrices E_λ:*

$$A = \sum_{\lambda \in M_k} \frac{p^{m_\lambda}}{2^{k-m_\lambda}} E_\lambda \ . \tag{5.30}$$

In particular, the matrix A is symmetric.

For example, if $k=3$ then the 15×15 matrix A equals to

$$A = p^3 \,\mathrm{Id} + \frac{p^2}{2}\, E_{2,1} + \frac{p}{4}\, E_3$$

and if $k=4$ then the 105×105 matrix A decomposes

$$A = p^4 \,\mathrm{Id} + \frac{p^3}{2}\, E_{2,1,1} + \frac{p^2}{4}\,(E_{2,2} + E_{3,1}) + \frac{p}{8} E_4.$$

Thus, in order to find the matrix A it is sufficient to determine the matrices E_λ.

The graph method based on Propositions 20 and 21 in Section 4 may be used to compute the E_λ's. When $k=3$ we obtain

$$E_{2,1} = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix} \tag{5.31}$$

and we compute easily $A = p^3 \,\mathrm{Id} + \frac{p^2}{2}\, E_{2,1} + \frac{p}{4}\,(\mathbf{1} - \mathrm{Id} - E_{2,1})$.

Kates([10]) used an analogous graph technique in double coset space $H \backslash \mathcal{S}_{2k}/H$ setting, in order to compute the spherical functions of the Gelfand pair $(\mathcal{S}_{2k}, H)$ and to obtain in this way the coefficients of zonal polynomials.

Let the $p(k) \times p(k)$ matrices C_λ(where $p(k)$ is the number of different partitions of k) be the following "contractions" of the matrices E_λ, passing from one-sided cosets gH to the double cosets HgH:
in order to compute the coefficient $c_{\mu\nu}$ of C_λ ($\mu, \nu \in M_k$) choose any column of E_λ, say the j-th, with $\mathbf{m}(\pi_j) = \nu$ and sum the coefficients of this column in the lines indexed by the i's such that $\mathbf{m}(\pi_i) = \nu$.

The eigenvectors of C_λ are the spherical functions on $(\mathcal{S}_{2k}, H)$ and Kates([10]) constructed the matrices C_λ using graph techniques.

Observe also that if we call $\tilde{A}$ the $p(k) \times p(k)$ matrix obtained by the above "contraction" from the matrix A then

$$\mathbb{E}(r_\mu(S)(m,\ldots,m)) = \sum_{\nu \in M_k} \tilde{a}_{\mu\nu} r_\nu(\sigma)(m,\ldots,m)$$

where we write r_μ instead of any r_π with π having the portait μ. The matrix $\tilde{A} = \sum_{\lambda \in M_k} \frac{p^{m_\lambda}}{2^{k-m_\lambda}} C_\lambda$ may be independently deduced from the results of [11]. We may understand the matrix A as an *éclatement* of the matrix $\tilde{A}$.

By an analogous convenient contraction of the matrix A we may compute the matrix B defined in (1.4). For $k = 3$ we may of course just use Proposition 1 and the formulas (4.26), (4.27), (4.28) and we obtain

$$B = \begin{pmatrix} p^3 & p^2 & p^2 & p^2 & p & p \\ \frac{p^2}{2} & p^3 + \frac{p^2}{2} & \frac{p}{2} & \frac{p}{2} & p^2 + \frac{p}{2} & p^2 + \frac{p}{2} \\ \frac{p^2}{2} & \frac{p}{2} & p^3 + \frac{p^2}{2} & \frac{p}{2} & p^2 + \frac{p}{2} & p^2 + \frac{p}{2} \\ \frac{p^2}{2} & \frac{p}{2} & \frac{p}{2} & p^3 + \frac{p^2}{2} & p^2 + \frac{p}{2} & p^2 + \frac{p}{2} \\ \frac{p}{4} & \frac{p}{4} + \frac{p^2}{2} & \frac{p}{4} + \frac{p^2}{2} & \frac{p}{4} + \frac{p^2}{2} & p^3 + p^2 + \frac{p}{4} & \frac{p^2}{2} + \frac{3p}{4} \\ \frac{p}{4} & \frac{p}{4} + \frac{p^2}{2} & \frac{p}{4} + \frac{p^2}{2} & \frac{p}{4} + \frac{p^2}{2} & \frac{p^2}{2} + \frac{3p}{4} & p^3 + p^2 + \frac{p}{4} \end{pmatrix}$$

where the permutations π in the vectors $[\mathbb{E}(r_\pi(S))]$ and $[r_\pi(\sigma)]$ are ordered as follows: id, (23), (13), (12), (123), (132).

Now we pass to the main subject of this section, the discussion of applications of the group and representation theory to the matrices E_λ and A.

Let $\mathcal{A}(\mathcal{S}_{2k})$ be the algebra of $\mathcal{S}_{2k}$, ie. the real linear space of functions from $\mathcal{S}_{2k}$ to $\mathbb{R}$. Define the convolution of functions on $\mathcal{S}_{2k}$ by

$$f * g(x) = \sum_{y \in \mathcal{S}_{2k}} f(xy^{-1}) g(y) = \sum_{y \in \mathcal{S}_{2k}} f(y) g(y^{-1}x)$$

and let $R(g)$ denote the (right) convolution operator $f \mapsto f * g$ on the group algebra $\mathcal{A}(\mathcal{S}_{2k})$.

Suppose that g is a right H-invariant function on $\mathcal{S}_{2k}$. Then

$$R(g) : \mathcal{A}(\mathcal{S}_{2k}) \ \to \ \mathcal{A}(\mathcal{S}_{2k}/H)$$

where we denote by $\mathcal{A}(\mathcal{S}_{2k}/H)$ the space of right H-invariant functions on $\mathcal{S}_{2k}$. It follows that if ϕ is an eigenvector of $R(g)$ corresponding to a non-zero eigenvalue then ϕ is right H-invariant.

