All invariant moments of the Wishart distribution

Gérard LETAC
Université Paul Sabatier
Toulouse, 31062 France

Hélène MASSAM
York University, Toronto
Canada, M3J 1P3

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Abstract

In this paper we compute the expectations $\mathbb{E}(Q(U))$ where U is a random matrix following the Wishart distribution $W_r(n,\Sigma)$ and where $Q(u)$ is a polynomial with respect to the entries of the symmetric matrix u, invariant in the sense that it depends only on the eigenvalues of the matrix u. We obtain in particular $\mathbb{E}(U^{\pm k})$ for $k = 1,2,\ldots$. For instance

$$\mathbb{E}(U^3) = n\Sigma(\text{tr }\Sigma)^2 + (n^2+n)(\Sigma\text{tr }(\Sigma^2) + 2\Sigma^2\text{tr }\Sigma) + (n^3+3n^2+4n)\Sigma^3.$$

Our results are extended to Hermitian matrices.

We give an extremely simple algorithm to compute these expected values, along with a Maple program to implement this algorithm.

1 Introduction

The Wishart distribution is the distribution of the sample variance-covariance matrix of a multivariate Gaussian model. As such, it is fundamental to inference in multivariate statistics. In particular its moments and the moments of its inverse are needed to approximate the distribution of many test statistics. For example, Krishnamoorty and Mathew (1999) use the first and second moments of the Wishart and the inverse Wishart in approximation methods for computing tolerance factors for a multivariate Gaussian population. The inverse Wishart distribution has also been used in discriminant analysis (see Das Gupta (1968) and Haff (1982)), in obtaining moments of the maximum likelihood estimator on a growth curve model (von Rosen (1988), (1991), (1997)) and in spectral estimation (see Shaman (1980)). In Bayesian statistics, the inverse Wishart distribution is used as a prior distribution for the covariance parameter when sampling from a multivariate Gaussian distribution (see Anderson (1984, §8.7.7) or Dawid (1981)). Some distributions related to the Wishart have been introduced in the more recent theory of graphical Markov models. The moments of these distributions are needed in evaluating the strength of the correlation between variables. They can be deduced quite easily from the moments of the Wishart distribution. It is therefore necessary to be able to compute the moments of the Wishart and inverse Wishart distribution.

In the last forty years, there has been a steady interest in the extension of the Gaussian distribution from $\mathbb{R}^r$ to $\mathbb{C}^r$ (see Goodman (1963)). To these Gaussian distribution on $\mathbb{C}^r$ are naturally associated complex Wishart distributions on the cone of non negative complex Hermitian (r,r) matrices. While the extension to complex numbers was originally motivated by the study of complex stationary Gaussian processes in signal processing (see for instance Maiwald and Kraus (2000)), a new motivation for the complex Wishart is found in the study of random matrices (Haagerup and Thorbjørsen (1998)) used in physics, in free probability and even in number theory, where explicit formulas for the moments of complex Wishart are extremely useful (See Graczyk, Letac and Massam (2003). Going from complex to quaternionic case as done by Andersson(1975) and Hanlon, Stanley and Stembridge (1992)) is a natural and relatively easy step.

Our aim in this paper is to give the explicit expression of certain moments of the Wishart matrix U and its inverse U^{-1}. We work here with real or complex matrices, but a simple change of parameter in the formulas will give also the quaternionic case. After the theoretical developpements in this paper, we give a simple Maple program to actually carry out the computations.

Let us describe the problem in the space of symmetric matrices with real entries. We denote by Ω the cone of positive definite elements of the space V of (r,r) real symmetric matrices and by $\overline{\Omega}$ the cone of the non negative elements of V, which is the closure of Ω. For p in

$$\Lambda = \{\frac{1}{2}, 2\frac{1}{2}, \ldots, (r-1)\frac{1}{2}\} \cup ((r-1)\frac{1}{2}, \infty) \tag{1.1}$$

and for σ in Ω, a random variable $U = (U_{ij})_{1\leq i,j\leq r}$ in $\overline{\Omega}$ is said to follow the Wishart distribution $\gamma_{p,\sigma}$ if, for all $s \in \Omega$, we have

$$\mathbb{E}(e^{-\text{tr }(sU)}) = (\det(I_r + \sigma s))^{-p}.$$

For $p > \frac{r-1}{2}$, the Wishart distribution has density

$$\gamma_{p,\sigma}(dx) = C\frac{(\det x)^{p-\frac{r+1}{2}}}{(\det \sigma)^p} \exp -\text{tr }(\sigma^{-1}x)dx,$$

where C is a multiple of the generalized Gamma function and where tr means trace. It is well known (see Muirhead (1982) or Eaton (1983)) that $\mathbb{E}(U) = p\sigma$ and that for $p > (r+1)/2$, $\mathbb{E}(U^{-1}) = \sigma^{-1}/(p - \frac{r+1}{2})$. This gives us the scalar moments $\mathbb{E}(U_{ij})$ and $\mathbb{E}((U^{-1})_{ij})$. Given a positive integer k, if for $l = 1, \ldots, k$, (i_l, j_l) is in $\{1, \ldots, r\}^2$, we call the scalar

$$\mathbb{E}(U_{i_1,j_1} \ldots U_{i_k,j_k}) \tag{1.2}$$

a *moment of* U of order k. Similarly, if $p > \frac{r-1}{2} + k$, we can see that $U^{-1} = ((U^{-1})_{ij})_{1\leq i,j,\leq r}$ exists and

$$\mathbb{E}((U^{-1})_{i_1 j_1} \ldots (U^{-1})_{i_k j_k}) \tag{1.3}$$

exists. The expected value (1.3) is called a *moment of* U^{-1}. We note that (1.2) and (1.3) can be expressed in a coordinate-free way if we consider the inner product on V defined by $\langle x, y\rangle = \text{tr }(xy)$. Indeed, if $h_1, \ldots, h_k$ are in V, by choosing h_l appropriately, we have $\langle U, h_l\rangle = U_{i_l j_l}$ and the scalars

$$\mathbb{E}(\langle U, h_1\rangle \ldots \langle U, h_k\rangle) \tag{1.4}$$

and

$$\mathbb{E}(\langle U^{-1}, h_1\rangle \dots \langle U^{-1}, h_k\rangle) \tag{1.5}$$

are moments of type (1.2) and (1.3) respectively. Any moment of the Wishart or its inverse can be expressed under these forms and therefore to compute (1.2) and (1.3), it suffices to find an expression for these quantities in function of the parameters p and σ and of the indices $(i_l, j_l), l = 1, \dots, k$. We give the explicit expression of (1.4) in Theorem 3.1 below. There we see that $\mathbb{E}(\text{tr } (Uh_1) \dots \text{tr } (Uh_k))$ is expressed in terms of quantities such as $\text{tr } (\sigma h_1) \dots \text{tr } (\sigma h_k)$ or $\text{tr } (\sigma h_1 \sigma h_2) \text{tr } (\sigma h_3 \sigma h_4 \sigma h_5) \text{tr } (\sigma h_6 \dots \sigma h_k)$ which, more generally, can be denoted by $r_\pi(\sigma)(h_1, \dots, h_k)$ defined as follows.

For a positive integer k, let $\mathcal{S}_k$ be the symmetric group on $\{1, \dots, k\}$. Any permutation π in $\mathcal{S}_k$ can be written in a unique way as a product of cycles. If $c = (j_1, j_2, \dots, j_l)$ is one of these cycles, then for $x_1, x_2, \dots, x_k$ in V, we use the notation $\text{tr } (\prod_{j \in c} x_j)$ to mean

$$\text{tr } (x_{j_1} x_{j_2} \dots x_{j_l}).$$

Let $\mathcal{C}(\pi)$ denote the set of cycles of π. We define

$$r_\pi(u)(h_1, h_2, \dots, h_k) = \prod_{c \in \mathcal{C}(\pi)} \text{tr } (\prod_{j \in c} u h_j). \tag{1.6}$$

To obtain the moments defined above, it is therefore natural to compute the expectation of $r_\pi(U)(h_1, h_2, \dots, h_k)$ when U is a Wishart random variable.

In Section 3, Theorem 3.1 gives the expectations of $r_\pi(U)(h_1, h_2, \dots, h_k)$ when π is the identity permutation. We then tackle in Section 4 the problem of the expectations of $r_\pi(U)(h_1, h_2, \dots, h_k)$ and $r_\pi(U^{-1})(h_1, h_2, \dots, h_k)$ in the special case when $h_1 = h_2 = \dots = h_k = h$. When h is the identity matrix e, the function $r_\pi(u)(e, \dots, e)$ is

$$r_{(i)}(u) = \prod_{(i)} (\text{tr } (u^j))^{i_j},$$

where (i) is a sequence of positive integers $(i) = (i_1, \dots i_k)$ such that $i_1 + 2i_2 + \dots + k i_k = k$. We will see that these $r_{(i)}$'s form a basis of the space $P(V)_k^K$ of homogeneous K-invariant polynomials of order k, that is in the real case, polynomials $Q(u)$ in the elements of u that are homogeneous of order k and such that for all $r \times r$ orthogonal matrices O, $Q(OuO^*) = Q(u)$. Thus the $Q(u)$'s are nothing but the polynomials with respect to the entries of u which depend only on the eigenvalues of u.

In Section 5, Theorem 5.1 gives the expression for $E(r_{(i)}(U))$ and $E(r_{(i)}(U^{-1}))$, which, of course, yields the expectation of any $Q(U)$ or $Q(U^{-1})$ when $Q \in P(V)_k^K$. Moreover when U is Wishart and h is a given matrix in Ω, $h^{1/2}Uh^{1/2}$ is also Wishart and we can deduce in Theorem 5.2 the expectations of $Q(h^{1/2}Uh^{1/2})$ and $Q((h^{1/2}Uh^{1/2})^{-1})$. This in turn will allow us, through a simple differentiation process to derive quantities such as

$$E(U^k) \text{ and } E(U^{-k}).$$

This will be given in Theorem 5.3. To prove Theorems 5.1 and 5.2, we use the basis of $P(V)_k^K$, formed by the spherical polynomials of order k. Though these tools are, even in the real case, rather sophisticated mathematical objects, the algorithm to obtain the moments is extremely simple and basically consists of the multiplication of three matrices.

In Section 6, we derive the explicit expression for moments of the form (1.2) in the real and complex case. We give an algorithm and the corresponding Maple program to obtain these moments.

2 Notations and definitions

Most readers might be mainly interested in the real Wishart distribution and may consider the complex and quaternionic cases as rather academic. However, as we shall see later, the calculation of moments is simplest in the complex case, and the real and quaternionic case are in a sense dual of each other (for further details see Graczyk, Letac, Massam (2003) for the complex case and Hanlon, Stanley, Stembridge (1992) for the real and quaternionic cases). It is thus interesting to treat the three cases simultaneously. To do so, we have to introduce some notation that fits the three cases. In particular we introduce the parameter d to be defined later and $f = d/2$. The reader who is definitely not interested in the complex and quaternionic cases should go immediately to 2.2 and read the remaining sections with $f = 1/2$.

To render the notation understandable in all three cases, we now have to recall some properties of Hermitian matrices.

2.1 Hermitian matrices

We use the parameter d which takes the values $d = 1$ in the real case, $d = 2$ in the complex case and $d = 4$ in the quaternionic case and we write $K_1 = \mathbb{R}$, $K_2 = \mathbb{C}$ and $K_4 = \mathbb{H}$. Recall that a quaternion is a pair $q = (q_0, \vec{q}) \in \mathbb{R} \times E = \mathbb{H}$ where E is a 3- dimensional oriented Euclidean space. The number $q_0 = \Re q$ is called the *real part* of q and the quaternion $\overline{q} = (q_0, -\vec{q})$ is called the *conjugate* of q. The product of two quaternions q and q' is

$$(q_0, \vec{q})(q_0', \vec{q}\,') = (q_0q_0' - \vec{q}.\vec{q}\,', q_0\vec{q}\,' + q_0'\vec{q} + \vec{q} \times \vec{q}\,'),$$

where $\vec{q}.\vec{q}'$ and $\vec{q}\times\vec{q}'$ are respectively the scalar and the cross product on E. The squared norm of q is $q_0^2+\vec{q}.\vec{q}$.

A square matrix x of order r with entries in K_d is said to be Hermitian if for $d=1$ it is symmetric, and if for $d=2,4$ it is equal to x^* (In general if y is any matrix with entries in K_d, then y^* denotes the transposed conjugate matrix $\overline{y}^T$). Throughout this paper the order r of the matrix x is fixed and we denote by V the real space of Hermitian matrices of order r on K_d. The parameters r, d and $f=d/2$ are sometimes called respectively the rank, the Peirce constant and the half Peirce constant of V. For $d=4$ a Hermitian matrix $x\in V$ is said to be non negative if the real number z^*xz is ≥ 0 for all $z\in\mathbb{H}^r$ and is said positive definite if $zxz^*>0$ for all $z\neq 0$. For $d=1,2,4$ we denote by Ω the open cone of positive definite elements of V and by $\overline{\Omega}$ the closed cone of the non negative ones. We denote by e the unit element of V. Define as $\mathbb{O}_d(r)$ the group of (r,r) matrices u with entries in $\mathbb{R},\mathbb{C}$ or $\mathbb{H}$ according to the values of d, such that $uu^*=e$. They are respectively the orthogonal, unitary and symplectic groups.

