Homogeneous cones and Gaussian models

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Cholevsky for a rooted tree. Let E be a finite dimensional real linear space and let $L(E)$ be the space of the linear endomorphisms of E. A homogeneous cone is an proper open convex cone C of E such that the group of the automorphisms of C, namely

$$\mathcal{G} = \{g \in L(E);\ g(C) = C\}$$

acts transitively on C. Note that $g \in \mathcal{G}$ is invertible since it transforms an open set into an open set.

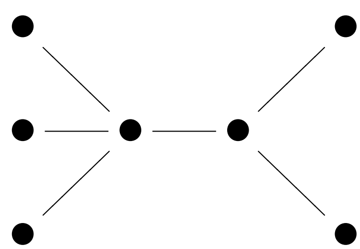
Example : the cone C of positive definite matrices as a subset of the space of symmetric matrices of order q. If a is a non singular matrix of order q with transposed a^* and $x \mapsto g_a(x) = a^*xa$ then $g_a \in \mathcal{G}$. Take $a = x^{-1/2}y^{1/2}$ for $g_a(x) = y$.

If P is a symmetric positive definite matrix of order q the Cholevsky decomposition writes $P = T^*T$ where T is an upper triangular matrix with positive diagonal and where T^* is the transposed matrix of T. Such a decomposition is unique. This result is generally considered as a consequence of the Schmidt orthonormalisation process. We are going to find it back as a consequence of a general theorem which considers a certain partial order on $\{1, \ldots, q\}$.

Trees and rooted trees.

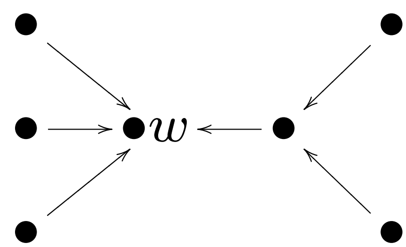
A tree is a connected graph (A, E) without cycles.

Example :

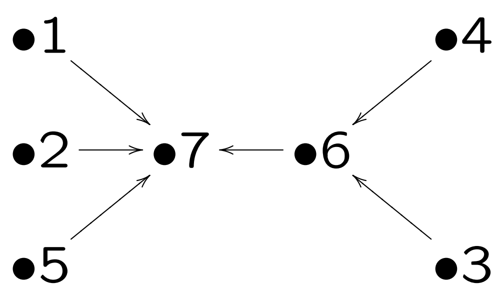


Now select a vertex w of the tree A and call it root. The pair (A, w) is called a rooted tree. The choice of a root endows the set A of vertices with the following structure of partial order $\preceq$: we write $x \preceq y$ if the unique path from x to w contains y. Clearly, this binary relation on A is a partial ordering.

Consider the example where a root has been chosen. We have $x \preceq y$ if one can travel from x to y while following arrows.



If A has q vertices, it is possible to number them with $(1, \ldots, q)$ such that $i \preceq j$ implies $i \leq j$. This obviously leads to $q = w$. Here is an acceptable numbering among others for the preceding example :



Here is the Cholevsky decomposition theorem for rooted trees :

Theorem 1.1. Let (A, w) be a rooted tree such that $A = \{1, \ldots, q\}$. Let $P = (p(x, y))_{x,y \in A}$ be a real symmetric matrix of order q. Let $\mathcal{G}$ be the set of $T = (t(x, y))_{x,y \in A}$ such that $t(x, y) \neq 0$ implies $x \preceq y$, such that $t(x, x) > 0$ for all $x \in A$. Then the following properties are equivalent

1. P is positive definite and such that if $p(x, y) \neq 0$ then either $x \preceq y$ or $y \preceq x$.
2. There exists a matrix $T \in \mathcal{G}$ such that $T^*T = P$.

Under these circumstances $T \in \mathcal{G}$ is unique. Furthermore $\mathcal{G}$ is a group. Finally if $w_1 \in A$ let

$$A_1 = \{x \in A;\ x \preceq w_1\},$$

and let P_1 and T_1 be the respective restrictions of P and T to $A_1 \times A_1$. Then $P_1 = T_1^*T_1$.

