Thomae $\mathcal{S}_5$ invariance, generalized beta distributions and random continued fractions

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A refreshment on hypergeometric functions.

The sequence of Pochhammer's symbols $((a)_n)_{n=0}^{\infty}$ is defined by $(a)_0 = 1$ and $(a)_{n+1} = (a+n)(a)_n$. For real numbers $a_1, \ldots, a_p$ and positive numbers $b_1, \ldots, b_q$ we denote by

$$_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$$

the sum of the power series

$$\sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{n!(b_1)_n \ldots (b_q)_n} z^n. \tag{1}$$

If $a > 0$ then $(a)_n \sim \frac{n^{a-1} n!}{\Gamma(a)}$ by the Stirling formula of the gamma function. For $p = q+1$ and for $a_j > 0$ for all j denote

$$c = b_1 + \cdots + b_q - a_1 - \cdots - a_{q+1}. \tag{2}$$

Then the coefficient of the general term of the series 1 is equivalent to n^{-1-c} up to a multiplicative constant and this implies that the series 1 converges for $z = 1$ if and only if $c > 0$.

Thomae invariance by $\mathcal{S}_5$. Let us state a stricking result due to Thomae (1879) :

Theorem. Consider the analytic function on $\mathbb{C}^5$ defined by the analytic continuation of

$$(a,b,c,d,e) \mapsto {}_3F_2(a,b,c;d,e;1) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{n!(d)_n(e)_n}.$$

Consider the (5,5) matrices

$$A = I_5 + J_5 - \begin{bmatrix} J_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad 3A^{-1} = 3I_5 + J_5 - 3\begin{bmatrix} J_3 & 0 \\ 0 & J_2 \end{bmatrix}$$

where J_k is the (k,k) matrix whose entries are 1. Define $(x,y,z,u,v) = (a,b,c,d,e)A^{-1}$. Then the function $E(x,y,z,u,v) =$

$$\frac{1}{\Gamma(d)\Gamma(e)\Gamma(d+e-a-b-c)}\ {}_3F_2(a,b,c;d,e;1)$$

is a symmetric function of (x,y,z,u,v).

Impressive, but not that much

In the previous theorem, actually the symmetry in x, y, z is more or less obvious because of the block structure of A and the obvious symmetry in a, b, c. Similarly the symmetry in u, v is inherited of the obvious symmetry in d, e. Since the group of permutations $\mathcal{S}_5$ is generated by the 4 transpositions $(1,2), (2,3), (3,4), (4,5)$ and since the Thomae result is obvious for $(1,2), (2,3), (4,5)$ the only thing to prove is the result for the transposition $(3,4)$, therefore to prove the formula

$$
{}_3F_2(a, b, c; d, e; 1) = \frac{\Gamma(d)\Gamma(e)\Gamma(d+e-a-b-c}{\Gamma(d+e-b-c)\Gamma(d+e-a-c)\Gamma(c)} {}_3F_2(d-c, e-c, d+e-a-b-c; d+e-a-c, d+e-b-c; 1)
$$

Generalized beta on (0,1). Let a, a' and $b > 0$.

As seen before ${}_3F_2(a, a, b; a + b, a + a'; 1)$ is finite. Define $\mu_{a,a',b}(dx) =$

$$Cx^{a-1}(1-x)^{b-1}\ {}_2F_1(a, b; a + a'; x)\mathbf{1}_{(0,1)}(x)dx.$$

where

$$\frac{1}{C} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\ {}_3F_2(a, a, b; a + b, a + a'; 1). \quad (3)$$

We shall see that $\mu_{a,a',b}$ is a probability and that C is a symmetric function of (a, a'). This strange distribution is actually a generalization of the beta distribution of the first kind since $\mu_{a,a',a+a'}(dx) =$

$$\frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')}x^{a-1}(1-x)^{a'-1}\mathbf{1}_{(0,1)}(x)dx = \beta_{a,a'}(dx).$$

The reason is that ${}_2F_1(a, a + a'; a + a'; x) = {}_1F_0(a; -; x) = (1-x)^{-a}$.