Let $\lambda \in M_k$. Denote by $\delta_\lambda \in \mathcal{A}(\mathcal{S}_{2k})$ the indicator function of the double coset $D_\lambda = \{w \in \mathcal{S}_{2k} \mid \Lambda(w) = \lambda\} \in H \backslash \mathcal{S}_{2k} / H$ (see Proposition 20(i)). We have $\delta_\lambda \in \mathcal{A}(H \backslash \mathcal{S}_{2k} / H)$. We write $\delta'_\lambda = \frac{1}{|H|} \delta_\lambda$.

Similarly, denote by $\delta_{C_i} \in \mathcal{A}(\mathcal{S}_{2k}/H)$ the indicator function of a coset $C_i \in \mathcal{S}_{2k}/H$, $i = 1, \ldots, N$, and by δ'_{C_i} this indicator function normalized by $\delta'_{C_i} = \frac{1}{|H|} \delta'_{C_i}$ (observe that $|C_i| = |H|$.) The algebra $\mathcal{A}(\mathcal{S}_{2k}/H)$ can be identified with $\mathbb{R}^N$ and $\{\delta'_{C_i}\}_{i=1,\ldots,N}$ is a canonical linear basis in it.

Proposition 25 *Let $\mathcal{R}_\lambda = R(\delta'_\lambda)|_{\mathcal{A}(\mathcal{S}_{2k}/H)}$. The matrix of the operator*

$$\mathcal{R}_\lambda : \mathcal{A}(\mathcal{S}_{2k}/H) \to \mathcal{A}(\mathcal{S}_{2k}/H)$$

in the basis $\{\delta'_{C_i}\}_{i=1,\ldots,N}$ is equal to the matrix E_λ ($\lambda \in M_k$).

Proof: In order to determine the coefficient m_{ij} of the matrix of the operator $\mathcal{R}_\lambda$ on $\mathcal{A}(\mathcal{S}_{2k}/H)$ we write

$$\mathcal{R}_\lambda(\delta'_{C_j}) = \sum_l m_{lj}\delta'_{C_l}$$

and we observe that

$$\sum_{z\in C_i} \mathcal{R}_\lambda(\delta'_{C_j})(z) = m_{ij}$$

which means that

$$\sum_{z\in C_i} \delta_{C_j} * \delta_{Hg_\lambda H}(z) = |H|^2 m_{ij}$$

where g_λ is any element of the double coset D_λ.

Let $T_i = (\epsilon_i, \pi_i) \in \mathbf{Z}_2^k \times \mathcal{S}_k$ be a particular pair such that the particular permutation $T'_i = \tau_{\epsilon_i} t_{\pi_i} \in C_i$. Then $C_i = T'_i H$ and we compute the left-hand side expression in the last formula. We have

$$\begin{aligned}\sum_{z\in C_i} \delta_{C_j} * \delta_{Hg_\lambda H}(z) &= |\{(x,y) \in T'_j H \times Hg_\lambda H \mid xy = z \in T'_i H\}| \\ &= |\{(x,z) \in T'_j H \times T'_i H \mid x^{-1}z = y \in Hg_\lambda H\}| \\ &= |H| \times |\{z \in T'_i H \mid T'^{-1}_j z \in Hg_\lambda H\}| \\ &= |H| \times |\{z \in T'_i H \mid z^{-1} T'_j \in Hg_\lambda H\}|.\end{aligned}$$

Now, if $T'^{-1}_i T'_j \in Hg_\lambda H$ then

$$|\{z \in T'_i H \mid z^{-1} T'_j \in Hg_\lambda H\}| = |H|$$

because all the elements $z = T'_i h$, $h \in H$, verify $z^{-1}T'_j \in Hg_\lambda H$.

For the same reason, if $T'^{-1}_i T'_j \notin Hg_\lambda H$ then

$$|\{z \in T'_i H \mid z^{-1} T'_j \in Hg_\lambda H\}| = 0.$$

This implies that $m_{ij} = 1$ if and only if $T'^{-1}_i T'_j \in D_\lambda$ which is equivalent, by Propositions 18(1) and 20 to $j = i \otimes l$ with $\Lambda(T'_l) = \Lambda(T'_{j\oslash i}) = \mathbf{m}(\pi_{j\oslash i}) = \lambda$. ∎

Remarks. 1. Kates([10, p.44-46]) showed in a similar way that the matrix of the operator $R(\delta'_\lambda)|_{\mathcal{A}(H\backslash\mathcal{S}_{2k}/H)} : \mathcal{A}(H\backslash\mathcal{S}_{2k}/H) \to \mathcal{A}(H\backslash\mathcal{S}_{2k}/H)$ in the basis $\{\frac{1}{|D_\lambda|}\delta_\lambda\}$ is equal to C_λ.

2. It is easy to verify that the operators $\mathcal{R}_\lambda$ are symmetric (ie. $\langle \mathcal{R}_\lambda f, g\rangle = \langle f, \mathcal{R}_\lambda g\rangle$, for all $f \in \mathcal{A}(\mathcal{S}_{2k}/H)$), which gives another proof of Proposition 23.

Let us resume Theorem 24 and Proposition 25.

Corollary 26 *The matrix $A(p)$ is a linear combination of matrices E_λ of right convolution operators with H-biinvariant functions $\frac{1}{|H|}\delta_\lambda$, in canonical basis $\{\frac{1}{|H|}\delta_{C_i}\}$ on $\mathcal{A}(\mathcal{S}_{2k}/H)$:*

$$A = \sum_{\lambda\in M_k} a(\lambda) E_\lambda, \quad \text{with } a(\lambda) = p^{m_\lambda}/2^{k-m_\lambda}.$$

The symmetric matrices $(E_\lambda)_{\lambda\in M_k}$ and $(A(p))_{p>0}$ commute. Hence, they have common eigenvectors.

Proof: The functions δ_λ are H-biinvariant and $(\mathcal{S}_{2k}, H)$ is a Gelfand pair, so the operators $R(\delta_\lambda)$ commute and so do the $\mathcal{R}_\lambda$. ∎

A new method of computation of the matrices E_λ and A may be deduced from Corollary 26. The coefficients of the matrix E_λ may be computed as

$$e_{ij} = \frac{1}{|H|}\delta_{C_j} * \delta_{Hg_\lambda H}(T_i').$$

This formula, tedious in direct computation, has the advantage of being very simple to be programmed on a computer.