Corollary 1.2. Let (A, w) be a rooted tree and let C be the set of positive definite matrices $P = (p(x, y))_{x,y \in A}$ such that if $p(x, y) \neq 0$ then either $x \preceq y$ or $y \preceq x$. Then C is a homogeneous cone.

Proof of the corollary. Let E be the linear space of symmetric matrices $S = (s(x, y))_{x,y \in A}$ such that $s(x, y) \neq 0$ imply that x and y are comparable. Clearly C is an open convex cone whose closure does not contain any line.

If $T \in \mathcal{G}$ the automorphism group $\mathcal{A} = \{g \in L(E);\ g(C) = C\}$ of C contains all g_T defined by $g_T(x) = T^*xT$. To see this one observes directly that $g_T \in L(E)$ (we skip this calculation) and that $g(T)(C) \subset C$, a rather obvious fact. For seeing that $g(T)(C) = C$ we choose an arbitrary element P of C. The theorem says that there exists $T \in \mathcal{G}$ such that $T^*T = P$ that we reformulate in $g_T(I_q) = P$: this shows $g_T \in \mathcal{A}$. Finally, to see that $\mathcal{A}$ acts transitively on C we select two points P and P_1 in C and we write them $P = T^*T$ and $P_1 = T_1^*T_1$ with T and T_1 in $\mathcal{G}$. Since $\mathcal{G}$ is a group then $S = T^{-1}T_1$ is in $\mathcal{G}$ and $g_S(P) = P_1$ proves the result.

Proof of Theorem 1.1. Have a look to

`http ://www.lsp.ups-tlse/Fp/Letac/deug105.pdf`
page 77, a second year course in linear algebra and in French.

$2 \Rightarrow 1$. Clearly $P = T^*T$ is a semi positive definite matrix since for all column vector X we have $X^*PX = (TX)^*(TX) \geq 0$. To see that P is positive definite one puts a numbering on A such that $x \preceq y$ implies $x \leq y$. With such a choice the matrix $T = (t(x, y))_{x,y \in A}$ is upper triangular since $t(x, y) \neq 0 \Rightarrow x \preceq y \Rightarrow x \leq y$. This implies $\det T = \prod_{x \in A} t(x, x) > 0$ hence $\det P = (\det T)^2 > 0$, hence the positive definiteness of P.

For all x and y of A, from the definition of the matrix product and from the definition of the transposed matrix T^* one has $p(x,y) = \sum_{z\in A} t(z,x)t(z,y)$. The definition of T implies

$$p(x,y) = \sum_{z\preceq x;\ z\preceq y} t(z,x)t(z,y). \qquad (1)$$

Suppose that there exists a pair (x,y) such that $p(x,y) \neq 0$ and such that $x \not\preceq y$ and $y \not\preceq x$. Then equality 1 would imply the existence of a $z \in A$ such that $t(z,x)t(z,y) \neq 0$, thus such that $z \preceq x$ and $z \preceq y$. The path from z to the root w being unique, such a path would use both x and y. In this case x et y would be comparable : a contradiction.

$1 \Rightarrow 2$. We proceed by induction on the size q of the tree. For $q = 1$ this is obvious. Suppose that the result is true for any tree of size $\leq q-1$. We chose a numbering of A such that $x \preceq y$ implies $x \leq y$. As a consequence there is no $x \neq 1$ such that $x \preceq 1$. In other terms 1 is minimal. Write $A' = A \setminus \{1\}$: it is a tree with root w. Let us now write the matrix P by blocks

$$\begin{aligned} P &= \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ b^* a^{-1} & I_{q-1} \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c - b^* a^{-1} b \end{bmatrix} \begin{bmatrix} 1 & a^{-1} b \\ 0 & I_{q-1} \end{bmatrix} \end{aligned}$$

where a is a number and where c is a square matrix of order $q-1$. The vector b is a row vector $b = (b(y))_{y \in A'}$. From the definition of P it satisfies $b(y) \neq 0$ only if $1 \preceq y$.