Another presentation of $\mu_{a,a',b}$.

It is based on the classical formula

$$ {}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z). \tag{4}$$

When applied to $(\alpha,\beta,\gamma) = (a,b,a+a')$ it gives

$$\mu_{a,a',b}(dx) = CB(a,a')\,{}_2F_1(a',a+a'-b;a+a';x)\beta_{a,a'}(dx). \tag{5}$$

This shows that when $d = a+a'-b$ is small, then $\mu_{a,a',b}$ appears to be a perturbation of $\beta_{a,a'}$. This also gives an other presentation of C as

$$\frac{1}{C} = B(a,a')\,{}_3F_2(a,a',a+a'-b;a+a',a+a';1)$$

where the symmetry between a and a' in the value of C appears clearly.

Consequence : the marginal distributions of

$$\frac{\beta_{a,a'}(dx)\beta_{a',a}(dx')}{K(1-xx')^d}$$

are $\mu_{a,a',b}(dx)$ and $\mu_{a',a,b}(dx')$.

The beta distribution of the second kind.

It is the probability on $(0,\infty)$ defined by

$$\beta^{(2)}_{b,a}(dw)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{w^{b-1}}{(1+w)^{a+b}}\mathbf{1}_{(0,\infty)}(w)dw.$$

In the following we will need the fact that

$$\int_0^\infty w^t\beta^{(2)}_{b,a}(dw)=\frac{\Gamma(a-t)\Gamma(b+t)}{\Gamma(a)\Gamma(b)},\quad -b<t<a.$$

.

The random continued fraction

Theorem 1.1. The measure $\mu_{a,a',b}$ is a probability with the following property : If $W' \sim \beta^{(2)}_{b,a'}$ and $X \sim \mu_{a,a',b}$ are independent then

$$\frac{1}{1+XW'} \sim \mu_{a',a,b}$$

Furthermore C defined by 3 is a symmetric function of (a,a').

Proof. The integration on $(0,1)$ of each term of the series

$$x^{a-1}(1-x)^{b-1}\ {}_2F_1(a,b;a+a';x)$$

gives a convergent series whose sum is

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\ {}_3F_2(a,a,b;a+b,a+a';1).$$

This shows that $\mu_{a,a',b}$ is a probability.

Suppose now that $X' \sim \mu_{a',a,b}$. We show that $\frac{1-X'}{X'} \sim XW'$. The density of $V = \frac{1-X'}{X'}$ is

$$f_V(v) = C' \frac{v^{b-1}}{(1+v)^{a'+b}}\ {}_2F_1(a',b;a+a';\frac{1}{1+v})\mathbf{1}_{(0,\infty)}(v)dv$$

where

$$\frac{1}{C'} = \frac{\Gamma(a')\Gamma(b)}{\Gamma(a'+b)}\ {}_3F_2(a',a',b;a'+b,a+a';1).$$

We now compute the density of $U = XW'$:
Writing $K = C\frac{\Gamma(a'+b)}{\Gamma(a')\Gamma(b)}$ the density $f_U(u)$ is

$$
\begin{aligned}
&= K\frac{1}{u}\int_0^1 x^{a-1}(1-x)^{b-1} \ {}_2F_1(a,b;a+a';x)(\frac{u}{x})^b \frac{dx}{(1+\frac{u}{x})^{a'}} \\
&= Ku^{b-1}\int_0^1 \frac{x^{a+a'-1}(1-x)^{b-1}}{(x+u)^{a'+b}} \ {}_2F_1(a,b;a+a';x)dx \\
&= Ku^{b-1}\int_0^1 \frac{t^{b-1}(1-t)^{a+a'-1}}{(1+u-t)^{a'+b}} \ {}_2F_1(a,b;a+a';1-t)dt \\
&= K\frac{u^{b-1}}{(1+u)^{a'+b}}\int_0^1 \frac{t^{b-1}(1-t)^{a+a'-1}}{(1-\frac{t}{1+u})^{a'+b}} \\
&\qquad {}_2F_1(a,b;a+a';1-t)dt
\end{aligned}
$$