Since the quaternionic algebra $\mathbb{H}$ is not commutative, the traditional definitions of the determinant for a quaternionic square matrix do not make sense and the statement $\mathrm{tr}\,(ab)=\mathrm{tr}\,(ba)$ is not true in general. Let us now therefore explain for $d=4$ what it meant by $\det x$ if x is in V and by $x^{1/2}$ if $x\in\Omega$. The idea is to consider $\mathbb{H}^r$ as a Euclidean space (with the squared norm of $(q_1,\ldots,q_r)$ equal to $\|q_1\|^2+\ldots+\|q_r\|^2$) and to use the fact that if q and q' are in $\mathbb{H}$ then $q'\mapsto qq'\stackrel{\mathrm{def}}{=}r_q(q')$ is a linear transformation of $\mathbb{H}$. Thus if $x\in V$ has entries $x=(q_{ij})$ with $q_{ij}\in\mathbb{H}$, x can be considered as associated to the linear transformation $r(x)$ of $\mathbb{H}^r$ as defined by blocks, namely $r(x)=(r(q_{ij}))$. One easily shows that $x=x^*$ implies that $r(x)$ is a symmetric transformation of the Euclidean space $\mathbb{H}^r$ and thus has real eigenvalues. Now it can be proved by induction on r that each of the eigenspaces of $r(x)$ has a dimension which is a multiple of 4, and that there exist u in the symplectic group $\mathbb{O}_4(r)$ and a real diagonal matrix $d=\mathrm{diag}(\lambda_1,\ldots,\lambda_r)$ such that $x=udu^*$. The $(\lambda_1,\ldots,\lambda_r)$ are naturally called the *eigenvalues* of x. By definition of the determinant of the element of $x\in V$ for $d=4$ will be the number $\det x=\lambda_1\ldots\lambda_r$. Note that if a and b are square quaternionic matrices (Hermitian or not), then as mentioned before $\mathrm{tr}\,(ab)\neq\mathrm{tr}\,(ba)$, but

$$\Re\mathrm{tr}\,(ab)=\Re\mathrm{tr}\,(ba). \tag{2.1}$$

We shall use (2.1) in three ways: first when $d=4$ to equip V with the scalar product

$$\langle x,y\rangle=\Re\mathrm{tr}\,(xy),$$

next to show that if $(a_1, \dots, a_k)$ is a sequence of square matrices of the same order with entries in $\mathbb{H}$, then

$$\Re\mathrm{tr}\,(a_1 a_2 \dots a_{k-1} a_k) = \Re\mathrm{tr}\,(a_2 a_3 \dots a_k a_1). \tag{2.2}$$

Finally we use (2.1) to prove that the trace of $x \in V$ is the sum of its eigenvalues: if $x = uDu^*$ with D diagonal and $u \in \mathbb{O}_4$ then

$$\mathrm{tr}\, x = \Re\mathrm{tr}\, x = \Re\mathrm{tr}\,(uDu^*) = \Re\mathrm{tr}\,(u^* u D) = \Re\mathrm{tr}\, D = \mathrm{tr}\, D.$$

2.2 The Wishart distribution

The Wishart distributions on V have been considered by Goodman (1954) and Carter (1975) for the complex case, and by Andersson (1975) for the quaternionic case. A more general framework for these Wishart distributions would be the framework of Jordan algebras, as introduced by Jensen (1988), and followed by Casalis and Letac (1994) and Massam (1994). We will not use the framework of Jordan algebras in the present paper but rather confine ourselves to the matrix case. We will however use on some circumstances Faraut and Koranyi (1995) (henceworth abreviated as FK) as a reference because we find that this is on some points the most accessible reference even in the matrix case.

Let us first define the Wishart distribution on V. Using the half Peirce constant $f = d/2$, the following subset of $\mathbb{R}^+$,

$$\Lambda = \{f, 2f, \dots, (r-1)f\} \cup ((r-1)f, \infty) \tag{2.3}$$

is called the *Gyndikin set* (compare with (1.1)). This set has the following remarkable property: For p positive, then there exists a positive measure μ_p on $\overline{\Omega}$ such that for all $\theta \in -\Omega$, we have

$$\int_{\Omega} e^{\langle \theta, u \rangle} \mu_p(du) = (\det(-\theta))^{-p} \tag{2.4}$$

if and only if p is in Λ (see Casalis and Letac (1994) or Massam (1994)). If $p > (r-1)f$, then μ_p is absolutely continuous with respect to the Lebesgue measure of V and is equal to

$$\mu_p(dx) = \frac{1}{\Gamma_{\Omega}(p)} (\det x)^{p-(1+f(r-1))} \mathbf{1}_{\Omega}(x) dx, \tag{2.5}$$

where the constant $\Gamma_{\Omega}(p)$ is defined by

$$\Gamma_{\Omega}(p) = \pi^{\frac{dr(r-1)}{4}} \prod_{j=1}^{r} \Gamma(p - f(j-1)) \tag{2.6}$$

and the Lebesgue measure dx on V is chosen such that (2.5) is a probability (note that our choice of the Lebesgue measure is different from FK page 123 and agree with the choice of statisticians and probabilists).

If $p = f, 2f, \dots, (r-1)f$, the measure μ_p is singular and is concentrated on the boundary $\partial\Omega = \overline{\Omega} \setminus \Omega$ of the cone. The measure μ_f can easily be described using the image of the Lebesgue measure on $\mathbb{R}^{rd}$ (suitably identified with $\mathbb{C}^r$ or $\mathbb{H}^r$ for $d = 2$ or 4) by the map $z \mapsto zz^*$ from $\mathbb{R}^{rd}$ to $\overline{\Omega}$. For any integer $n = 2, \dots, r-1$ the measure μ_{nf} is obtained as the n th power of convolution of the above μ_f.

The natural exponential family on V generated by μ_p is the set of probabilities on $\overline{\Omega}$ of the form $(\det(-\theta))^p \exp\langle\theta, x\rangle\mu_p(dx)$ where $\theta \in -\Omega$. A more convenient parametrization of this family of distributions is given by the map from $-\Omega$ to Ω defined by

$$\theta \mapsto \sigma(\theta) = (-\theta)^{-1}. \tag{2.7}$$

For $\sigma \in \Omega$ and $p \in \Lambda$, the distribution

$$\gamma_{p,\sigma}(du) = \frac{1}{(\det\sigma)^p} e^{-\langle\sigma^{-1}, u\rangle}\mu_p(du) \tag{2.8}$$

will be called the *Wishart distribution* with *shape* parameter p and *scale* parameter σ. When $d = 1$, $p = n/2$ where n is a positive integer, and when $\Sigma = 2\sigma$ then $\gamma_{p,\sigma}$ is the classical Wishart distribution $W_r(n, \Sigma)$. Since $2p$ is not necessarily an integer in our definition (2.8) of the Wishart distribution, it is is slightly more general than the usual one.

The next proposition shows that the image of a Wishart distribution $\gamma_{p,\sigma}$ by the linear transformations g of V which preserve Ω is again a Wishart distribution with parameters $(p, g(\sigma))$. We also introduce a basic operator T_p. The proof is deferred to the Appendix.

Proposition 2.1 *Let a be an invertible (r, r) matrix with entries in K_d. Consider the automorphism g of Ω defined by $g(x) = axa^*$. If U follows the Wishart distribution $\gamma_{p,\sigma}$ then the distribution of $g(U)$ is $\gamma_{p,g(\sigma)}$. Furthermore, if $Q : V \to \mathbb{R}$ is any function such that*

$$T_p(Q)(\sigma) \stackrel{\text{def}}{=} \int_V Q(u)\gamma_{p,\sigma}(du) \tag{2.9}$$

exists, then we have for all g as above

$$T_p(Q \circ g)(\sigma) = T_p(Q)(g(\sigma)). \tag{2.10}$$

Finally, if Q is a homogeneous polynomial with degree k, then $T_p(Q)$ has the same property.

3 Expectation of product of traces.

As mentioned in Section 1, expected values of the type (1.2) can be computed from expected values of the type (1.4) by choosing the h_i's appropriately. In this section, we compute the expected value (1.4) when the Wishart random variable U belongs to the cone $\overline{\Omega}$ of Hermitian non negative matrices for $d = 1, 2, 4$. From now on, $h_1, \dots, h_k$ are fixed given elements of the space V of Hermitian (r, r) matrices. In the introduction we have already denoted by $\mathcal{S}_k$ the group of permutations of $\{1, \dots, k\}$ and we have considered the set $C(\pi)$ of cycles of $\pi \in \mathcal{S}_k$. To prove the theorem in this section, we state without proof the following differentiation result.

Proposition 3.1 *Let Ω be the cone of positive definite matrices. For $\theta \in \Omega$ define $\sigma(\theta) = (-\theta)^{-1}$. Then*

1. *The differential of $\theta \mapsto \sigma(\theta) = (-\theta)^{-1}$ is $h \mapsto \sigma h \sigma$.*
2. *For $d = 1, 2$ the differential of $\theta \mapsto \log\det\sigma(\theta)$ is $h \mapsto tr\,(\sigma h)$.*
3. *For $d = 2, 4$ the differential of $\theta \mapsto \log\det\sigma(\theta)$is $h \mapsto \Re tr\,(\sigma h)$.*

Recall that for $d = 2$ we have $\mathrm{tr}\,(\sigma h) = \Re\mathrm{tr}\,(\sigma h)$ for σ and h in V.

Theorem 3.1 *Let U be a random variable on V following the Wishart $\gamma_{p,\sigma}$ distribution. Then for $d = 1, 2$ we have*

$$\mathbb{E}(\langle U, h_1\rangle \dots \langle U, h_k\rangle) = \mathbb{E}(tr\,(Uh_1) \dots tr\,(Uh_k)) = \sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma h_1, \dots, \sigma h_k) \tag{3.1}$$

where $r_\pi(x_1, \dots, x_k) = \prod_{c \in C(\pi)} tr\,(\prod_{j \in c} x_j)$ and where $m(\pi)$ is the number of cycles of π. For $d = 2, 4$ we have

$$\mathbb{E}(\langle U, h_1\rangle \dots \langle U, h_k\rangle) = \mathbb{E}(\Re tr\,(Uh_1) \dots \Re tr\,(Uh_k)) = \sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} R_\pi(\sigma h_1, \dots, \sigma h_k) \tag{3.2}$$

where $R_\pi(x_1, \dots, x_k) = \prod_{c \in C(\pi)} \Re tr\,(\prod_{j \in c} x_j)$.

Remarks. As an example of (3.1) and (3.2) above let us consider the case where $k = 6$ and the cycle decomposition of $\pi \in \mathcal{S}_6$ is

$$(3, 1, 5)(2, 6)(4).$$

Then $m(\pi) = 3$ and

$$r_\pi(x_1, \dots, x_6) = \mathrm{tr}\,(x_3x_1x_5)\mathrm{tr}\,(x_2x_6)\mathrm{tr}\,x_4, \ \ R_\pi(x_1, \dots, x_6) = \Re\mathrm{tr}\,(x_3x_1x_5)\Re\mathrm{tr}\,(x_2x_6)\Re\mathrm{tr}\,x_4.$$

Note that the functions r_π and R_π make sense since $\mathrm{tr}\,(h_1h_2 \dots h_{k-1}h_k) = \mathrm{tr}\,(h_2h_3 \dots h_kh_1)$ is well known for $d = 1, 2$ and since $\Re\mathrm{tr}\,(h_1h_2 \dots h_{k-1}h_k) = \Re\mathrm{tr}\,(h_2h_3 \dots h_kh_1)$ follows from

(2.2) for $d = 4$: thus the terms tr $(\prod_{j\in c} x_j)$ and $\Re$tr $(\prod_{j\in c} x_j)$ appearing in the definitions of r_π and R_π do not depend on a particular parametrization of the cycle c. For $d = 2$ the functions R_π and r_π do not coincide since for $k \geq 3$ the number tr $(h_1h_2 \ldots h_{k-1}h_k)$ needs not be real for complex Hermitian matrices. Thus for $d = 2$, formulas (3.1) and (3.2) provide different expressions of the same left hand side. Finally, Theorem 7.1 below gives a simpler form of these formulas when $h_1 = \ldots = h_k$.