The fact that P is positive definite implies that $a > 0$ and that the symmetric matrix $c - b^* a^{-1} b$ is positive definite. Now, the important remark is the following : if the entry (x, y) of the square matrix $b^* a^{-1} b$ is not zero then $b(x)b(y) \neq 0$, which implies $1 \preceq x$ and $1 \preceq y$. Since A is a rooted tree this implies that either $x \preceq y$ or $y \preceq x$. As a consequence one can apply the induction hypothesis to A' and to the positive definite matrix $c - b^* a^{-1} b$. We write it $(T')^* T'$ where $T' = (t(x, y))_{x,y \in A'}$ satisfies $t(x, x) \neq 0$. Finally we define

$$T = \begin{bmatrix} a^{1/2} & a^{-1/2} b \\ 0 & T' \end{bmatrix}$$

which satifies $P = T^* T$ by an immediate calculation as well as the other asked properties.

We have now to show the uniqueness of T. We show it first in the particular case $P = I_q$. Then $T^*T = I_q$ implies that T is orthogonal and upper triangular (assuming that the numbering of A is such that $x \preceq y$ implies $x \leq y$). Such a matrix T is necessarily diagonal with entries ± 1 on the diagonal. However $t(x,x) > 0$ for all $x \in A$ implies that T is the identity matrix I_q and uniqueness is therefore shown in this particular case $P = I_q$.

To reach the general case we now introduce the set $\mathcal{G}$ of all matrices $T = (T(x,y))_{x,y \in A}$ such that $T(x,y) \neq 0$ implies $x \preceq y$ and such that $T(x,x) > 0$ for all $x \in A$. We show that $\mathcal{G}$ is a group namely that $TS \in \mathcal{G}$ and that $T^{-1} \in \mathcal{G}$ if T and S are in $\mathcal{G}$. This will lead to the desired conclusion since $P = S^*S = T^*T$ implies $(ST^{-1})^*ST^{-1} = I_q$ and therefore $ST^{-1} = I_q$ by the preceding remark.

With obvious notations we have

$$(TS)(x,y) = \sum_{z \in A} T(x,z)S(z,y) = \sum_{x \preceq z \preceq y} T(x,z)S(z,y)$$

and $(TS)(x,x) = T(x,x)S(x,x)$. This shows that TS is in $\mathcal{G}$. Now $S = T^{-1}$ does exist since $\det T \neq 0$ as we have seen before. However, checking that S is in $\mathcal{G}$ is by no means obvious. Actually this is seen by induction and by writing (assuming that 1 is minimal)

$$T = \begin{bmatrix} a & b \\ 0 & T' \end{bmatrix}, \;\; S = T^{-1} = \begin{bmatrix} a^{-1} & -a^{-1}b(T')^{-1} \\ 0 & (T')^{-1} \end{bmatrix}.$$

This allows us to make our induction argument and complete the proof of uniqueness of T.

To conclude the proof let w_1, A_1, P_1 and T_1 as in the statement of the theorem. We write the matrices P and T by blocks corresponding to A_1 and $A \setminus A_1$ as follows :

$$P = \begin{bmatrix} P_1 & P_{12} \\ P_{21} & P_2 \end{bmatrix}, \quad T = \begin{bmatrix} T_1 & T_{12} \\ T_{21} & T_2 \end{bmatrix}.$$

One observes that for $x \notin A_1$ and $y \in A_1$ then $t(x,y) = 0$ from the definition of T and A_1. This implies $T_{21} = 0$. Hence

$$\begin{aligned} P = T^*T &= \begin{bmatrix} T_1^* & 0 \\ T_{12}^* & T_2^* \end{bmatrix} \begin{bmatrix} T_1 & T_{12} \\ 0 & T_2 \end{bmatrix} \\ &= \begin{bmatrix} T_1^*T_1 & T_1^*T_{12} \\ T_{12}^*T_1 & T_{12}^*T_{12} + T_2^*T_2 \end{bmatrix}. \end{aligned}$$

This shows $P_1 = T_1^*T_1$. The proof is now complete.