We conclude the proof of the theorem by a lemma :

Lemma 1.2. For $a, a', b > 0$ and for $0 < z < 1$ we have

$$\int_0^1 \frac{t^{b-1}(1-t)^{a+a'-1}}{(1-zt)^{a'+b}} \; {}_2F_1(a, b; a+a'; 1-t)dt$$
$$= \frac{\Gamma(a')\Gamma(b)}{\Gamma(a'+b)} \; {}_2F_1(a', b; a+a'; z).$$

This lemma when applied to $z = 1/(1+u)$ shows that the densities of U and V are proportional. Since their integrals are 1 one even gets that $C = C'$ or that

$$B(a, b) \; {}_3F_2(a, a, b; a+b, a+a'; 1)$$
$$= \; B(a', b) \; {}_3F_2(a', a', b; a'+b, a+a'; 1)$$

This concludes the proof.

Proof of the lemma.

$$\int_0^1 \frac{t^{b-1}(1-t)^{a+a'-1}}{(1-zt)^{a'+b}}\ {}_2F_1(a,b;a+a';1-t)dt$$

$$\overset{(1)}{=} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(a+a')_n} \int_0^1 \frac{t^{b-1}(1-t)^{a+a'+n-1}}{(1-zt)^{a'+b}} dt$$

$$\overset{(2)}{=} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(a+a')_n} \frac{\Gamma(b)\Gamma(a+a'+n)}{\Gamma(a+a'+b+n)}$$

$${}_2F_1(a'+b,b;a+a'+b+n;z)$$

$$\overset{(3)}{=} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(a+a')_n} \frac{\Gamma(b)\Gamma(a+a'+n)}{\Gamma(a+a'+b+n)}$$

$$\sum_{k=0}^{\infty} \frac{(a'+b)_k(b)_k}{k!(a+a'+b+n)_k} z^k$$

$$\overset{(4)}{=} \sum_{k=0}^{\infty} \frac{(a'+b)_k(b)_k}{k!} z^k$$

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(a+a')_n} \frac{\Gamma(b)\Gamma(a+a'+n)}{\Gamma(a+a'+b+n+k)}$$

$$\overset{(5)}{=} \sum_{k=0}^{\infty} \frac{(a'+b)_k(b)_k}{k!} z^k \frac{\Gamma(b)\Gamma(a+a')}{\Gamma(a+a'+b+k)}$$

$${}_2F_1(a,b;a+a'+b+k;1)$$

$$\overset{(6)}{=} \sum_{k=0}^{\infty} \frac{(a'+b)_k(b)_k}{k!} z^k \frac{\Gamma(b)\Gamma(a+a')}{\Gamma(a+a'+b+k)}$$

$$\frac{\Gamma(a+a'+b+k)\Gamma(a'+k)}{\Gamma(a'+b+k)\Gamma(a+a'+k)}$$

$$\overset{(7)}{=} \frac{\Gamma(a')\Gamma(b)}{\Gamma(a'+b)} \sum \frac{(a')_k(b)_k}{k!(a+a')_k} z^k$$

Corollary 1.3.

1. If $W \sim \beta^{(2)}_{b,a}$ $W' \sim \beta^{(2)}_{b,a'}$ and $X > 0$ are three independent random variables, then
$$X \sim \frac{1}{1 + \frac{W}{1+W'X}}$$
if and only if $X \sim \mu_{a,a',b}$.

2. If $W \sim \beta^{(2)}_{b,a}$ and if $X > 0$ are two independent random variables, then
$$X \sim \frac{1}{1 + WX}$$
if and only if $X \sim \mu_{a,a,b}$.