Our objective now is to determine the eigenvectors and the eigenvalues of the matrix A and to provide in this way an alternative method of its construction.

Note that the functions $f = \delta_\lambda$ are not central. Consider for instance $k = 3$ and 2 permutations $\pi_1 = (123)(456)$ and $\pi_2 = (145)(236)$ from $\mathcal{S}_{2k}$. Then $\Lambda(\pi_1) = (1,1,1)$ (π_1 is in the double coset of identity) whereas $\Lambda(\pi_2) = (3)$. We have $\delta_{(3)}(\pi_1) = 0 \neq 1 = \delta_{(3)}(\pi_2)$ but π_1 and π_2 have the same portrait $(3,3)$.

The knowledge of the character theory and the character tables of the group $\mathcal{S}_{2k}$ is not sufficient for our problem. A dimension argument confirms it: for example, for $k = 3$ we have 11 characters on $\mathcal{S}_{2k}$ but the matrix A has $N = 15$ eigenvectors.

The functions $f = \delta'_\lambda$ form a basis of $\mathcal{A}(H\backslash\mathcal{S}_{2k}/H)$, so the eigenvectors of the matrices E_λ are the coefficients of eigenfunctions of all convolution operators $R(f)$, $f \in \mathcal{A}(H\backslash\mathcal{S}_{2k}/H)$, in the basis $\{\delta'_{C_i}\}$.

We need some information from the representation theory, taking into account the fact that $(\mathcal{S}_{2k}, H)$ is a Gelfand pair(see e.g. [1], [16]).

Let (π, V_π, u_π) be an irreducible H-spherical representation of $\mathcal{S}_{2k}$. This means that there exists a vector $u_\pi \in V_\pi$, $u_\pi \neq 0$, which is invariant by H: $\pi(h)u_\pi = u_\pi$ for all $h \in H$. Such a vector is called spherical and is unique up to a scalar factor. One also knows that u_π is cyclic, ie. the vector space generated by its orbit $\pi(\mathcal{S}_{2k})u_\pi$ is equal to V_π. Denote by $(\cdot,\cdot)$ the scalar product on V_π such that all the operators $\pi(g)$ are unitary on V_π.

Denote by $\hat{\mathcal{S}}_{2k}^H$ the set of all irreducible H-spherical representations of $\mathcal{S}_{2k}$.

Theorem 27 *All eigenfunctions of the operators $R(f)$, $f \in \mathcal{A}(H\backslash\mathcal{S}_{2k}/H)$ are obtained as the functions*

$$\phi_v^\pi(g) = (\pi(g^{-1})v\,,\, u_\pi), \qquad v \in V_\pi$$

where (π, V_π, u_π) is an irreducible H-spherical representation of $\mathcal{S}_{2k}$. The functions ϕ_v^π are right H-invariant.

We have $R(f)\phi_v^\pi = \phi_v^\pi * f = \hat{f}(\pi)\phi_v$ and the eigenvalue $\hat{f}(\pi)$ is given by

$$\hat{f}(\pi) = \frac{1}{|\mathcal{S}_{2k}|}\sum_{g\in\mathcal{S}_{2k}} \varphi^\pi(g^{-1})f(g),$$

where $\varphi^\pi(g) = \frac{1}{\|u_\pi\|^2}(\pi(g^{-1})u_\pi\,,\, u_\pi)$ is the H-biinvariant spherical function induced by π. In particular, all the eigenfunctions ϕ_v^π of $R(f)$ correspond to the same eigenvalue $\hat{f}(\pi)$.

Note that if $\{v_i\}_{i=1,\dots,d_\pi}$ is a basis of V_π(we write $\dim V_\pi = d_\pi$, the dimension of the representation π) then the fact that u_π is cyclic implies that the eigenfunctions $\phi^\pi_{v_i}$ are independent. On the other hand any function ϕ^π_v is a linear combination of the $\phi^\pi_{v_i}$'s, so each irreducible spherical representation π gives rise to d_π independent common eigenfunctions of the operators $\mathcal{R}_\lambda$.

The orthogonality of the decomposition of the regular representation of $\mathcal{S}_{2k}$ on $\mathcal{A}(\mathcal{S}_{2k})$ into irreducible representations guarantees that the union of eigenvectors coming from different representations is an independent family. In particular

$$N = \sum_{\pi \in \hat{\mathcal{S}}^H_{2k}} d_\pi. \tag{5.32}$$

If $f \in \mathcal{A}(\mathcal{S}_{2k}/H)$ then denote by $c(f) = [c_1, \dots, c_N]^\top$ the vector of coefficients of f in the basis δ'_{C_i}:

$$f = \sum_{i=1}^{N} c_i \, \delta'_{C_i}.$$

Theorem 27, the remarks after it and Corollary 26 imply the following theorem which is the main result of this section.

Theorem 28 *The matrix A has N independent eigenvectors*

$$c(\phi^\pi_v), \quad \pi \in \hat{\mathcal{S}}^H_{2k}, \quad v \in B_\pi,$$

where B_π is a linear basis of V_π.

The eigenvalue of A corresponding to the vector $c(\phi^\pi_v)$ is equal to

$$e_\pi = \sum_{\lambda \in M_k} a(\lambda) \hat{\delta}'_\lambda(\pi), \quad \pi \in \hat{\mathcal{S}}^H_{2k}$$

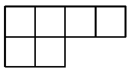
where $a(\lambda) = p^{m(\lambda)}/2^{k-m(\lambda)}$.

We will see that the eigenvalues e_π are non zero and pairwise different.

Applications of the theory of representations of the symmetric group. In order to go further with our representation-theoretic approach of the matrices A and E_λ and to give examples we will now use some information on the construction of the irreducible representations of $\mathcal{S}_k$.

Apart classical books on this theory ([8], [14]) let us cite some other very accessible references ([4], [13], [15]).