Posets and their Hasse diagrams. A poset is a finite set V with a binary relation $\preceq$ which is transitive : $x \preceq y$ and $y \preceq z$ implies $x \preceq z$ and anti reflexive : $x \preceq y$ and $y \preceq x$ implies $x = y$. Traditionally one writes $x \prec y$ for $x \preceq y$ and $x \neq y$. The Hasse diagram $H = (V, E)$ associated to the poset $(V, \preceq)$ is the directed graph with V as set of vertices and edges $x \to y$ when the pair (x, y) is such that $x \prec y$ and such that $x \preceq z \preceq y$ implies either $z = x$ or $z = y$. The knowledge of the Hasse diagram of a poset gives obviously the knowledge of the poset since $x \preceq y$ if and only if there exists $n \geq 0$ and a path $x = x_0 \to x_1 \to \cdots \to x_{n-1} \to x_n = y$.

For example, the Hasse diagram of the poset diamond

$$\begin{tikzcd} \bullet 1 \arrow[r] \arrow[d] \arrow[dr] & \bullet 2 \arrow[d] \\ \bullet 3 \arrow[r] & \bullet 4 \end{tikzcd} \quad \text{is} \quad \begin{tikzcd} \bullet 1 \arrow[r] \arrow[d] & \bullet 2 \arrow[d] \\ \bullet 3 \arrow[r] & \bullet 4 \end{tikzcd}$$

Two important examples of posets are given by the rooted trees (A, w). The root w induces on A a natural partial order which is defined by $x \preceq y$ if and only if the unique path from w to y contains x. The Hasse diagram of this order is obtained by the following set of arrows : $i \to j$ if $i \sim j$ and if the unique path from w to j contains i. In this case, w is the minimum point of the partial order. The opposite partial order induced by a root w of a tree is simply called "the opposite natural order" (that is $x \preceq y$ if the unique path from w to x contains y.) The root w is now the maximum point of the partial order. In section 1 we have been using the opposite natural order.

Sandglass posets. A poset $(V,\preceq)$ is a sandglass poset if for all $w \in V$ the Hasse diagrams of the two subposets

$$A_w^+ = \{x \in V; w \preceq x\}, \quad A_w^- = \{x \in V; x \preceq w\}$$

are rooted trees with root w such that A_w^+ has the natural order and A_w^- has the opposite natural order. For instance

$$\begin{tikzcd} \bullet 1 \arrow[r] \arrow[d] & \bullet 2 \\ \bullet 3 & \bullet 4 \arrow[l] \arrow[u] \end{tikzcd}$$

is an sandglass poset, while the preceding example is not. Of course a rooted tree either with its natural order or with its opposite natural order is a sandglass poset.

Proposition 2.1. Let $(V,\preceq)$ be a poset. Then it is a sandglass poset if and only if for all w and w_1 in V such that $w \preceq w_1$ then $\{x \in V;\ w \preceq x \preceq w_1\} = \{x_1,\ldots,x_k\}$ is a chain, that is has a numerotation such that

$$w = x_1 \prec x_2 \prec \ldots \prec x_n = w_1.$$

Some examples :

$$
\begin{tikzcd}
\bullet 4 & & \\
\bullet 1 \arrow[u] \arrow[r] & \bullet 2 \arrow[r] & \bullet 3
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet 4 \arrow[d] & & \\
\bullet 1 & \bullet 2 \arrow[l] & \bullet 3 \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet 1 & \bullet 2 & \bullet 3 \\
\bullet 4 \arrow[u] \arrow[ur] & \bullet 5 \arrow[ul] \arrow[u] & \bullet 6 \arrow[u] \\
\bullet 7 \arrow[u] & \bullet 8 \arrow[u] \arrow[ur] & \bullet 9 \arrow[ul] \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \bullet 1 & \bullet 2 & & \\
\bullet 3 \arrow[ur] & \bullet 4 \arrow[u] \arrow[ur] & \bullet 5 \arrow[ul] \arrow[u] & \bullet 6 & \bullet 7 \\
 & & \bullet 8 \arrow[u] \arrow[ur] & \bullet 9 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

The algebra $\mathcal{A}$ and the groups $\mathcal{G}$ and $\mathcal{G}_1$ of a poset

Given an arbitrary poset $(V, \preceq)$ denote by $\mathcal{A}$ the algebra of this poset, namely the linear space of functions $a : V \times V \to \mathbb{R}$ such that $a(x, y) = 0$ if $x \not\preceq y$. With the product

$$c(x, y) = \sum_{z \in V} a(x, z) b(z, y) = \sum_{x \preceq z \preceq y} a(x, z) b(z, y)$$

it is clear that $\mathcal{A}$ is an associative algebra which is a subalgebra of the algebra of real (n, n) matrices, by writing $A = (a(x, y))_{x, y \in V}$.