Proof: We will prove (3.1) and (3.2) by induction on k. Clearly, (3.1) and (3.2) are true for $k = 1$. Let us assume that they are true for k and show they also hold for $k + 1$. From the expression of the Laplace transform of the measure μ_p, i.e. $L_p(\theta) = \int_\Omega e^{\langle\theta,u\rangle}\mu_p(du) = (\det(-\theta))^{-p} = \exp(p \log\det(\sigma(\theta)))$ where $\sigma(\theta) = (-\theta)^{-1}$, we see immediately that the k-th differential of L_p for $d = 1, 2$ is

$$L_p^{(k)}(\theta)(h_1, \ldots, h_k) = L_p(\theta)\mathbb{E}(\text{tr } (Uh_1) \ldots \text{tr } (Uh_k)). \tag{3.3}$$

For $d = 2, 4$ it is

$$L_p^{(k)}(\theta)(h_1, \ldots, h_k) = L_p(\theta)\mathbb{E}(\Re\text{tr } (Uh_1) \ldots \Re\text{tr } (Uh_k)). \tag{3.4}$$

Let us now take the differential of these expressions in the direction h_{k+1} and use our induction assumption. The right-hand sides of (3.3) and (3.4) are then the sum of products of functions of θ. The differential of $L_p(\theta)$ is $p\langle\sigma, h\rangle L_p(\theta)$. Using Proposition 3.1 we find that the differentials of $\theta \mapsto$ tr $(\sigma h_1\sigma \ldots \sigma h_k)$ and $\theta \mapsto \Re$tr $(\sigma h_1\sigma \ldots \sigma h_k)$ in the direction h_{k+1} are respectively for the case $d = 1, 2$ and for the case $d = 2, 4$

$$\sum_{j=1}^{k} \text{tr } (\sigma h_1\sigma \ldots \sigma h_{j-1}\sigma h_{k+1}\sigma h_j \ldots \sigma h_k),$$

$$\sum_{j=1}^{k} \Re\text{tr } (\sigma h_1\sigma \ldots \sigma h_{j-1}\sigma h_{k+1}\sigma h_j \ldots \sigma h_k),$$

from which we deduce the differential of $\theta \mapsto$ tr $\prod_{j\in c}(\sigma h_j)$ and of $\theta \mapsto \Re$tr $\prod_{j\in c}(\sigma h_j)$. From this we get the differentials of $r_\pi(\sigma h_1, \ldots, \sigma h_k)$ and of $R_\pi(\sigma h_1, \ldots, \sigma h_k)$. Combining these differentials we can see that (3.1) and (3.2) hold true for $k + 1$. $\square$

The complete extension of Theorem 3.1 to Jordan algebras (for non necessarily equal $h_i's$) is treated in Letac and Massam (2001).

4 Expectation of invariant polynomials in a Wishart matrix

Let U be a Wishart variable on V. Our aim in this section is to compute the expectations of $Q(U)$ where Q is a polynomial with respect to the entries of U and depending on U only through its eigenvalues. Examples of polynomials $Q(U)$ are $\det U$, any polynomial with respect to the variables $\mathrm{tr}\,(U^j)$, $r_\pi(U)(e,\dots,e)$ or $R_\pi(U)(e,\dots,e)$. Such polynomials Q are said to be *invariant*. As we shall see in Proposition 4.1, they can also be considered as polynomials with respect to the eigenvalues themselves. Being able to compute $T_p(Q)(\sigma)$, i.e. the expectation of $Q(U)$ when Q is invariant and when U follows the $\gamma_{p,\sigma}$ distribution enables us using Proposition 2.1 to compute the expectation of $Q(aUa^*)$ if a is an invertible matrix of order r with coefficients in K_d : for this we simply use Proposition 2.1. Applying this principle to $Q(u) = r_\pi(u,\dots,u)$, $h \in \Omega$ and $a = h^{1/2}$, for $d = 1,2$ we will obtain

$$r_\pi(U)(h,\dots,h) = r_\pi(U^{1/2}hU^{1/2})(e,\dots,e).$$

For $d = 2,4$ using (2.1) we have

$$\Re tr((Uh)^k) = \Re \mathrm{tr}\,((U^{1/2}hU^{1/2})^k) = \mathrm{tr}\,((U^{1/2}hU^{1/2})^k),$$

and therefore

$$R_\pi(U)(h,\dots,h) = r_\pi(U^{1/2}hU^{1/2})(e,\dots,e).$$

Thus the computation of $\mathbb{E}(r_\pi(U)(e,\dots,e))$ will lead us to $\mathbb{E}(r_\pi(U)(h,\dots,h))$ and to $\mathbb{E}(R_\pi(U)(h,\dots,h))$, an extension of Theorem 3.1 in the particular case $h_1 = \dots = h_k$. (Theorem 3.1 was the particular case $\pi=$ identity.) Details are given in Theorem 5.2, which covers also the case $\mathbb{E}(r_\pi(U^{-1})(h,\dots,h))$.

Note that since the multilinear forms on V^k defined by $(h_1,\dots,h_k) \mapsto r_\pi(u)(h_1,\dots,h_k)$ are not necessarily symmetric, there is no hope to recover $\mathbb{E}(r_\pi(U)(h_1,\dots,h_k)$ from the knowledge of the $h \mapsto \mathbb{E}(r_\pi(U)(h,\dots,h)$ by a polarisation process. At the end of Section 5 we give $\mathbb{E}(\langle U,h\rangle \mathrm{tr}\ (U^2))$ as an example of a quantity that cannot be computed by the methods of the present paper.

In the remainder of this section, we will introduce the reader to the main tools of our computations: the concept of K invariant polynomials and two bases in $P(V)_k^K$, the $r_{(i)}$'s and the spherical polynomials $\Phi_{\mathbf{m}}$.

4.1 Invariance

Recall that $\mathbb{O}_d(r)$ is the group of (r,r) matrices u with entries in $\mathbb{R},\mathbb{C}$ or $\mathbb{H}$ such that $uu^* = e$. For $u \in \mathbb{O}_d(r)$ the endomorphism k_u of V defined by $k_u(x) = uxu^*$ maps Ω onto itself and satisfies $k_u(e) = e$. One can easily prove that any endomorphism k of V with these two properties has the form $k = k_u$ with $u \in \mathbb{O}_d(r)$. Note that k_u is an orthogonal transformation of the Euclidean space V. We define K as the group of isomorphisms of

V of the form k_u fo some $u \in \mathbb{O}_d(r)$ (the correspondance is almost one to one, since the kernel of the group homorphism $u \mapsto k_u$ is $\pm e$).

It is important for our subject to understand what the orbits of K on V are. Clearly when $d = 1$ or $d = 2$ and for x and y given in V one has $y = k(x)$ for some $k \in K$ if and only if the Hermitian matrices x and y have the same eigenvalues. A similar characterization holds for $d = 4$ (see FK).

Let us now define real invariant homogeneous polynomials of degree k on V. We denote the set of such polynomials as $P(V)_k^K$. Let $u_1, \dots, u_n$ be the components of $u \in V$ in a given basis, where $n = r + fr(r-1)$ is the dimension of the real space V. A mapping $Q : V \mapsto \mathbb{R}$ is said to be a homogeneous polynomial of degree k if the mapping $(u_1, \dots, u_n) \mapsto Q(u)$ is a homogeneous polynomial of degree k. It is trivial to check that this property does not depend upon the chosen basis.

Finally, we will say that a homogeneous polynomial Q is *K-invariant* (or simply that Q is invariant) if, for all $k \in K$ and all $x \in V$, we have

$$Q(k(x)) = Q(x).$$

Thus Q is invariant if and only if for all $x \in V$, then $Q(x)$ depends only on the eigenvalues of x.

4.2 The basis of the $r_{(i)}$'s

Let us now introduce the polynomials $r_{(i)}$. In Section 3, for $d = 1$ or 2 we have considered the polynomials

$$u \mapsto r_\pi(u)(h_1, \dots, h_k)$$

for π in the symmetric group $\mathcal{S}_k$ and for fixed $h_1, \dots, h_k$ in V. When all the h_i's are equal to the unit element $h = e \in V$, $r_\pi(u)(e, \dots, e)$ becomes a polynomial $Q(u) = r_{(i)}(u)$ such that $Q \in P(V)_k^K$, that we are going to describe now.

Let I_k be the set of all sequences $(i) = (i_1, \dots, i_k)$ of nonnegative integers such that $i_1 + 2i_2 + \ldots + ki_k = k$. If $\pi \in \mathcal{S}_k$, let i_j be the number of cycles of π of length j. The corresponding element (i) of I_k is called the *portrait* of π. If π and π' have the same portrait, we clearly have $r_\pi(u)(h, \dots, h) = r_{\pi'}(u)(h, \dots, h)$. In the cases $d = 1$ and $d = 2$, for a given $(i) \in I_k$, it makes sense to consider

$$r_{(i)}(h) = \prod_{j=1}^{k} (\text{tr } h^j)^{i_j}. \tag{4.1}$$

For instance $r_{(0,0,\ldots,0,1)}(h) = \text{tr }(h^k)$, $r_{(k)}(h) = (\text{tr } h)^k$. We can extend definition (4.1) to the quaternionic case: indeed, if $d = 4$ and h is in V then h^i is in V thus (4.1) makes sense and is real.

For simplicity, in the sequel, for $d = 1, 2, 4$ we write $r_{(i)} = r_{(i_1,\ldots,i_n)}$ where i_n is the last non zero integer of the sequence (i). These $r_{(i)}$ are important because, as we are going to see in Proposition 4.2,that they form a basis of $P(V)_k^K$. To show this, we need Proposition 4.1 which tells us that the invariant polynomials of $u \in V$ are in $1-1$ correspondence with the symmetric homogeneous polynomials in the eigenvalues of u.

Proposition 4.1 *Let $P(V)_k^K$ be the space of K-invariant homogeneous polynomials $Q : V \to \mathbb{R}$ of degree k and let $\mathcal{S}_{k,r}$ be the space of symmetric homogeneous polynomials $S : \mathbb{R}^r \mapsto \mathbb{R}$ of degree k. Then, for all $Q \in P(V)_k^K$, there exists a unique S_Q in $\mathcal{S}_{k,r}$ such that for all $u \in V$,*

$$Q(u) = S_Q(\lambda_1, \ldots, \lambda_r)$$

where $(\lambda_1, \ldots, \lambda_r)$ are the eigenvalues of u. Furthermore $Q \mapsto S_Q$ is an isomorphism between the two spaces.

Proof: For $i = 1, \ldots, r$ let $a_i \in V$ defined as the (r, r) matrix whose all entries are 0 except the entry (i, i) taken as equal to 1. Call a *frame* in V a sequence $(c_1, \ldots, c_r)$ in V such that there exists $k \in K_d$ with $c_i = k(a_i)$ for $i = 1, \ldots, r$. Given two frames $(c_1, \ldots, c_r)$ and $(c'_1, \ldots, c'_r)$, both $u = \lambda_1 c'_1 + \ldots + \lambda_r c'_r$ and $\lambda_1 c_1 + \ldots + \lambda_r c_r$ are in V. Since K acts transitively on frames, there exists $k \in K$ such that $k(u) = \lambda_1 c_1 + \ldots + \lambda_r c_r$. We define $S_Q(\lambda_1, \ldots, \lambda_r) = Q(\lambda_1 c_1 + \ldots + \lambda_r c_r)$. Then, clearly, S_Q is a polynomial in $\mathcal{S}_{k,r}$ such that $Q(u) = S_Q(\lambda_1, \ldots, \lambda_r)$ and the injectivity of $Q \mapsto S_Q$ is obvious. Conversely, if S is a symmetric and homogeneous polynomial from $\mathbb{R}^r$ to $\mathbb{R}$, define the function Q on V by $Q(u) = S(\lambda_1, \ldots, \lambda_r)$ where $(\lambda_1, \ldots, \lambda_r)$ are the eigenvalues of u. To see that Q is a polynomial we recall that S being symmetric is a polynomial in the elementary symmetric functions

$$e_1(\lambda_1, \ldots, \lambda_r) = \lambda_1 + \ldots + \lambda_r, \;\; e_2(\lambda_1, \ldots, \lambda_r) = \sum_{1 \le i \le j \le r} \lambda_i \lambda_j, \ldots. \tag{4.2}$$

Since the e_j's are polynomial functions of u then Q is a polynomial (see FK p. 28, but this is also easy to verify directly). The homogeneity of Q is easy to check. Therefore $Q \mapsto S_Q$ is also surjective. □

In the next proposition, we show that the set of the $r_{(i)}$'s forms a basis of $P(V)_k^K$.

Proposition 4.2 *Let $I_{k,r}$ be the set of $(i) = (i_1, \ldots, i_k)$ in I_k such that $i_j = 0$ when $j > r$, r being the rank of V. Then $(r_{(i)})_{(i)\in I_{k,r}}$ is a basis of $P(V)_k^K$.*

Proof: To show this result, we need to show that the $(r_{(i)})_{(i)\in I_{k,r}}$ are linearly independent and that the cardinality of this set is equal to the dimension of $P(V)_k^K$. If $u \in V$ has eigenvalues $(\lambda_1, \ldots, \lambda_r)$ then

$$r_{(i)}(u) = \prod_{i=1}^{r} (\lambda_1^j + \cdots . \lambda_r^j)^{i_j}$$

and therefore from the previous proposition, showing that the $(r_{(i)})_{(i)\in I_{k,r}}$ are independent is equivalent to showing that the

$$p_i(\lambda_1, \ldots, \lambda_r) = S_{r_{(i)}}(\lambda_1, \ldots, \lambda_r) = \prod_{j=1}^{r} (\lambda_1^j + \ldots + \lambda_r^j)^{i_j} \tag{4.3}$$

are independent for $(i) \in I_{k,r}$. This is a classical result and can be found for example in Macdonald (1995) p.24.