Recall briefly that all the irreducible representations of $\mathcal{S}_k$ are indexed by the partitions $\mathbf{m} \in M_k$. To each partition $\mathbf{m}$ there corresponds a Young diagram. For example, if $\mathbf{m} = (4, 2)$ then the corresponding Young diagram is



$$\begin{array}{|c|c|c|c|}\hline 6&4&1&3\\ \hline 5&2\\ \cline{1-2}\end{array}$$

If we put a permutation of $\{1, \ldots, k\}$ into such a diagram we obtain a Young tableau, for example

If t is a Young tableau, denote by R_t the isotropy group of its lines and by C_t the isotropy group of its columns. For example, if t is as above, then $R_t = \mathcal{S}_{\{1,3,4,6\}} \times \mathcal{S}_{\{2,5\}}$ and $C_t = \mathcal{S}_{\{5,6\}} \times \mathcal{S}_{\{2,4\}}$. One calls *tabloids* the equivalence classes of tableaux with respect to the equivalence relation

$$t_1 \sim t_2 \quad \Leftrightarrow \pi t_1 = t_2$$

for a permutation $\pi \in R_t$. In other words, two tableaux correspond to the same tabloid if and only if the elements of their lines are identical. For example, the tableau above belongs to the tabloid

$$\begin{array}{cccc}\hline 1&3&4&6\\ \hline 2&5\\ \cline{1-2}\end{array}$$

We denote by $\{t\}$ the tabloid of t. Let $\mathcal{T}_{\mathbf{m}}$ be the set of all tabloids corresponding to the Young diagram of $\mathbf{m}$. The symmetric group acts on $\mathcal{T}_{\mathbf{m}}$ and $\pi\{t\} = \{\pi t\}$ for $\pi \in \mathcal{S}_k$.

We still denote by $\mathcal{A}(\mathcal{T}_{\mathbf{m}})$ the vector space of all real functions on $\mathcal{T}_{\mathbf{m}}$. Of course we may embed $\mathcal{A}(\mathcal{T}_{\mathbf{m}}) \subset \mathcal{A}(\mathcal{S}_k)$.

The irreducible representation $\pi_{\mathbf{m}}$ of $\mathcal{S}_k$ is the (left) regular representation of $\mathcal{S}_k$ restrained to the subspace $F_{\mathbf{m}} \subset \mathcal{A}(\mathcal{T}_{\mathbf{m}}) \subset \mathcal{A}(\mathcal{S}_k)$ defined as the linear span of all the functions

$$e_t = \sum_{\pi \in C_t} \epsilon_\pi \, \delta_{\pi\{t\}}$$

where t is any Young tableau corresponding to $\mathbf{m}$ and $\delta_{\pi\{t\}} \in \mathcal{A}(\mathcal{T}_{\mathbf{m}})$ is the indicator function of the tabloid $\pi\{t\}$. The functions e_t are sometimes called *polytabloids* and the space $F_{\mathbf{m}}$ the *Specht module* corresponding to the partition $\mathbf{m}$.

A linear basis of the irreducible space $F_{\mathbf{m}}$ is given by so called standard polytabloids e_t, ie. the functions e_t corresponding to the tableaux t with elements increasing in all lines and columns of the diagram.

Let us illustrate all these concepts on our example $k = 6$ and $\mathbf{m} = (4, 2)$. There are 15 tabloids in $\mathcal{T}_{(4,2)}$ (we may identify them with the elements of the second line of the Young diagram of $(4, 2)$ and one may choose 2 elements among 6 in 15 different ways). We will write δ_{ij} for the indicator function of the tabloid containing i and j in the second line.

The standard tableaux are the following:

$$\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\ \hline 2&4\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\ \hline 2&5\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\ \hline 2&6\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\ \hline 3&4\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\ \hline 3&5\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\ \hline 3&6\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\ \hline 4&5\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\ \hline 4&6\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 5&6\\ \cline{1-2}\end{array}$$

Let us denote these standard tableaux by $t_{24}, \ldots, t_{56}$ and the corresponding polytabloid functions by $e_{ij} = e_{t_{ij}}$. In order to compute e_{24} we remark that the column isotropy group $C_{t_{24}} = \mathcal{S}_{\{1,2\}} \times \mathcal{S}_{\{3,4\}}$ has 4 elements. The definition of e_t then yields

$e_{24} = \delta_{24} - \delta_{14} - \delta_{23} + \delta_{13}$. We compute in this way the standard linear basis of the Specht module $F_{(4,2)}$:

$$\begin{aligned}
e_{24} &= \delta_{24} - \delta_{14} - \delta_{23} + \delta_{13}, & e_{25} &= \delta_{25} - \delta_{15} - \delta_{23} + \delta_{13}, & e_{26} &= \delta_{26} - \delta_{16} - \delta_{23} + \delta_{13},\\
e_{34} &= \delta_{34} - \delta_{14} - \delta_{23} + \delta_{12}, & e_{35} &= \delta_{35} - \delta_{15} - \delta_{23} + \delta_{12}, & e_{36} &= \delta_{36} - \delta_{16} - \delta_{23} + \delta_{12},\\
e_{45} &= \delta_{45} - \delta_{15} - \delta_{24} + \delta_{12}, & e_{46} &= \delta_{46} - \delta_{16} - \delta_{24} + \delta_{12}, & e_{56} &= \delta_{56} - \delta_{16} - \delta_{25} + \delta_{12}.
\end{aligned}$$

Let us come back to the study of the H-spherical representations of $\mathcal{S}_{2k}$.

Theorem 29 *The H-spherical representations of the symmetric group $\mathcal{S}_{2k}$ are exactly the representations $\pi_{\mathbf{m}} \in \hat{\mathcal{S}}_{2k}$ such that*

$$\frac{1}{2}\mathbf{m} \in M_k.$$

Equivalently, all terms of the partition $\mathbf{m}$ are even numbers. In particular, $|\hat{\mathcal{S}}_{2k}^H| = p(k)$.

