A question of Lutz Mattner.

Dear Gérard,

here is a question (Question 1 below) which occurred to me when teaching examples of explicitly obtainable convolutions of laws on the real line. I haven't thought very seriously about it yet, because the answer should be well known if "yes" and probably trivial if "no", and I thought you are likely to know it. Anyway, here it is :

Call a subset $\mathcal{P}$ of the set of all laws on the real line $\mathbb{R}$ d-parametric, if it is the image of an injective and continuous function on an open subset of $\mathbb{R}^d$. (Here "continuous" refers to the Levy topology on the set of laws.)

Call $\mathcal{P}$ a semigroup, if it is closed under convolution.

Let now $\mathcal{P}$ be 2-parametric as well as a semigroup.

Question 1 : Is then $\mathcal{P}$ necessarily a subset of a set of the form

$$\{P_1 * P_2 : P_1 \in \mathcal{P}_1, P_2 \in \mathcal{P}_2\}$$

with 1-parametric semigroups $\mathcal{P}_1$ and $\mathcal{P}_2$?

Question 2 : If the answer to Question 1 is "no", are there any NATURAL counterexamples ?

Best regards, Lutz (*January 22d, 2005*)

Gérard Letac : Answer to Lutz's question :

A little dent.

After Paul Lévy, consider the polynomial P

$$P(z) = 1 + z - z^2/2 + z^3 + z^4.$$

It has the property that P^2 and P^3 have non negative coefficients. This implies that P^n has non negative coefficients for $n \geq 2$ since every $n \geq 2$ can be written $n = 2a+3b$. Thus if we consider the entire series

$$e^{xP(z)} = \sum_{n=0}^{\infty} a_n(x) z^n$$

then for $x > 0$ all $a_n(x)$ are non negative except possibly for $a_2(x) = \frac{1}{2}x(x-1)$. Thus for $x \geq 1$ there exists a random variable ζ_x concentrated on the set $\mathbb{N}$ of non negative integers such that

$$\mathbb{E}(z^{\zeta_x}) = e^{x(P(z)-P(1))}.$$

Now for $y > 0$ consider the Poisson rv ξ_y independent of ζ_x with $\mathbb{E}(z^{\xi}) = e^{y(z-1)}$. The generating function of $\xi_y + \zeta_x$ is therefore

$$\mathbb{E}(z^{\xi_y+\zeta_x}) = e^{y(z-1)+x(P(z)-P(1))} = e^{-y-xP(1)} \sum_{n=0}^{\infty} a_n(x,y) z^n$$

Now let us wonder for which values of $(x, y) \in (0, \infty)^2$ the above function of z is the generating function of a distribution $\mu_{x,y}$ on $\mathbb{N}$. Denote by J the set of such (x, y). Of course the construction shows that $[1, \infty) \times (0, \infty) \subset J$ but can J be bigger ?

J is nothing but the set of $(x, y) \in (0, \infty)^2$ such that $a_n(x, y) \geq 0$ for all $n \geq 0$. But it is easily seen that $a_n(x, y) \geq 0$ for all $(x, y) \in (0, \infty)^2$ and for all $n \neq 2$. Now

$$a_2(x,y) = \frac{1}{2}((y + x)^2 - x)$$

We now draw the parabola $a_2(x, y) = 0$ in the plane. It cuts the x axis in 0 and 1. We draw the domain $a_2(x, y) < 0$. We intersect this set with $(0, \infty)^2$ obtaining a little dent J_1. The set J is $(0, \infty)^2 \setminus J_1$ and is a semigroup. The family

$$\{\mu_{x,y};\ (x,y) \in J\}$$

provides the elementary counter example that you need.

Comments. Actually behind this example there is a little game. Let μ be a distribution on the real line having Laplace transform and which is not infinitely divisible. Denote by Λ its Jorgensen set, that is the set of $x > 0$ such that there exists a probability measure λ_x such that its Laplace transform satisfies $L_{\lambda_x} = L_\lambda^x$. Now consider an infinitely divisible distribution ν on the real line, also having Laplace transform. We denote by ν_y the distribution such that $L_{\nu_y} = L_\nu^y$. We denote by J the set of $(x, y) \in (0, \infty)^2$ such that there exists a probability $\mu_{x,y}$ on the real line with Laplace transform $L_\nu^y L_\lambda^x$. Clearly J contains $\Lambda \times (0, \infty)$ but it can be bigger. A good tool of investigation is the introduction of the function f defined on $(0, \infty)$ by

$$f(x) = \inf\{y;\ (x,y) \in J\},$$

with the convention that $f(x) = \infty$ if this set is empty. The knowledge of this function f is the knowledge of the additive semigroup J since $J = \{(x, y);\ y \geq f(x)\}$. We have $f(x) = 0$ if and only if $x \in \Lambda$.

The preceding example has given $f(x) = \sqrt{x} - x$ for $0 \leq x \leq 1$ and $f(x) = 0$ for $x \geq 1$. Other examples can be more challenging. For instance in Letac-Malouche-Maurer, PAMS 2000 vol 130 pages 2107-2114 we have considered the case where μ is Bernoulli with expectation $< 1/2$ and ν is negative binomial. The function f is never ∞, is made of elegant festoons -an infinite number of smaller and smaller little dents- and is explicitly computed. An other question, what is f if λ is Bernoulli and ν is normal ?