3. Let $(W_n)_{n\geq 1}$ and $(W'_n)_{n\geq 1}$ be two iid sequence of random variables with respective distributions $\beta^{(2)}_{b,a}$ and $\beta^{(2)}_{b,a'}$. Then $\mu_{a,a',b}$ is the distribution of the random continued fraction
$$\frac{1}{1 + \frac{W_1}{1+\frac{W'_1}{1+\frac{W_2}{1+\dots}}}}.$$

Some particular cases for $\mu_{a,a',b}$ We list some values of (a,a',b) for which $\mu_{a,a',b}$ has a remarkable form.

1)As seen before $b=a+a'$ gives $\mu_{a,a',a+a'}(dx)=\beta_{a,a'}(dx)$. If more generally $b=a+a'+k$ where k is a non negative integer, then the hypergeometric series ${}_2F_1(a',a+a'-b;a+a';x)$ terminates and is a polynomial with degree $\leq k$. Thus $\mu_{a,a',a+a'+k}(dx)$ has a simple expression from 5. In particular the normalizing constant and the moments of $\mu_{a,a',a+a'+k}(dx)$ are computable, since the integral of a polynomial by $\beta_{a,a'}$ is elementary. For instance

$$\mu_{a,a',a+a'+1}(dx)=\frac{a+a'}{a^2+aa'+a'^2}(a+a'(1-x))\beta_{a,a'}(dx).$$

2) For $a = a' = b = 1/2$ then

$$\mu_{1/2,1/2,1/2}(dx) = \frac{2}{\pi x^{1/2}} \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x \sin^2 t}} \ \mathbf{1}_{(0,1)}(x)dx$$

3) For $a = a' = b + \frac{1}{2}\epsilon$ with $\epsilon = \pm 1$ we get

$$\begin{aligned} & \mu_{a,a,a+\frac{1}{2}}(dx) \\ = \ & C_1(1 - \sqrt{1-x})^{2a-1} x^{-a} (1-x)^{2a-\frac{3}{2}} \ \mathbf{1}_{(0,1)}(x)dx \\ & \mu_{a,a,a-\frac{1}{2}}(dx) \\ = \ & C_2(1 - \sqrt{1-x})^{2a-1} x^{-a} (1-x)^{2a-1} \ \mathbf{1}_{(0,1)}(x)dx \end{aligned}$$

4) A elegant example is

$$\mu_{1,1,1}(dx) = \frac{6}{\pi^2} \log \frac{1}{1-x} \mathbf{1}_{(0,1)}(x) \frac{dx}{x}. \qquad (6)$$

This is the distribution of

$$\frac{U_1}{U_1 + (1-U_1)\frac{U_2}{U_2+(1-U_2)\frac{U_3}{U_3+\dots}}}$$

where $U_1, U_2, \dots$ are independent and uniform on (0,1), a rather unexpected result.

Thus if $(1-X)/X \sim WX$ for X and W independent, the latter with the Pareto distribution function $\Pr(W > x) = 1/(1+x)$, for $x > 0$, then X has distribution (6). Note that here $\mathbb{E}(X^t)$ can be made explicit when $t > -1$ and $t \neq 0$ by expanding $x \mapsto \log(1-x)$ in power series :

$$\mathbb{E}(X^t) = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{t+n-1} dx = \frac{6}{\pi^2} \frac{1}{t} \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+t})$$

When t is a positive integer we get

$$\mathbb{E}(X^t) = \frac{6}{t\pi^2} \sum_{n=1}^{t} \frac{1}{n}. \qquad (7)$$

Mellin's transforms Observe that the Mellin transform of $\mu_{a,a',b}$ can be easily computed from the definition. More generally if $X \sim \mu_{a,a',b}$ we can easily compute $\mathbb{E}(X^t(1-X)^s)$ and from this Theorem 1.1 will give us an other expression of the Mellin transform of X :

Corollary 1.4. For real t and s the integral $I = \int_0^1 x^t(1-x)^s \mu_{a,a',b}(dx)$ converges if and only if $t > -a$ and $s > -\min(a', b)$. In this case

$$I = C\frac{\Gamma(a+t)\Gamma(b+s)}{\Gamma(a+b+t+s)}\,{}_3F_2(a+t, a, b; a+b+t+s, a+a'; 1) \tag{8}$$