Let us now show that the cardinality of $I_{k,r}$ is the dimension of $\mathcal{S}_{k,r}$, that is, from the previous proposition, the dimension of $P(V)_k^K$. Any element of $\mathcal{S}_{k,r}$ is a polynomial with respect to the elementary symmetric functions as given in (4.2). Since monomials of $e_1, \ldots, e_r$ in $\mathcal{S}_{k,r}$ must be of the form $e_1^{i_1} e_2^{i_2} \ldots e_r^{i_r}$ with $i_1 + \ldots + ri_r = k$, it is clear that the cardinality of $I_{k,r}$ is the dimension of $\mathcal{S}_{k,r}$ and therefore of $P(V)_k^K$. $\square$

4.3 The basis of spherical polynomials

We now introduce another basis of $P(V)_k^K$. Let $M_{k,r}$ to be the set of sequences of integers $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ such that $m_1 \geq m_2 \geq \cdots m_r \geq 0$ and $m_1 + \cdots + m_r = k$. The sets $M_{k,r}$ and $I_{k,r}$ as defined in Proposition 4.2 have the same cardinality, since the correspondence $\mathbf{m} \mapsto (i)$ given by $i_j = \#\{l; m_l = j\}$ is clearly one-to-one. It is well known (see Macdonald 1995, chap VI, specially section 1 and 10) that for a given number $a > 0$, one can define a set of polynomials $(J_{\mathbf{m}}^{(a)}, \mathbf{m} \in M_{k.r})$ in $\lambda_1, \ldots, \lambda_r$, called the Jack polynomials of order $\mathbf{m}$ that form an orthogonal basis of $\mathcal{S}_{k,r}$ for a special scalar product depending on a (see Macdonald (1995) VI formulas (1.4) and (4.7), Stanley (1989) pages 76- 79 and Lassalle (1992) page 224 and 225 for a more compact presentation. For $a = 2$, the $J_{\mathbf{m}}^{(a)}$'s are the zonal polynomials, well known to statisticians; for $a = 1$ they are the Schur polynomials (see Takemura (1984) and James (1964)). For $a = 1/2$ they can be obtained from the zonal polynomials by a reciprocity formula given in Stanley (1989) (Theorem 3.3, page 85).

Coming back to V with rank r and Peirce constant $d = 2f = 1, 2, 4$, following the notation of Proposition 4.1 we define on V the *spherical polynomial* $u \mapsto \Phi_{\mathbf{m}}(u)$ of order $\mathbf{m} \in M_{k,r}$ by

$$J_{\mathbf{m}}^{(1/f)} = S_{\Phi_{\mathbf{m}}},$$

that is, $J_{\mathbf{m}}^{(1/f)}$ is the polynomial in $\mathcal{S}_{k,r}$ corresponding to the symmetric polynomial $\Phi_{\mathbf{m}}$ in $P(V)_k^K$. (To see that they coincide with the spherical polynomials as defined in FK p.228, see FK p.239 and the references given therein).

As mentioned before, our aim is to compute $\int_V Q(u)\gamma_{p,\sigma}(du)$ for $Q \in P(V)_k^K$. Since the $r'_{(i)}$s and the $\Phi'_{\mathbf{m}}$s form two different bases of $P(V)_k^K$, if we know the expectations of each element of one basis, we can compute the expectation of any element of the other basis provided we know the matrix of passage from one basis to the other. We can then compute the expectation of any $Q(u)$ with Q in $P(V)_k^K$. This is exactly what we are going to do. The expected value of $\Phi_{\mathbf{m}}(U)$ is known (Proposition 4.3) and we will compute the expected value of $\Phi_{\mathbf{m}}(U^{-1})$ (Proposition 4.4).

Recall the notation introduced in Proposition 2.1. For any function $Q :\to \mathbb{R}$, we write

$$T_p(Q)(\sigma) = \int_V Q(u)\gamma_{p,\sigma}(du). \tag{4.4}$$

It follows from Proposition 2.1 that the mapping T_p defined by $Q \mapsto T_p(Q)$ is an endomorphism of $P(V)_k^K$. It has already been considered by Kushner and Meixner (1980). The expression of $T_p(\Phi_{\mathbf{m}})(\sigma)$ is particularly simple since as stated in Proposition 4.3 below the $\Phi'_{\mathbf{m}}$s are in fact the eigenvectors of T_p.

To describe the eigenvalues of T_p we have to recall the definition of the Gamma function of V (see FK p. 233, with a slight modification here): for $s_j > (j-1)f$,

$$\Gamma_\Omega(s_1, \ldots, s_r) = \pi^{\frac{dr(r-1)}{4}} \prod_{j=1}^{r} \Gamma(s_j - (j-1)f). \tag{4.5}$$

If p is a real number and if $\mathbf{s} = (s_1, \ldots, s_r)$ we write $\Gamma_\Omega(p + \mathbf{s})$ for $\Gamma_\Omega(p + s_1, \ldots, p + s_r)$ and $\Gamma_\Omega(p)$ for $\Gamma_\Omega(p, \ldots, p)$ (this symbol was used in (2.6)). If $p > (r-1)f$ and $\mathbf{m} \in M_{k,r}$ we define the generalized Pochhammer symbol as

$$(p)_{\mathbf{m}} = \frac{\Gamma_\Omega(p + \mathbf{m})}{\Gamma_\Omega(p)}$$

(see FK p.230). If p belongs to the singular part $\{f, 2f, \ldots, (r-1)f\}$ of the Gyndikin set, the generalized Pochhammer symbol is

$$(lf)_{\mathbf{m}} = \prod_{j=1}^{l} \frac{\Gamma(m_j + (l-j+1)f)}{\Gamma((l-j+1)f)}$$

for $l = 1, 2, \dots, r-1$ (see Letac, Massam and Richards (2001)).

At this point, we should mention that the Pochhammer $(p)_{\mathbf{m}}$ depends on $\mathbf{m}$ and p, but does not depend on the rank r, in the following sense. Given a sequence $\mathbf{m} = (m_1, \dots, m_c, 0, 0, \dots)$ of integers such that $m_1 \geq m_2 \geq \cdots \geq m_c > 0$ and $m_1 + \cdots + m_c = k$, then $\mathbf{m}$ can be considered as a member of $M_{k,r}$ for any $r \geq c$ and the value of $(p)_{\mathbf{m}}$ will not change with r. Let us momentarily denote $(p)_{\mathbf{m}}$ by $((p)_{\mathbf{m}})_r$. We have to check that $((p)_{\mathbf{m}})_r = ((p)_{\mathbf{m}})_c$. This is easily done by considering separately the case $p = f, 2f, \dots, (c-1)f$, the case $p = cf, \dots, (r-1)f$, and the case $p > (r-1)f$. Our remark is thus proved.

The following proposition is most important since it gives us the eigenvectors and eigenvalues of T_p. For $p > (r-1)f$, its proof is essentially in FK p.226. For the singular case, see Letac, Massam and Richards (2001) Theorem 3.6. When p/f is an integer, it is also proved by beautiful combinatorial methods in Hanlon, Stanley and Stembridge (1992), formula (5.3) page 166.

Proposition 4.3 *Let U be $\gamma_{p,\sigma}$ distributed on V, where p is in the Gyndikin set (2.3). For $\mathbf{m} \in M_{k,r}$ we have*

$$T_p(\Phi_{\mathbf{m}})(\sigma) = \mathbb{E}(\Phi_{\mathbf{m}}(U)) = (p)_{\mathbf{m}} \Phi_{\mathbf{m}}(\sigma) \tag{4.6}$$

where $(p)_{\mathbf{m}}$ is the generalized Pochhammer symbol.

Thus $\Phi_{\mathbf{m}}$ is an eigenvector of T_p associated to the eigenvalue $(p)_{\mathbf{m}}$. We have therefore solved, at least in theory, the problem of the computation of the expectation of the $\mathbb{E}(r_{(i)}(U))$'s. The only thing left to do is to write the $r_{(i)}$'s as a linear combination of the $\Phi_{\mathbf{m}}$'s, or equivalently, according to Proposition 4.1, to write $p_{(i)}$ as defined in (4.3 as a linear combination of the $J^{(1/f)}_{\mathbf{m}}$'s. To simplify our notation, we denote by $\mathtt{p}$ the basis of the $p_{(i)}$'s in $\mathcal{S}_{k,r}$ and by $\mathtt{J}$ the basis of the $J^{(1/f)}_{\mathbf{m}}$'s. With some abuse of language we also identify $\mathtt{p}$ and $\mathtt{J}$ to the corresponding basis $(r_{(i)}, (i) \in I_{k,r})$ and $(\Phi_{\mathbf{m}}, \mathbf{m} \in M_{k,r})$. Let $A_{k,r}$ be the passage matrix from the basis $\mathtt{p}$ to the basis $\mathtt{J}$, i.e. the matrix representative of the identity endomorphism of $\mathcal{S}_{k,r}$ from the basis $\mathtt{J}$ to the basis $\mathtt{p}$. This matrix $A_{k,r} = (P_{(i),\mathbf{m}}(a))$ is a matrix of polynomials with respect to one variable $a = 1/f$ (see Stanley (1989), page 97) whose rows are indexed by $I_{k,r}$ and columns are indexed by $M_{k,r}$. The inverse matrix $A^{-1}_{k,r} = (Q_{\mathbf{m},(i)}(a))$ is the passage matrix from the basis $\mathtt{J}$ to the basis $\mathtt{p}$, i.e. the matrix representative of the identity endomorphism of $\mathcal{S}_{k,r}$ from the basis $\mathtt{p}$ to the basis $\mathtt{J}$. We also consider the square diagonal matrix $D_{k,r}(p)$ whose rows and columns are indexed by $M_{k,r}$ and with diagonal element corresponding to $\mathbf{m}$ the Pochhammer symbol $(p)_{\mathbf{m}}$. With these notations Proposition 4.3 implies that the matrix representative of T_p in the basis $\mathtt{J}$ is

$$[T_p]_{\mathtt{J}}^{\mathtt{J}} = D_{k,r}(p),$$

and the representative matrix of T_p in the basis $\mathtt{p}$ is

$$[T_p]_{\mathtt{p}}^{\mathtt{p}} = A_{k,r} D_{k,r}(p) A_{k,r}^{-1}. \tag{4.7}$$

If we write $[r_{(i)}(u)]_{I_{k,r}}$ for the column vector made with the polynomials $r_{(i)}(u); (i) \in I_{k,r}$, (4.7) is equivalent to

$$[\mathbb{E}(r_{(i)}(U))]_{I_{k,r}} = [A_{k,r} D_{k,r}(p) A_{k,r}^{-1}]^t [r_{(i)}(\sigma)]_{I_{k,r}} \tag{4.8}$$

where M^t is the transposed matrix of the matrix M.

It is now time to investigate the moments of U^{-1} when U has a Wishart distribution. This is considered in the literature as a difficult problem (see Haff (1982), von Rosen (1988), Lauritzen (1996) page 259, Maiwald and Kraus (2000)). However the expectation of $E(\Phi_{\mathbf{m}}(U^{-1}))$, and therefore of $E(r_{(i)}(U^{-1}))$ appears to be relatively easy. For $q > k - f$ and $\mathbf{m} \in M_{k,r}$ we denote

$$(q)_{\mathbf{m}}^* = \frac{\prod_{j=1}^r \Gamma(-m_j + jf + q)}{\prod_{j=1}^r \Gamma(jf + q)} \tag{4.9}$$

Note that $q \mapsto (q)_{\mathbf{m}}^*$ is a rational function whose coefficients depend only on $\mathbf{m}$. A remark similar to the one made before Proposition 4.3 is in order: For fixed q and $\mathbf{m}$, the number $(q)_{\mathbf{m}}^*$ does not depend on r. We have the following result.

Proposition 4.4 *Let U be $\gamma_{p,\sigma}$ distributed on V, where $p > (r-1)f + k$. For $\mathbf{m} \in M_{k,r}$ we have*

$$\mathbb{E}(\Phi_{\mathbf{m}}(U^{-1})) = ((p - rf)^*)_{\mathbf{m}} \Phi_{\mathbf{m}}(\sigma^{-1}) \tag{4.10}$$

where $(q)_{\mathbf{m}}^*$ is defined by (4.9).

Proof:

To show that (4.10) is true, we need the following proposition, whose proof is given in the Appendix.

Proposition 4.5 *Let $\mathbf{m} = (m_1, \ldots, m_r)$ with $m_j \in \mathbb{R}$ and $p > f(r-1)$ such that $p - m_j > f(r-j)$ for $j = 1, \ldots, r$. Let $\mathbf{m}^* = (m_r, \ldots, m_1)$ and for $s = (s_{lk})_{1 \le l,k \le r}$ in Ω, and for*

$j = 1, \dots, r$, let $\Delta_j(s)$ be the determinant of the (j,j) Hermitian matrix $(s_{lk})_{1\leq l,k\leq j}$. We write

$$\Delta_{\mathbf{m}}(s) = (\Delta_1(s))^{m_1-m_2}(\Delta_2(s))^{m_2-m_3}\cdots(\Delta_r(s))^{m_r}.$$

Then, if the random variable S has the Wishart distribution $\gamma_{p,\sigma}$, we have

$$\mathbb{E}(\Delta_{\mathbf{m}}(S^{-1})) = \frac{\Gamma_\Omega(p-\mathbf{m}^*)}{\Gamma_\Omega(p)}\Delta_{\mathbf{m}}(\sigma^{-1}).$$

Proposition 4.5 enables us to complete the proof of Proposition 4.4.