Proof: This theorem is known and used in the statistical literature(see eg.[10]) and may be proved using eg. the results in [3, Ch.12.10] (in particular Lemma 12.10.6).

Let us give here an elementary constructive proof of sufficiency of the condition $\frac{1}{2}\mathbf{m} \in M_k$. We will construct a spherical vector for the representation of $\mathcal{S}_{2k}$ corresponding to a fixed partition $\mathbf{m}$ of $2k$ with all terms even.

Consider the set ω of $k!$ tableaux $\mathbf{t}_\pi$ obtained by filling up the Young diagram of $\mathbf{m}$ by the ordered pairs $\{j, j^+\}$, $j = 1, \ldots, k$, in all possible ways according to all permutations $\pi \in \mathcal{S}_k$, line after line, from the left to the right and from the top to the bottom of the diagram.

Denote by $\mathcal{T}_o$ the set of tabloids corresponding to tableaux $\mathbf{t} \in \omega$.

The group H is generated by the transpositions (j, j^+) and by the permutations s_σ, $\sigma \in \mathcal{S}_k$. The action of (j, j^+) on a tabloid $\{\mathbf{t}\} \in \mathcal{T}_o$ does not change it. The action of s_σ:

$$s_\sigma(\{\mathbf{t}_\pi\}) = \{\mathbf{t}_{\sigma\pi}\}$$

can change the tabloid, but the image of $\mathcal{T}_o$ in this action is equal to $\mathcal{T}_o$. It follows that the indicator function $\delta_{\mathcal{T}_o} \in \mathcal{A}(\mathcal{T}_{\mathbf{m}})$ is an H-invariant function:

$$\pi(h)\delta_{\mathcal{T}_o} = \delta_{\mathcal{T}_o}, \quad \forall h \in H.$$

Let $\mathbf{t}_o \in \omega$ be the tableau corresponding to $\pi = e$ (identity). It is easy to see that the scalar product

$$(e_{\mathbf{t}_o}, \delta_{\mathcal{T}_o}) \neq 0. \tag{5.33}$$

Actually, if the indicator function $\delta_{\{\mathbf{t}\}}$ of an element $\{\mathbf{t}\} \in \mathcal{T}_o$, $\mathbf{t} \in \omega$, appears in the sum defining $e_{\mathbf{t}_o}$ then the corresponding permutation $\pi \in C_{\mathbf{t}_o}$ has positive signature (π must permute the whole pairs $\{j, j^+\}$).

Let u be the orthogonal projection of the function $\delta_{\mathcal{T}_o}$ on the Specht module $F_{\mathbf{m}}$. It follows from (5.33) that $u \neq 0$.

If the orthogonal decomposition of $\delta_{\mathcal{T}_o}$ is $\delta_{\mathcal{T}_o} = w+u$, $w \in F_{\mathbf{m}}^{\perp}$, then $\langle \pi(h)w, \pi(h)u\rangle = 0$. As $u \in F_{\mathbf{m}}$, a representation space for $\mathcal{S}_{2k}$, we have $\pi(h)u \in F_{\mathbf{m}}$. Now the H-invariance of $\delta_{\mathcal{T}_o}$ and the unicity of orthogonal decomposition imply that $\pi(h)u = u$ for all $h \in H$. Thus the irreducible representation $\pi_{\mathbf{m}}$ of $\mathcal{S}_{2k}$ on $F_{\mathbf{m}}$ is H-spherical.

In order to see the necessity of the condition $\frac{1}{2}\mathbf{m} \in M_k$ one uses a dimensionality argument(see [3]; cf. (5.32)). ∎

By Theorems 28 and 29 we see that the matrix A may have at most $p(k)$ different eigenvalues $e_{\pi_{2\mathbf{m}}}$, $\mathbf{m} \in M_k$. One may say more about these eigenvalues. Recall the definition of the gamma function of the real symmetric cone, equal, up to a factor,

$$\Gamma_{\Omega}(\mathbf{s}) = \prod_{j=1}^{r} \Gamma(s_j - \frac{1}{2}(j-1)).$$

Theorem 30 *The eigenvalues of the matrix A are given by*

$$e_{\pi_{2\mathbf{m}}} = \frac{\Gamma_{\Omega}(p+\mathbf{m})}{\Gamma_{\Omega}(p)}, \quad \mathbf{m} \in M_k.$$

In particular the matrix A has exactly $p(k)$ eigenvalues and they are all different from zero when $\frac{\Gamma_{\Omega}(p+\mathbf{m})}{\Gamma_{\Omega}(p)} \neq 0$. The matrix A is then non-singular.

Proof: It follows from the duality theorem of Kates ([10, Th.8, p.38]) that if $\mathbf{m} \in M_k$ and $\varphi_{2\mathbf{m}}$ is the spherical function on $(\mathcal{S}_{2k}, H)$ with $c(\varphi_{2\mathbf{m}}) = [c_1, \ldots, c_N]^\top$, that is $\varphi_{2\mathbf{m}} = \sum_{j=1}^{N} c_j \delta'_{C_j}$, then for any $\sigma \in \Omega$ the spherical polynomial $\Phi_{\mathbf{m}}$ on $\Omega = GL(n, \mathbb{R})/O(n)$ verifies

$$\Phi_{\mathbf{m}}(\sigma) = \sum_{j=1}^{N} c_j r_{\pi_j}(\sigma)(e, \ldots, e). \tag{5.34}$$

By Theorem 28, as the function $\varphi_{2\mathbf{m}}$ is a linear combination of the functions $\phi_v^{\pi_{2\mathbf{m}}}$, $v \in B_\pi$, $A\,c(\varphi_{2\mathbf{m}}) = e_{\pi_{2\mathbf{m}}} c(\varphi_{2\mathbf{m}})$ holds. It means that for all $j = 1, \ldots, N$

$$\sum_{k=1}^{N} a_{jk} c_k = e_{\pi_{2\mathbf{m}}} c_j.$$

If we multiply the last equality by $r_{\pi_j}(\sigma)(e, \ldots, e)$ and sum over j we get, using the symmetry of the matrix A and (5.34)

$$\mathbb{E}\Phi_{\mathbf{m}}(S) = e_{\pi_{2\mathbf{m}}} \Phi_{\mathbf{m}}(\sigma).$$

On the other hand we know([2]) that $\mathbb{E}\Phi_{\mathbf{m}}(S) = \frac{\Gamma_{\Omega}(p+\mathbf{m})}{\Gamma_{\Omega}(p)} \Phi_{\mathbf{m}}(\sigma)$. ∎

For any k the 1-dimensional trivial representation of $\mathcal{S}_{2k}$ corresponding to the partition $(2k)$ of $2k$ yields the constant eigenvector $[1, \ldots, 1]$ of the matrix A which corresponds to the eigenvalue $e_{\pi_{(2k)}} = \Gamma(p+k)/\Gamma(p) = p(p+1)\ldots(p+k-1)$ of A.