Proposition 2.1. Let $a \in \mathcal{A}$. Then a is invertible in $\mathcal{A}$ if and only if for all $x \in V$ one has $a(x,x) \neq 0$.

Proof. Adapt the proof given inside the proof of Theorem 1.1 above from a rooted tree with its opposite natural order to an arbitrary poset.

The set of invertible a is therefore a group that we denote by $\mathcal{G}$. For instance if V is

$$\bullet 1 \rightarrow \bullet 2 \rightarrow \ldots \rightarrow \bullet n$$

then $\mathcal{G}$ is the group of UPPER triangular matrices.

The subgroup of $\mathcal{G}$ such that furthermore for all $x \in V$ one has $a(x,x) = 1$ is denoted by $\mathcal{G}_1$.

The homogeneous cone C of a sandglass poset. Let $(V, \preceq)$ be a fixed sandglass poset. Given a (n, n) symmetric matrix $S = (s(x,y))_{x,y \in V}$ we denote $P = \pi(S) = (p(x,y))_{x,y \in V_0}$ the matrix defined by $p(x,y) = s(x,y)$ when $x \preceq y$ or $y \preceq x$ If $x \not\preceq y$ and $y \not\preceq x$ then $p(x,y) = 0$. Consider now the cone C_1 of positive definite matrices $S = (s(x,y))_{x,y \in V}$ of the special form $S = T^*T$ where $T \in \mathcal{G}$ and its image $C = \pi(C_1)$ by π. Here, T^* denotes the transposed matrix.

Example. We start from the sandglass poset :

$$\begin{tikzcd} \bullet 2 & \bullet 4 \\ \bullet 1 \arrow[u] & \bullet 3 \arrow[ul] \arrow[u] \end{tikzcd}$$

The group $\mathcal{G}$ is the set of matrices of real numbers such that $t_i > 0$, $i = 1, \ldots, 4$:

$$T = \begin{bmatrix} t_1 & t_{12} & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & t_{32} & t_3 & t_{34} \\ 0 & 0 & 0 & t_4 \end{bmatrix}$$

The cone C_1 is the set of symmetric matrices of the form

$$T^*T = \begin{bmatrix} t_1^2 & t_{12}t_1 & 0 & 0 \\ t_{12}t_1 & t_{12}^2 + t_2^2 + t_{32}^2 & t_{32}t_3 & t_{32}t_{34} \\ 0 & t_{32}t_3 & t_3^2 & t_{34}t_3 \\ 0 & t_{32}t_{34} & t_{34}t_3 & t_{34}^2 + t_4^2 \end{bmatrix}$$

The cone C is the cone of matrices of the form

$$\pi(T^*T) = \begin{bmatrix} t_1^2 & t_{12}t_1 & 0 & 0 \\ t_{12}t_1 & t_{12}^2 + t_2^2 + t_{32}^2 & t_{32}t_3 & 0 \\ 0 & t_{32}t_3 & t_3^2 & t_{34}t_3 \\ 0 & 0 & t_{34}t_3 & t_{34}^2 + t_4^2 \end{bmatrix}$$

Remark 3.1. Thus the π operation passes from S to P simply by erasing entries (x, y) such that x and y are not comparable and by replacing them by 0. For creating the cone C we apply this process to special symmetric matrices, namely the T^*T when $T \in \mathcal{G}$. The present remark points out that if the poset is a rooted tree with its opposite natural order, then $\pi(T^*T) = T^*T$. The proof has been given in the part $1 \Rightarrow 2$ of the proof of Theorem 1.1. Thus for this special poset $C = C_1$.