We equip the compact group $\mathbb{O}_d(r)$ with its Haar measure du. Then

$$\begin{aligned}
\mathbb{E}(\phi_{\mathbf{m}}(U)) &\overset{(1)}{=} \mathbb{E}\left(\int_{\mathbb{O}_d(r)} \Delta_{\mathbf{m}}((uUu^*)^{-1})du\right)\\
&\overset{(2)}{=} \mathbb{E}\left(\int_{\mathbb{O}_d(r)} \Delta_{\mathbf{m}}(uU^{-1}u^*)du\right)\\
&\overset{(3)}{=} \int_{\mathbb{O}_d(r)} \mathbb{E}(\Delta_{\mathbf{m}}(uU^{-1}u^*))du\\
&\overset{(4)}{=} \frac{\Gamma_\Omega(p-\mathbf{m}^*)}{\Gamma_\Omega(p)}\int_{\mathbb{O}_d(r)} \Delta_{\mathbf{m}}(u\sigma^{-1}u^*)du\\
&\overset{(5)}{=} \frac{\Gamma_\Omega(p-\mathbf{m}^*)}{\Gamma_\Omega(p)}\phi_{\mathbf{m}}(\sigma^{-1}).
\end{aligned}$$

In this sequence of equalities, (1) and (5) comes from a property of spherical functions as given in FK page 304, Theorem XIV.3.1, (2) is clear, (3) comes from Fubini and (4) from Proposition 4.5. The proof of Proposition (4.4) is therefore complete.

□

Even the case $k = 1$ provides a non trivial example of application of (4.10) since it yields $\mathbb{E}(U^{-1}) = \sigma^{-1}/(p-1-(r-1)f)$ if $p > (r-1)f+1$, since $(q)_1^* = 1/(q+f-1)$ and $\Phi_1(u) = \operatorname{tr} u$. If the polynomial Q is in $P(V)_k^K$, we introduce the symbol $T_p^*(Q)$ as follows

$$T_p^*(Q)(\sigma^{-1}) = \int_V Q(u^{-1})\gamma_{p,\sigma}(du) = \mathbb{E}(Q(U^{-1})). \tag{4.11}$$

Proposition 4.4 shows that T_p^* is an endomorphism of $P(V)_k^K$, and that the eigenvectors are the $\Phi_{\mathbf{m}}$'s again, with eigenvalues $((p-rf)^*)_{\mathbf{m}}$. Denoting by $D_{k,r}^*(q)$ the diagonal matrix whose entry corresponding to $\mathbf{m}$ is $(q)_{\mathbf{m}}^*$, we obtain the matrix representative of T_p^* in the basis $\mathtt{p}$:

$$[T_p^*]_{\mathtt{p}}^{\mathtt{p}} = A_{k,r}D_{k,r}^*(p-rf)A_{k,r}^{-1}. \tag{4.12}$$

This is equivalent to

$$[\mathbb{E}(r_{(i)}(U^{-1}))]_{I_{k,r}} = [A_{k,r}D_{k,r}^*(p-rf)A_{k,r}^{-1}]^t[r_{(i)}(\sigma^{-1})]_{I_{k,r}}. \tag{4.13}$$

5 The expectations of $r_{(i)}(U^{\pm 1})$ and of $U^{\pm k}$

5.1 The expectation of $r_{(i)}(U^{\pm 1})$.

In equations (4.8) and (4.13) above, we have already given the expected values of $r_{(i)}(U)$ and $r_{(i)}(U^{-1})$ for $k \leq r$. We want to obtain these expectations for all k. We observe that for $k \leq r$ we have the following four equalities:

$$\begin{aligned} I_{k,r} &= I_{k,k} \\ D_{k,r}(p) &= D_{k,k}(p) \\ D^*_{k,r}(q) &= D^*_{k,k}(q) \\ A_{k,r} &= A_{k,k}. \end{aligned}$$

The first equality is obvious from the definition of $I_{k,r}$ given in Proposition 4.2, the second and the third are consequences of the remarks made just before Proposition 4.3 and Proposition 4.4 respectively, and the fourth is a consequence of the properties of symmetric polynomials. A detailed proof of this last statement would imitate the proof of Proposition 5.1 below, and so we skip it. We now denote for simplicity

$$D_{k,k}(p) = D_k(p), \ D^*_{k,k}(q) = D^*_k(q), \ A_{k,k} = A_k, B_k = A^t_k. \tag{5.1}$$

Recall that I_k has already been defined before (4.1). Clearly $I_{k,k} = I_k$.

The next theorem extends (4.8) and (4.13) to be valid for any k. If $r < k$ the polynomials $r_{(i)}$ are not independent, thus formulas (4.8) and (4.13) do not give $\mathbb{E}(r_{(i)}(U))$ and $\mathbb{E}(r_{(i)}(U^{-1}))$ for $(i) \in I_k \setminus I_{k,r}$ in general. But we are going to see that the same results do hold true for any k. We imitate the notation used in (4.8) by considering the column vector $[r_{(i)}(\sigma)]_{I_k}$. Note that from now on, we use the transposed matrix $B_k = A^t_k$ rather than A_k, since it is B_k rather than A_k which will appear in the numerical calculations of Section 5.

Theorem 5.1 *Let U follow the $\gamma_{p,\sigma}$ distribution. Then for $h \in \Omega$ and with the notation given in (5.1) we have*

$$\begin{aligned} \text{for } p \in \Lambda,\ [\mathbb{E}(r_{(i)}(U))]_{I_k} &= B_k^{-1} D_k(p) B_k [r_{(i)}(\sigma)]_{I_k} && (5.2) \\ \text{for } p \geq k + (r-1)f,\ [\mathbb{E}(r_{(i)}(U^{-1}))]_{I_k} &= B_k^{-1} D^*_k(p - rf) B_k [r_{(i)}(\sigma^{-1})]_{I_k} && (5.3) \end{aligned}$$

Proof: For $r \geq k$ there is nothing to prove since the desired results are identical to (4.8) and (4.13). When $r = k$, (5.2) and (5.3) are identities in $P(V)^K_k$ which can be translated into identities on $\mathcal{S}_{k,k}$ by the map $Q \mapsto S_Q$ as defined in Proposition 4.1. Thus we obtain identities between homogeneous symmetric polynomials of degree k with respect to $\lambda_1, \ldots, \lambda_k$. Let us now consider the case $r < k$ for some V with Peirce constant d and rank r. Setting $\lambda_{r+1} = \ldots = \lambda_k = 0$, in (5.2) and (5.3), thus obtaining identities in $\mathcal{S}_{k,r}$. and translate these identities through the isomorphism $Q \mapsto S_Q$ into identities in $P(V)_k^K$. These are nothing but (4.8) and (4.13) for V and our result is proved. □

The actual computation of $B_k, B_k^{-1}, B_k^{-1}D_k(p)B_k$, and $B_k^{-1}D_k^*(p-rf)B_k$ is detailed in Section 6.

5.2 Extended Results

As explained at the beginning of Section 4, the results of Theorem 5.1 can be extended by considering the expected value of $r_{(i)}(h^{1/2}Uh^{1/2})$ As seen in Proposition 2.1, if U follows a $\gamma_{p,\sigma}$ distribution, then for $h \in V$, $h^{1/2}Uh^{1/2}$ follows a $\gamma_{p,h^{1/2}\sigma h^{1/2}}$ distribution. We can therefore deduce from Theorem 5.1 the following.

Theorem 5.2 *Let U follow the $\gamma_{p,\sigma}$ distribution. Then for $h \in \Omega$ and with the notation given in (5.1) we have*

$$for\ p \in \Lambda,\ [\mathbb{E}(r_{(i)}(h^{1/2}Uh^{1/2}))]_{I_k} = B_k^{-1}D_k(p)B_k[r_{(i)}(h^{1/2})\sigma h^{1/2})]_{I_k} \tag{5.4}$$

and for $p \geq k + (r-1)f$,

$$[\mathbb{E}(r_{(i)}(h^{1/2}U^{-1}h^{1/2}))]_{I_k} = B_k^{-1}D_k^*(p-rf)B_k[r_{(i)}(h^{1/2}\sigma^{-1}h^{1/2})]_{I_k}. \tag{5.5}$$

For the real and complex case, we thus obtain the expected values of

$$r_{(i)}(h^{1/2}Uh^{1/2}) = \prod_j (tr\ (Uh)^j)^{i_j}, \quad r_{(i)}(h^{-1/2}U^{-1}h^{-1/2}) = \prod_j (tr\ (Uh)^{-j})^{i_j}.$$

For the complex and quaternionic case, we thus obtain the expected values of

$$r_{(i)}(h^{1/2}Uh^{1/2}) = \prod_j (\Re tr\ (Uh)^j)^{i_j}, \quad r_{(i)}(h^{-1/2}U^{-1}h^{-1/2}) = \prod_j (\Re tr\ (Uh)^{-j})^{i_j}.$$

It is then clear that we can also compute the expectation of any $Q(h^{1/2}Uh^{1/2})$ for Q in $\mathbb{P}(V)_k^K$. It will be useful to view $Q(h^{1/2}Uh^{1/2})$ as both a polynomial in U and in h, which we will denote as $\tilde{Q}(U,h)$. From Proposition XIV.1.1 of FK, also given in the Appendix as Proposition 7.1, we know that the correspondance between Q and $\tilde{Q}$ is an isomorphism. The polynomial $\tilde{Q}$ is used in the next section to obtain such results as $\mathbb{E}(U^{\pm k})$.

5.3 Previous results

Results (5.4) when $k = 3$ and (5.5) for $k = 2$ have already been obtained by Letac and Massam (1998) in a paper devoted to the characterization of the Wishart distribution. Indeed, for

$$M(p) = \begin{bmatrix} p^2 & p \\ pf & p - pf + p^2 \end{bmatrix}$$

Letac and Massam (1998) have shown that (Proposition 3.2, p.584)

$$\begin{bmatrix} \mathbb{E}(U \otimes U) \\ \mathbb{E}(\mathbb{P}(U)) \end{bmatrix} = M(p) \begin{bmatrix} \sigma \otimes \sigma \\ \mathbb{P}(\sigma) \end{bmatrix}. \tag{5.6}$$

Here, $u \otimes u$ is the symmetric endomorphism of V defined by $x \mapsto u\langle u, x\rangle$ and $\mathbb{P}(u)$ is the symmetric endomorphism of V defined by $x \mapsto uxu$. Due to the relation between symmetric endomorphisms and quadratic forms on a Euclidean space, (5.6) is equivalent to

$$\begin{bmatrix} \mathbb{E}(\langle U, h\rangle^2) \\ \mathbb{E}(\langle h, UhU\rangle) \end{bmatrix} = M(p) \begin{bmatrix} \langle \sigma, h\rangle^2 \\ \langle h, \sigma h \sigma\rangle \end{bmatrix},$$

for any h in V. From Proposition 7.1 in the Appendix, this is in turn equivalent to

$$\begin{bmatrix} \mathbb{E}(r_{(2)}(U)) \\ \mathbb{E}(r_{(01)}(U)) \end{bmatrix} = M(p) \begin{bmatrix} r_{(2)}(\sigma)) \\ r_{(01)}(\sigma) \end{bmatrix}. \tag{5.7}$$

This formula says that the matrix $M(p)$ is the representative matrix of the endomorphism T_p (as defined by (4.4)) of the space $P(V)_2^K$ when written in the basis $(r_{(2)}, r_{(01)})$. The first formula of (5.7), namely $\mathbb{E}(r_{(2)}(U)) = p^2 r_{(2)}(\sigma) + pr_{(01)}(\sigma)$ is the result given by Theorem 7.1 when $k = 2$. The second formula

$$\mathbb{E}(r_{(01)}(U)) = pfr_{(2)}(\sigma) + (p - pf + p^2)r_{(01)}(\sigma)$$

is somewhat more difficult to obtain. The method of proof (let us call it the ψ method) used in Letac and Massam (1998) is not elementary but can be extended to case $k = 3$:

$$\begin{bmatrix} \mathbb{E}(r_{(3)}(U)) \\ \mathbb{E}(r_{(11)}(U)) \\ \mathbb{E}(r_{(001)}(U)) \end{bmatrix} = \tag{5.8}$$
$$\begin{bmatrix} p^3 & 3p^2 & 2p \\ p^2 f & p^3 + p^2(1-f) + 2pf & 2p^2 + 2p(1-f) \\ pf^2 & 3p^2 f + 3pf(1-f) & p^3 + 3p^2(1-f) + p(2 - 3f + 2f^2) \end{bmatrix} \begin{bmatrix} r_{(3)}(\sigma) \\ r_{(11)}(\sigma) \\ r_{(001)}(\sigma) \end{bmatrix}.$$

Similarly, using the ψ method, Letac and Massam (2000), Theorem 6.1 gives the equivalent of (5.7) for U^{-1}. Writing $q = p - rf$, they obtain

$$\begin{bmatrix} \mathbb{E}(r_{(2)}(U^{-1})) \\ \mathbb{E}(r_{(01)}(U^{-1})) \end{bmatrix} = \frac{1}{(q+f-1)(q+f-2)(q+2f-1)} \begin{bmatrix} q+2f-2 & 1 \\ f & q+f-1 \end{bmatrix} \begin{bmatrix} r_{(2)}(\sigma^{-1})) \\ r_{(01)}(\sigma^{-1}) \end{bmatrix} \quad (5.9)$$

A look at the proof in Letac and Massam (2000) will convince the reader that the ψ method is too difficult to be extended to higher k. A second method of computation of the $\mathbb{E}(r_{(i)}(U))$ could be extracted from Hanlon, Stanley and Stembridge (1992). This elegant paper provides a general formula for $\mathbb{E}(r_{(i)}(U))$. In this paper, we have used a third method. Our tool, as we have seen, are the two different bases of $P(V)_k^K$ and the eigenvectors of T_p.