In order to construct other eigenvectors of A we can use Theorem 28, by constructing spherical vectors $u_{\pi_{2\mathbf{m}}}$ according to the method given in the proof of Theorem 29. Theorem 30 provides the corresponding eigenvalues of A.

$$\begin{array}{l} \hline 1\ 1^+ 2\ 2^+ \\ \hline 3\ 3^+ \\ \hline \end{array} \quad \begin{array}{l} \hline 1\ 1^+ 3\ 3^+ \\ \hline 2\ 2^+ \\ \hline \end{array} \quad \begin{array}{l} \hline 2\ 2^+ 3\ 3^+ \\ \hline 1\ 1^+ \\ \hline \end{array}$$

Example. Consider $k = 3$ and $\mathbf{m} = (2, 1) \in M_3$. Then $2\mathbf{m} = (4, 2)$. The subset $\mathcal{T}_o$ of $\mathcal{T}_{(4,2)}$ (we refer to the proof of Theorem 29) contains 3 tabloids

and the H-invariant function $\delta_{\mathcal{T}_o}$ is equal to

$$\delta_{\mathcal{T}_o} = \delta_{14} + \delta_{25} + \delta_{36}.$$

Its orthogonal projection on the Specht module $F_{(4,2)}$ is equal, up to a scalar factor $-\frac{1}{5}$, to

$$u = [1, 1, -4, 1, 1, 1, 1, -4, 1, 1, 1, -4, 1, 1, 1]^\top$$

in the basis $\delta_{12}, \delta_{13}, \ldots, \delta_{56}$ of the space $\mathcal{A}(\mathcal{T}_{(4,2)})$ (this is very quick to find with XMaple, using the *GramSchmidt* function).

Note that the orthogonal decomposition of $\delta_{\mathcal{T}_o}$ is $\delta_{\mathcal{T}_o} = \frac{1}{5}[1, \ldots, 1] - \frac{1}{5}u$; the $F_{(4,2)}^\perp$ component of $\delta_{\mathcal{T}_o}$ is a constant function. The coefficients of the vector u in the standard basis $\{e_{24}, \ldots, e_{56}\}$ of $F_{(4,2)}$ are $[3, -3, 1, 1, 1, -4, 1, 1, 1]$ and do not present any special regularity.

Let us compute the eigenvector $c(\phi_{e_{24}}^{\pi(4,2)})$ of the matrix A, corresponding to the eigenvalue $e_{\pi_{(4,2)}} = \frac{\Gamma_\Omega(p+(2,1))}{\Gamma_\Omega(p)} = p(p+1)(p-\frac{1}{2})$. Up to a factor $|H|$, the N coefficients of this vector are given by

$$(e_{t_{\pi_j}^{-1}\tau_{\epsilon(j)}\mathbf{t}}, u), \quad j = 1, \ldots, N$$

where $(\epsilon(j), \pi_j)$ are particular pairs and $\mathbf{t} = t_{24}$, the first standard tableau. We obtain an eigenvector

$$v_{24} = 5 \times [1, 1, 2, 0, -1, -1, 0, 0, 1, 0, 0, -1, -2, 1, -1]^\top.$$

Similarly we compute

$$\begin{aligned}
v_{25} &= \ 5 \times [-1, 0, 1, -1, -2, 1, 0, 1, 2, 0, 0, 0, -1, 1, -1]^\top \\
v_{26} &= \ 5 \times [0, -1, 1, 1, -1, 0, 0, -1, 1, 0, 0, 1, -1, 2, -2]^\top \\
v_{34} &= \ 5 \times [1, 1, 2, -1, 0, 0, -1, -1, 1, -2, -1, 0, 0, 1, 0]^\top \\
v_{35} &= \ 5 \times [0, -1, 1, 0, 0, 1, -1, 1, 2, -1, -2, -1, 0, 1, 0]^\top \\
v_{36} &= \ 5 \times [-1, 0, 1, 1, 0, -1, -2, 0, 1, -1, -1, 1, 0, 2, 0]^\top \\
v_{45} &= \ 5 \times [0, 0, 0, 0, 0, 2, -2, 1, 1, -1, -1, 1, 1, -1, -1]^\top \\
v_{46} &= \ 5 \times [0, 0, 0, 1, -1, 1, -1, 0, -1, -1, -2, 2, 1, 1, 0]^\top \\
v_{56} &= \ 5 \times [1, 0, -1, 2, 1, 0, -1, 0, 0, -2, -1, 1, -1, 1, 0]^\top.
\end{aligned}$$

The 9 vectors $v_{24}, \ldots, v_{56}$ are a basis of the eigenspace of A corresponding to the eigenvalue $p(p+1)(p-\frac{1}{2})$.