Theorem 3.1. Let $(V, \preceq)$ be a sandglass poset. Let $S = (s(x,y))_{x,y \in V}$ be a symmetric matrix. The following facts are equivalent

1. $S = T^*T$ where $T \in \mathcal{G}$
2. If $P = \pi(S)$ then for any maximal point $w \in V$ the restriction of P to the set $A_w^- \times A_w^-$ where $A_w^- = \{x \in V;\ x \preceq w\}$ is positive definite.

Furthermore the map π from C_1 to C is injective.

Comments. This is the fundamental theorem, and $2 \Rightarrow 1$ is the hard point. Consider the particular case where $(V, \preceq)$ is a rooted tree with its opposite natural order. As pointed before, then $\pi(T^*T) = T^*T$. This shows that Theorem 3.1 is a generalization of Theorem 1.1. We use Theorem 1.1 in the proof of Theorem 3.1.

Example. :

$$\begin{tikzcd} \bullet 2 & \bullet 4 \\ \bullet 1 \arrow[u] & \bullet 3 \arrow[ul] \arrow[u] \end{tikzcd}$$

$$\pi(T^*T) = \begin{bmatrix} t_1^2 & t_{12}t_1 & 0 & 0 \\ t_{12}t_1 & t_{12}^2 + t_2^2 + t_{32}^2 & t_{32}t_3 & 0 \\ 0 & t_{32}t_3 & t_3^2 & t_{34}t_3 \\ 0 & 0 & t_{34}t_3 & t_{34}^2 + t_4^2 \end{bmatrix}$$

Since the $t_i > 0$ one easily sees that if $X \in C$ then the equation in $T \in \mathcal{G}$ written as $\pi(T^*T) = X$ or as :

$$\begin{bmatrix} t_1^2 & t_{12}t_1 & & \\ t_{12}t_1 & t_{12}^2 + t_2^2 + t_{32}^2 & t_{32}t_3 & \\ & t_{32}t_3 & t_3^2 & t_{34}t_3 \\ & & t_{34}t_3 & t_{34}^2 + t_4^2 \end{bmatrix}$$
$$= \begin{bmatrix} x_1 & x_{12} & & \\ x_{12} & x_2 & x_{32} & \\ & x_{32} & x_3 & x_{34} \\ & & x_{34} & x_4 \end{bmatrix}$$

has only one solution in t.

Condition 2 of the theorem 3.1 says that in this example X is in the cone C if and only if the two matrices

$$\begin{bmatrix} x_1 & x_{12} & 0 \\ x_{12} & x_2 & x_{32} \\ 0 & x_{32} & x_3 \end{bmatrix}, \quad \begin{bmatrix} x_3 & x_{34} \\ x_{34} & x_4 \end{bmatrix}$$

are positive definite. Let us insist on the fact that C_1 here is not convex, while C is :

Corollary 3.2. C is a convex homogeneous open cone.

Proof. Convexity and openness come from property 2. Homogeneity comes from the fact that $\mathcal{G}$ operates on C by $g_{T_1}(P) = \pi(T_1^* S T_1)$ if $P = \pi(S)$ and this is clearly transitive.

Proof of Theorem 3.1. 1 $\Rightarrow$ 2. Let w be a maximal point and consider the rooted tree $A_w^- = \{x \preceq w\}$. From the remark 3.1 the restriction of $S = T^*T$ to $A_w^- \times A_w^-$ is not modified by π and therefore is positive definite.

2 $\Rightarrow$ 1. Let $P = \pi(S)$ such that the restriction P_w of P to $A_w^- \times A_w^-$ is positive definite for all maximal w of V. By Theorem 1.1 there exists a matrix

$$T_w = (t_w(x,y))_{x,y \in A_w^-}$$

such that $t_w(x,y) \neq 0$ implies $x \preceq y$ and such that $P_w = T_w^* T_w$. We use it to determine a matrix $T \in \mathcal{G}$ such that $P = \pi(T^*T)$. For this, consider an other maximal point w' and suppose that x and y are both in A_w^- and $A_{w'}^-$. Let us show that under this condition we have $t_w(x,y) = t_{w'}(x,y)$. This is true if $x \not\preceq y$ since in this case $t_w(x,y) = t_{w'}(x,y) = 0$.