5.4 The expectation of $U^{\pm k}$ and other animals

For each polynomial $\widetilde{r_{(i)}}(u,h) = r_{(i)}(h^{1/2}uh^{1/2})$, we can derive by a process of polarization and differentiation a quantity that we are going to denote as

$$L_{r_{(i)}}(u) = r_{(i)}(u) \sum_{j=1}^{k} j i_j \frac{u^j}{\mathrm{tr}\ (u^j)}. \quad (5.10)$$

We can think of $L_{r_{(i)}}(u)$ as "nearly a derivative" of $r_{(i)}(u)$ with respect to u. Note that $L_{r_{(i)}}(u)$ is not a number, but an element of V. We will call it the *lifting* of $r_{(i)}(u)$. Its expectation is given in the following theorem, which allows us to compute the expectations of random variables like U^k, U^{-k} and more generally $L_{r_{(i)}}(U)$, the "other animals" of the title. The Appendix considers in general the lifting L_Q of $Q \in P(V)_k^K$: see Propositions 7.3 and 7.4.

Theorem 5.3 *Let U follow the $\gamma_{p,\sigma}$ distribution. Let $[L_{r_{(i)}}]_{I_k}$ denote the column vector of the $L_{r_{(i)}}$'s. Then*

$$\begin{aligned} for\ p \in \Lambda,\ [\mathbb{E}(L_{r_{(i)}}(U))]_{I_k} &= B_k^{-1} D_k(p) B_k [L_{r_{(i)}}(\sigma)]_{I_k} \quad (5.11) \\ for\ p \geq k + (r-1)f,\ [\mathbb{E}(L_{r_{(i)}}(U^{-1}))]_{I_k} &= B_k^{-1} D_k^*(p - rf) B_k [L_{r_{(i)}}(\sigma^{-1})]_{I_k}. \quad (5.12) \end{aligned}$$

In particular the last line of (5.11) gives $k\mathbb{E}(U^k)$ and the last line of (5.12) gives $k\mathbb{E}(U^{-k})$.

Proof: From Propositions 5.1 and 7.5 we have

$$[T_p(L_{r_{(i)}})]_{I_k} = [L_{T_p(r_{(i)})}]_{I_k} = L_{B_k^{-1} D_k(p) B_k [r_{(i)}]_{I_k}} = B_k^{-1} D_k(p) B_k [L_{r_{(i)}}]_{I_k}.$$

This proves (5.11). The proof of (5.12) is similar. From Proposition 7.4 it follows that for $(i) = (0, \ldots, 0, 1)$, $L_{r_{(i)}(u)} = ku^k$ and this proves the last part of the theorem. $\square$

For instance, when $k = 1$, we obtain $\mathbb{E}(U^{-1}) = \sigma^{-1}/(p - (r-1)f)$ (see Muirhead (1982) page 103, with a misprint). Formulas (5.7) and (5.9) combined with Proposition 7.4 and Theorem 5.3 give

$$\begin{aligned}
\mathbb{E}(U\, tr\, U) &= p^2\sigma \mathrm{tr}\ \sigma + p\sigma^2, \\
\mathbb{E}(U^2) &= pf\sigma \mathrm{tr}\ \sigma + (p - pf + p^2)\sigma^2, \\
\mathbb{E}(U^{-1}\, tr\, U^{-1}) &= \frac{1}{(q+f-1)(q+f-2)(q+2f-1)}((q+2f-2)\sigma^{-1}\mathrm{tr}\ \sigma^{-1} + \sigma^{-2}), \\
\mathbb{E}(U^{-2}) &= \frac{1}{(q+f-1)(q+f-2)(q+2f-1)}(f\sigma^{-1}\mathrm{tr}\ \sigma^{-1} + (q+f-1)\sigma^{-2})
\end{aligned}$$

(Recall that $q = p - rf$ as usual). Similarly, formula (5.8) gives

$$\mathbb{E}(U^3) = pf^2\sigma(\mathrm{tr}\ \sigma)^2 + (p^2f + pf(1-f))(\sigma \mathrm{tr}\ (\sigma^2) + 2\sigma^2 \mathrm{tr}\ \sigma) + (p^3 + 3p^2(1-f) + p(2 - 3f + 2f^2))\sigma^3.$$

Using the traditional notations $p = n/2$ and $\sigma = 2\Sigma$, for $f = 1/2$ (real case), this becomes the formula given in the Abstract.

Formulas parallel to the ones obtained above, for $\mathbb{E}(U^{-3}), \mathbb{E}(U^4)$ and $\mathbb{E}(U^{-4})$ are given below in (6.3), (6.4), (6.5) and (6.6). Note that, since $L_{r_{(1,1)}}(u) = u\,\mathrm{tr}\ u^2 + 2u^2\,\mathrm{tr}\ u$, formula (5.8) leads to

$$\begin{aligned}
&\mathbb{E}(U\,\mathrm{tr}\ U^2 + 2U^2\,\mathrm{tr}\ U) = \\
&\quad 3p^2f\sigma(\mathrm{tr}\ \sigma)^2 + (p^3 + p^2(1-f) + 2pf)(\sigma \mathrm{tr}\ (\sigma^2) + 2\sigma^2 \mathrm{tr}\ \sigma) + 3(2p^2 + 2p(1-f))\sigma^3,
\end{aligned}$$

but the methods of the present paper cannot yield the individual computation of $\mathbb{E}(U\ \mathrm{tr}\ U^2)$.

6 Explicit computations

6.1 The algorithm

The expressions given in (5.2) and (5.3) can actually be computed explicitely, for any k. Here, we use Maple in order to take advantage of the Maple package called SF developped by Stembridge (1998). The package SF does a number of things. The function that we are going to be most interested in is the function "top" which allows the passage from the basis `J` to the basis `p`.

According to the theory developped in Section 4, in order to obtain the matrix A_k for a given k, it is sufficient to obtain the coefficient of $J_{\mathbf{m}}^{(1/f)}$, for all $\mathbf{m} \in M_k$ in the basis `p`.

These coefficients will form the columns of the matrix A_k, or equivalently, the rows of the matrix $B_k = A_k^t$. The matrices B_k and B_k^{-1} are therefore obtained immediately.

In order to compute all the expectations in (5.2) and (5.3), we need the diagonal matrices D_k and D_k^*. The diagonal element of D_k corresponding to the line indexed by $\mathbf{m}$ is

$$(p)_{\mathbf{m}} = \frac{\Gamma_\Omega(p+\mathbf{m})}{\Gamma_\Omega(p)} = \prod_{j=1}^{l_{\mathbf{m}}}\prod_{s=1}^{m_j}(p+s-1-(j-1)f), \tag{6.1}$$

where $l_{\mathbf{m}}$ is the number of nonzero m_j's in $\mathbf{m}$. Similarly, for $q = p - rf$ and $\mathbf{m}^* = (-m_k, \dots, -m_1)$, according to (4.9), the diagonal element of D_k^* corresponding to row $\mathbf{m}$ is equal to

$$(q)^*_{\mathbf{m}} = [\prod_{j=k-l_{\mathbf{m}}+1}^{k}\ \prod_{s=1}^{m_{k-j+1}}(q+(k-j+1)f-s)]^{-1}. \tag{6.2}$$

We can write an elementary program in Maple to obtain each diagonal element (6.1) and (6.2).

Once the matrices $B_k(f), B_k^{-1}(f), D_k(p), D_k^*(q)$ have been obtained in this fashion, the expected values are calculated following the formula in (5.2) and (5.3).

The algorithm to compute the matrix $M(p) = B_k^{-1}D_k(p)B_k$ representative of T_p (in the basis $\mathtt{p}$) and the matrix $M^*(q) = B_k^{-1}D_k^*(q)B_k$ representative of T_p^* for each $k \in \mathbb{N}$ is therefore very simple:

step 1 Using the function "top" in SF, build the matrix B_k,

step 2 Using formulae (6.1) and (6.2), build the matrices $D_k(p)$ and $D_k^*(q)$,

step 3 Compute $M(p) = B_k^{-1}D_k(p)B_k$ and $M^*(q) = B_k^{-1}D_k^*(q)B_k$.

In the rest of this section, we are going to give the Maple commands as well as the results given by Maple for (5.2) and (5.3) in the case $k = 2$. Similar programs can be written for any k. The results obtained allow us to obtain $\mathbb{E}(U^{\pm k})$, for any k. For example, using Theorem 5.3 and the last line of $M_3^*(q)$, as given in Subsection 6.2, our algorithm yields immediately

$$\begin{aligned} d_3\mathbf{E}(U^{-3}) &= 2f^2\sigma^{-1}(\mathrm{tr}\ \sigma^{-1})^2 + f(q+f-1)(\sigma^{-1}\mathrm{tr}\ \sigma^{-2} \\ &\quad +2\sigma^{-2}\mathrm{tr}\ \sigma^{-1}) + (q+f-1)^2\sigma^{-3}, \end{aligned} \tag{6.3}$$

where $d_3 = (q-1+3f)(q-1+2f)(q-1+f)(q-2+f)(q-3+f)$.

In the case $d = 2f = 1$ of the real Wishart distributions, with the classical notations $p = n/2$, $\sigma = 2\Sigma$ this gives

$$C\mathbf{E}(U^{-3}) = \tag{6.4}$$

$$2\Sigma^{-1}(\text{tr } \Sigma^{-1})^2 + c(\Sigma^{-1}\text{tr } \Sigma^{-2} + 2\Sigma^{-2}\text{tr } \Sigma^{-1}) + c^2\Sigma^{-3},$$

where $c = n - r - 1$ and $C = c(c+1)(c+2)(c-2)(c-4)$.

Similarly for $k = 4$, we obtain $\mathbf{E}(U^4)$ and $\mathbf{E}(U^{-4})$ respectively, as follows:

$$\begin{aligned}
&\mathbf{E}(U^4) = \\
&\quad [pf^3]\sigma(\text{tr } \sigma)^3 \qquad (6.5)\\
&\quad +[3p^2f^2 + 3(-f+1)f^2p](\sigma\text{tr } \sigma\text{tr } (\sigma)^2 + (\sigma)^2(\text{tr } \sigma)^2)\\
&\quad +[p^3f + (-3f+3)fp^2 + (2f^2-3f+2)fp](\sigma\text{tr } (\sigma)^3 + 3(\sigma)^3\text{tr } (\sigma))\\
&\quad +[2p^3f + (-5f+5)fp^2 + (3f^2-5f+3)fp]\sigma^2\text{tr } (\sigma^2)\\
&\quad +[p^4 + (-6f+6)p^3 + (-17f+11+11f^2)p^2 + (-6f^3-11f+11f^2+6)p]\sigma^4
\end{aligned}$$

$$\begin{aligned}
&d_4\mathbf{E}(U^{-4}) = \qquad (6.6)\\
&[5f^3q + 11(-1+f)f^3]\sigma^{-1}(\text{tr } \sigma^{-1})^3\\
&+[5f^2q^2 + 16(-1+f)f^2q + 11(-1+f)^2f^2](\sigma^{-1}\text{tr } \sigma^{-1}\text{tr } (\sigma^{-2}) + (\sigma^{-2})(\text{tr } \sigma^{-1})^2)\\
&+[fq^3 + 4f(-1+f)q^2 + (3f(-1+f)^2 + f(2f^2-3f+2))q + f(-1+f)(2f^2-3f+2)]\\
&(\sigma^{-1}\text{tr } (\sigma^{-1})^3 + 3(\sigma^{-1})^3\text{tr } (\sigma^{-1}))\\
&+[2fq^3 + 9(f-1)fq^2 + (7(-1+f)^2 + 5f^2 - 13f + 5)fq +\\
&(-1+f)(5f^2-13f+5)f]\sigma^{-2}\text{tr } (\sigma^{-2})\\
&+[q^4 + 5(-1+f)q^3 + (4(-1+f)^2 + 5f^2 + 5 - 9f)q^2 +\\
&((-1+f)(5f^2+5-9f) + 2f^3 - 5f^2 + 5f - 2)q + (-1+f)(2f^3-5f^2+5f-2)]\sigma^{-4}
\end{aligned}$$

where $d_4 = (q-1+4f)(q-1+3f)(q-2+2f)(q-1+2f)(q-4+f)(q-3+f)(q-2+f)(q-1+f)$.