Furthermore for $t > -a$

$$\int_0^1 x^t \mu_{a,a',b}(dx) = \frac{{}_3F_2(a', a'-t, b; a'+b, a+a'; 1)}{{}_3F_2(a', a', b; a'+b, a+a'; 1)}. \tag{9}$$

Proof. We skip the easy first part. For the second part, we consider the two independent random variables $X \sim \mu_{a,a',b}$ and $W' \sim \beta^{(2)}_{b,a'}$. Theorem 1.1 says that $X' = (1 + XW')^{-1} \sim \mu_{a',a,b}$. Since $\frac{1-X'}{X'} = XW'$ we write for $t > a$

$$\mathbb{E}(X^t) = \frac{1}{\mathbb{E}((W')^t)} \mathbb{E}((\frac{1 - X'}{X'})^t).$$

We now apply the first part of the proposition by replacing (a, a', b, t, s) by $(a', a, b, -t, t)$ and we get the result 9 for $-\min(b, a) < t < a'$. Extension to a larger interval is standard.

Explanation by Thomae's formula

The last equality of Cor 1.4 has been obtained as a consequence of Theorem 1.1 and is actually equivalent to it. Thus, doing $s = 0$ in 8 our main result is equivalent to

$$C\frac{\Gamma(a+t)\Gamma(b)}{\Gamma(a+b+t)}\,{}_3F_2(a+t,a,b;a+b+t,a+a';1)$$
$$= \frac{{}_3F_2(a',a'-t,b;a'+b,a+a';1)}{{}_3F_2(a',a',b;a'+b,a+a';1)}$$

The Thomae's theorem gives an other proof : Consider the coefficients occurring in ${}_3F_2(...;1)$ in the preceding formula. Let us compute the two vectors $(a+t, a, b; a+b+t, a+a')3A^{-1}$ and $(a', a'-t, b; a'+b, a+a')3A^{-1}$. We get respectively

$$\begin{bmatrix} 2t-b+a+a' \\ -t-b+a+a' \\ -t+2b-2a+a' \\ 2t+2b+a-2a' \\ -t-b+a+a' \end{bmatrix} \quad \begin{bmatrix} 2t-b+a+a' \\ -t-b+a+a' \\ 2t+2b+a-2a' \\ -t+2b-2a+a' \\ -t-b+a+a' \end{bmatrix}$$

and we observe that the second vector is a permutation of the first (the transposition (3,4)). We omit the details about the gamma factors.

Injectivity of $(a, a', b) \mapsto \mu_{a,a',b}$

Proposition 1.6. If (a, a', b) and (a_1, a'_1, b_1) in $(0, \infty)^3$ are such that $\mu_{a,a',b} = \mu_{a_1,a'_1,b_1}$ then $(a, a', b) = (a_1, a'_1, b_1)$.

Proof. We denote by C_1 the constant corresponding to (a_1, a'_1, b_1). We see easily that $a = a_1$ and that $C = C_1$ since when $x \to 0$ of the density of $\mu_{a,a',b} = \mu_{a_1,a'_1,b_1}$ when $x \to 0$ is equivalent to $Cx^{a-1} = C_1 x^{a_1-1}$. Thus writing the equality of densities leads to

$$(1-x)^b \, {}_2F_1(a, b; a+a'; x) = (1-x)^{b_1} \, {}_2F_1(a, b_1; a+a'_1; x)$$

for all $x \in (0, 1)$. Now use a classical formula to obtain

$${}_2F_1(a', b; a+a'; -x/(1-x)) = {}_2F_1(a'_1, b_1; a+a'_1; -x/(1-x))$$

Finally observe that if for z in a neighborhood of 0 we have

$${}_2F_1(\alpha, \beta; \gamma; z) = {}_2F_1(\alpha_1, \beta_1; \gamma_1; z)$$

then watching the coefficients of z, z^2 and z^3 will give

$$\frac{\alpha\beta}{\gamma} = \frac{\alpha_1\beta_1}{\gamma_1}$$
$$\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} = \frac{\alpha_1(\alpha_1+1)\beta_1(\beta_1+1)}{\gamma_1(\gamma_1+1)}$$
$$\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}$$
$$= \frac{\alpha_1(\alpha_1+1)(\alpha_1+2)\beta_1(\beta_1+1)(\beta_1+2)}{\gamma_1(\gamma_1+1)(\gamma_1+2)}$$