If $x \preceq y$ we are going to apply the last part of Theorem 1.1 to $y = w_1$. Denote

$$A_1 = \{t \in A_w^-;\ t \preceq y\},\ A_1' = \{t \in A_{w'}^-;\ t \preceq y\}$$

and observe that actually $A_1 = A_1'$. Denote by $P_1,\ T_1$ and by $P_1',\ T_1'$ the respective restrictions of $P_w,\ T_w$ and $P_{w'},\ T_{w'}$ to $A_1 \times A_1$. Note that actually $P_1 = P_1'$. Theorem 1.1 says that $P_1 = T_1^* T_1 = (T_1')^* T_1'$. Now, we have proved in Theorem 1.1 that such a decomposition is unique. Therefore $T_1 = T_1'$ and finally $t_w(x, y) = t_{w'}(x, y)$ also when $x \preceq y$.

We now define the matrix $T = (t(x, y))_{x,y \in V}$ by $t(x, y) = 0$ if $x \not\preceq y$ and by $t(x, y) = t_w(x, y)$ if $x \preceq y \preceq w$ if w is a maximal element. We have shown that actually it does not depend on a particular maximal element w of V. Since $P_w = T_w^* T_w$ for all w this implies that $P = \pi(T^* T)$ and proves $2 \Rightarrow 1$.

We finally observe that $\pi : C_1 \to C$ is injective. This comes from the formula $P_w = T_w^* T_w$ which shows that the knowledge of P gives the knowledge of P_w thus the knowledge of T_w and therefore the knowledge of T and the knowledge of $S = T^* T$.

Why the sandglass? A natural question is the following : in Section 2 we started with a certain poset : a sandglass poset. What happens if we relax the sandglass condition? The answer is the cone C is not convex. To see this we start from the following poset :

$$\begin{tikzcd} \bullet 1 \arrow[r] \arrow[d] \arrow[dr] & \bullet 2 \arrow[d] \\ \bullet 3 \arrow[r] & \bullet 4 \end{tikzcd}$$

We consider similarly the algebra $\mathcal{A}$ of this poset, namely the set of real matrices of the form

$$T = \begin{bmatrix} t_1 & t_{12} & t_{13} & t_{14} \\ 0 & t_2 & 0 & t_{24} \\ 0 & 0 & t_3 & t_{34} \\ 0 & 0 & 0 & t_4 \end{bmatrix}$$

The group $\mathcal{G}$ is the set of $T \in \mathcal{A}$ such that $t_i > 0$ with $i = 1, \ldots, n$. The projection π of the symmetric matrices $S = (s_{ij})$ of order 4 simply erases the entries s_{23} and s_{32}. Thus if $T \in \mathcal{G}$ an easy calculation gives $\pi(T^*T) =$

$$\left[\begin{array}{cccc} t_1^2 & t_{12}t_1 & t_{13}t_1 & t_{14}t_1 \\ t_{12}t_1 & t_2^2+t_{12}^2 & 0 & t_{12}t_{14}+t_{24}t_2 \\ t_{13}t_1 & 0 & t_3^2+t_{13}^2 & t_{13}t_{14}+t_{34}t_3 \\ t_{14}t_1 & t_{12}t_{14}+t_{24}t_2 & t_{13}t_{14}+t_{34}t_3 & t_4^2+t_{14}^2+t_{24}^2+t_{34}^2 \end{array}\right.$$

Denote

$$C = \{\pi(T^*T);\ T \in \mathcal{G}\}.$$

Clearly C is stable by dilations. But we are going to see that the sum of two elements of C is not necessarily in C. In other terms, C is not a convex cone. The production of a particular pair P, P' of C such that $P + P' \notin C$ is a painful calculation.

The link with Gaussian graphical models

Let us recall a simple fact about a centered Gaussian variable $X = (X_v)_{v \in V}$ where V is a finite set. Denote by Σ the covariance of X. If $I \subset V$ we write $X_I = (X_v)_{v \in I}$. If I and J are subsets of V we write $\Sigma_{I \times J}$ the restriction of Σ to $I \times J$. We rather denote Σ_I instead of $\Sigma_{I \times I}$.