6.2 Example: the case k=2

We first call the SF library and introduce the $J^{1/f}_{\mathbf{m}}$ polynomials as a basis.

```
> restart:with(SF):
> add_basis(J, proc(mu) zee(mu,1/f) end, proc(mu) hooks(mu,1/f)
> end);
```

We then build the matrix B_2.

```
> l1:=top(J[2]);
```

$$l1 := p1^2 + \frac{p2}{f}$$

```
> with(linalg):
```

```
> l12:=vector([coeff(l1,p1^2),coeff(l1,p2)]);
```

$$l12 := \left[1,\ \frac{1}{f}\right]$$

```
> l2:=top(J[1,1]);
```

$$l2 := -p2 + p1^2$$

```
> l22:=vector([coeff(l2,p1^2),coeff(l2,p2)]);
```

$$l22 := [1,\ -1]$$

The matrix B_2 is therefore equal to

```
> B2:=matrix(2,2,[l12,l22]);
```

$$B2 := \begin{bmatrix} 1 & \frac{1}{f} \\ 1 & -1 \end{bmatrix}$$

We now obtain the matrix B_2^{-1}.

```
> IB2:=inverse(B2);
```

$$IB2 := \begin{bmatrix} \frac{f}{f+1} & \frac{1}{f+1} \\ \frac{f}{f+1} & -\frac{f}{f+1} \end{bmatrix}$$

The diagonal matrix $D_2(p)$ is easily obtained:

```
> D2:=diag(d12(p),d22(p,f));
```

$$\mathrm{D2} := \begin{bmatrix} p\,(p+1) & 0 \\ 0 & p\,(p-f) \end{bmatrix}$$

The matrix $M_2(p) = B_2^{-1} D_2(p) B_2$:

```
> M2:=simplify(multiply(IB2,D2,B2));
```

$$M2 := \begin{bmatrix} p^2 & p \\ f\,p & (-f+p+1)\,p \end{bmatrix}$$

We now write the entries of $M_2(p)$ as polynomials in decreasing powers of p.

```
> MM2:=array(1..2,1..2):for i to 2 do for j to 2 do
> MM2[i,j]:=collect(M2[i,j],[p^2,p]) od od :print (MM2);
```

$$\begin{bmatrix} p^2 & p \\ f\,p & p^2 + (-f+1)\,p \end{bmatrix},$$

which is exactly the matrix $M_2(p)$ obtained in (5.7). For the computation of the moments of the inverse Wishart, we now need to know the elements of $D_2^{-1}(q)$. The diagonal elements of this matrix are equal to

$$[\prod_{j=1}^{j=r} \prod_{i=1}^{i=m_j} (q + jf - i)]^{-1}$$

where $q = p - rf$.

The matrix $D_2^{-1}(q)$, denoted here `SD2` is:

```
> SD2:=diag(ds12(q,f),ds22(q,f));
```

$$SD2 := \begin{bmatrix} \frac{1}{(q+f-1)\,(q+f-2)} & 0 \\ 0 & \frac{1}{(q+f-1)\,(q+2\,f-1)} \end{bmatrix}$$

The matrix $M_2^*(q)$, denoted here `SM2` is therefore equal to

```
> SM2:=simplify(multiply(IB2,SD2,B2));
```

$$SM2 := \begin{bmatrix} \frac{2\,f+q-2}{(q+2\,f-1)\,(q+f-2)\,(q+f-1)} & \frac{1}{(q+f-1)\,(q+f-2)\,(q+2\,f-1)} \\ \frac{f}{(q+2\,f-1)\,(q+f-2)\,(q+f-1)} & \frac{1}{(q+2\,f-1)\,(q+f-2)} \end{bmatrix}$$

which is, of course, the matrix obtained in (5.9).

7 Appendix

7.1 Proof of Proposition 2.1

We write $C = (\mathrm{D}et\ g)^{r/n}$, where $n = r + r(r-1)f$ is the dimension of V. The image $g_*\mu_p$ of the measure μ_p by g has the following Laplace transform:

$$\begin{aligned} \int_V e^{\langle\theta,y\rangle} g_*\mu_p(dy) &\overset{(1)}{=} \int_V e^{\langle\theta,g(u)\rangle}\mu_p(du) \overset{(2)}{=} \int_V e^{\langle g^*(\theta),u\rangle}\mu_p(du) \\ &\overset{(3)}{=} (\det(-g^*(\theta))^{-p} \overset{(4)}{=} C^{-p}(\det(\theta))^{-p}. \end{aligned}$$

where $g^*(\theta) = a^*\theta a$. Equality (1) is due to the change of variable $u \mapsto g(u)$, (2) is due to the definition of the adjoint on the Euclidean space V, (3) comes from (2.4) and for (4) see FK p.53. This implies that $g_*\mu_p$ is $C^{-p}\mu_p$. Thus the image of $\gamma_{p,\sigma}(du)$ by the map $u \mapsto y = g(u)$ is

$$\frac{C^{-p}}{(\det\sigma)^p} e^{-\langle\sigma^{-1},g^{-1}(y)\rangle}\mu_p(dy) = \frac{C^{-p}}{(\det\sigma)^p} e^{-\langle(g(\sigma))^{-1},y\rangle}\mu_p(dy) = \gamma_{p,g(\sigma)}(dy),$$

since $(g^{-1})^*(\sigma^{-1}) = (g(\sigma))^{-1}$ (see FK p.16). Formula (2.10) is now evident. Applying it to g in K shows that $T_p(Q)$ is K-invariant. From

$$\int_V Q(u)e^{\langle\theta,u\rangle}\mu_p(du) = Q(\frac{\partial}{\partial\theta})\int_V e^{\langle\theta,u\rangle}\mu_p(du)$$

it follows easily that if Q is a homogeneous polynomial of degree k, then so is $T_p(Q)$.

7.2 Proof of Proposition 4.5

For $s = (s_{lk})_{1\leq l,k\leq r}$ in Ω, denote by $\Delta_j^*(s)$ the determinant of $(s_{lk})_{r-j+1\leq l,k\leq r}$ and

$$\Delta_{\mathbf{m}}^*(s) = \Delta_1^*(s)^{m_1-m_2}\Delta_2^*(s)^{m_2-m_3}\cdots\Delta_r^*(s)^{m_r}.$$

From FK VII.1.5, (ii), we know that

$$\Delta_{\mathbf{m}}(s^{-1}) = \Delta_{-\mathbf{m}^*}^*(s). \tag{7.1}$$

We also introduce the (r,r) matrix $u = (u_{ij})_{1\leq i,j\leq r}$ defined by $u_{i,r+1-i} = 1$ for all $i = 1,\dots,r$ and $u_{ij} = 0$ otherwise. Thus $usu = (s_{r+1-i,r+1-j})$, $\mathrm{tr}\,(\sigma^{-1}usu) = \mathrm{tr}\,((u\sigma u)^{-1}s)$ and

$$\Delta_{\mathbf{m}}^*(s) = \Delta_{\mathbf{m}}(usu) \tag{7.2}$$

Therefore, since $\Delta_r^*(s) = \Delta_r(s) = \det s$, we have

$$\begin{aligned}
&\mathbb{E}(\Delta_{\mathbf{m}}(S^{-1}))\\
&\overset{(1)}{=} \int_\Omega \Delta_1^*(s)^{-m_r+m_{r-1}}\Delta_2^*(s)^{-m_{r-1}+m_{r-2}}\cdots\Delta_r^*(s)^{-m_1}\Delta_r^*(s)^{p-r}e^{-\mathrm{tr}\,(\sigma^{-1}s)}\frac{ds}{(\det\sigma)^p\Gamma_\Omega(p)}\\
&\overset{(2)}{=} \int_\Omega \Delta_1^*(s)^{p-m_r-(p-m_{r-1})}\Delta_2^*(s)^{p-m_{r-1}-(p-m_{r-2})}\cdots\Delta_r^*(s)^{p-m_1}\Delta_r^*(s)^{-r}e^{-\mathrm{tr}\,(\sigma^{-1}s)}\frac{ds}{(\det\sigma)^p\Gamma_\Omega(p)}\\
&\overset{(3)}{=} \int_\Omega \Delta_{p-m^*}^*(s)(\det s)^{-r}\frac{ds}{(\det\sigma)^p\Gamma_\Omega(p)}\\
&\overset{(4)}{=} \int_\Omega \Delta_{p-m^*}(s')(\det s')^{-r}e^{-\mathrm{tr}\,((u\sigma u)^{-1}s')}\frac{ds'}{(\det\sigma)^p\Gamma_\Omega(p)}\\
&\overset{(5)}{=} \Delta_{p-m^*}(u\sigma u)\frac{\Gamma(p-m^*)}{(\det\sigma)^p\Gamma_\Omega(p)}\\
&\overset{(6)}{=} \Delta_{p-m^*}^*(\sigma)\frac{\Gamma(p-m^*)}{(\det\sigma)^p\Gamma_\Omega(p)}\\
&\overset{(7)}{=} (\det\sigma)^p\Delta_{-m^*}^*(\sigma)\frac{\Gamma(p-m^*)}{(\det\sigma)^p\Gamma_\Omega(p)}\\
&\overset{(8)}{=} \frac{\Gamma(p-m^*)}{\Gamma_\Omega(p)}\Delta_{\mathbf{m}}(\sigma^{-1})
\end{aligned}$$

Equalities (1), (2) and (7) are due to definitions, equalities (3) and (8) are due to (7.1), equality (4) is obtained by the change of variables $s' = usu$ which has Jacobian 1, (5) is true by FK VII.1.2 while equality (6) is due to (7.2). The proof of the proposition is thus complete.

7.3 Lifting

Now, let k be a nonnegative integer and let $P(V)_k^K$ be the space of homogeneous polynomials $Q : V \mapsto \mathbb{R}$ of degree k which are K-invariant, i.e. which satisfy $Q(k(u)) = Q(u)$

for all $k \in K$ and all $u \in V$. As explained in section 4.2, these invariant polynomials Q will serve to create new polynomials $\tilde{Q} : V \times V \mapsto \mathbb{R}$. The following proposition gives the correspondence between Q and $\tilde{Q}$. Since this is essentially shown in FK p.291, we do not give a proof.

Proposition 7.1 *Let Q be in $P(V)^K_k$. There exists a unique polynomial $\tilde{Q} : V \times V \mapsto \mathbb{R}$ such that for all h and u in Ω, one has*

$$Q(h^{\frac{1}{2}}uh^{\frac{1}{2}}) = Q(u^{\frac{1}{2}}hu^{\frac{1}{2}}) = \tilde{Q}(u,h). \tag{7.3}$$

Furthermore, $\tilde{Q}$ is homogeneous of degree $2k$ and for all $g \in G$, $\tilde{Q}(g(u),h) = \widetilde{Q}(u, g^*(h))$, where g^* is the adjoint of g.
Conversely, if $Q_1 : V \times V \mapsto \mathbb{R}$ is a homogeneous polynomial of degree $2k$ such that, for all $g \in G$

$$Q_1(g(u),h) = Q_1(u, g^*(h)), \tag{7.4}$$

then, there exists a unique $Q \in P(V)^K_k$ such that $Q_1 = \widetilde{Q}$, namely $Q(h) = Q_1(e,h) = Q_1(h,e)$.

The actual computation of $\tilde{Q}$ given Q is not straightforward. We do it in the next proposition when $Q(u) = \mathrm{tr}\ (u^k)$. We must first introduce the triple product $\{x, y, z\}$ of x, y and z in V,

$$\{x,y,z\} = \frac{1}{2}(xyz + zyx), \tag{7.5}$$

which satisfies the following lemma.

Lemma 7.1 *If a is an invertible matrix of order r with coefficients in K_d, denote $g(x) = axa^*$ and $g^*(x) = a^*xa$ for $x \in V$. Then*

$$g^*\{x, g(y), z\} = \{g^*(x), y, g^*(z)\} \tag{7.6}$$

Proof: Easy.

□

Proposition 7.2 *Let u and h be in V. Define the sequence $(q_k)_{k\geq 1}$ in V by $q_1 = h$ and $q_k = \{h, u, q_{k-1}\}$. Then, for $Q(u) = tr\ u^k$, we have $\tilde{Q}(u,h) = \langle u, q_k\rangle$.*

Proof: We write $q_k = q_k(u,h)$ and we show by induction on k that, for all $g \in G$, one has

$$g^*(q_k(g(u),h)) = q_k(u, g^*(h)). \tag{7.7}$$

The result is obvious for $k=1$. Assuming it is true for $k-1$, we obtain (7.7) immediately by applying (7.6) to q_k. It follows from (7.7) that $Q_1(u,h) = \mathrm{tr}\ (u.q_k(u,h))$ satisfies (7.4). Clearly Q_1 is a homogeneous polynomial of degree $2k$. Moreover, since $\{x,e,z\} = \frac{1}{2}(xz+zx)$, we have $q_k(e,h) = h^k$. Thus if $Q(h) = \mathrm{tr}\ h^k$, it follows from Proposition 7.1 that $Q_1 = \widetilde{Q}$. □

With the notations of Proposition 7.1, for $d=1,2$, and for $d=2,4$ we have respectively

$$\begin{aligned}\widetilde{r_{(i)}}(u,h) &= \prod_{j=1}^{k} (\mathrm{tr}\ (uh)^j)^{i_j} \qquad (7.8)\\ &= \prod_{j=1}^{k} (\Re\mathrm{tr}\ (uh)^j)^{i_j},\end{aligned}$$

that is $\widetilde{r_{(i)}}(u,h)$ is the value of $r_\pi(u)(h,\ldots,h)$ or$R_\pi(u)(h,\ldots,h)$ when π has portrait (i). We mention a more compact form of Theorem 3.1 for all $h_1,\ldots,h_k$ equal.