From this we get $(\alpha, \beta, \gamma) = (\alpha_1, \beta_1, \gamma_1)$ or $(\beta_1, \alpha_1, \gamma_1)$. Here are the details : introduce the numbers

$$\lambda_0 = \frac{\alpha\beta}{\gamma},\ \lambda_1 = \frac{(\alpha+1)(\beta+1)}{\gamma+1},$$

$$\lambda_2 = \frac{(\alpha+2)(\beta+2)}{\gamma+2}.$$

Clearly these numbers do not change when replacing (α, β, γ) by $(\alpha_1, \beta_1, \gamma_1)$. We deduce from this that

$$\gamma(2\lambda_1 - \lambda_0 - \lambda_2) + 2 + \lambda_1 - 2\lambda_2 = 0,$$

$$\alpha + \beta = (\gamma+1)\lambda_1 - \gamma\lambda_0 - 1,\ \alpha\beta = \gamma\lambda_0.$$

Thus the knowledge of $(\lambda_0, \lambda_1, \lambda_0)$ gives the knowledge of γ and the knowledge of the pair $\{\alpha, \beta\}$.

We apply this to $\alpha = a'$, $\beta = b$, $\gamma = a + a' = a + \alpha$ etc and we get easily $a' = a_1$ and $b = b_1$. The proposition is proved.

Image of $\mu_{a,a',b}$ by $x \mapsto 1-x$. We investigate here the question : when $\mu_{a,a',b}$ is symmetric with respect to 1/2 ? The answer is in Corollary 1.9 below. We show first a lemma :

Lemma 1.7. Let (α, β, γ) and $(\alpha_1, \beta_1, \gamma_1)$ be real numbers such that $\gamma > 0$ and $\gamma_1 > 0$. Consider the two functions y and y_1 on $(0,1)$ defined by

$$y(x) = {}_2F_1(\alpha, \beta; \gamma; 1-x), \;\; y_1(x) = {}_2F_1(\alpha_1, \beta_1; \gamma_1; x)$$

and assume that $Dy = y_1$ for some $D \neq 0$. Then y_1 is the constant 1, that is $\alpha\beta = \alpha_1\beta_1 = 0$.

Proof. The functions y and y_1 are solutions of the differential equations

$$x(1-x)y_1''(x) +$$
$$(\gamma_1 - (\alpha_1 + \beta_1 + 1)x)y_1'(x) - \alpha_1\beta_1\, y_1(x) = 0$$
$$x(1-x)y''(x) +$$
$$(-\gamma + \alpha + \beta + 1 - (\alpha + \beta + 1)x)y'(x) - \alpha\beta\, y(x) = 0$$

Since $Dy = y_1$ multiplying both sides of the first by D we get another differential equation for y_1 which subtracted from the second yields $(A - Bx)y_1'(x) = Cy_1(x)$ where A, B, C are suitable constants. Remember that $y_1(0+) = 1$ by definition. Now suppose that y_1 is not constantly equal to 1. Then $C \neq 0$ and $(A, B) \neq (0, 0)$. If $B = 0$ then $y_1(x) = e^{Cx/A}$, which is not a hypergeometric function as seen by the power series expansion. If $A = 0$ then $y_1(0+) = 1$ is impossible. Thus without loss of generality we assume $A = 1$ and we get $y_1(x) = (1-Bx)^{-a}$ with $a = C/B \neq 0$. Expanding $x \mapsto (1-Bx)^{-a}$ in power series $\sum_{n=0}^{\infty} \frac{(a)_n}{n!} (B)^n x^n$ shows that it is a hypergeometric function only if $B = 1$. To conclude the proof

$$ {}_2F_1(\alpha, \beta; \gamma; x) = y(1-x) = D^{-1} y_1(x) = D^{-1} x^{-a} \tag{10} $$

which is impossible since the limit of function at the r.h.s. on 0 is not finite for $a > 0$ and is zero if $a < 0$. □