Proposition 4.1 Let I, J, K be three disjoint non empty subsets of V. Then X_I and X_J are independent knowing X_K if and only if

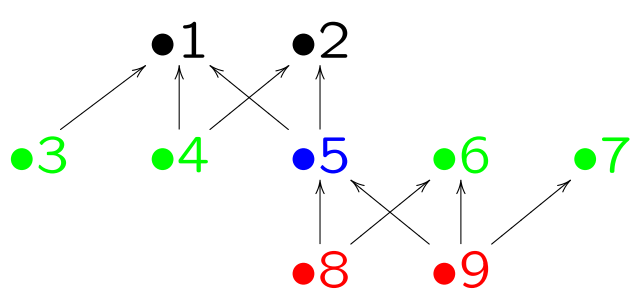
$$(\Sigma_{I \cup J \cup K})^{-1}_{I \times J} = 0$$

and if and only if

$$\Sigma_{I \times J} = \Sigma_{I \times K} (\Sigma_K)^{-1} \Sigma_{K \times J}$$

We apply this principle to the particular case where V is a irreducible sandglass (that means that for each $w \in V$ then each of the trees $V_w^{\pm}$ has either two points or has no vertex with only one child). In this case the triplets (I, J, K) that we consider are associated to a vertex $w \in V$ and therefore denoted (I_w, J_w, K_w) and they have the form

1. The set I_w is the singleton $\{w\}$ (BLUE)
2. The set J_w is $V \setminus (V_w^- \cup V_w^+)$. (GREEN)
3. The set K_w is $V_w^- \setminus \{w\}$ (RED)

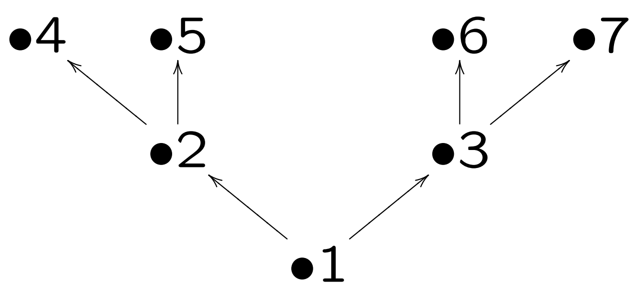


We have now the fundamental result

Theorem 4.2. Let $(V, \preceq)$ be an irreducible sandglass. Consider the set $\mathcal{M}$ of invertible covariance matrices Σ such that the Gaussian distribution $N(0, \Sigma)$ is such that for all $w \in V$ the random variables X_{I_w} and X_{J_w} are conditionally independent knowing X_{K_w}. Then $\Sigma \mapsto P = \pi(\Sigma)$ is a bijective mapping between $\mathcal{M}$ and the homogeneous cone C associated to $(V, \preceq)$.

In the previous statement, if J_w is empty then the condition of conditional independence disappears. If K_w is empty then X_{I_w} and X_{J_w} are independent.

Example :



This means that for $w = 2$ that X_2 and (X_3, X_6, X_7) are conditionally independent knowing X_1. For $w = 4$ this means that X_4 and (X_3, X_5, X_6, X_7) are conditionally independent knowing (X_1, X_2). There is no condition for $w = 1$ since J_w is empty.

A remark : given an undirected graph $G = (V, E)$ the Dempster model is built with the triplets (I, J, K) where $I = \{i\}$ and $J = \{j\}$ are all possible singletons such that $i \not\sim j$ and $K = V \setminus \{i, j\}$. When the poset $(V, \preceq)$ is a rooted tree with natural order as in the example, consider the undirected graph where $i \sim j$ if and only if either $i \preceq j$ or $j \preceq i$ for $i \neq j$. Then the Dempster model and the homogeneous cone model coincide. Here the homogeneous cone is the cone is Q_G the set of incomplete matrices $x = (x_{ij})$ such that x_S is positive definite for all maximal complete subsets S of V for the undirected graph structure $G = (V, E)$.

Duality

Theorem 5.1 Let $(V, \preceq)$ be a sandglass poset and let C be the associated homogeneous cone. Consider the sandglass poset $(V, \preceq_1)$ obtained by reversing the partial order $\preceq$ and consider the corresponding homogeneous cone. Then C_1 is the dual cone of C.