Theorem 7.1 *If U has the Wishart distribution $\gamma_{p,\sigma}$ on V, then for all $h \in V$, one has*

$$\mathbb{E}((\langle U,h\rangle^k) = \sum_{(i)\in I_k} a_i\ p^{i_1+\ldots i_k}\widetilde{r_{(i)}}(\sigma,h) \qquad (7.9)$$

where $\widetilde{r_{(i)}}$ is defined by (7.8) and $a_{(i)} = \frac{k!}{i_1!\ldots i_k!1^i\ldots k^{i_k}}$.

Proof: This is an immediate consequence of Theorem 3.1, of (7.8) and of the fact that $a_{(i)}$ is the number of permutations π in $\mathcal{S}_k$ with portrait (i). □

Let us now turn to the definition of the *lifting* $L_{r_{(i)}}(u)$ of $r_{(i)}(u)$. Given a real homogeneous polynomial Q of degree k on V, we construct its *polarized* form as the unique symmetric k-linear form $F_Q(h_1,\ldots,h_k)$ on V^k such that $Q(h) = F_Q(h,\ldots,h)$. This process is familiar for the quadratic forms and the associated symmetric bilinear forms. For instance, if $Q(h) = \mathrm{tr}\ (h^k)$ we have

$$F_Q(h_1,\ldots,h_k) = \frac{1}{k!}\sum_{\pi\in\mathcal{S}_k} \mathrm{tr}\ (h_{\pi(1)}\ldots h_{\pi(k-1)}h_{\pi(k)}). \qquad (7.10)$$

We use the following proposition

Proposition 7.3 *If Q is a homogeneous polynomial on V of degree k. Then for all h_0 and h in V one has*

$$F_Q(h_0,\ldots,h_0,h) = \frac{1}{k}\langle Q'(h_0),h\rangle.$$

Proof: Among the many ways to express F_Q, we choose the following: $F_Q(h_1, \ldots, h_k)$ is the coefficient of $x_1 \cdots x_k$ in the polynomial on $\mathbb{R}^k$ defined by

$$(x_1, \ldots, x_k) \mapsto \frac{1}{k!} Q(x_1 h_1 + \cdots + x_k h_k).$$

Suppose now that $h_1 = \ldots = h_{k-1} = h_0$ and $h_k = h$. Using the homogeneity of Q and its Taylor's expansion, we write

$$\begin{aligned}
&\frac{1}{k!} Q((x_1 + \cdots + x_{k-1})h_0 + x_k h) \\
&\quad = \frac{1}{k!}(x_1 + \cdots + x_{k-1})^k Q(h_0 + \frac{x_k}{x_1 + \cdots + x_{k-1}} h) \\
&\quad = \frac{1}{k!}(x_1 + \cdots + x_{k-1})^k [Q(h_0) + \frac{x_k}{x_1 + \cdots + x_{k-1}} \langle Q'(h_0), h\rangle + \ldots].
\end{aligned}$$

The coefficient of $x_1 \ldots x_k$ is therefore the coefficient of $x_1 \ldots x_k$ in

$$\frac{1}{k!}(x_1 + \cdots + x_{k-1})^{k-1} x_k \langle Q'(h_0), h\rangle,$$

and this gives the desired result.

$\square$

When furthermore Q is a K-invariant polynomial, using the polarized form F_Q, we are going to define the *lifting* of Q. To do so, we fix u in V and consider the polynomial $h \mapsto \widetilde{Q}(u, h)$ as defined in Proposition 7.1. The lifting $L_Q(u)$ of Q is the element of V defined by

$$\langle L_Q(u), h\rangle = k F_{\widetilde{Q}(u,.)}(e, \ldots, e, h). \tag{7.11}$$

For instance, we are going to see that for $Q(u) = \text{tr }(u^k)$, then $L_Q(u) = k u^k$ and more generally we are going to calculate $L_{r_{(i)}}$.

Proposition 7.4 *For $(i) \in I_k$ we have*

$$L_{r_{(i)}}(u) = r_{(i)}(u) \sum_{j=1}^{k} j i_j \frac{u^j}{tr\ (u^j)},$$

Proof: We show first the result for $(i) = (0, \ldots, 0, 1)$. From Proposition 7.3, this is equivalent to showing that $\frac{\partial}{\partial h} \widetilde{Q}(u, e) = k u^k$ for $Q(h) = \text{tr }(h^k)$. To see this, we observe first that the differential of the map $x \mapsto x^k$ from V to V is

$$s \mapsto s x^{k-1} + x s x^{k-2} + x^2 s x^{k-3} + \cdots + x^{k-1} s.$$

Thus the differential of $x \mapsto \text{tr }(x^k)$ is $s \mapsto k\langle s, x^{k-1}\rangle$: this is clear for $d = 1, 2$ and uses the fact that $\operatorname{tr} x^k = \Re tr x^k$ for $d = 4$. Taking $h = e$ in this last formula we obtain, as desired, $\langle \frac{\partial}{\partial h}\widetilde{Q}(u,e), s\rangle = k\langle u^k, s\rangle$. From this, we compute the differential of $h \mapsto \tilde{Q}(u,h) = \operatorname{tr} u^{1/2}hu^{1/2}$ by composition of differentiation and we get the differential at h as

$$s \mapsto k\langle u^{1/2}su^{1/2}, u^{1/2}hu^{1/2}\rangle.$$

Letting $h = e$ in this last formula gives $\langle \frac{\partial}{\partial h}\widetilde{Q}(u,e), s\rangle = k\langle u^k, s\rangle$ as desired.

Having proved the result of the proposition in this particular case leads us immediately to the general case, since from Proposition 7.3 we have for any Q_1, Q_2 in $P(V)_k^K$

$$L_{Q_1Q_2} = Q_1L_{Q_2} + Q_2L_{Q_1}.$$

Since $r_{(i)}$ is a product of functions of type $Q(h) = \text{tr } (h^k)$, the general result is obtained. $\square$

In the next proposition we use in an obvious way the notations $T_p(F)$ and $T_p^*(F)$ for $\mathbb{E}(F(U))$ and $\mathbb{E}(F(U^{-1}))$ respectively, when F is not necessarily a real function of u but takes its values in a finite dimensional real linear space.

Proposition 7.5 *Let U follow the $\gamma_{p,\sigma}$ distribution and let Q be in $P(V)_k^K$. Then with the notation of (2.9) and (4.11) respectively we have*

$$\begin{aligned} for\ p \in \Lambda,\ T_p(L_Q) &= L_{T_p(Q)} &\quad (7.12)\\ for\ p \geq k + (r-1)f,\ T_p^*(L_Q) &= L_{T_p^*(Q)} &\quad (7.13)\end{aligned}$$

Proof: This an immediate consequence of the linearity of $Q \mapsto L_Q$. $\square$

8 Bibliographical comments

A first draft of the present paper is Letac and Massam (1999). The idea of considering the operator T_p on various spaces of functions goes back to Kushner and Meisner (1980).

First and second moments can be found in the literature (see for example Muirhead (1982), p.90 for the mean and covariance of a Wishart variable U). Proofs for the expressions of the first and second moments for the inverse Wishart distribution can be found in Eaton (1983). Haff (1979) gives the mean, covariance of U^{-1} and also $\mathbb{E}(U^2)$ and $\mathbb{E}(U^{-2})$. Wong and Liu (1995) have considered moments for generalized Wishart distribution, including non central Wishart distributions. Their methods are different from those used by previous authors in the sense that they use differential forms and permutations instead of matrix derivatives and commutation matrices. In that sense, their methods are closer to ours than to any other method used so far. They obtain all moments of the type

$\mathbb{E}(\mathrm{tr}\ (Uh_1))\ldots\mathrm{tr}\ (Uh_k))$, thus generalizing our Theorem 3.1 to these non central Wishart ditributions.

The inverse Wishart distribution has also been used in discriminant analysis (see Das Gupta (1968) and Haff (1982)), in obtaining moments of the maximum likelihood estimator on a growth curve model (see von Rosen (1988,1991 and 1997a)) and in spectral estimation (see Shaman (1980)). Moments of the Wishart and inverse Wishart are also used in approximating various statistics: for example, Krishnamoorty and Mathew (1999) use the first and second moments of the Wishart and the inverse Wishart in approximation methods for computing tolerance factors for a multivariate Gaussian population. Lauritzen (1996, p.141 and 259) uses the moments of the inverse Wishart when computing the asymptotic variance of the maximum likelihood estimate of the canonical parameter in covariance selection models. The inverse Wishart distribution is also used as a prior when sampling from a multivariate Gaussian distribution (see Anderson (1984, §8.7.7) or Dawid (1981)) and many more recent papers.

Let us now consider the paper by Hanlon, Stanley,Stembridge (1992) and reformulate some of their results with the notations of the present paper. Recall that the field K_d is endowed with its natural Euclidean structure. A standard normal variable Z in K_d is a centered Gaussian variable invariant by rotation such that $\mathbb{E}(\|Z\|^2) = 1$. A standard normal matrix $Z = (Z_{i,j})_{1\leq i,j\leq r}$ is a Gaussian random matrix such that the $Z_{i,j}$'s are independent standard normal and taking their values in K_d. We consider the space V of (r, r) Hermitian matrices on K_d, two elements A and B of V and the random matrix $AZBZ^*$. If A is in the cone Ω of positive definite elements of V, then the polynomial in A and B defined by $\mathrm{tr}\ ((AZBZ^*)^k)$ can be written as the trace of an element of V, namely $\mathrm{tr}\ ((A^{1/2}ZBZ^*A^{1/2})^k)$. Hanlon et al. (1992) prove the existence of, and compute the numbers $c((i),(j),(l))$ such that for all (i) in I_k we have

$$\mathbb{E}(r_{(i)}(A^{1/2}ZBZ^*A^{1/2})) = \sum_{(j),(l)\in I_k} c((i),(j),(l))r_{(i)}(A)r_{(i)}(B). \tag{8.1}$$

They do this with the help of the classical representation theory of the symmetric group $\mathcal{S}_k$ for $d = 2$ and $\mathcal{S}_{2k}$ for $d = 1$ and 4. If we take B to be the identity matrix e of dimension r in the field K_d, then

$$ZBZ^* = ZZ^* = (\sum_{k=1}^{r} Z_{ik}\overline{Z}_{jk})_{1\leq i,j\leq r}$$

is Wishart distributed with mean the multiple re of the identity and shape parameter $p = rf$. Thus the scale parameter of ZZ^* is e/f. Therefore, from Proposition 3.1, the distribution of $U = A^{1/2}ZZ^*A^{1/2}$ is $\gamma_{rf,A/f}$. Since $r_{(i)}(e) = r^{\sum_j i_j}$, then (8.1) implies that

$$T_{rf}(r_{(i)}(\sigma)) = \mathbf{E}(r_{(i)}(U)) = \sum_{(j)\in I_k} c((i),(j),.)r^{\sum_j i_j} r_{(i)}(\sigma)$$

where $c((i),(j),.) = \sum_{(l)\in I_k} c((i),(j),(l))$. This is a particular case of our formula (5.2) for $p = rf$ and $d = 1, 2, 4$. For $d = 1$ and $d = 4$ explicit expressions for $c((i),(j),.)$ are difficult to obtain. On the other hand, if in Hanlon et al. (1992), we take B to be an arbitrary Hermitian matrix, the random variable $U = A^{1/2}ZBZ^*A^{1/2}$ becomes the sum of r independent Wishart variables with shape parameter f but with possibly different scale parameters. The same paper makes also a thorough study of the particular case $B = e$ and A idempotent.

Another group of related results can be found in Graczyk, Letac, Massam (2003) where only the complex case $d = 2$ is considered. This paper extends the results of Maiwald and Strauss (2000). For the r_π's defined here in Theorem 3.1, the above paper by Graczyk et al. gives explicitely the expectation of $r_\pi(Uh_1, \ldots, Uh_k)$ as a linear combination of the $r_{\pi'}(\sigma h_1, \ldots, \sigma h_k)$ for $\pi' \in \mathcal{S}_k$ and it also gives a similar result for the expectation of $r_\pi(U^{-1}h_1, \ldots, U^{-1}h_k)$. The theory of these non invariant moments is not well understood yet for other values of d. The paper Letac and Massam (2001) gives an analogue to Theorem 3.1 for an arbitrary Jordan algebra V by replacing r_π by an other closely related k linear form R_π on V. Interesting informations about the complex case are in Section 5 of Haagerup and Thorbjørnsen (1998).

Interestingly enough, all results found in the literature on the moments of the inverse Wishart use Stokes formula, including Maiwald and Kraus (2000) and Graczyk, Letac, Massam (2003), while the calculations of the present paper involving the inverse Wishart (in particular Proposition 4.4) do not.

9 Conclusion and acknowlegments.

In this paper we have designed an algorithm for computing the expectation of polynomials of the entries of a Wishart variable U or its inverse U^{-1} which are functions of the eigenvalues of U alone. The detailed formula for polynomials which are monomials with respect to the $\mathrm{tr}\,(U^j)$'s or $\mathrm{tr}\,(U^{-j})$'s is given in Theorem 5.1. This leads to the calculation of $\mathbb{E}(U^{\pm k})$. We have also provided an explicit formula for the expectation of $\mathrm{tr}\,(Uh_1)\ldots\mathrm{tr}\,(Uh_k)$ where $h_1, \ldots h_k$ are arbitrary symmetric matrices (Theorem 3.1). Our formulas are purely algebraic but can be a starting point of the asymptotic analysis of the eigenvalues of the Wishart variable U when the order r of the matrix U is large.

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