In order to find the eigenspace for the eigenvalue $e_{\pi_{(2,2,2)}} = \frac{\Gamma_\Omega(p+(1,1,1))}{\Gamma_\Omega(p)} = p(p-1)(p-\frac{1}{2})$ we could work with the representation $\pi_{(2,2,2)}$ of $\mathcal{S}_6$ but it is simpler to determine the orthogonal complement of the linear space generated by the constant vector $l = [1, \ldots, 1]^\top$ and the vectors $v_{24}, \ldots, v_{56}$. An orthogonal basis of the eigenspace for the eigenvalue $p(p-1)(p-\frac{1}{2})$ is given by

$$
\begin{aligned}
w_1 &= [\ 4, -2, -2, -2, -2, -2, -2, 1, 1, 1, 1, 1, 1, 1, 1]^\top \\
w_2 &= [\ 0, 2, -2, 0, 0, 0, 0, -1, 1, 1, -1, -1, 1, 1, -1]^\top \\
w_3 &= [\ 0, 0, 0, 2, -2, 0, 0, -1, 1, -1, 1, -1, 1, -1, 1]^\top \\
w_4 &= [\ 0, 0, 0, 0, 0, 2, -2, -1, -1, 1, 1, -1, -1, 1, 1]^\top \\
w_5 &= [\ 0, 0, 0, 0, 0, 0, 0, 1, -1, -1, 1, -1, 1, 1, -1]^\top.
\end{aligned}
$$

Now, in order to compute the matrix A, put the vectors $l, v_{24}, \ldots, v_{56}, w_1, \ldots, w_5$ into the columns of a matrix P. Writing $e_1 = e_{\pi_{(6)}} = p(p+1)(p+2)$, $e_2 = e_{\pi_{(4,2)}} = p(p+1)(p-\frac{1}{2})$ and $e_3 = e_{\pi_{(2,2,2)}} = p(p-1)(p-\frac{1}{2})$ and defining $D = \mathrm{diag}(e_1, e_2, \ldots, e_2, e_3, \ldots, e_3)$ where e_2 appears 9 times and e_3 5 times, we have

$$A = P\, D\, P^{-1}.$$

Let us also observe that the contraction procedure of the eigenvectors of the matrix A, obtained in a similar way as the matrix B is obtained from the matrix A, ie. by summing up the terms indexed by particular pairs (ϵ, π) with the same permutations $\pi \in \mathcal{S}_k$, yields the eigenvectors of the matrix B, without changing the eigenvalue. For $k = 3$ we get in this way one eigenvector $[1, 2, 2, 2, 4, 4]^\top$ for e_1, 4 independent eigenvectors $[1, 3, -1, -1, 1, -3]^\top, [1, -1, 3, -1, -3, 1]^\top, [1, 3, -1, -1, -3, 1]^\top, [-1, 1, 1, -3, -1, 3]^\top$ for e_2 and one eigenvector $[4, -4, -4, -4, 4, 4]^\top$ for the eigenvalue e_3. This allows us to construct the matrix B as $B = R\,\mathrm{diag}(e_1, e_2, e_2, e_2, e_2, e_3)\, R^{-1}$ where R is the matrix with the above eigenvectors in the columns.

Remark. Another method of determining the eigenvectors of the matrix A corresponding to the eigenvalue $e_{\pi_{2\mathbf{m}}} = \frac{\Gamma_\Omega(p+\mathbf{m})}{\Gamma_\Omega(p)}$, could be proposed. It is based on the knowledge of the spherical functions $\varphi_{2\mathbf{m}}$ on $(\mathcal{S}_{2k}, H)$ which are equal to the vectors of coefficients of zonal polynomials $\Phi_{\mathbf{m}}$, which are in turn accessible in tables.

If $\Phi_{\mathbf{m}}(\sigma) = \sum_{i=1}^{p(k)} d_i\, r_i(\sigma)(e, \ldots, e)$ then, by [10], $\varphi_{2\mathbf{m}} = \sum_{j=1}^{N} c_j \delta'_{C_j}$ with

$$c_j = d_i / n_i \quad \text{when } C_j \subset D_i$$

and n_i is equal to the number of cosets $C_j = T'_j H$ contained in the double coset D_i.

For example, when $k = 3$ and $\mathbf{m} = (2, 1)$ then the coefficients of $\Phi_{(2,1)}$ are 1,1,-2 and the numbers n_i are equal to $1, 6, 8$. It follows that

$$c(\varphi_{2\mathbf{m}}) = [1, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{-2}{8}, \frac{-2}{8}, \frac{-2}{8}, \frac{-2}{8}, \frac{-2}{8}, \frac{-2}{8}, \frac{-2}{8}, \frac{-2}{8}].$$

This is one eigenvector of the matrix A corresponding to the eigenvalue $e_2 = \frac{\Gamma_\Omega(p+(2,1))}{\Gamma_\Omega(p)}$. We should find 9 independent eigenvectors.

The irreducible representation space for the representation $\pi_{2\mathbf{m}}$ may be realised as the linear span of all $\mathcal{S}_{2k}$-translates of $\varphi_{2\mathbf{m}}$:

$$\varphi^g_{2\mathbf{m}} : \ x \to \varphi_{2\mathbf{m}}(gx), \quad g \in \mathcal{S}_{2k}$$

(in the preceding considerations it was realized as the Specht module $F_{2\mathbf{m}}$). As the functions $\varphi^g_{2\mathbf{m}}$ are right H-invariant, it would be sufficient to find $d_{\pi_{2\mathbf{m}}}$ permutations g such that the functions $\varphi^g_{2\mathbf{m}}$ are independent.

What we did using Theorem 28 was to find $d_{\pi_{2\mathbf{m}}}$ independent eigenvectors using Theorem 27 and the explicit construction of the standard basis of $F_{2\mathbf{m}}$. What we suggest in this Remark is that if we find $d_{\pi_{2\mathbf{m}}}$ permutations $g \in \mathcal{S}_{2k}$ such that the functions $\varphi^g_{2\mathbf{m}}$ are independent then we reach our objective, too.

Let us come back to the example $k = 3$ and $\mathbf{m} = (2,1)$. If we take $g = t_{(23)}$ then the functions $\varphi_{2\mathbf{m}}$ and $\varphi^g_{2\mathbf{m}}$ are independent since, up to a factor $|H|$ we have $\varphi_{2\mathbf{m}}(T_1') = 1$, $\varphi_{2\mathbf{m}}(T_2') = \frac{1}{6}$ and $\varphi^g_{2\mathbf{m}}(T_1') = \frac{1}{6}$, $\varphi^g_{2\mathbf{m}}(T_2') = 1$. One may try to add more and more permutations to $\{id, t_{(23)}\}$ until one obtains 9 independent functions $\varphi^g_{2\mathbf{m}}$. We do not think a systematic method of choosing the g's may be found. This is the intrinsic knowledge of the structure of the representation $\pi_{2\mathbf{m}}$ which must intervene.