Theorem 1.8. Let $X \sim \mu_{a,a',b}$ Then there exists (a_1, a'_1, b_1) in $(0,\infty)^3$ such that $X_1 = 1 - X \sim \mu_{a_1,a'_1,b_1}$ if and only if $a_1 = a'$, $a'_1 = a_1$ and $b_1 = b = a + a'$.

Corollary 1.9. Let $X \sim \mu_{a,a',b}$ and $X' \sim \mu_{a',a,b}$. Then $X \sim 1 - X'$ if and only if $b = a + a'$ and $X \sim 1 - X$ if and only if $a = a' = b/2$.

Proof of Theorem 1.8. The if part is obvious since $\mu_{a,a',a+a'} = \beta_{a,a'}$. Conversely, suppose that $X_1 \sim 1 - X$ with $X \sim \mu_{a,a',b}$ and $X_1 \sim \mu_{a_1,a'_1,b_1}$. For all $x \in (0,1)$ since the densities are continuous we have :

$$\begin{aligned}&C_1 x^{a_1-1}(1-x)^{b_1-1}\ {}_2F_1(a_1, b_1; a_1 + a'_1; x)\\ &= C(1-x)^{a-1}x^{b-1}\ {}_2F_1(a, b; a + a'; 1-x)\end{aligned}$$

or

$$\begin{aligned}&C_1(1-x)^{b_1-a}\ {}_2F_1(a_1, b_1; a_1 + a'_1; x)\\ &= Cx^{b-a_1}\ {}_2F_1(a, b; a + a'; 1-x)\end{aligned} \tag{11}$$

Introduce $d = a + a' - b$ and $d_1 = a_1 + a'_1 - b_1$. Applying 4 to both sides of 11 (specifically $(\alpha, \beta, \gamma, z) = (a_1, b_1; a_1 + a'_1; x)$ and $(a, b; a + a'; 1 - x)$) we get

$$C_1(1-x)^{a'_1 - a} \, {}_2F_1(a'_1, d_1; a_1 + a'_1; x) = Cx^{a' - a_1} \, {}_2F_1(a', d; a + a'; 1 - x). \tag{12}$$

Note for all $t \in \mathbb{R}$ we have

$$\mathbb{E}((\frac{X_1}{1 - X_1})^t) = \mathbb{E}((\frac{1 - X}{X})^t).$$

Now application of Corollary 1.4 shows that the left hand side is finite if and only if

$$-\min(a'_1, b) < t < a_1$$

and that the right hand side is finite if and only if $-a < t < \min(a', b)$. This leads to

$$a_1 = \min(a', b), \quad a = \min(a'_1, b_1)$$

It remains to discuss several cases :

If $a_1 = a' \leq b$ and $a = a'_1 \leq b_1$ then one applies Lemma 1.7 to 12 and we get $d = d_1 = 0$ as desired.

If $a_1 = b < a'$ and $a = b_1 < a'_1$ then one applies Lemma 1.7 to 11 and we get the absurd statement $ab = a_1 b_1 = 0$.

If $a_1 = a' \leq b$ and $a = b_1 < a'_1$ then mixing 11 and 12 we get

$$C_1 \, {}_2F_1(a_1, b_1; a_1 + a'_1; x) = C \, {}_2F_1(a', d; a + a'; 1 - x)$$

leading via Lemma 1.7 to the absurd statement $a_1 b_1 = 0$. The case $a = a'_1 \leq b_1$ and $a_1 = b < a'$ is similar. The theorem is proved.