6 Appendix

Theorem 31 *Let Ω be a symmetric cone of positive definite real symmetric or complex hermitian matrices. Let V be the associated Jordan algebra.*

Let $(A_1, ..., A_l) \in \Omega^l$ and $(A_1', ..., A_l') \in \Omega^l$. Then

$$\exists u \in K \quad A_j' = u.A_j, \ j = 1, ..., l \ \Leftrightarrow \ \forall m_1, ..., m_j \in \{1, ..., l\} \quad \mathrm{tr}(A_{m_1} ... A_{m_j}) = \mathrm{tr}(A'_{m_1} ... A'_{m_j})$$

Proof: $\Rightarrow$ is evident.

The $\Leftarrow$ direction follows from Specht's Theorem in the real case (cf. [9], Theorems 63 and 65). ∎

Corollary 32 *The algebra $I(V)^K_k$ of non-commutative K-invariant polynomials in k variables is generated by the traces of the words in $h_1, \ldots, h_k \in V$.*

Theorem 33 *Let $\mathcal{E}_k$ be the space of k-multilinear forms on V spanned by the forms*

$$(h_1, ..., h_k) \in V^k \mapsto r_\pi(e)(h_1, ..., h_k), \quad \pi \in \mathcal{S}_k.$$

Then the space $\mathcal{E}_k$ is equal to the space $\mathcal{F}_k$ of K-invariant k-multilinear forms on V.

Remark. We talk about the K-invariance in the following sense:

denoting $u.h = uhu^*$, $h \in V$, $u \in K$, if $f \in \mathcal{F}_k$ then

$$f(u.h_1, ..., u.h_k) = f(h_1, ..., h_k).$$

Proof: Obviously $\mathcal{E}_k \subset \mathcal{F}_k$. In order to prove the other inclusion we use the Corollary 32. THE SECOND AUTHOR PROMISED TO EXPLAIN THIS PROOF IN DETAILS. ∎

Now we come back to the space $\mathcal{P}_k$ defined in the Introduction.

Corollary 34 *The linear space $\mathcal{P}_k$ is generated by $(r_\pi)_{\pi\in\mathcal{S}_k}$, where*

$$r_\pi(x, h_1, \ldots, h_k) = r_\pi(x)(h_1, \ldots, h_k).$$

Proof: THE SECOND AUTHOR PROMISED TO EXPLAIN THIS PROOF IN DETAILS. ∎

We finish by the formal definition of the Wishart expectation operators T_p and by showing directly the commutativity of T_p's for different p's. This is more general than Corollary 26 because $(r_\pi)_{\pi\in\mathcal{S}_k}$ may not be independent (they are always a generating family for $\mathcal{P}_k$). When they are independent, the Corollary 26 implies the Theorem 36 (the matrices $A(p)$ are then the matrices of the operators T_p in the basis $(r_\pi)_{\pi\in\mathcal{S}_k}$).

Definition 35 (of "Wishart expectation" operators).
Let T_p be an operator on $\mathcal{P}_k$ defined by

$$T_pF(\sigma; h_1, ..., h_k) = \mathbb{E}F(W_{p,\sigma}, h_1, ..., h_k)$$

where $W_{p,\sigma}$ is a random variable with Wishart distribution with parameters p, σ.

Theorem 36 *If p and $p' \in \Lambda$(= Gindikin set) then the operators T_p and $T_{p'}$ commute.*

Proof: Let $f(x) = f_\theta(x) = e^{\langle\theta,x\rangle}$. It is sufficient to show that for all $\theta \in -\Omega$

$$T_{p'} \circ T_p(f)(\sigma) = T_p \circ T_{p'}(f)(\sigma)$$

(we skip $h_1, ..., h_k$ in the notation). We have

$$T_{p'} \circ T_p(f)(\sigma) = \int_\Omega \int_\Omega e^{\langle\theta,x\rangle}\gamma_{p',y}(dx)\gamma_{p,\sigma}(dy).$$

Consider first $\sigma = e$ and use $y = P(y^{1/2})(e)$. Knowing that $g.\gamma_{p,\sigma} = \gamma_{p,g.\sigma}$ we get

$$\begin{aligned} T_{p'} \circ T_p(f)(e) &= \int\int e^{\langle\theta,x\rangle}\gamma_{p',P(y^{1/2})(e)}(dx)\gamma_{p,e}(dy) \\ &= \int\int e^{\langle\theta,P(y^{1/2})x\rangle}\gamma_{p',e}(dx)\gamma_{p,e}(dy). \end{aligned}$$

It is well known that if X and Y are two independent K-invariant random variables then the laws of $P(X^{1/2})Y$ and $P(Y^{1/2})X$ are equal. We can apply this to our last formula because the measures $\gamma_{p',e}$ and $\gamma_{p,e}$ are K-invariant. It follows that

$$T_{p'} \circ T_p(f)(e) = T_p \circ T_{p'}(f)(e).$$

In the general case

$$\begin{aligned} T_{p'} \circ T_p(f)(\sigma) &= \int\int e^{\langle\theta,x\rangle}\gamma_{p',y}(dx)\gamma_{p,P(\sigma^{1/2})(e)}(dy) \\ &= \int\int e^{\langle\theta,x\rangle}\gamma_{p',P(\sigma^{1/2})y}(dx)\gamma_{p,e}(dy) = \int\int e^{\langle\theta,P(\sigma^{1/2})x\rangle}\gamma_{p',y}(dx)\gamma_{p,e}(dy) \\ &= T_{p'} \circ T_p(f_{P(\sigma^{1/2})\theta})(e) \end{aligned}$$

and we conclude thanks to the first step of the proof. ∎

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